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DIXMIER SETS, NUMERICAL RANGES, COMMUTATION AND SPLITTING THEOREMS

BY ŞERBAN STRĂTILĂ*) AND LÁSZLÓ ZSIDÓ**)

ABSTRACT

The aim of this expository article is to connect, in the simplest case – that of factors – and with the simplest arguments, works done in early 1970 years focussing on the Dixmier sets ([C], [S-Z2], [H2]) and algebraic reduction theory ([S-Z1], [S-Z2], [H1], [H2]) with commutation theorems and splitting theorems for operator algebras obtained in the last years ([G-K], [S-Z3]). A more detailed discussion is given in Section 5°.

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1° TERMINOLOGY AND NOTATION

Let $M \subset B(H)$ be an <u>infinite</u> factor, $Z := \mathbb{C} \cdot 1_M$ its center, $U(M)$ the unitaries in M, $P(M)$ the projections in M. The weak (resp. strong) operator topology on $B(H)$ is denoted as the wo (resp. so) topology and the natural („ultra-weak") topology on M, which is preserved by $*$-isomorphisms, is denoted as the w-topology.

Let $B(M) = \{$bounded linear operators on $M\}$ endowed with the w_p-topology (pointwise w-convergence). The usual application of the TIKHONOV's Theorem shows that

the unit ball $B(M)_1$ is w_p-compact

Consider the „DIXMIER maps":

$$\mathcal{F} := \{ f \in B(M) ; \ f(x) = \sum_{k=1}^{m} \alpha_k u_k x u_k^*, \ \alpha_k \geq 0, \sum_{k=1}^{m} \alpha_k = 1, \ u_k \in U(M) \}$$

It is clear that

$f \in \mathcal{F} \Rightarrow f$ linear, positive, $\|f\| = 1$, and f is (w, wo, so)-continuous

$f, g \in \mathcal{F} \Rightarrow f \circ g \in \mathcal{F}$

$f \in \mathcal{F} \Rightarrow \mathcal{F}(f(x)) \subset \mathcal{F}(x)$, $(\forall)\ x \in M$.

The set $\mathcal{T} := \overline{\mathcal{F}}^{w_p} \subset B(M)_1$ is w_p-compact

and

$f \in \mathcal{T} \Rightarrow f$ linear, positive, $\|f\| = 1$

$f \in \mathcal{T} \Rightarrow f(z) = z \quad (\forall)\ z \in Z$.

$f \in \mathcal{F}, g \in \mathcal{T} \Rightarrow f \circ g \in \mathcal{T}$, and hence $f, g \in \mathcal{T} \Rightarrow f \circ g \in \mathcal{T}$

Recall the definition of the DIXMIER sets $\mathcal{C}(x)$ and $\mathcal{K}(x)$ for $x \in M$ and the AVERAGING DIXMIER THEOREM:

$$\mathcal{C}(x) := \overline{\mathcal{F}(x)}^{w} \cap Z \ \supset \ \mathcal{K}(x) := \overline{\mathcal{F}(x)}^{NORM} \cap Z \neq \emptyset \qquad (\forall)\ x \in M$$

These are <u>nonvoid</u> closed convex sets of Z (that is of $\mathbb{C}$).

In what follows we deal separately but similarly with these two sets, identifying them with certain „numerical ranges".

2° THE SET $\mathcal{C}(x)$

2.1. $\mathcal{C}(x) = \mathcal{T}(x) \cap Z$

<u>Proof</u>. If $z \in \mathcal{C}(x)$, then $z \in Z$ and $z = w\text{-}\lim f_n(x)$ with $f_n \in \mathcal{F} \subset \mathcal{T}$. As $\mathcal{T}$ is w_p-compact, by passing to a subnet we may assume that $f_n \xrightarrow{w_p} f \in \mathcal{T}$, and then $z = f(x) \in \mathcal{T}(x)$.

If $z \in \mathcal{T}(x) \cap Z$, then $z \in Z$ and $z = f(x)$ for some $\mathcal{T} \ni f = w_p\text{-}\lim f_n$ with $f_n \in \mathcal{F}$, hence $z = w\text{-}\lim f_n(x) \in \overline{\mathcal{F}(x)}^w$

2.2. $(\forall)\, x_0, x_1, \dots, x_N \in M$, $(\forall)\, z_0 \in \mathcal{C}(x_0)$, $(\exists)\, f \in \mathcal{T}$ such that:

$$f(x_0) = z_0 \quad \text{and} \quad f(x_k) \in \mathcal{C}(x_k) \quad (\forall)\, k = 1, \dots, N$$

<u>Proof</u>. Choose inductively $g_0, g_1, \dots, g_N \in \mathcal{T}$ such that $g_0(x_0) = z_0$, $g_1 \circ g_0(x_1) = z_1 \in \mathcal{C}(x_1), \dots, g_N \circ \dots \circ g_1 \circ g_0(x_N) = z_N \in \mathcal{C}(x_N)$; this is possible by the DIXMIER THEOREM and 2.1. Then $f = g_N \circ \dots \circ g_1 \circ g_0 \in \mathcal{T}$ will fulfill the statement.

2.3. $\mathcal{C}(x) = \{ f(x);\ f \in \mathcal{T},\ f(y) \in \mathcal{C}(y)\ (\forall)\, y \in M \}$

<u>Proof</u>. Let $z \in \mathcal{C}(x)$ and $y_1, \dots, y_N \in M$. Then the sets

$$\mathcal{T}(z; y_1, \dots, y_N) := \{ f \in \mathcal{T};\ f(x) = z,\ f(y_k) \in \mathcal{C}(y_k)\ (\forall)\, k = 1, \dots, N \}$$

are non-empty (by 2.2), w_p-compact and decreasing (as the sets $\{y_1, \dots, y_N\}$ increase), hence

$$\bigcap_{y_1, \dots, y_N} \mathcal{T}(z; y_1, \dots, y_N) \neq \emptyset$$

and this proves the statement.

2.4. (1) $\mathcal{C}(x_1 + x_2) \subset \mathcal{C}(x_1) + \mathcal{C}(x_2)$, hence $\mathcal{C}(x_1) = \{0\}, \mathcal{C}(x_2) = \{0\} \Rightarrow \mathcal{C}(x_1 + x_2) = \{0\}$

(2) $x_1 \leq x_2$, $z_1 \in \mathcal{C}(x_1) \Rightarrow (\exists)\, z_2 \in \mathcal{C}(x_2)$, $z_1 \leq z_2$, hence

$$0 \leq x_1 \leq x_2,\ \mathcal{C}(x_2) = \{0\} \Rightarrow \mathcal{C}(x_1) = \{0\}$$

(3) $\| x_1 - x_2 \| < \varepsilon$, $z_1 \in \mathcal{C}(x_1) \Rightarrow (\exists)\, z_2 \in \mathcal{C}(x_2)$, $\| z_1 - z_2 \| < \varepsilon$, hence

$$\text{distance}(\mathcal{C}(x_1), \mathcal{C}(x_2)) \leq \| x_1 - x_2 \|$$

Proof. (1) Let $z \in \mathcal{C}(x_1+x_2)$. Using 2.3, there is $f \in \mathcal{T}$ such that $f(x_1+x_2)=z$, $f(x_1)=z_1 \in \mathcal{C}(x_1)$, $f(x_2)=z_2 \in \mathcal{C}(x_2)$. Then

$$z = f(x_1+x_2) = f(x_1)+f(x_2) = z_1+z_2 \in \mathcal{C}(x_1)+\mathcal{C}(x_2).$$

(2) Using 2.3, there is $f \in \mathcal{T}$ such that $f(x_1)=z_1$, $f(x_2)=z_2 \in \mathcal{C}(x_2)$. Then

$$z_1 = f(x_1) \leq f(x_2) = z_2 \in \mathcal{C}(x_2).$$

(3) Using 2.3, there is $f \in \mathcal{T}$ such that $f(x_1)=z_1$, $f(x_2)=z_2 \in \mathcal{C}(x_2)$. Then

$$\|z_1-z_2\| = \|f(x_1)-f(x_2)\| = \|f(x_1-x_2)\| \leq \|f\|\,\|x_1-x_2\| \leq \|x_1-x_2\| < \varepsilon$$

2.5. (1) $\mathcal{C}(f(x)) \subset \mathcal{C}(x)$ $(\forall)\ f \in \mathcal{T}$

(2) $\mathcal{C}(uxu^*) = \mathcal{C}(x)$ $(\forall)\ u \in U(M)$

These facts are obvious since $\mathcal{T} \circ \mathcal{T} \subset \mathcal{T}$ and $x = u^*(uxu^*)u$.

From 2.4 and 2.5 we see that

$M \ni x \longmapsto \mathcal{C}(x)$ is a norm-continuous „multi-valued trace" on M

2.6. $\tilde{I} := \{x \in M ;\ \mathcal{C}(x^*x)=\{0\}\} \subset M$ is a norm-closed two-sided ideal in M.

Proof. For $x, y \in \tilde{I}$ we have $(x+y)^*(x+y) \leq 2(x^*x+y^*y)$, hence $x+y \in \tilde{I}$ follows by using 2.4.(1) and (2). If $u \in U(M)$, then $(ux)^*(ux) = x^*x$, hence $ux \in \tilde{I}$, and $(xu)^*(xu) = u^*x^*xu$, hence $xu \in \tilde{I}$ by using 2.5.(2). Finally, let $x \in \overline{\tilde{I}}$, $\varepsilon > 0$ and $a \in \tilde{I}$ such that $\|x-a\| < \varepsilon$. Using 2.4.(3) we infer that $d(\mathcal{C}(x^*x), 0) < \|x^*x - a^*a\| \longrightarrow 0$ as $\varepsilon \to 0$, hence $\mathcal{C}(x^*x)=\{0\}$ and $x \in \tilde{I}$.

2.7. $e \in P(M)$ finite projection $\Longrightarrow \mathcal{C}(e)=\{0\}$ (Recall: M = infinite factor)

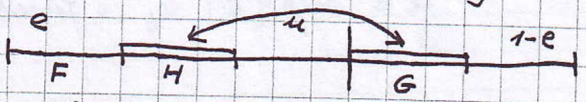
Proof. This is obvious for type III factors since then $e=0$, so we may assume M is semifinite. Then let τ be the (essentially unique) normal semifinite faithful trace on M. Let $\lambda \in \mathcal{C}(e)$ and $f_n \in \mathcal{F}$ such that $\lambda = w\text{-}\lim f_n(e)$. The net $\{\varphi_n(\cdot) = \tau(f_n(e)\,\cdot)\} \subset M_+^*$ is norm-bounded by $\tau(e) < \infty$ (since e is a finite projection), and hence it has a weak*-limit point $\varphi \in M_+^*$, $\|\varphi\| \leq \tau(e)$. Then, for any finite projection $a \in M$ we have

$$\|\varphi\| \geq \varphi(a) = \lim \varphi_n(a) = \lim z(f_n(e)a) = \lambda z(a),$$

hence $\lambda = 0$, since there are finite projections $a \in M$ with $z(a) \longrightarrow \infty$

2.8. $e \in P(M)$ infinite projection $\Rightarrow \mathcal{C}(e) \ni 1$.

Proof. In this case we can write $e = \sum_{n \in A} e_n$, $(1-e) = \sum_{m \in B} f_m$ with $e_n \sim f_m$ $((\forall)\, n, m)$ finite projections and the index set A infinite. For any finite subsets $F \subset A$, $G \subset B$ choose a finite subset $H \subset A$, $H \cap F = \emptyset$, $\operatorname{card}(H) = \operatorname{card}(G)$. There is a unitary $u \in U(M)$ interchanging $\sum_{n \in H} e_n$ with $\sum_{m \in G} f_m$ and acting identically on all other e_n's and f_m's. Then:



$$u e u^* \geq \sum_{n \in F} e_n + \sum_{m \in G} f_m \xrightarrow{w} 1 \text{ when } F \text{ and } G \text{ increase}.$$

This proves the statement.

2.9. From 2.7 and 2.8 it follows that, for a projection $e \in P(M)$ we have

e finite $\Rightarrow \mathcal{C}(e) = \{0\}$, $1-e$ finite $\Rightarrow \mathcal{C}(e) = \{1\}$

e and $(1-e)$ both infinite $\Rightarrow \mathcal{C}(e) = [0,1]$

since, in the last situation, $\mathcal{C}(e) \ni 1$, $\mathcal{C}(1-e) \ni 1$ hence $\mathcal{C}(e) \ni 0$, and $\mathcal{C}(e) \subset [0,1]$ is a convex set.

Now let $I :=$ the norm-closed two-sided ideal in M generated by $\{e \in P(M)$, finite projections$\}$.

Notice that any projection $p \in I$ is finite. Indeed let a be an element in the two-sided ideal generated by $\{e \in P(M)$ finite projections$\}$ such that $\|p - a\| < 1$. Then the left support $l(a)$ of a is a finite projection and $1 - p + a$ is invertible with left support $1 = (1-p) \vee l(a)$. By the parallelogramm rule it follows that $p = 1 - (1-p) \prec l(a)$, hence p is a finite projection

On the other hand, recall that, by spectral theory, any closed two-sided ideal in M is generated by the projections it contains.

2.10. $I = \tilde{I} = \{x \in M ;\ \mathcal{C}(x^*x) = \{0\}\}$

Indeed, 2.7, 2.8 and the above discussion show that I and $\tilde{I}$ contain the same projections – the finite projections.

Let $x = x^* \in M$ with spectrum $\sigma(x) \subset [m, M]$, $m \in \sigma(x)$, $M \in \sigma(x)$ and spectral scale $e_t = \chi_{(-\infty, t]}(x)$, and denote

$$a = \sup\{t;\ e_t \text{ is finite}\} \quad , \quad b = \inf\{t;\ 1 - e_t \text{ is finite}\}$$

2.11. (1) $m \leq a \leq b \leq M$

(2) $\mathcal{C}(x) \subset [a, b]$

<u>Proof</u>. (1) $t < m \Rightarrow e_t = 0$ hence $m \leq a$

$t > M \Rightarrow 1 - e_t = 0$ hence $M \geq b$

$b < t < a \Rightarrow$ both e_t and $1 - e_t$ are finite $\Rightarrow 1 = e_t + (1 - e_t)$ is finite, a contradiction, hence $a \leq b$.

(2) $t < a \Rightarrow e_t$ is finite $\Rightarrow \mathcal{C}(e_t) = \{0\}$; but $0 \leq (x - m)e_t \leq (t - m)e_t$ hence $\mathcal{C}((x - m)e_t) = \{0\}$, hence $\mathcal{C}(xe_t) = \{0\}$

$s > b \Rightarrow \mathcal{C}(x(1 - s)) = \{0\}$, similarly, hence

$t < a \leq b < s \Rightarrow \mathcal{C}(x) \subset \mathcal{C}(x(e_s - e_t))$.

Now $t(e_s - e_t) \leq x(e_s - e_t) \leq s(e_s - e_t)$ and $1 - (e_s - e_t) = (1 - e_s) + e_t$ is a finite projection, hence $\mathcal{C}(e_s - e_t) = \{1\}$. It follows that

$$\mathcal{C}(x) \subset \mathcal{C}(x(e_s - e_t)) \subset [t, s].$$

As $t < a \leq b < s$ were arbitrary, we conclude $\mathcal{C}(x) \subset [a, b]$.

2.12. $\mathcal{C}(x) = [a,b]$

<u>Proof.</u> a) We first show that, for $x = x^* \in M$, the following holds:

(1) $(\forall)\ \delta > 0\ (\exists)\ e \in P(M)$, e infinite, $(1-e)$ infinite, $\|xe\| < \delta \implies 0 \in \mathcal{C}(x)$.

Notice that $exe + (1-e)x(1-e) = \frac{1}{2}[(2e-1)x(2e-1) + x] \in \mathcal{F}(x)$.

Write $e = \sum_{n \in A} e_n$, $1-e = \sum_{m \in B} f_m$ with $e_n \sim f_m$ $((\forall)\, m,n)$ finite projections and both index sets infinite. Let $F \subset A$, $G \subset B$ be arbitrary finite sets and $H \subset A$, $H \cap F = \emptyset$, $\mathrm{card}(H) = \mathrm{card}(G)$. There is a unitary $u \in M$ interchanging $\sum_{n \in H} e_n$ with $\sum_{m \in G} f_m$ and acting identically on all other e_n's and f_m's. Then

$$\mathcal{F}(x) \ni u^* exe u + u^*(1-e)x(1-e)u$$

where $\|u^* exe u\| \leq \|xe\| < \delta$ and $|u^*(1-e)x(1-e)u| \leq \|x\|\, u^*(1-e)u$. But $u^*(1-e)u = \sum_{n \in H} e_n + \sum_{m \in B \setminus G} f_m \leq \sum_{n \in A \setminus F} e_n + \sum_{m \in B \setminus G} f_m \xrightarrow{w} 0$ as F and G increase. This proves that indeed $0 \in \mathcal{C}(x)$.

b) Let $\delta > 0$ and $e = e_{a+\delta} - e_{a-\delta}$: $a-\delta \quad a \quad a+\delta$. By the definition of a, both e and $(1-e)$ are infinite projections, and we clearly have $\|(x-a)e\| < 2\delta$. By the first part of the proof, this shows that $a \in \mathcal{C}(x)$. Similarly, $b \in \mathcal{C}(x)$. Hence $[a,b] \subset \mathcal{C}(x)$ since $\mathcal{C}(x)$ is convex, and, with 2.11, $\mathcal{C}(x) = [a,b]$.

We introduce some more notation and terminology:

$S(M) := \{\varphi \in M_+^* \,;\, \|\varphi\| = 1\}$ = the states of M

For any norm-closed two-sided ideal $J \subset M$ and any $x \in M$:

$\sigma_J(x) :=$ the (J-essential) spectrum of $x/_J$ in $M/_J$

$\overline{V}_J(x) := \{\varphi(x) \,;\, \varphi \in S(M), \varphi|_J = 0\}$, the closed J-essential numerical range of $x \in M$.

Then: $\sigma_J(x) \subset \overline{V}_J(x)$, $(\forall)\ x \in M$

$\overline{V}_J(x) = \mathrm{co}\, \sigma_J(x)$, $(\forall)\ x = x^* \in M$ (even $(\forall)\ x \in M$ normal)

[These are well known standard facts (see e.g. HALMOS - A Hilbert space problem book). We briefly recall the arguments:

– for x normal, $\sigma_J(x) \subset \overline{V}_J(x)$ since any „character" $t \in \sigma_J(x)$ can be extended to a state $\varphi \in S(M)$, $\varphi|_J = 0$ and hence $\varphi(x-t) = 0$

– for x arbitrary: $t \in \sigma(x) \Rightarrow$ either t is an eigenvalue of x, or $\bar{t}$ is an eigenvalue of x^*, or $(\exists)$ vectors ξ_n, $\|\xi_n\| = 1$, such that $((x-t)\xi_n | \xi_n) \longrightarrow 0$ and take φ a weak limit of the vector states ω_{ξ_n}.

– the equality for x normal is obtained using the functional model of x and the Hahn-Banach theorem ... both sets are in the same half-spaces.]

With these notation, 2.12 rewrites as

$$\mathcal{C}(x) = co\, \sigma_I(x), \quad (\forall)\ x = x^* \in M$$

that is

$$\mathcal{C}(x) = \overline{V}_I(x), \quad (\forall)\ x = x^* \in M.$$

2.13. THEOREM. For any infinite factor M

$$\{f \in \mathcal{T};\ f(y) \in \mathcal{C}(y)\ (\forall)\ y \in M\} = \{\varphi \in S(M);\ \varphi|_I = 0\}$$

<u>Proof</u>. Let $f \in \mathcal{T}$ such that $f(y) \in \mathcal{C}(y)$ $(\forall)$ $y \in M$. Let $y \in I$. Then $y^*y \in I$, hence $f(y^*y) \in \mathcal{C}(y^*y) = \{0\}$ by 2.10 and hence $f(y) = 0$ by the Cauchy-Schwarz inequality.

To prove the converse inclusion assume the contrary holds: there is $\varphi \in S(M)$, $\varphi|_I = 0$ which does not belong to the left hand side of the stated equality. Then, by HAHN-BANACH, there is $a = a^* \in M$ such that:

$$\varphi(a) > \sup\{f(a);\ f \in \mathcal{T},\ f(y) \in \mathcal{C}(y)\ (\forall)\ y \in M\} = \sup\{\lambda;\ \lambda \in \mathcal{C}(a)\}$$

the last equality being due to 2.3. However

$$\varphi(a) \in \overline{V}_I(a) = \mathcal{C}(a)$$

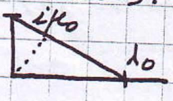
and this contradiction proves the desired result.

2.14. COROLLARY. $\mathcal{C}(x) = \overline{V}_I(x)$ $(\forall)$ $x \in M$

This follows obviously from 2.13, 2.3 and the definition of $\overline{V}_I(x)$. Here is an alternate proof without using 2.13. The inclusion $\mathcal{C}(x) \subset \overline{V}_I(x)$ is obtained with the above argument, i.e. the inclusion „$\subset$" in 2.13. Now, assume $0 \in \overline{V}_I(x)$ and let $\varphi \in S(M)$, $\varphi|_I = 0$, $\varphi(x) = 0$. Let $\lambda_0 \in \mathcal{C}(x)$ such that $|\lambda_0| = \inf\{|\lambda|\,;\, \lambda \in \mathcal{C}(x)\}$. By using a rotation, we may assume $\lambda_0 \geq 0$. We have:

$$\varphi(\mathrm{Re}\, x) = \frac{1}{2}(\varphi(x) + \varphi(x^*)) = \frac{1}{2}(\varphi(x) + \overline{\varphi(x)}) = 0,$$

hence $0 \in \overline{V}_I(\mathrm{Re}\, x) = \mathcal{C}(\mathrm{Re}\, x)$ by 2.12. Thus, there are $f_n \in \mathcal{F}$ with $f_n(\mathrm{Re}\, x) \xrightarrow{w} 0$, $f_n(\mathrm{Im}\, x) \xrightarrow{w} \mu_0 \in \mathcal{C}(\mathrm{Im}\, x)$, and then $f_n(x) \xrightarrow{w} i\mu_0$, hence $i\mu_0 \in \mathcal{C}(x)$. As also $\lambda_0 \in \mathcal{C}(x)$ and as $\mathcal{C}(x)$ is convex, the whole segment $[\lambda_0, i\mu_0] \subset \mathcal{C}(x)$. If $\lambda_0 > 0$ this obviously contradicts the definition of λ_0:



Hence $\lambda_0 = 0$. We have thus proved: $0 \in \overline{V}_I(x) \Rightarrow 0 \in \mathcal{C}(x)$. By translation this means that $\overline{V}_I(x) \subset \mathcal{C}(x)$.

2.15. COROLLARY. If M is a type $\underline{\mathrm{III}}$ factor, then

$$S(M) \subset \mathcal{T}$$

i.e. any state of M can be w_p-approximated by DIXMIER maps.

This follows cleary from the Theorem 2.13 since if M is type $\underline{\mathrm{III}}$ there are no non-zero finite projections, hence $I = \{0\}$.

Notice that there are sufficiently many <u>normal</u> states in $S(M)$. On the opposite, if M is a semifinite infinite factor, then the ideal I is w-dense in M, hence there are no normal non-zero states in $S(M)$ with restiction to I equal to zero.

3° THE SET $\mathcal{K}(x)$

3.1. $(\forall)\ x_1,\dots,x_N \in M$, $(\exists)\ z_1,\dots,z_N \in Z$, $(\exists)\ \{f_n\} \subset \mathcal{F}$ such that:

$$\| f_n(x_k) - z_k \| \xrightarrow[n\to\infty]{} 0 \qquad (\forall)\ k = 1,\dots,N$$

Proof. Using the DIXMIER THEOREM we choose inductively $g_n \in \mathcal{F}$ and $z_1^n,\dots,z_N^n \in Z$ such that:

$$\| g_n \circ g_{n-1} \circ \cdots \circ g_1(x_k) - z_k^n \| < 2^{-n} \qquad (\forall)\ k = 1,\dots,N$$

With $f_n = g_n \circ g_{n-1} \circ \cdots \circ g_1$ we have $\| f_n(x_k) - z_k^n \| < 2^{-n}$ and it follows that:

$$\| z_k^{n+1} - z_k^n \| \leq \| z_k^{n+1} - f_{n+1}(x_k) \| + \| g_{n+1} \circ f_n(x_k) - z_k^n \| < 2^{-n+1}.$$

Thus all $\{z_k^n\}_n$ are Cauchy sequences and therefore convergent:

$(\forall)\ k = 1,\dots,N$ $(\exists)\ z_k \in Z$ such that $\| z_k^n - z_k \| \xrightarrow[n\to\infty]{} 0$.

It follows that $\| f_n(x_k) - z_k \| \xrightarrow[n\to\infty]{} 0$.

3.2. $(\forall)\ x_0, x_1,\dots,x_N \in M$, $(\forall)\ z_0 \in \mathcal{K}(x_0)$, $(\exists)\ \{f_n\} \subset \mathcal{F}$ such that:

$$f_n(x_k) \xrightarrow[n\to\infty]{NORM} z_k \in \mathcal{K}(x_k) \qquad (\forall)\ k = 0,1,\dots,N.$$

Proof. Using 3.1 one choses $g_n, h_n \in \mathcal{F}$ such that

$$\| g_n(x_0) - z_0 \| < 2^{-n} \quad \text{and} \quad \| h_n \circ g_n(x_k) - z_k \| < 2^{-n} \qquad (\forall)\ k = 1,\dots,N$$

for some $z_k \in \mathcal{K}(x_k)$, $(k = 1,\dots,N)$. Then the statement is satisfied with $f_n = h_n \circ g_n$.

3.3. $\mathcal{K}(x) = \{ f(x);\ f \in \mathcal{T},\ f(y) \in \mathcal{K}(y)\ (\forall)\ y \in M \}$

The proof is completely similar to the proof of 2.3.

3.4. (1) $\mathcal{K}(x_1 + x_2) \subset \mathcal{K}(x_1) + \mathcal{K}(x_2)$, hence $\mathcal{K}(x_1) = \{0\}, \mathcal{K}(x_2) = \{0\} \Rightarrow \mathcal{K}(x_1 + x_2) = \{0\}$

(2) $x_1 \leq x_2$, $z_1 \in \mathcal{K}(x_1) \Rightarrow (\exists)\ z_2 \in \mathcal{K}(x_2)$, $z_1 \leq z_2$, hence

$$0 \leq x_1 \leq x_2,\ \mathcal{K}(x_2) = \{0\} \Rightarrow \mathcal{K}\{x_1\} = \{0\}$$

(3) $\| x_1 - x_2 \| < \varepsilon$, $z_1 \in \mathcal{K}(x_1) \Rightarrow (\exists)\ z_2 \in \mathcal{K}(x_2)$, $\| z_1 - z_2 \| < \varepsilon$, hence

$$\text{distance}(\mathcal{K}(x_1), \mathcal{K}(x_2)) \leq \| x_1 - x_2 \|$$

Proof. A proof completely similar to the proof of 2.4 works. But here we can use also the definition of $\mathcal{K}(x) = \overline{\mathcal{F}(x)}^{NORM} \cap Z$.

For instance, the proof of (1) runs as follows. Let $z \in \mathcal{K}(x_1+x_2)$. By 3.2 there is a sequence $\{f_n\} \subset \mathcal{F}$ such that

$$f_n(x_1+x_2) \xrightarrow[n\to\infty]{NORM} z \quad , \; f_n(x_1) \xrightarrow[n\to\infty]{NORM} z_1 \in \mathcal{K}(x_1),\; f_n(x_2) \xrightarrow[n\to\infty]{NORM} z_2 \in \mathcal{K}(x_2).$$

It follows that $z = z_1 + z_2 \in \mathcal{K}(x_1) + \mathcal{K}(x_2)$ since each f_n is linear. The proofs of (2) and (3) are similar.

3.5. It is obvious that: (1) $\mathcal{K}(f(x)) \subset \mathcal{K}(x)$ $(\forall)\, f \in \mathcal{F}$

(2) $\mathcal{K}(uxu^*) = \mathcal{K}(x)$ $(\forall)\, u \in U(M)$

3.6. $\tilde{J} := \{x \in M ;\, \mathcal{K}(x^*x) = \{0\}\} \subset M$ is a norm-closed two-sided ideal in M

Same proof as for 2.6.

In what follows, in comparison with Section 2°, instead of infinite projections we deal with projections equivalent to 1 in M, $e \sim 1$, and instead of finite projections we deal with projections non equivalent to 1 in M, $e \nsim 1$.

3.7. $e \in P(M)$, $e \nsim 1 \Rightarrow \mathcal{K}(e) = \{0\}$

3.8. $e \in P(M)$, $e \sim 1 \Rightarrow \mathcal{K}(e) \ni 1$

3.9. $e \nsim 1 \Rightarrow \mathcal{K}(e) = \{0\}$, $1-e \nsim 1 \Rightarrow \mathcal{K}(e) = \{1\}$

$e \nsim 1,\; 1-e \nsim 1 \Rightarrow \mathcal{K}(e) = [0,1]$

Now let $J :=$ the norm-closed two-sided ideal in M generated by $\{e \in P(M) ;\, e \nsim 1\}$.

3.10 $J = \tilde{J} = \{x \in M ;\, \mathcal{K}(x^*x) = \{0\}$

<u>Proof</u> <u>of</u> <u>3.7 – 3.10</u> Clearly, $J \cap Z = \{0\}$ since $1 \notin J$. Thus

$$x \in J \Rightarrow x^*x \in J \Rightarrow \mathcal{K}(x^*x) \subset J \cap Z = \{0\} \Rightarrow x \in \tilde{J}.$$

This proves also 3.7. If $e \sim 1$ and $(1-e) \nsim 1$, then $\mathcal{K}(1-e) = \{0\}$ and therefore $\mathcal{K}(e) = \{1\} \neq \{0\}$. Finally let $e \in P(M)$ such that $e \sim 1$ and $(1-e) \sim 1$. Then, for each $n \in \mathbb{N}$ we can write:

$1 = e_1 + \cdots + e_n$, $e = v_k^* v_k \sim v_k v_k^* = e_k$ $(\forall)\, k = 2, \ldots, n$ and $v_1 = e_1 = e$

Then $u_k := v_k + v_k^* + (1 - e - e_k) \in U(M)$ $(\forall)\, k = 2, \ldots, n$, $u_1 = 1 \in U(M)$

and we have:

$$\frac{1}{n} \sum_{k=1}^{n} u_k e u_k^* = \frac{1}{n} \sum_{k=1}^{n} e_k = \frac{1}{n}$$

This shows that $\mathcal{K}(e) \ni \frac{1}{n} \neq 0$. Similarly, $\mathcal{K}(1-e) \ni \frac{1}{n}$, that is $\mathcal{K}(e) \ni 1 - \frac{1}{n}$. As $\mathcal{K}(e)$ is convex, we get $\mathcal{K}(e) \supset [\frac{1}{n}, 1 - \frac{1}{n}]$, $(\forall) n \in \mathbb{N}$ and, as $\mathcal{K}(e)$ is closed, we conclude that $\mathcal{K}(e) = [0,1]$.

Thus the two ideals $\mathcal{J}$ and $\tilde{\mathcal{J}}$ have the same projections and therefore $\mathcal{J} = \tilde{\mathcal{J}}$.

Let $x = x^* \in M$ with spectrum $\sigma(x) \subset [m, M]$, $m \in \sigma(x)$, $M \in \sigma(x)$, and spectral scale $e_t = \chi_{(-\infty, t]}(x)$, and denote

$$A = \sup\{t \,;\, e_t \nsim 1\} , \qquad B = \inf\{t \,;\, 1 - e_t \nsim 1\}.$$

Then, with proofs completely similar to those in Section 2°, we get successively:

3.11. (1) $m \leq A \leq B \leq M$ and (2) $\mathcal{K}(x) \subset [A, B]$

3.12. $\mathcal{K}(x) = [A, B]$ $(\forall)\, x = x^* \in M$

3.13. THEOREM. For any infinite factor M

$$\{ f \in \mathcal{T} \,;\, f(y) \in \mathcal{K}(y) \ (\forall)\, y \in M \} = \{ \varphi \in S(M) \,;\, \varphi|_{\mathcal{J}} = 0 \}$$

3.14. COROLLARY. $\mathcal{K}(x) = \overline{V}_{\mathcal{J}}(x)$, $(\forall)\, x \in M$.

The alternate proof for 2.14 works also for proving 3.14.

There is no analogue of 2.15 since the absence of non-zero projections not equivalent to 1 does not single out a specific class of factors.

Actually if the infinite factor M is of <u>countable type</u> (e.g. if M acts on a separable Hilbert space) then, for $e \in M$, we have

$$e \text{ is infinite} \iff e \sim 1$$

hence

$$e \text{ is finite} \iff e \nsim 1$$

and therefore $I = J$. By 2.14 and 3.14 we conclude that

3.15. If M is an infinite factor of countable type then
$$\mathcal{C}(x) = \mathcal{K}(x) \qquad (\forall)\ x \in M.$$

4°. THE GE-KADISON THEOREM

4.1. THEOREM (GE, KADISON) Let M be a factor, H a Hilbert space and R a von Neumann algebra such that
$$M \otimes 1 \subset R \subset M \bar{\otimes} B(H)$$
Then there is a von Neumann algebra $N \subset B(H)$ such that
$$R = M \bar{\otimes} N$$

<u>Proof.</u> Our first remark is that <u>we may assume that</u> M <u>is a factor of type</u> III. Indeed, there is a factor P of type III and then also $P \bar{\otimes} M$ is a factor of type III and
$$P \bar{\otimes} M \otimes 1 \subset P \bar{\otimes} R \subset P \bar{\otimes} M \otimes B(H)$$
Assuming the Theorem proved for factors of type III, this would imply the existence of a von Neumann algebra $N \subset B(H)$ such that
$$P \bar{\otimes} R = P \bar{\otimes} M \bar{\otimes} N$$
which in turn implies $R = M \bar{\otimes} N$ just by applying to the above equality a slice map of the form $\psi \bar{\otimes} id_R$ with ψ any normal state on P.

If the Theorem is true, then an obvious candidate for N will be:
$$N \equiv 1 \otimes N = \{(\varphi \bar{\otimes} id)(R) \,;\, \varphi \in M_*^+, \varphi(1) = 1\}''$$
To prove the equality $R = M \bar{\otimes} N$ means to prove two reciprocal inclusions: $R \subset M \bar{\otimes} N$ and $R \supset M \bar{\otimes} N$.

(1) $R \subset M \bar{\otimes} N \iff M' \bar{\otimes} N' \overset{!}{=} (M \bar{\otimes} N)' \subset R'$ which is equivalent to the following two conditions:

a) $M' \otimes 1 \subset R'$ which means $R \subset M \bar{\otimes} B(H)$, assured by the assumption

b) $1 \otimes N' \subset R'$ which is simple since $(\forall)\, X \in R$, $y' \in N'$, $\varphi \in M_*$ we have

$$(\varphi \bar{\otimes} id)(X(1 \otimes y')) = [(\varphi \bar{\otimes} id)(X)](1 \otimes y')$$
$$= (1 \otimes y') \cdot [(\varphi \bar{\otimes} id)(X)] = (\varphi \bar{\otimes} id)((1 \otimes y')X)$$

hence $X(1 \otimes y') = (1 \otimes y')X$.

(2) $R \supset M \bar{\otimes} N$ is equivalent to the following two conditions:

a) $R \supset M \otimes 1$, assured by the assumption

b) $R \supset 1 \otimes N$, that is $R \supset (\varphi \bar{\otimes} id)(R)$ $(\forall)\, \varphi \in M_*^+$, $\varphi(1) = 1$

As $M \otimes 1 \subset R$ we have $(u \otimes 1) R (u^* \otimes 1) \subset R$ $(\forall)\, u \in U(M)$ and hence $(f \otimes id)(R) \subset R$ for any DIXMIER map $f \in \mathcal{F}$. As we assumed that M is a factor of type III, by THEOREM 2.13 (actually COROLLARY 2.15) we know that any $\varphi \in M_*^+$, $\varphi(1) = 1$ is a w_p-limit of DIXMIER maps and this enables us to conclude that $(\varphi \bar{\otimes} id)(R) \subset R$ using the Corollary of the simple lemma below, and thus completing the proof.

LEMMA (GE-KADISON) Let $\Phi : M \to M$ be a normal completely positive map and $\Theta : M \bar{\otimes} N \to M \bar{\otimes} N$ be an arbitrary map such that:

(i) Θ commutes with all slice maps $id_M \otimes \psi$, $\psi \in N_*$

(ii) Θ coincides with $\Phi \bar{\otimes} id_N$ on $M \otimes 1_N$

Then $\Theta = \Phi \bar{\otimes} id_N$.

Proof. $(\forall)\, X \in M \bar{\otimes} N$, $(\forall)\, \varphi \in M_*$, $(\forall)\, \psi \in N_*$ we have:

$$(\varphi \bar{\otimes} \psi)(\Theta(X)) = (\varphi \otimes id_N) \circ (id_M \otimes \psi) \circ \Theta(X) = \quad \text{by (i)}$$
$$= (\varphi \otimes id_N) \circ \Theta \circ (id_M \otimes \psi)(X) = \quad \text{by (ii)}$$
$$= (\varphi \otimes id_N) \circ (\Phi \otimes id_N) \circ (id_M \otimes \psi)(X) = (\varphi \bar{\otimes} \psi)(\Phi \otimes id_N)(X).$$

Since φ, ψ were arbitrary, we conclude $\Theta(X) = (\Phi \bar{\otimes} id_N)(X)$ for all $X \in M \bar{\otimes} N$.

COROLLARY (GE-KADISON) Let $\Phi_n, \Phi \in B(M)$ be normal completely positive maps such that $\Phi_n \xrightarrow{w_p} \Phi$ on M. Then

$$\Phi_n \bar{\otimes} id_N \xrightarrow{w_p} \Phi \bar{\otimes} id_N \text{ on } M \bar{\otimes} N$$

Proof. By compactness we may assume that $\Phi_n \bar{\otimes} id_N \xrightarrow{w_p} \Theta$ and, applying the LEMMA, it follows that $\Theta = \Phi \bar{\otimes} id_N$.

Notice that, besides THEOREM 2.13, a highly non-trivial result was used, namely the TOMITA COMMUTATION THEOREM: $(M \bar{\otimes} N)' = M' \bar{\otimes} N'$!

5° COMMENTS

The results in Corollaries 2.14 and 3.14 for infinite factors were first obtained in [C] with a different proof. Although they easy imply the Theorems 2.13 and 3.13, these statements were not explicit in [C].

In [S-Z 2] and [H2] the results in Corollaries 2.14 and 3.14 were extended to properly infinite von Neumann algebras using the factor case and the algebraic reduction theory developed in ([S-Z1], [S-Z2], [H1], [H2]). The corresponding extension of Theorems 2.13, 3.13 appears explicitely only in [S-Z2, Theorem 7.10), while in [H2] the extension of Corollaries 2.14, 3.14 is obtained using an extension of the alternate proof we presented.

In [G-K] there is a quite laborious proof of the approximation of normal states with Dixmier maps (apparently the authors did not notice the above mentioned papers), which in turn is used to prove the remarkable Theorem 4.1. Earlier only the case of type I factors and the result of Alain Connes quoted below were known:

M, R factors, A commutative, $M \otimes 1 \subset R \subset M \bar{\otimes} A \implies R = M \otimes 1$

For the latter result the proof made an essential use of the modular theory.

Our contribution to the proof of Theorem 4.1 is the simple remark that we may assume M to be a factor of <u>type III</u>, which allowed a direct use of Corollary 2.15.

Besides the approximation with Dixmier maps, a main technical tool in the proof of Theorem 4.1 is the Tomita Commutation Theorem for tensor products. So, wanting to extend the Ge-Kadison Theorem for M an arbitrary von Neumann algebra, we had already one of the main ingredients – the extension of Theorem 2.13 ([S-Z2]) – but we needed also an appropriate extension of the Tomita Commutation Theorem. This led us to introduce the notion of tensor product over a subalgebra and the main result in [S-Z3] is a general commutation theorem which gives in particular both the Tomita Commutation Theorem and the extension of the Ge-Kadison Theorem for an arbitrary von Neumann algebra M with the conclusion that the intermediate subalgebra R is a tensor product of M <u>over its center</u> Z.