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THE OSOFSKY-SMITH THEOREM FOR MODULAR LATTICES, AND APPLICATIONS (II)

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Abstract

This is the second part of the paper with the same title published in *Communications in Algebra* **39** (2011), 4488-4506. It contains applications of the Latticial Osofsky-Smith Theorem to Grothendieck categories and module categories equipped with a torsion theory. Various many different meanings spread in the literature of the relative concepts with respect to a hereditary torsion theory τ on Mod-R like τ-essential submodule, τ-complement submodule, τ-CS module, etc. are also discussed.

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Key words: The Latticial Osofsky-Smith Theorem, The Categorical Osofsky-Smith Theorem, The Relative Osofsky-Smith Theorem, Grothendieck category, finitely generated object, locally finitely generated category, torsion theory, τ-CS module, τ-compact module, τ-compactly generated module, τ-finitely generated module.

4 Applications to Grothendieck categories

In this section we apply the lattice-theoretical results established in the previous sections to Grothendieck categories.

Throughout this section $\mathcal{G}$ will denote a fixed *Grothendieck category*, that is, an Abelian category with exact direct limits and with a generator. For any object $X \in \mathcal{G}$, $\mathcal{L}(X)$ will denote the lattice of all subobjects of X. It is well-known that $\mathcal{L}(X)$ is an upper continuous modular lattice (see e.g., Stenström [31, Chapter 4, Proposition 5.3, and Chapter 5, Section 1]). For any subobjects Y and Z of X we denote by $Y \cap Z$ their meet and by $Y + Z$ their join in the lattice $\mathcal{L}(X)$.

Recall that an object $X \in \mathcal{G}$ is said to be *finitely generated* if the lattice $\mathcal{L}(X)$ is compact. The category $\mathcal{G}$ is called *locally finitely generated* if it has a family of finitely generated generators, or equivalently if the lattices $\mathcal{L}(X)$ are compactly generated for all objects X of $\mathcal{G}$ (see Stenström [31, p. 122]). We say that an object $X \in \mathcal{G}$ is *locally finitely generated* if the lattice $\mathcal{L}(X)$ of all its subobjects is compactly generated.

Observe that unlike the category Mod-R whose objects are all locally finitely generated, a quotient category Mod-$R/\mathcal{T}$ of Mod-R modulo a localizing subcategory $\mathcal{T}$ may not have this property. Indeed let R be an infinite direct product of copies of a field and let $\mathcal{L}$ the localizing subcategory of Mod-R consisting of all semi-Artinian R-modules. Then, as observed in Albu [1, Remark 1.4(1)], the quotient category $\mathcal{C}_0 :=$ Mod-$R/\mathcal{L}$ has no simple object, in particular it has no nonzero finitely generated object because any nonzero finitely generated object must have maximal proper subobjects, so at least a simple factor object. Observe also that for any nonzero object X of $\mathcal{C}_0$, the lattice $\mathcal{L}(X)$ is not compactly generated.

For all undefined notation and terminology on Abelian categories the reader is referred to Albu and Năstăsescu [8] and/or Stenström [31].

For any object $X \in \mathcal{G}$, we denote

$C(X) := C(\mathcal{L}(X)) =$ the set of all *closed* elements of $\mathcal{L}(X)$,

$D(X) := D(\mathcal{L}(X)) =$ the set of all *complement* elements of $\mathcal{L}(X)$.

If $\mathbb{P}$ is any property on lattices, we say that an object $X \in \mathcal{G}$ is/has $\mathbb{P}$ if the lattice $\mathcal{L}(X)$ is/has $\mathbb{P}$. Similarly, a subobject Y of an object $X \in \mathcal{G}$ is/has $\mathbb{P}$ if the element Y of the lattice $\mathcal{L}(X)$ is/has $\mathbb{P}$. Thus, we obtain the concepts of an *uniform* object, *compact* object, *CC* object, *completely CC* object, *CEK* object, *pseudo-complement* subobject of an object, *essential* subobject of an object, *closed* subobject of an object, *complement* subobject of an object, *irreducible subobject* of an object, *essentially compact* subobject of an object, etc. For a complement (resp. compact) subobject of an object $X \in \mathcal{G}$ one uses the well-established term of a *direct summand* (resp. *finitely generated*) subobject of X, and for this reason instead of saying that X is a CC object we will say that X is a CS object (acronym for *C*losed subobjects are direct *S*ummands). For the same reason, instead of using the term of essentially compact subobject (resp. CEK object) we will use the term of *essentially finitely generated* subobject (resp. *CEF* object).

If we specialize Lemma 2.1 (this means Lemma 2.1 from the first part Albu [4] of this paper) for $L = \mathcal{L}(X)$, we obtain at once

Lemma 4.1. *Let X be a finitely generated, locally finitely generated object of a Grothendieck category $\mathcal{G}$. Assume that all finitely generated subfactors Z/Y of $X, Y \subseteq Z \subseteq X$, are* CEF*, i.e., every $U \in C(Z/Y)$ is an essentially finitely generated subobject of Z/Y. Then X satisfies the ACC on direct summands, i.e., the poset $D(X)$ of all direct summands of X is Noetherian.* □

By Lemma 4.1, or applying Theorem 3.4 to the lattice $L = \mathcal{L}(X)$, we deduce immediately:

Theorem 4.2. (The Categorical Osofsky-Smith Theorem). *Let $\mathcal{G}$ be a Grothendieck category, and let $X \in \mathcal{G}$ be a finitely generated, locally finitely generated object such that every finitely generated subfactor object of X is* CS*. Then X is a finite direct sum of uniform objects.* □

Remarks 4.3. (1) It is not clear whether the hypothesis “X is a locally finitely generated object of $\mathcal{G}$” in both Lemma 4.1 and Theorem 4.2 can be removed. In fact, according to Remarks 2.2 (2), only the following property

If $Z \subseteq Y \subseteq X$ are such that such that Y/Z is finitely generated then $\exists U \subseteq X$ such that U is finitely generated and $Y = Z + U$,

implied by this hypothesis is used in their proofs.

(2) In Albu and Van Den Berg [10, Proposition 5], a triangular 2×2-matrix ring S is constructed and a hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ on Mod-S is considered, having the following two properties:

- the quotient category $\mathcal{C} :=$ Mod-$S/\mathcal{T}$ has only one simple object;
- the canonical image U of the module S_S in the quotient category $\mathcal{C}$ has only one nonzero finitely generated subobject.

It follows that the lattice $\mathcal{L}(U)$ of all subobjects of U is not compactly generated, in other words, U is not locally finitely generated, and consequently the Grothendieck category $\mathcal{C}$ is not locally finitely generated. □

Following Dung [20], a right R-module M is said to be *CF* if every closed submodule of M is finitely generated, and *completely CF* provided every quotient of M is also CF. Similarly, an object X of a Grothendieck category $\mathcal{G}$ is called *CF* (acronym for *C*losed are *F*initely generated) if every closed subobject of X is finitely generated, and *completely CF* if every quotient object of X is CF. Clearly, any CF object of $\mathcal{G}$ is CEF, therefore, using Lemma 3.1 specialized for the lattice $L = \mathcal{L}(X)$, the next result is a particular case of Lemma 4.1.

Corollary 4.4. *Let X be a finitely generated, locally finitely generated object of a Grothendieck category $\mathcal{G}$ such that every finitely generated subobject of X is completely* CF*. Then X is a finite direct sum of indecomposable subobjects.* □

Remark 4.5. Corollary 4.4 is the categorical counterpart of the module-theoretical result Dung [20, Theorem 2.5], which in turn, can be viewed as a generalization of the Osofsky-Smith Theorem. □

More generally, we say that a lattice L is *CK* (acronym for *C*losed are *K*ompact) if every closed element of L is compact, i.e., $C(L) \subseteq K(L)$. Clearly, any CK lattice is also CEK, so the following result, which is a latticial version of Corollary 4.4, is an immediate consequence of Lemmas 2.1 and 3.1:

Proposition 4.6. *Let L be a compact, compactly generated, modular lattice. Assume that all compact subfactors of L are* CK*. Then $D(L)$ is a Noetherian poset, in particular* 1 *is a finite direct join of indecomposable elements of L.* □

Denote by $\mathcal{H}$ the class of all finitely generated objects of $\mathcal{G}$, and let $\mathcal{A}$ be a subclass of $\mathcal{H}$ satisfying the following three conditions:

(A_1) If $X \in \mathcal{A}$, $X' \in \mathcal{G}$ and $X \simeq X'$ then $X' \in \mathcal{A}$.

(A_2) If $X \in \mathcal{A}$ then $X/X' \in \mathcal{A}, \forall X' \subseteq X$.

(A_3) If $X \in \mathcal{A}$ and $Z \subseteq Y \subseteq X$ with $Y/Z \in \mathcal{A}$, then $\exists U \subseteq X$ such that $U \in \mathcal{A}$ and $Y = Z + U$.

As we have noticed above, the class $\mathcal{H}$ could be empty, and in this case everything that follows makes no sense.

Similarly with the latticial case, we say that an object $X \in \mathcal{G}$ is *essentially* $\mathcal{A}$ if there exists an essential subobject Y of X with $Y \in \mathcal{A}$. Further, X is called *CE*$\mathcal{A}$ if any closed subobject of X is essentially $\mathcal{A}$.

Lemma 4.7. *Let $\mathcal{A}$ be a class of finitely generated objects of a Grothendieck category $\mathcal{G}$ satisfying the conditions $(A_1) - (A_3)$ above, and let $X \in \mathcal{A}$. Assume that all subfactors of X are* CE$\mathcal{A}$. *Then $D(X)$ is a Noetherian poset.* □

Proof. Apply Lemma 3.6 to the lattice $L = \mathcal{L}(X)$. □

By Lemma 4.7 or applying Theorem 3.7 to the lattice $L = \mathcal{L}(X)$ we deduce immediately:

Theorem 4.8. (The Categorical $\mathcal{A}$-Osofsky-Smith Theorem). *Let $\mathcal{A}$ be a class of finitely generated objects of a Grothendieck category $\mathcal{G}$ satisfying the conditions $(A_1) - (A_3)$ above, and let $X \in \mathcal{A}$. Assume that all subfactors of X in $\mathcal{A}$ are* CS. *Then X is a finite direct sum of uniform objects of $\mathcal{G}$.* □

An $\mathcal{A}$-version of Corollary 4.4, which is an easy consequence of Proposition 4.6, also holds:

Corollary 4.9. *Let $\mathcal{A}$ be a class of finitely generated objects of a Grothendieck category $\mathcal{G}$ satisfying the conditions $(A_1) - (A_3)$ above, and let $X \in \mathcal{A}$. Assume that every finitely generated subobject of X in $\mathcal{A}$ is completely* CF. *Then X is a finite direct sum of indecomposable subobjects.* □

We are now going to present a consequence, involving injective objects, of the Categorical Osofsky-Smith Theorem. Note that because we did not have handy a good latticial substitute of the concept of an injective object in a category we could not obtain in Albu [4] such a result for lattices. However, using the concept of a linear morphism of lattices recently introduced by Albu and Iosif [5], we expect to provide a consequence, involving injective lattices, of the Latticial Osofsky-Smith Theorem.

Recall that for any Grothendieck category one can define as in Mod-R the concepts of an *M-injective object, self-injective object,* and *semi-simple object* (see, e.g., Albu and Năstăsescu [8, p. 9]). For any object X of a Grothendieck category we denote by $E(X)$ its injective hull.

Lemma 4.10. *Let $\mathcal{A}$ be an arbitrary Abelian category, $A, B \in \mathcal{A}$, and $u, v \in \mathrm{Hom}_{\mathcal{A}}(A, B)$. Then*

$$\mathrm{Im}\,(u + v) \subseteq \mathrm{Im}\,(u) + \mathrm{Im}\,(v).$$

Proof. Consider the diagram:

$$A \xrightarrow{\Delta} A \times A \xrightarrow{f} B,$$

where Δ is the diagonal morphism, and f denotes the morphism (u, v) determined by the two morphisms $u,\, v : A \longrightarrow B$. Then, by Mitchell [27, Lemma 18.3, Chap. I], we have

$$u + v = f \circ \Delta.$$

Clearly $\mathrm{Im}\,(u + v) = \mathrm{Im}\,(f \circ \Delta) \subseteq \mathrm{Im}\,(f)$.

Next, in order to calculate $\mathrm{Im}\,(f)$, we write $A \times A = A \oplus A$ as $A \oplus A = (A \oplus 0) + (0 \oplus A)$, and use Mitchell [27, Proposition 11.2, Chap. I] to deduce that

$$\mathrm{Im}\,(f) = f(A \oplus A) = f((A \oplus 0) + (0 \oplus A)) = f(A \oplus 0) + f(0 \oplus A) = u(A) + v(A) = \mathrm{Im}\,(u) + \mathrm{Im}\,(v).$$

□

The next result is the categorical version of Albu and Năstăsescu [8, Proposition 2.5].

Proposition 4.11. *Let $\mathcal{A}$ be an Abelian category, and let $U,\, M \in \mathcal{A}$. Assume that U has an injective hull $E(U)$ in $\mathcal{A}$ (this holds always when $\mathcal{A}$ is a Grothendieck category). Then, the following two assertions are equivalent.*

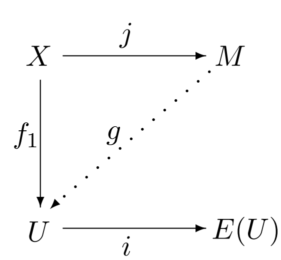
(1) *U is M-injective.*

(2) $\mathrm{Im}(f) \subseteq U, \forall\, f \in \mathrm{Hom}(M, E(U))$.

In particular, U is self-injective $\Longleftrightarrow$ $\mathrm{Im}(f) \subseteq U, \forall\, f \in \mathrm{End}(E(U))$.

Proof. (2) $\Longrightarrow$ (1): This implication is exactly as for modules, cf. Albu and Năstăsescu [8, Proposition 2.5].

(1) $\Longrightarrow$ (2): Of course, we may assume that both U and M are non-zero. We adapt the proof in the module case by avoiding the use of elements. Let $f \in \mathrm{Hom}(M, E(U)))$, and set $X := f^{-1}(U)$. Then $X \subseteq M$. Let $f_1 := f|X$. Since U is M-injective, there exists a morphism g making commutative the following diagram:



where j and i are the canonical injections. Observe that $\mathrm{Ker}\,(f - ig) = X$.

We claim that $X = M$, which will imply that $\mathrm{Im}(f) \subseteq U$, as desired. Assume that $X \neq M$. Then $(f - ig)(N) \neq 0$ for some $N \subseteq M$ with $N \not\subseteq X$. Since the extension $U \subseteq E(U)$ is essential, it follows that $(f - ig)(N) \cap U \neq 0$. For simplicity, set

$$\alpha := f - ig,\; V := \alpha(N) \cap U,\; Y := \alpha^{-1}(V).$$

Since $Y = \alpha^{-1}(V) = (f - ig)^{-1}(V)$, we have $(f - ig)(Y) \subseteq V \subseteq U$, so

$$f(Y) = ((f - ig) + ig)(Y) \subseteq (f - ig)(Y) + (ig)(Y) \subseteq U + U = U$$

in view of Lemma 4.10. Thus $f(Y) \subseteq U$, and then $Y \subseteq f^{-1}(U) = X$. We deduce that $\alpha(Y) \subseteq \alpha(X) = 0$.

On the other hand, we claim that $\alpha(Y) \neq 0$, which will produce a contradiction. Indeed, denote by $\beta : N \twoheadrightarrow \alpha(N)$ the canonical epimorphism induced by α. Since $V \subseteq \alpha(N)$, let $Z := \beta^{-1}(V)$. Then, by Mitchell [27, Corollary 16.3, Chap. I] applied to the pullback diagram

$$\begin{array}{ccc} Z & \xrightarrow{\gamma} & V \\ \downarrow & & \downarrow \\ N & \xrightarrow{\beta} & \alpha(N) \end{array}$$

where γ is the canonical epimorphism induced by α and the vertical arrows are the canonical injections, one deduces that γ is an epimorphism, so $\gamma(Z) = V$. On the other hand, clearly $Z \neq 0$ because $0 \neq V = \gamma(Z)$, so $0 \neq Z = \beta^{-1}(V) \subseteq \alpha^{-1}(V) = Y$, and then necessarily $0 \neq V = \gamma(Z) = \alpha(Z) \subseteq \alpha(Y)$, as claimed. Consequently, our assumption that $X \neq M$ fails, and then $X = M$, which finishes the proof. □

Lemma 4.12. *If X is a self-injective object of a Grothendieck category and $E(X) = E_1 \oplus E_2$, then $X = (X \cap E_1) \oplus (X \cap E_2)$.*

Proof. Clearly $(X \cap E_1) + (X \cap E_2) \subseteq X$. Denote by i_1, i_2, p_1, p_2 the canonical injections and projections defining the direct sum $E(X) = E_1 \oplus E_2$, and by $i : X \hookrightarrow E(X)$ the canonical inclusion morphism. Then

$$1_{E(X)} = i_1 p_1 + i_2 p_2.$$

For $k = 1, 2$, the morphism $g_k := i_k p_k i$ can be extended to an endomorphism f_k of $E(X)$. By Proposition 4.11, $f_k(X) = i_k p_k(X) \subseteq X \cap E_k$, so

$$X = 1_{E(X)}(X) = (i_1 p_1 + i_2 p_2)(X) \subseteq (X \cap E_1) + (X \cap E_2),$$

by Lemma 4.10. □

Lemma 4.13. *Any self-injective object of a Grothendieck category $\mathcal{G}$ is a* CS *object.*

Proof. Let X be a self-injective object of $\mathcal{G}$, and let $Y \subseteq X$. Consider a pseudo-complement Z of Y in X. Then $Y \oplus Z$ is an essential subobject of X because this happens in any upper continuous modular lattice by Stenström [31, Proposition 6.4, p. 75], so in particular in the lattice $\mathcal{L}(X)$ of all subobjects of X. By taking injective hulls, we deduce that

$$E(X) = E(Y) \oplus E(Z).$$

By Lemma 4.12, we have

$$X = (X \cap E(Y)) \oplus (X \cap E(Z),$$

which implies that Y is an essential subobject of the direct summand $X \cap E(Y)$ of X, i.e., X is a CS object, as desired. □

The next result, a categorical counterpart of Osofsky and Smith [29, Corollary 2] and Dung, Huynh, Smith, and Wisbauer [21, Corollary 7.14], is more general than Crivei, Năstăsescu Torrecillas [16, Corollary 2.9]:

Proposition 4.14. *The following assertions are equivalent for a locally finitely generated object X of a Grothendieck category $\mathcal{G}$.*

(1) *X is semi-simple.*

(2) *Every finitely generated subfactor of X is X-injective.*

Proof. (1) $\Longrightarrow$ (2): If X is semi-simple then so is any subfactor of X, which is clearly X-injective.

(2) $\Longrightarrow$ (1): Let $V = Y/Z,\ Z \subseteq Y \subseteq X,$ be a finitely generated subfactor of X. Then V is X-injective by hypothesis. It follows that V is X/Z-injective, and so, also Y/Z-injective by the well known properties of X-injective objects (see, e.g., Albu and Năstăsescu [8, Proposition 1.11]). Thus, V is self-injective, and consequently CS by Lemma 4.13.

Now let F be a finitely generated subobject of X. By the Categorical Osofsky-Smith Theorem, F is a finite direct sum of uniform objects. Let U be a uniform direct summand of F. Then, by hypothesis, any finitely generated subobject U' of U is X-injective, so it is a direct summand of X. Clearly U' is also a direct summand of the uniform object U. It follows that either $U' = 0$ or $U' = U$. On the other hand, because X has been supposed to be locally finitely generated, for any $0 \neq W \subseteq U$, W is the sum of all its nonzero finitely generated subobjects, all of them being equal to U. Thus, U is a simple object of $\mathcal{G}$, and consequently F is a semi-simple object of $\mathcal{G}$. Using again the fact that X is locally finitely generated, we conclude that X is a sum of simple objects, i.e., is semi-simple. □

Remark 4.15. Our proof of Proposition 4.14 has used essentially the hypothesis that the object X is locally finitely generated. The result fails for objects that are not locally finitely generated. Indeed, let $\mathcal{C}_0$ be the Grothendieck category considered at the beginning of this section. We have seen that there are no nonzero finitely generated objects in $\mathcal{C}_0$, so assertion (2) of Proposition 4.14 is vacuously satisfied for any nonzero object X of $\mathcal{C}_0$, but X is not semi-simple. □

Recall that an object Q of $\mathcal{G}$ is called *completely injective* if every quotient object of Q is injective, and the category $\mathcal{G}$ is said to be *semi-simple* or *discrete spectral* if every its object is semi-simple.

Corollary 4.16. (Crivei, Năstăsescu, and Torrecillas [16, Theorem 2.10]). *Let $\mathcal{G}$ be a Grothendieck category having a family of completely injective finitely generated generators. Then $\mathcal{G}$ is semi-simple.*

Proof. Let X be an arbitrary object of $\mathcal{G}$. Then X is the sum of all its finitely generated subobjects because $\mathcal{G}$ is locally finitely generated. In order to show that X is semi-simple, it is sufficient to prove that so is any finitely generated subobject of X. Consequently, without loss of generality, we may as assume that X is finitely generated. Then X is an epimorphic image of a direct sum of finitely many completely injective generators of $\mathcal{G}$, so it is itself completely injective by Crivei, Năstăsescu, Torrecillas [16, Proposition 2.2]. Therefore, any finitely generated subfactor of X is injective, and a fortiori X-injective. Apply now Proposition 4.14 to conclude that X is semi-simple. □

Remark 4.17. A nice result due to Okado [28] states that a unital ring R is right Noetherian if and only if every CS right R-module can be expressed as a direct sum of indecomposable (or uniform) modules. We guess that the following categorical version of Okado's Theorem holds:

A Grothendieck category $\mathcal{G}$ is locally Noetherian (*this means that $\mathcal{G}$ has a set of Noetherian generators*) *if and only if every* CS *object of $\mathcal{G}$ can be expressed as a direct sum of indecomposable* (*or uniform*) *objects.* □

We end this section by referring to some statements related to the Categorical Osofsky-Smith Theorem from Osofsky and Smith [29] and Crivei, Năstăsescu, and Torrecillas [16] ,that seem not to be in order.

First of all, Osofsky and Smith claim in the introduction of their paper [29] that the proof of their main theorem also works in any Abelian category $\mathcal{C}$ having arbitrary direct sums and exact direct limits (i.e., satisfying the Grothendieck's AB5 condition) and any class $\mathcal{A}$ of finitely generated objects of $\mathcal{C}$ having the following two properties:

(A'_1) If $X \in \mathcal{A}$ then $X' \in \mathcal{A}$ for any direct summand X' of X.

(A'_2) If $X \in \mathcal{A}$ and $Z \subseteq Y \subseteq X$ with $Y/Z \in \mathcal{A}$ is a direct summand of X/Z, then $\exists U \subseteq X$ such that $U \in \mathcal{A}$ and $Y = Z + U$.

Note that though in Crivei, Năstăsescu, and Torrecillas [16] it is claimed that their [16, Theorem 2.7] can be proved in the same manner as in the module case, there are some parts of the original proof in Osofsky and Smith [29] that cannot be transferred mutatis-mutandis to Grothendieck categories; in particular, [16, Corollary 2.8], which is a consequence [16, Theorem 2.7], is not true in case the considered object M is not locally finitely generated, as this has been pointed out in Remark 4.15.

A thorough analysis of the proof of the Latticial Osofsky-Smith Theorem specialized to the lattice $\mathcal{L}(X)$ of all subobjects of an object X of a Grothendieck category $\mathcal{G}$ suggests us to assert that the Osofsky and Smith's conditions (A'_1) and (A'_2) are not sufficient to prove the Categorical $\mathcal{A}$-Osofsky-Smith Theorem (Theorem 4.8) for $\mathcal{G}$; they should be replaced by our conditions $(A_1) - (A_3)$ presented above just after Proposition 4.6. Unfortunately we do not have handy an example of such a class $\mathcal{A}$ of finitely generated objects of $\mathcal{G}$ satisfying the conditions (A'_1) and (A'_2) above for which the Categorical $\mathcal{A}$-Osofsky-Smith Theorem fails.

5 Applications to module categories equipped with a hereditary torsion theory

In this section we present the relative version with respect to a hereditary torsion theory of the module-theoretical Osofsky-Smith Theorem [29], as well as some of its consequences. Their proofs are easy applications of the corresponding lattice-theoretical results of sections 1-3.

Throughout this section R denotes an associative ring with nonzero identity, Mod-R the category of all unital right R-modules, $\tau = (\mathcal{T}, \mathcal{F})$ a fixed hereditary torsion theory on Mod-R, and $\tau(M)$ the τ-torsion submodule of a right R-module M.

We shall use the notation M_R to emphasize that M is a right R-module. For any M_R we denote $\mathrm{Sat}_\tau(M) := \{\, N \mid N \leqslant M,\, M/N \in \mathcal{F} \,\}$, and for any $N \leqslant M$ we denote by $\overline{N} := \bigcap \{\, C \mid N \leqslant C \leqslant M,\, M/C \in \mathcal{F} \,\}$ the *τ-saturation* of N in M; N is called *τ-saturated* if $N = \overline{N}$. Note that $\overline{N}/N = \tau(M/N)$ and

$$\mathrm{Sat}_\tau(M) = \{\, N \mid N \leqslant M,\, N = \overline{N} \,\},$$

so $\mathrm{Sat}_\tau(M)$ is the set of all *τ-saturated* submodules of M, which explains the notation. It is known that $\mathrm{Sat}_\tau(M)$ is an upper continuous modular lattice for any M_R (see Stenström [31, Chapter 9, Proposition 4.1]).

For all undefined notation and terminology on torsion theories the reader is referred to Albu and Năstăsescu [8], Golan [23], and/or Stenström [31].

We say that a module M_R is *τ-CC* (or *τ-extending*) if the lattice $\mathrm{Sat}_\tau(M)$ is CC (or extending). More generally, if $\mathbb{P}$ is any property on lattices, we say that a module M_R is/has τ-$\mathbb{P}$ if the lattice $\mathrm{Sat}_\tau(M)$ is/has $\mathbb{P}$. Since the lattices $\mathrm{Sat}_\tau(M)$ and $\mathrm{Sat}_\tau(M/\tau(M))$ are canonically isomorphic, we deduce that M_R is τ-$\mathbb{P}$ if and only if $M/\tau(M)$ is τ-$\mathbb{P}$. Thus, we obtain the concepts of a *τ-Artinian* module, *τ-Noetherian* module, *τ-uniform* module, *τ-compact* module, *τ-compactly generated* module, *τ-CEK* module, etc. We say that a submodule N of M_R is/has τ-$\mathbb{P}$ if its τ-saturation $\overline{N}$, which is an element of $\mathrm{Sat}_\tau(M)$, is/has $\mathbb{P}$. Thus, we obtain the concepts of a *τ-pseudo-complement* submodule of a module, *τ-complement* submodule of a module, *τ-essential* submodule of a module, *τ-closed* submodule of a module, *τ-essentially compact* submodule of a module, etc. Since $\overline{N} = \overline{\overline{N}}$, it follows that N is/has τ-$\mathbb{P}$ if and only if $\overline{N}$ is/has τ-$\mathbb{P}$. In the sequel we shall use the well-established term of a *τ-CS* module (resp. *τ-direct summand* of a module) instead of that of a *τ-CC* module (resp. *τ-complement submodule* of a module).

For any module M_R we denote

$C_\tau(M) = C(\mathrm{Sat}_\tau(M))$ = the set of all closed elements of the lattice $\mathrm{Sat}_\tau(M)$,

$D_\tau(M) = D(\mathrm{Sat}_\tau(M))$ = the set of all complement elements of the lattice $\mathrm{Sat}_\tau(M)$,

$K_\tau(M) = K(\mathrm{Sat}_\tau(M))$ = the set of all τ-compact elements of $\mathrm{Sat}_\tau(M)$.

We denote by Mod-$R/\mathcal{T}$ the quotient category of Mod-R by its localizing subcategory $\mathcal{T}$ and by T_τ the canonical functor Mod-$R \longrightarrow$ Mod-$R/\mathcal{T}$. Note that Mod-$R/\mathcal{T}$ is a

Grothendieck category, and actually, any Grothendieck category is equivalent to such a category in view of the renown *Gabriel-Popescu Theorem* (see, e.g., Stenström [31, Chapter 10, Theorem 1.6]). Moreover, for any M_R, the map

$$\mathrm{Sat}_\tau(M) \longrightarrow \mathcal{L}(T_\tau(M)), \;\; N \mapsto T_\tau(N),$$

is an isomorphism of lattices by Albu and Năstăsescu [8, Proposition 7.10]; so, for any property on lattices $\mathbb{P}$, the module M_R is/has τ-$\mathbb{P}$ if and only if the object $T_\tau(M)$ in the quotient Grothendieck category Mod-$R/\mathcal{T}$ is/has $\mathbb{P}$.

We are now going to provide intrinsic characterizations, that is, without explicitly referring to the lattice $\mathrm{Sat}_\tau(M)$, of the relative concepts appearing in the *Relative Osofsky-Smith Theorem* we shall prove in the final part of this section.

Lemma 5.1. *The following assertions hold for submodules N, P of a module M_R.*

(1) $\overline{\overline{N}+P} = \overline{N+P}$.

(2) *If $N \subseteq P$, then $\overline{N} = \overline{P} \Longleftrightarrow P/N \in \mathcal{T}$.*

Proof. (1) The result is certainly known, but because we did not find a reference in the literature, for reader's convenience we include below its proof.

Since $N+P \subseteq \overline{N}+P$, we have $\overline{N+P} \subseteq \overline{\overline{N}+P}$. For the opposite inclusion, it is sufficient to show that $(\overline{\overline{N}+P})/(N+P) \in \mathcal{T}$ because $\overline{X}/X = \tau(M/X)$ for any $X \leqslant M$. To do that, consider the exact sequence in Mod-R:

$$0 \longrightarrow (\overline{N}+P)/(N+P) \longrightarrow (\overline{\overline{N}+P})/(N+P) \longrightarrow (\overline{\overline{N}+P})/(\overline{N}+P) \longrightarrow 0.$$

The nonzero edges of this sequence are both in $\mathcal{T}$: this is clear for the right edge, and for the left one, consider the canonical epimorphism $p : M/N \longrightarrow M/(N+P)$. Since $\overline{N}/N \in \mathcal{T}$, we deduce that $p(\overline{N}/N) = (\overline{N}+P)/(N+P) \in \mathcal{T}$. Because $\mathcal{T}$ is closed under extensions, we deduce that the middle term $(\overline{\overline{N}+P})/(N+P)$ of the sequence is also a member of $\mathcal{T}$, as desired.

(2) follows from Albu and Smith [9, Lemma 3.4] applied to the lattice $L = \mathrm{Sat}_\tau(M)$. □

Lemma 5.2. *The following statements hold for a module M_R and $X \in \mathrm{Sat}_\tau(M)$.*

(1) *For any $N \leqslant M$ with $N \subseteq X$, the τ-saturation $\overline{N}$ of N in M coincides with the τ-saturation $\overline{N}_X$ of N in X.*

(2) $\mathrm{Sat}_\tau(X)$ *is exactly the interval $[\,\tau(M), X\,]$ of* $\mathrm{Sat}_\tau(M)$.

Proof. (1) By definition,

$$\overline{N}/N = \tau(M/N) \quad \text{and} \quad \overline{N}_X/N = \tau(X/N).$$

Since $X/N \leqslant M/N$, we have $\overline{N}_X/N = \tau(X/N) \leqslant \tau(M/N) = \overline{N}/N$, so $\overline{N}_X \subseteq \overline{N}$.

In order to prove the opposite inclusion $\overline{N} \subseteq \overline{N}_X$, let $x \in \overline{N}$. Then, there exists a right ideal I of R such that $R/I \in \mathcal{T}$ $xI \subseteq N \subseteq X$. But $x \in M$ and $M/X \in \mathcal{F}$, so $x \in X$. Because $xI \subseteq N$, we have $x + N \in \tau(X/N) = \overline{N}_X/N$, and then $x \in \overline{N}_X$, as desired.

(2) If $Y \in \mathrm{Sat}_\tau(X)$ then $Y \leqslant X$ and $\tau(M) \leqslant Y$. Moreover, $Y \in \mathrm{Sat}_\tau(M)$, i.e., $M/Y \in \mathcal{F}$, because the nonzero edges of the exact sequence

$$0 \longrightarrow X/Y \longrightarrow M/Y \longrightarrow M/X \longrightarrow 0$$

are both in $\mathcal{F}$. So, $\mathrm{Sat}_\tau(X) \subseteq [\,\tau(M), X\,]$.

Conversely, let $Y \in [\,\tau(M), X\,]$. Then $Y \in \mathrm{Sat}_\tau(M)$. We have to show that $X/Y \in \mathcal{F}$. Indeed, because $Y \in \mathrm{Sat}_\tau(M)$ we have $M/Y \in \mathcal{F}$, and so $X/Y \in \mathcal{F}$, as desired. □

Proposition 5.3. *The following assertions hold for a module M_R and $N \leqslant M$.*

(1) *N is τ-essential in $M \Longleftrightarrow (\forall\, P \leqslant M,\, P \cap N \in \mathcal{T} \Longrightarrow P \in \mathcal{T}\,)$.*

(2) *M is τ-uniform $\Longleftrightarrow (\forall\, P, K \leqslant M,\, P \cap K \in \mathcal{T} \Longrightarrow P \in \mathcal{T}$ or $K \in \mathcal{T}\,)$.*

(3) *N is a τ-pseudo-complement in $M \Longleftrightarrow \exists\, P \leqslant M$ such that $N \cap P \in \mathcal{T}$ and N is maximal among the submodules of M having this property; in this case $N \in \mathrm{Sat}_\tau(M)$ and $N \cap \overline{P} = \tau(M)$.*

(4) *N is τ-closed in $M \Longleftrightarrow$ for any $P \leqslant M$ such that $N \subseteq P$ and N is a τ-essential submodule of P one has $P/N \in \mathcal{T}$. If additionally $N \in \mathrm{Sat}_\tau(M)$, then N is τ-closed in $M \Longleftrightarrow N$ has no proper τ-essential extension in M.*

(5) *N is a τ-direct summand in $M \Longleftrightarrow \exists\, P \leqslant M$ such that $M/(N+P) \in \mathcal{T}$ & $N \cap P \in \mathcal{T}$.*

(6) *M is τ-complemented $\Longleftrightarrow \forall\, N \leqslant M, \exists\, P \leqslant M$ such that $M/(N+P) \in \mathcal{T}$ & $N \cap P \in \mathcal{T}$.*

(7) *M is τ-compact $\Longleftrightarrow \forall\, N \leqslant M$ with $M/N \in \mathcal{T}$, $\exists\, N' \leqslant N$ such that N' is finitely generated and $M/N' \in \mathcal{T}$, in other words, the filter $F(M) := \{\, N \leqslant M \mid M/N \in \mathcal{T}\,\}$ has a basis consisting of finitely generated submodules.*

(8) *M is τ-CEK $\Longleftrightarrow$ any τ-closed submodule of M is a τ-essential submodule of a τ-compact submodule of M.*

(9) *M is τ-compactly generated $\Longleftrightarrow \forall\, N \leqslant M, \exists\, I_N$ a set and a family $(C_i)_{i\in I_N}$ of τ-compact submodules of M such that $\sum_{i\in I_N} C_i \subseteq N$ and $N/\big(\sum_{i\in I_N} C_i\big) \in \mathcal{T}$.*

Proof. By our definitions, for any property $\mathbb{P}$ on lattices, N is/has τ-$\mathbb{P}$ if and only if the element $\overline{N}$ of $\mathrm{Sat}_\tau(M)$ is/has $\mathbb{P}$.

(1) is a part of Gómez Pardo [24, Proposition 2.2].

(2) follows immediately from (1) (see also Albu [3, Corollary 2.10]).

(3) is exactly Gómez Pardo [24, Propositions 2.8] by observing that his concept of a τ-complement submodule coincides with our concept of a τ-pseudo-complement submodule.

(4) Assume that N is τ-closed in M, i.e., $\overline{N}$ is a closed element of the lattice $\mathrm{Sat}_\tau(M)$. Let $P \leqslant M$ be such that $N \subseteq P$ and N is a τ-essential submodule of P. We are going to show that $P/N \in \mathcal{T}$. First note that any submodule T of M with $M/T \in \mathcal{T}$ is τ-essential in M because $\overline{T} = M$ is an essential element of $\mathrm{Sat}_\tau(M)$. In particular, P is a τ-essential submodule of $\overline{P}$. It follows easily that N is a τ-essential submodule of $\overline{P}$. By definition, this means that the τ-saturation $\overline{N}_{\overline{P}}$ of N in $\overline{P}$ is an essential element of the lattice $\mathrm{Sat}_\tau(\overline{P})$. By Lemma 5.2 (1), $\overline{N}_{\overline{P}} = \overline{N}$, so $\overline{N}$ is an essential element of $\mathrm{Sat}_\tau(\overline{P})$. By Lemma 5.2 (2), $\mathrm{Sat}_\tau(\overline{P})$ is exactly the interval $[\,\tau(M), \overline{P}\,] = [\,\overline{0}, \overline{P}\,]$ of $\mathrm{Sat}_\tau(M)$, so $\overline{N}$ is an essential element of the interval $[\,\overline{0}, \overline{P}\,]$ of $\mathrm{Sat}_\tau(M)$. Since $\overline{N}$ is a closed element of the lattice $\mathrm{Sat}_\tau(M)$ by assumption, we deduce that $\overline{N} = \overline{P}$. Therefore $P/N \leqslant \overline{P}/N = \overline{N}/N \in \mathcal{T}$, and then $P/N \in \mathcal{T}$, as desired.

Conversely, assume that for any $P \leqslant M$ such that $N \subseteq P$ and N is a τ-essential submodule of P one has $P/N \in \mathcal{T}$, and prove that N is τ-closed, i.e., $\overline{N}$ is a closed element of the lattice $\mathrm{Sat}_\tau(M)$. To do that, let $X \in \mathrm{Sat}_\tau(M)$ with $\overline{N} \leqslant X$ and $\overline{N}$ is an essential element of the interval $[\,\overline{0}, X\,]$ of $\mathrm{Sat}_\tau(M)$, so of the sublattice $\mathrm{Sat}_\tau(X)$ of $\mathrm{Sat}_\tau(M)$. By definition, this means that the $\overline{N}$ is a τ-essential submodule of X. As we have observed above, N is a τ-essential submodule of $\overline{N}$, so N is a τ-essential submodule of X. By hypothesis, it follows that $X/N \in \mathcal{T}$, which implies that $X = \overline{X} = \overline{N}$. This shows that N is a τ-closed submodule of M, as desired.

In case $N \in \mathrm{Sat}_\tau(M)$, the implication "$\Longleftarrow$" is clear. For implication "$\Longrightarrow$", assume that $P \leqslant M$ is such that $N \subseteq P$ and N is a τ-essential submodule of P. Then $P/N \in \mathcal{T}$, and so $P/N \subseteq \tau(M/N) = \overline{N}/N = N/N$, i.e., $P = N$, as desired.

(5) By definition, N is a τ-direct summand in M if and only if there exists $P \in \mathrm{Sat}_\tau(M)$ such that $\overline{N} \vee P = M$ and $\overline{N} \wedge P = \tau(M)$, where "$\vee$" and "$\wedge$" are the join and meet, respectively, in the lattice $\mathrm{Sat}_\tau(M)$, i.e., $\overline{\overline{N} + P} = M$ and $\overline{N} \cap P = \tau(M)$. By Lemma 5.1 (1), $\overline{\overline{N} + P} = \overline{N + P}$, so we deduce that $M/(N+P) \in \mathcal{T}$ and $N \cap P \in \mathcal{T}$.

Conversely, if $M/(N+P) \in \mathcal{T}$ and $N \cap P \in \mathcal{T}$ for some $P \leqslant M$, then we also have $M/(\overline{N} + \overline{P}) \in \mathcal{T}$, i.e., $M = \overline{\overline{N} + \overline{P}} = \overline{N} \vee \overline{P}$. Since $\overline{N} \wedge \overline{P} = \overline{N} \cap \overline{P} = \tau(M)$ we deduce that N is a τ-direct summand in M.

(6) By definition, M is τ-complemented if and only if for every $A \in \mathrm{Sat}_\tau(M)$ there exists $B \in \mathrm{Sat}_\tau(M)$ such that $A \vee B = M$ and $A \wedge B = \tau(M)$, i.e., $M/(A+B) \in \mathcal{T}$ and $A \cap B \in \mathcal{T}$. Continue now as in (5).

(7) is exactly the equivalence $(b) \Longleftrightarrow (c)$ in Stenström [31, Proposition 1.1, Chap. XXIII].

(8) Assume that M is τ-CEK, and let N be a τ-closed submodule of M, i.e., $\overline{N}$ is a closed element of the lattice $\mathrm{Sat}_\tau(M)$. By definition, there exists $P \in \mathrm{Sat}_\tau(M)$, $P \leqslant \overline{N}$, such that P is an essential element of the interval $[\,\overline{0}, \overline{N}\,]$ of $\mathrm{Sat}_\tau(M)$ and P is a compact element of $\mathrm{Sat}_\tau(M)$. By Lemma 5.2 (2), $[\,\overline{0}, \overline{N}\,] = \mathrm{Sat}_\tau(\overline{N})$, so P is a τ-essential submodule of $\overline{N}$. Since

N is a τ-essential submodule of $\overline{N}$, it follows that $P\cap N$ is a τ-essential submodule of $\overline{N}$, so also a τ-essential submodule of N. Now, observe that $P\cap N$ is a τ-compact submodule of M because $\overline{P\cap N}=\overline{P}\cap\overline{N}=P\cap\overline{N}=P$ is a compact element of the lattice $\mathrm{Sat}_\tau(M)$.

Conversely, assume that any τ-closed submodule of M is a τ-essential submodule of a τ-compact submodule of M, and let $X\in\mathrm{Sat}_\tau(M)$ be a closed element. Then X is a τ-closed submodule of M, so, by hypothesis, there exists a τ-compact submodule P of M such that P is a τ-essential submodule of X. Therefore, $\overline{P}$ is a compact element of $\mathrm{Sat}_\tau(M)$ and an essential element of the interval $[\,\overline{0},X\,]$ of $\mathrm{Sat}_\tau(M)$. This shows that $\mathrm{Sat}_\tau(M)$ is a CEK lattice, i.e., M is τ-CEK.

(9) Assume that M is τ-compactly generated, i.e., $\mathrm{Sat}_\tau(M)$ is a compactly generated lattice. Then, for any $N\leqslant M$, there exists a set I_N and a family $(B_i)_{i\in I_N}$ of compact elements of $\mathrm{Sat}_\tau(M)$, i.e., of τ-compact submodules of M, such that $\overline{N}=\bigvee_{i\in I_N}B_i=\overline{\sum_{i\in I_N}B_i}$. We claim that $C_i:=B_i\cap N$ is a τ-compact submodule of M for each $i\in I$. Indeed, we have $\overline{C_i}=\overline{B_i\cap N}=\overline{B_i}\cap\overline{N}=B_i\cap\overline{N}=B_i$, so $\overline{C_i}$ is a compact element of $\mathrm{Sat}_\tau(M)$, i.e., C_i is a τ-compact submodule of M for all $i\in I$, as claimed. Since $\sum_{i\in I_N}C_i\subseteq N$ and

$$\overline{N}=\bigvee_{i\in I_N}B_i=\bigvee_{i\in I_N}\overline{C_i}=\overline{\sum_{i\in I_N}C_i},$$

by Lemma 5.1 (2), we deduce that $N/\big(\sum_{i\in I_N}C_i\big)\in\mathcal{T}$, which proves the implication "$\Longrightarrow$".

Conversely, let $X\in\mathrm{Sat}_\tau(M)$. By assumption, there exists a set I_X and a family $(C_i)_{i\in I_X}$ of τ-compact submodules of M such that $X/\sum_{i\in I_X}C_i\in\mathcal{T}$. Then, $E_i:=\overline{C_i}$ is a compact element of $\mathrm{Sat}_\tau(M)$, and $X=\overline{X}=\overline{\sum_{i\in I_X}C_i}=\bigvee_{i\in I_X}\overline{C_i}=\bigvee_{i\in I_X}E_i$, which shows that $\mathrm{Sat}_\tau(M)$ is a compactly generated lattice, i.e., M is τ-compactly generated, and proves the implication "$\Longleftarrow$". □

Remark 5.4. We are going to clarify the relations between the concepts of a τ-compact, τ-compactly generated, and τ-finitely generated module.

As in Albu and Năstăsescu [7], a module M is said to be τ-*finitely generated* if there exists a finitely generated submodule M' of M such that $M/M'\in\mathcal{T}$. Note that a τ-finitely generated module is not necessarily τ-compact. To see this, let R be an infinite direct product of copies of a field, let $\mathcal{L}$ be the localizing subcategory of Mod-R consisting of all semi-Artinian R-modules, and let τ_0 be the hereditary torsion theory on Mod-R defined by $\mathcal{L}$. We have seen at the beginning of Section 4 that the quotient category $\mathcal{C}_0:=$ Mod-$R/\mathcal{L}$ has no simple object, so, in particular $\mathrm{Sat}_{\tau_0}(R_R)$ is not a compact lattice, i.e., R_R is not τ_0-compact, though R_R is a finitely generated R-module, in particular τ_0-finitely generated. Conversely, a τ-compact module is necessarily τ-finitely generated: indeed, $M\in F(M)$ because $M/M\in\mathcal{T}$, so, there exists a finitely generated submodule M' of M such that $M/M'\in\mathcal{T}$, i.e., M is τ-finitely generated.

A τ-compactly generated module is not necessarily τ-compact: indeed, any module M_R which is not finitely generated is clearly ξ-compactly generated but not ξ-compact, where $\xi=$

$(\{0\}, \text{Mod-}R)$ is the trivial torsion theory on Mod-R. Conversely, we guess that a τ-compact module is not necessarily τ-compactly generated, but do not have any counterexample. $\square$

The following example considered in Albu [2, Example 1.16] is very helpful to illustrate some of the relative concepts discussed above and to relate them with the ones in the literature and having the same names.

Example 5.5. Let R be an infinite direct product of copies of a commutative and let $\mathfrak{m}$ be a maximal ideal of R which is not principal (e.g., $R = \mathbb{Q}^{\mathbb{N}}$ and $\mathfrak{m}$ is a maximal ideal of $\mathbb{Q}^{\mathbb{N}}$ including its ideal $\mathbb{Q}^{(\mathbb{N})}$). Denote by $\mathcal{T}_{\mathfrak{m}}$ the localizing subcategory of Mod-R consisting of all modules M_R such that their localizations $M_{\mathfrak{m}}$ at $\mathfrak{m}$ are zero and by $\tau_{\mathfrak{m}}$ the hereditary torsion theory on Mod-R defined by $\mathcal{T}_{\mathfrak{m}}$.

Then $\tau_{\mathfrak{m}}(R) = \mathfrak{m}$ and $\text{Sat}_{\tau_{\mathfrak{m}}}(R) = \{\mathfrak{m}, R\}$, so R_R is $\tau_{\mathfrak{m}}$-simple but R_R does not contain any $\tau_{\mathfrak{m}}$-cocritical submodule. Moreover, $\mathfrak{m}$ is an essential submodule of R_R but not an essential element of the lattice $\text{Sat}_{\tau_{\mathfrak{m}}}(R)$. It follows that

$$C_{\tau_{\mathfrak{m}}}(R) = \text{Sat}_{\tau_{\mathfrak{m}}}(R) = \{\mathfrak{m}, R\} = D_{\tau_{\mathfrak{m}}}(R).$$

Consequently, the lattice $\text{Sat}_{\tau_{\mathfrak{m}}}(R)$ is both complemented and CC, in other words, R_R is both $\tau_{\mathfrak{m}}$-complemented and $\tau_{\mathfrak{m}}$-CS. $\square$

Remarks 5.6. We are going to discuss below in a chronological order how our relative τ-$\mathbb{P}$ concepts are related with the ones spread in the literature.

(1) 1985: Gómez Pardo defines in [24] the relative concepts of essential, complement, and essentially closed submodule of a module. By [24, Proposition 2.2], his concept of *τ-essential* submodule coincides with ours, by [24, Propositions 2.8] his concept of *τ-complement* submodule is exactly our concept of *τ-pseudo-complement* submodule, and his concept of a *τ-essentially closed* submodule is exactly our concept of a *τ-closed* submodule. Because our relative concepts defined above have been introduced as natural specializations to lattices of type $\text{Sat}_\tau(M_R)$ of well established latticial concepts, we preferred to use these latticial concepts preceded by "τ-", and this is the reason to talk, e.g., about "τ-pseudo-complement" instead of "τ-complement", which corresponds to the latticial concept of "complement".

(2) 1997: Smith, Viola-Prioli, and Viola-Prioli [30] say that a module M_R is *τ-complemented* if for every submodule N of M there exists a direct summand K of M such that $N \leqslant K$ and $K/N \in \mathcal{T}$, and show in [30, Proposition 1.6] that this is equivalent with the condition that every $P \in \text{Sat}_\tau(M)$ is a direct summand of M. As we will see immediately, this concept is different form ours, and for this reason we will call such modules *strongly τ-complemented.*

Clearly, any strongly τ-complemented module is τ-complemented but not conversely as Example 5.5 shows: $\mathfrak{m} \in \text{Sat}_\tau(R_R)$ is not a direct summand of R_R, so R_R is not strongly $\tau_{\mathfrak{m}}$-complemented, but R_R is $\tau_{\mathfrak{m}}$-complemented as we have seen above.

(3) 1998: López-Permouth, Oshiro, and Rizvi [26] introduce among others the relative concept of a *$\mathcal{A}$-CS* or *$\mathcal{A}$-extending* module, where $\mathcal{A}$ is a given nonempty subclass of Mod-R,

as being a module M_R such that any $A \in \mathcal{A}$ is essential in a direct summand of M. This concept has no connection with our relative concept of a CS-module.

(4) 1998: Doğruöz and Smith [19] define for any nonempty subclass $\mathcal{A}$ of Mod-R closed under isomorphisms and containing the zero module two concepts of CS modules relative to $\mathcal{A}$: M_R is called a *type 1 $\mathcal{A}$-extending* module (resp. *type 2 $\mathcal{A}$-extending*) if for any $N \leqslant M$ with $N \in \mathcal{A}$, every $C \leqslant M$ maximal with respect to $C \cap N = 0$ is a direct summand of M (resp. every essential closure of N in M is a direct summand of M). These two concepts have no relations with our relative concept of a CS-module.

(5) 2006: Al-Takhman, Lomp, and Wisbauer [11] define a completely different concept of τ-complemented module, we will not present here.

(6) 2007: Charalambides and Clark [13] define several relative notions of Module Theory in their attempt to relativize the concept of a CS-module. We warn the reader that their concepts of τ-essential, τ-complement, and τ-compact are different from ours and do not agree with the natural expectation that M_R is/has τ-$\mathbb{P}$ if and only if the object $T_\tau(M)$ of the quotient Grothendieck category Mod-$R/\mathcal{T}$ is/has $\mathbb{P}$. For instance, they call a module τ-compact if it is either τ-torsion or τ-cocritical, that is far away from what is expected to be such a module. Further, with our definition, any submodule of any module M_R has a τ-pseudo-complement (because in any upper continuous modular lattice, in particular in the lattice $\mathrm{Sat}_\tau(M)$, any element has a pseudo-complement) but not with their definition which, in fact, should have been called so and not τ-complement, as we explained this in (1). Note also that the concept of a τ-CS module as defined in Charalambides and Clark [13] is different from ours and does not agree with the property that M_R is τ-CS if and only if the object $T_\tau(M)$ of the quotient Grothendieck category Mod-$R/\mathcal{T}$ is CS. In their setting, a module M_R is τ-CS in case any essentially closed submodule N of M with $M/N \in \mathcal{T}$ is a direct summand of M. If we denote by $\xi = (\{0\}, \text{Mod-}R)$ the trivial torsion theory on Mod-R, then any module M_R is ξ-CS with their definition, but, in our setting, a module M_R is ξ-CS if and only M_R is a CS module, so their τ-CS concept is far away from ours. It is strange that Charalambides and Clark [13] make no reference to the paper of Gómez Pardo [24] published more than 20 years earlier than theirs, where the natural concepts of τ-essential, τ-complement, and τ-essentially closed have been introduced and investigated.

(7) 2008: Doğruöz [18] call a module M to be a *type 2 τ-extending*, where $\tau = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory on Mod-R, if every essentially closed submodule N of M with $M/N \in \mathcal{T}$ is a direct summand of M. Note that these are exactly the τ-CS modules considered by Charalambides and Clark [13].

(8) 2008-2009: Crivei [14], [15] defines the concept of an $\mathbb{E}$-*$\mathcal{A}$-extending* module, where $\mathcal{A}$ is a nonempty subclass $\mathcal{A}$ of Mod-R closed under isomorphisms and containing the zero module and $\mathbb{E}$ is a so called *proper class* of short exact sequences in Mod-R in the sense of Buchsbaum, as being a module A_R such that for every $B \leqslant A$ there exists $C \leqslant A$ with

$B \subseteq C$, $C/B \in \mathcal{A}$ and the canonical exact sequence $0 \to C \to A \to C/A \to 0$ is a member of $\mathbb{E}$. If $\mathbb{E}_s$ denotes the class of all splitting short exact sequences in Mod-R, then the $\mathbb{E}_s$-$\mathcal{A}$-*extending* modules are simply called $\mathcal{A}$-*extending*. Note that for any hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$, the $\mathcal{T}$-extending modules are exactly the $\mathcal{T}$-complemented modules in the sense of Smith, Viola-Prioli, and Viola-Prioli [30] we discussed and called *strongly* τ-*complemented*.

(9) 2012: Çeken and Alkan [17] use the concepts of τ-essential, τ-complement, and τ-essentially closed as introduced by Gómez Pardo [24] in order to define the relative concept of a τ-*CS* module: a module M_R is called τ-CS if every τ-(essentially) closed submodule of M is a direct summand of M. Because, as we will see right away, this concept is different of ours, we will call such modules *strongly* τ-*CS*. The module R_R in Example 5.5 is τ-CS but not strongly τ-CS because $\mathfrak{m}$ is τ-(essentially) closed but not a direct summand of R.

Note that the authors use the confusing related terms of a τ-complement submodule of a module as given in Gómez Pardo [24] and that of a τ-complemented module as given in Smith, Viola-Prioli, and Viola-Prioli [30]. Also, note that many of the results of Section 2 in Çeken and Alkan [17] are simple consequences of the more general results true in any upper continuous modular lattice. □

The next result is needed in the proof of the Relative Osofsky-Smith Theorem as an easy specialization of the Latticial Osofsky-Smith Theorem (Theorem 3.4) for the particular case of lattice $\mathrm{Sat}_\tau(M)$.

Lemma 5.7. *The following statements hold for a module M_R and submodules N, P of M_R such that $P \subseteq N$.*

(1) *The mapping*

$$\alpha : \mathrm{Sat}_\tau(N/P) \longrightarrow \mathrm{Sat}_\tau(\overline{N}/\overline{P}),\ X/P \mapsto \overline{X}/\overline{P},$$

is a lattice isomorphism.

(2) *If N, $P \in \mathrm{Sat}_\tau(M)$, then the assignment $X \mapsto X/P$ defines a lattice isomorphism from the interval $[P, N]$ of the lattice $\mathrm{Sat}_\tau(M)$ onto the lattice $\mathrm{Sat}_\tau(N/P)$.*

Proof. (1) is exactly Albu [2, Lemma 1.6], and (2) is a specialization of Albu and Smith [9, Proposition 3.9] for the lattice $L = \mathrm{Sat}_\tau(M)$. □

We say that a finite family $(N_i)_{1 \leqslant i \leqslant n}$ of submodules of a module M_R is τ-*independent* if $N_i \notin \mathcal{T}$, $\forall\, i$, $1 \leqslant i \leqslant n$, and

$$N_{k+1} \cap \sum_{1 \leqslant j \leqslant k} N_j \subseteq \tau(M),\ \forall\, k,\ 1 \leqslant k \leqslant n-1,$$

or, equivalently

$$\overline{N_{k+1}} \cap \overline{\sum_{1 \leqslant j \leqslant k} N_j} = \overline{N_{k+1}} \wedge \big(\bigvee_{1 \leqslant j \leqslant k} \overline{N_j} \,\big) = \tau(M),$$

in other words, the family $(\overline{N_i}\,)_{1 \leqslant i \leqslant n}$ of elements of the lattice $\mathrm{Sat}_\tau(M)$ is independent (see, e.g., Grätzer [25, Theorem 11, Chapter 4]).

Theorem 5.8. (The Relative Osofsky-Smith Theorem). *Let M_R be a τ-compact, τ-compactly generated module. Assume that all τ-compact subfactors of M are τ-CS. Then there exists a finite τ-independent family $(U_i)_{1\leqslant i\leqslant n}$ of τ-uniform submodules of M such that $M/(\sum_{1\leqslant i\leqslant n} U_i) \in \mathcal{T}$.*

Proof. Let N/P, $P \leqslant N \leqslant M$, be a τ-compact subfactor of M. Observe that, in view of Lemma 5.7, the interval $[\overline{P}, \overline{N}]$ of $\mathrm{Sat}_\tau(M)$ is isomorphic to the compact lattice $\mathrm{Sat}_\tau(N/P)$, which, by hypothesis is CC. So, we can specialize the Latticial Osofsky-Smith Theorem (Theorem 3.4) for the compact, compactly generated, modular lattice $L = \mathrm{Sat}_\tau(M)$ to deduce that there exists a finite independent family $(U_i)_{1\leqslant i\leqslant n}$ of uniform elements of L such that $M = \dot{\bigvee}_{1\leqslant i\leqslant n} U_i$ is the direct join in L of the family $(U_i)_{1\leqslant i\leqslant n}$. Thus, $(U_i)_{1\leqslant i\leqslant n}$ is a τ-independent family of τ-uniform submodules of M. Since

$$M = \dot{\bigvee}_{1\leqslant i\leqslant n} U_i = \overline{\sum_{1\leqslant i\leqslant n} U_i}\,,$$

it follows that $M/(\sum_{1\leqslant i\leqslant n} U_i) \in \mathcal{T}$, as desired. □

As we remember, the key point in proving the Latticial Osofsky-Smith Theorem was Lemma 2.1, whose immediate specialization for the lattice $L = \mathrm{Sat}_\tau(M)$ can be proven similarly to that of Theorem 5.8 by using Lemma 5.1 (2), Proposition 5.3, and Lemma 5.7:

Lemma 5.9. *Let M_R be a τ-compact, τ-compactly generated module. Assume that all τ-compact subfactor modules N/P of M, $P \leqslant N \leqslant M$, are τ-CEK. Then, for any ascending chain*

$$D_1 \subseteq D_2 \subseteq \ldots \subseteq \ldots$$

of τ-direct summands of M, there exists a positive integer k such that $D_{n+1}/D_n \in \mathcal{T}$ for each $n \geqslant k$. □

As we have stressed in Remarks 2.2 (2), the condition that the lattice L in the Latticial Osofsky-Smith Theorem is compactly generated can be replaced by the following weaker condition:

$$(*) \qquad \forall\, a < b \ \text{ in } L, \forall\, k \in K(b/a), \exists\, c \in K(L) \ \text{ with } k = c \vee a.$$

where, remember that $K(L)$ denotes the set of all compact elements of L. Observe that the condition $(*)$ trivially holds for $a = b$.

If we specialize the condition $(*)$ for the lattice $L = \mathrm{Sat}_\tau(M)$, we obtain the condition:

$$(\tau\text{-}*) \qquad \forall\, P \leqslant N \ \text{ in } \mathrm{Sat}_\tau(M), \forall\, K \in K([P,N]), \exists\, C \in K_\tau(M) \ \text{ with } K = C \vee P,$$

which, in view of Lemma 5.7 can be also expressed as

$$(\tau\text{-}*) \qquad \forall\, P \leqslant K \leqslant N \leqslant M, \ \text{ with } K/P \ \tau\text{-compact in } N/P, \exists\, C \leqslant M \\ \tau\text{-compact in } M \text{ with } K = \overline{C + P}.$$

Thus, the condition that the module M_R in the Relative Osofsky-Smith Theorem is τ-compactly generated may be replaced by the weaker one (τ-$*$).

We are now going to state a more simplified version of the Relative Osofsky-Smith Theorem in case the given module M_R is τ-torsion-free. To do that, we need the following results.

Lemma 5.10. (Albu [3, Lemma 2.22]). *Let $M_R \in \mathcal{F}$ be a module, and let $(N_i)_{i\in I}$ be a family of submodules of M. Then $(N_i)_{i\in I}$ is an independent family of submodules of M if and only if $(\overline{N_i})_{i\in I}$ is an independent family of elements of the lattice* $\mathrm{Sat}_\tau(M)$. □

Lemma 5.11. (Albu [3, Corollary 2.10]). *If $M_R \in \mathcal{F}$, then M is τ-uniform $\Longleftrightarrow$ M is uniform.* □

Theorem 5.12. (The Torsion-free Relative Osofsky-Smith Theorem). *Let $M_R \in \mathcal{F}$ be a τ-compact, τ-compactly generated module. Assume that all τ-compact subfactors of M are τ-*CS. *Then there exists a finite independent family $(U_i)_{1\leqslant i\leqslant n}$ of uniform submodules of M such that $M/(\sum_{1\leqslant i\leqslant n} U_i) \in \mathcal{T}$.*

Proof. Use Lemmas 5.10 and 5.11 in Theorem 5.8. □

Since M is τ - $\mathbb{P}$ if and only if $M/\tau(M)$ is so, in view of 5.12 we can of course formulate the Relative Osofsky-Smith Theorem 5.8 in terms of essentiality and independence in the lattice $\mathcal{L}(M/\tau(M))$ instead of the relative ones in the lattice $\mathcal{L}(M)$:

Theorem 5.13. *Let M_R be a τ-compact, τ-compactly generated module. If all τ-compact subfactors of M are τ-*CS, *then there exists a finite family $(U_i)_{1\leqslant i\leqslant n}$ of submodules of M, all containing $\tau(M)$, such that $(U_i/\tau(M))_{1\leqslant i\leqslant n}$ is an independent family of uniform submodules of $M/\tau(M)$ and $M/(\sum_{1\leqslant i\leqslant n} U_i) \in \mathcal{T}$.* □

For a hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ on Mod-R we denote by

$$F_\tau := \{\, I \leqslant R_R \mid R/I \in \mathcal{T}\}$$

the so called *Gabriel filter* associated with τ. With the notation in Proposition 5.3 (7), we have $F_\tau = F(R_R)$. Recall that by a *basis* of the Gabriel filter F_τ we mean a subset B of F_τ such that every right ideal in F_τ contains some $J \in B$. By Stenström [31, Corollary 2.5, Chap. XIII], if R is τ-Noetherian, then F_τ has a basis consisting of finitely generated right ideals of R. Of course, the converse is, in general, not true by looking at the trivial torsion theory $\xi = (\{0\}, \text{Mod-}R)$ on Mod-R for any non-Noetherian ring R: the Gabriel filter $F_\xi = \{R\}$ has the basis $B = \{R\}$ having a single finitely generated ideal R, but R is not ξ-Noetherian.

By Albu, Iosif, and Teply [6, Proposition 2.12], a Grothendieck category $\mathcal{G}$ has a finitely generated generator if and only if there exists a unital ring A and a hereditary torsion theory $\rho = (\mathcal{H}, \mathcal{E})$ on Mod-A such that $\mathcal{G} \simeq$ Mod-$A/\mathcal{H}$ and the Gabriel filter F_ρ has a basis consisting of finitely generated right ideals of A. Therefore, in case F_τ has a basis consisting of finitely generated right ideals of R, then the Grothendieck category Mod-$R/\mathcal{T}$ is locally finitely generated, and so, any module M_R is τ-compactly generated. Therefore, the next result is an immediate consequence of Theorem 5.13.

Theorem 5.14. *Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on* Mod-R *such that its Gabriel filter F_τ has a basis consisting of finitely generated right ideals of R (in particular, this holds when R is τ-Noetherian), and let M_R be a τ-compact module. If all τ-compact subfactors of M are τ-*CS*, then there exists a finite family $(U_i)_{1 \leqslant i \leqslant n}$ of submodules of M, all containing $\tau(M)$, such that $(U_i/\tau(M))_{1 \leqslant i \leqslant n}$ is an independent family of uniform submodules of $M/\tau(M)$ and $M/(\sum_{1 \leqslant i \leqslant n} U_i) \in \mathcal{T}$.* □

The policy of this paper is to provide for most categorical results of Section 4 their relativizations with respect to a hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ on Mod-R. Therefore, we will discuss now the relative versions of the consequences of the Categorical Osofsky-Smith Theorem that involve the concept of an injective object of a Grothendieck category, namely Proposition 4.14 and Corollary 4.16. To do that, we need first of all to characterize the modules M_R such that $T_\tau(M)$ is an injective object of the quotient category Mod-$R/\mathcal{T}$. An expected candidate for such modules are the well-studied τ-injective modules. Recall that a module M_R is said to be *τ-injective* if for every module B_R and every submodule A of B with $B/A \in \mathcal{T}$, any morphism $f \in \mathrm{Hom}_R(A, M)$ can be extended to a morphism $\overline{f} \in \mathrm{Hom}_R(B, M)$.

By Gabriel [22, Chapitre 3], the canonical functor $T_\tau : \text{Mod-}R \longrightarrow \text{Mod-}R/\mathcal{T}$ has a right adjoint $S_\tau : \text{Mod-}R/\mathcal{T} \longrightarrow \text{Mod-}R$, and the canonical functorial morphism

$$\Phi : T_\tau \circ S_\tau \longrightarrow 1_{\text{Mod-}R/\mathcal{T}}$$

is a functorial isomorphism.

Proposition 5.15. *The following statements are equivalent for a τ-injective module $M_R \in \mathcal{F}$.*

(1) *$T_\tau(M)$ is an injective object of* Mod-$R/\mathcal{T}$.

(2) *M is an injective R-module.*

Proof. (1) $\Longrightarrow$ (2): Assume that $T_\tau(M)$ is an injective object of Mod-$R/\mathcal{T}$. By Bucur and Deleanu [12, Proposition 6.3], $S_\tau(Q)$ is an injective R-module for any injective object Q of Mod-$R/\mathcal{T}$, so $S_\tau(T_\tau(M))$ is an injective R-module. By hypothesis, M_R is a τ-injective and τ-torsion-free module, i.e., M_R is a τ-closed module. By Gabriel [22, Corollaire, p. 371], the canonical morphism $\Psi(M) : M \longrightarrow (S_\tau \circ T_\tau)(M)$ is an isomorphism, so that

$$M \simeq (S_\tau \circ T_\tau)(M) = S_\tau(T_\tau(M))$$

is an injective R-module, as desired.

(2) $\Longrightarrow$ (1): If M_R is an injective module, then, by Gabriel [22, Corollaire 2, p.375], $M \simeq E \oplus S_\tau(X)$, where E is the injective hull of an injective R-module from $\mathcal{T}$ and X is an injective object of Mod-$R/\mathcal{T}$. Since $M \in \mathcal{F}$ by hypothesis, it follows that $E = 0$, and so, $M \simeq S_\tau(X)$. Therefore, $T_\tau(M) \simeq T_\tau(S_\tau(X)) = (T_\tau \circ S_\tau)(X) \simeq X$ is an injective object of Mod-$R/\mathcal{T}$. □

Proposition 5.15. shows that the usual concept of a τ-injective module does not agree with the natural expectation that its canonical image in Mod-$R/\mathcal{T}$ is an injective object.

In order to obtain a τ-version of Proposition 4.14 we should define the relative corresponding concept of an X-injective object of a Grothendieck category as follows: if M_R, N_R are modules, then N is said to be *τ-M-injective* if $T_\tau(N)$ is a $T_\tau(M)$-injective object of Mod-$R/\mathcal{T}$. Unfortunately we could not find a characterization of such modules similar to that in Proposition 5.15, so that, we are unable to state a relative version of Proposition 4.14; however, we can do this for Corollary 4.16 as we will see right away.

Recall that a module U_R is said to be *τ-simple* if $U \notin \mathcal{T}$ and $\mathrm{Sat}_\tau(U) = \{\tau(U), U\}$, and *$\tau$-cocritical* if it τ-simple and $U \in \mathcal{F}$. The *τ-socle* of a module M_R, denoted by $\mathrm{Soc}_\tau(M)$, is defined as the τ-saturation of the sum of all τ-simple (or τ-cocritical) submodules of M, and M is said to be *τ-semi-simple* if $M = \mathrm{Soc}_\tau(M)$. By Albu [2, Proposition 1.15], $\mathrm{Soc}_\tau(M)$ is exactly the socle of the lattice $\mathrm{Sat}_\tau(M)$, and so,

the module M_R is *τ-semi-simple* $\Longleftrightarrow$ the lattice $\mathrm{Sat}_\tau(M)$ is semi-simple
$\Longleftrightarrow$ $T_\tau(M)$ is a semi-simple object of the quotient category Mod-$R/\mathcal{T}$.

We are now in a position to state the relative version of Corollary 4.16.

Corollary 5.16. *Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on* Mod-R *such that its Gabriel filter F_τ has a basis consisting of finitely generated right ideals of R (in particular, this holds when R is τ-Noetherian). Assume that R/I is an injective R-module for any $I \in \mathrm{Sat}_\tau(M)$. Then, any right R-module is τ-semi-simple.*

Proof. First, note that $T_\tau(R_R)$ is a generator of Mod-R/τ by Gabriel [22, Lemme 4, p. 373], which is finitely generated by Stenström [31, Proposition 1.1, Chap. XXIII] (or Proposition 5.3 (7)). Moreover, every quotient object of $T(R_R)$ is isomorphic to $T(R_R)/T(I) \simeq T(R/I)$ for some $I \in \mathrm{Sat}_\tau(M)$, i.e., is an injective object of Mod-R/τ by Proposition 5.15. Apply now Corollary 4.16 to deduce that Mod-$R/\mathcal{T}$ is a semi-simple category, i.e., any right R-module is τ-semi-simple. □

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