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Solutions for some nonlinear SPDEs via irregular perturbations of the characteristic systems

I. Molnar and C. Varsan

Abstract: The initial value problem of some nonlinear SPDEs are solved involving irregular perturbations of the corresponding characteristic system associated to a deterministic Hamilton-Jacobi (H-J) equations.

Key Words: Stochastic characteristic systems and irregular perturbations of nonlinear (H-J) equations

Math. Subject Classification 2010: 58F39

1 Introduction

The standard characteristic system associated to an initial value problem of nonlinear (H-J) equations can be described as a system of ODEs

$$\frac{d\hat{x}}{dt}(t,\lambda) = f(\hat{x}(t,\lambda),\varphi(\lambda)),\ t\in[0,T],\ \hat{x}(0,\lambda)=\lambda\in\mathbb{R}^n,$$

where the smooth mappings $f(x,u):\mathbb{R}^n\times\mathbb{R}^N\to\mathbb{R}^n$ and $\varphi=(\varphi_1,\ldots,\varphi_N)$ fulfil some regularity conditions (see $f\in(\mathcal{C}_b^1\cap\mathcal{C}^2)(\mathbb{R}^n\times\mathbb{R}^N;\mathbb{R}^n)$, $\varphi\in(\mathcal{C}_b^1\cap\mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^N)$). The smooth solution $u(t,x):[0,\hat{T}]\times\mathbb{R}^n\to\mathbb{R}^N$ of the given system of nonlinear (H-J) equations will be obtained as a composition $u(t,x)=\varphi(\hat{\psi}(t,x))$, $t\in[0,\hat{T}]$, $x\in\mathbb{R}^n$, where $0<\hat{T}\le T$ is sufficiently small such that the flow equations $\hat{x}(t,\lambda)=x\in\mathbb{R}^n$, $t\in[0,\hat{T}]$, can be solved leading us to a unique smooth solution $\{\lambda=\hat{\psi}(t,x)\in\mathbb{R}^n, t\in[0,\hat{T}], x\in\mathbb{R}^n\}$ (Banach fixed point theorem will be the main ingredient). The irregular perturbations of the flow solution

$\{\hat{x}(t,\lambda) \in \mathbb{R}^n : t \in [0,\hat{T}]\}$ will be generated in a simple case when smooth composition of commuting deterministic flows $y(p,z) := G_1(t_1) \circ \cdots \circ G_m(t_m)[z]$, $p = (t1,\ldots,t_m)$, $z \in \mathbb{R}^n$ are used. Define o primary perturbation of the flow $\{\hat{x}(t,\lambda)\}$ considering the following smooth mapping $\hat{y}(p,t;\lambda) := y(p,\hat{x}(t,\lambda))$, $p \in \mathbb{R}^n$, $t \in [0,\hat{T}]$, $\lambda \in \mathbb{R}^n$. Notice that the smooth mapping $\{\hat{y}(p,t;\lambda)\}$ is the solution of the following gradient system

$$\begin{aligned}&\partial_{t_1}\hat{y} = g_1(\hat{y}),\ldots,\partial_{t_m}\hat{y} = g_m(\hat{y}), \partial_t\hat{y} = f(\hat{y},\varphi(\lambda)),\\&\hat{y}(0,0;\lambda) = \lambda \in \mathbb{R}^n,\ t \in [0,\hat{T}],\end{aligned}$$

provided the vector field $f(\cdot,u) \in \mathcal{C}_b^1 \cap \mathcal{C}^2(\mathbb{R}^n;\mathbb{R}^n)$ commutes with any $g_i \in \{g_1,\ldots,g_m\}$, i.e. $[f(\cdot,u),g_i] = 0$, $i \in \{1,\ldots,m\}$, $u \in \mathbb{R}^N$ (see $[\cdot,\cdot]$ as the Lie bracket). By definition, $y(p,x)$, $p \in \mathbb{R}^m$ shares the group property and the unique solution of the perturbed flow equation $\hat{y}(p,t;\lambda) = x \in \mathbb{R}^n$, $t \in [0,\hat{T}]$, $p \in \mathbb{R}^m$ will be given by $\lambda = \tilde{\psi}(t,p;x) := \hat{\psi}(t,y(-p,x))$, where $\lambda = \hat{\psi}(t,z)$ is the unique solution of the original flow equation. Let $\{w(t) \in \mathbb{R}^m : t \in [0,\hat{T}]\}$ be standard m-dimensional Wiener process associated with the complete filtered probability space $\{\Omega,\mathcal{F} \supset \{\mathcal{F}_t\},P\}$. The nonlinear SPDEs involved in this paper will be obtained as the stochastic dynamics satisfied by the mappings $\psi(t,x) := \tilde{\psi}(t,w(t);x) \in \mathbb{R}^n$ and $u(t,x) = \varphi(\psi(t,x)) \in \mathbb{R}^N$, $t \in [0,\hat{T}]$, $x \in \mathbb{R}^n$, where $\psi(0,x) = x \in \mathbb{R}^n$ and $\tilde{\psi}(t,p;x) \in \mathbb{R}^n$ stands for the corresponding smooth test deterministic function. They will appear as solutions for problems (P4) and (P5) correspondingly as it is contained in section 3 of this paper. We shall recall them as follows

$$\begin{cases} d_t\psi(t,x) + [\partial_x\psi(t,x)]f(x,\varphi(\psi(t,x)))dt + \displaystyle\sum_{k=1}^m \partial_x\psi(t,x)g_k(x)\hat{\circ}dw_k(t) = 0,\\ \psi(0,x) = x \in \mathbb{R}^n,\ t \in [0,\hat{T}],\end{cases}$$

and

$$\begin{cases} d_t u(t,x) + [\partial_x u(t,x)]f(x,u(t,x)))dt + \displaystyle\sum_{k=1}^m [\partial_x u(t,x)]g_k(x)\hat{\circ}dw_k(t) = 0,\\ u(0,x) = h(x),\ x \in \mathbb{R}^n,\ t \in [0,\hat{T}],\end{cases}$$

Here, the Stratonovich integral "$\hat{\circ}$" is computed by

$$h_k(t,x)\hat{\circ}dw_k(t) = h_k(t,x)\cdot dw_k(t) - \frac{1}{2}[\partial_x h_k(t,x)]g_k(x)dt,\ k \in \{1,\ldots,m\},$$

using Itô integral "$\cdot$".

A direct inspection of the last two nonlinear SPDEs show us that the nonlinearity are contained in the drift part of both equations, and diffusion part is linear but contain the gradient with respect to $x \in \mathbb{R}^n$ of the unknown functions. Some subjects of this paper have been analysed also in [2], [3], [4] and [7], but under stronger hypotheses. Regarding some applications, one may notice that the drift in both SPDEs contains a nonlinearity $\langle \partial_x u_j(t,x), u(t,x)\rangle$, $j \in \{1,\dots,n\}$, as in Burger's equations from mechanics, provided $N = n$ and $f(x,u) = u \in \mathbb{R}^n$. If this is the case, the nonlinear system of SPDEs includes a Laplacian of each u_j ($\Delta_x u_j(t,x)$) in the drift, provided $m = n$ and $g_k = e_k$, $k \in \{1,\dots,n\}$, where $\{e_1,\dots,e_n\} \subset \mathbb{R}^n$ is the canonical basis. This allows us to say that the second SPDEs mentioned above stands for Burger's equations with stochastic martingale perturbations for which a classical solution can be constructed. The investigation of evolution equations with stochastic perturbations serves a large variety of applicability. It is well known the applicability of backward SDEs (BSDEs) in mathematical finance as in [1], [4] and [9], and in stochastic control with partial information as it is specified in [8]. Other applications of SPDEs, including finance, may be found in Da Prato and Tubaso (see [3]). In Buckdahn and Ma (see [2]) the authors consider a system of nonlinear SPDEs driven by Fisk-Stratonovich integrals with the diffusion term independent of the gradiendt of the solution, for which they prove the existence and uniqueness of the so called stochastic viscosity solution, introduced in [8] and relying on Doss-Sussmann type transformations applied in the corresponding stochastic characteristic system. This paper contains section 2 where the problems (P1)-(P5) are stated. Solutions for the problems (P1)-(P5) given in section 2 will be analysed in section 3 (see Lemmas 1, 2 and Theorems 1,2 and 3). The main result including the above given nonlinear SPDEs is contained in Theorems 2 and 3 solving the problems (P4) and (P5).

2 Quasilinear (H-J) equations with irregular perturbations. Statement of the problems

There are given smooth mappings $\varphi = (\varphi_1, \ldots, \varphi_N) \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n; \mathbb{R}^N)$, $f \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n \times \mathbb{R}^N; \mathbb{R}^n)$ and a finite set of smooth vector fields $\{g_1, \ldots, g_m\} \subset (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n; \mathbb{R}^n)$.

Associate two types of characteristic systems. According to some scalar bounded variation functions $\mu_k(t) : [0, T] \to \mathbb{R}$, which are piecewise continuous and $\mu_k(0) = 0$, we introduce the characteristic system

$$\begin{cases} d_t x(t; \lambda) = f(x(t; \lambda), \varphi(\lambda))dt + \sum_{k=1}^{m} g_k(x(t-; \lambda))d\mu_k(t), \ t \in [0, T], \\ x(0; \lambda) = \lambda \in \mathbb{R}^n. \end{cases} \tag{1}$$

For a given standard m-dimensional Wiener process $w(t) = (w_1(t), \ldots, w_m(t)) \in \mathbb{R}^m$, $t \in [0, T]$, over the complete filtered probability space $\{\Omega, \mathcal{F} \supset \{\mathcal{F}^t\}, P\}$ we consider the following characteristic system

$$\begin{cases} d_t x(t; \lambda) = f(x(t; \lambda), \varphi(\lambda))dt + \sum_{k=1}^{m} g_k(x(t; \lambda)) \circ dw_k(t), \ t \in [0, T], \\ x(0; \lambda) = \lambda \in \mathbb{R}^n, \end{cases} \tag{2}$$

where the Stratonovich integral "$\circ$" is computed by

$$g_k(x) \circ dw_k(t) = g_k(x) \cdot dw_k(t) + \frac{1}{2}[\partial_x g_k(x)]g_k(x)dt$$

using Itô integral "$\cdot$". The main assumptions which allow us to get classical solutions for our problems are the following:

(a) $\{g_1, \ldots, g_m\} \subset (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n; \mathbb{R}^n)$ commute using Lie bracket $[\cdot, \cdot]$;

(b) $[f_u, g_k] = 0$, $u \in \mathbb{R}^N$, $k \in \{1, \ldots, m\}$, where $f_u(x) := f(x, u)$ and $f \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n \times \mathbb{R}^N; \mathbb{R}^n)$ is given.

Denote by $\{G(\sigma)[x] : \sigma \in \mathbb{R}^m, x \in \mathbb{R}^n\}$ the unique smooth solution of the gradient system

$$\begin{cases} \partial_{\sigma_i} G(\sigma)[x] = g_i(G(\sigma)[x]), \ i \in \{1, \ldots, m\}, \ \sigma \in \mathbb{R}^m, \\ G(0)[x] = x \in \mathbb{R}^n, \end{cases} \tag{3}$$

and let $F_\varphi(t;\lambda) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ be the unique smooth solution satisfying ODEs

$$\begin{cases} \dfrac{dF_\varphi(t;\lambda)}{dt} = f(F_\varphi(t;\lambda), \varphi(\lambda)),\ t \in [0,T], \\ F_\varphi(0;\lambda) = \lambda \in \mathbb{R}^n. \end{cases} \tag{4}$$

Define a smooth deterministic mapping

$$S_\varphi(t,\sigma;\lambda) := G(\sigma)[F_\varphi(t;\lambda)],\ t \in [0,T],\ \lambda \in \mathbb{R}^n,\ \sigma \in \mathbb{R}^m \tag{5}$$

and consider two types of mappings. Let

$$x^1_\varphi(t;\lambda) := S_\varphi(t,\mu(t);\lambda),\ t \in [0,T] \tag{6}$$

be a bounded variation piecewise continuous application for each $\lambda \in \mathbb{R}^n$, and define

$$x^2_\varphi(t;\lambda) := S_\varphi(t,w(t);\lambda),\ t \in [0,T] \tag{7}$$

as a continuous $\mathcal{F}^t$-adapted process for each $\lambda \in \mathbb{R}^n$.

The problems we are going to study are the following:

(P1) Under the hypothesis (a)-(b), show that $\{x^1_\varphi(t;\lambda) \in \mathbb{R}^n : t \in [0,T]\}$ (see (6)) is the unique solution of the characteristic system (1) and $\{x^2_\varphi(t;\lambda) \in \mathbb{R}^n : t \in [0,T]\}$ is the unique continuous and $\mathcal{F}^t$-adapted solution of SDEs (2) (see (7)). The characteristic systems (1) and (2) will be called gradient characteristic systems (see the gradient representation (5) of the flow solution).

(P2) Under the hypothesis (a)-(b), find $0 < \hat{T} \leq T$ and a smooth mapping $\lambda = \hat{\psi}(t,z) : [0,\hat{T}] \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying functional equations $F_\varphi(t;\hat{\psi}(t,z)) = z \in \mathbb{R}^n$, $t \in [0,\hat{T}]$, $z \in \mathbb{R}^n$ (see ODEs (4)). In addition, show (if possible) that $\lambda = \hat{\psi}(t,z)$ satisfies the following quasilinear (H-J) equations

$$\begin{cases} \partial_t \hat{\psi}(t,z) + [\partial_z \hat{\psi}(t,z)] f(z, \varphi(\hat{\psi}(t,z))) = 0,\ t \in [0,\hat{T}], \\ \hat{\psi}(0,z) = z \in \mathbb{R}^n. \end{cases} \tag{8}$$

(P3) Under the hypothesis (a)-(b), prove (if possible) that $\lambda = \hat{\psi}(t, z_1(t,x)) := \psi^1(t,x)$, (see $z_1(t,x) = G(-\mu(t))[x]$, $t \in [0,\hat{T}]$, $x \in \mathbb{R}^n$), is a solution of the flow equation $x^1_\varphi(t;\lambda) =$

$x \in \mathbb{R}^n$, $t \in [0, \hat{T}]$ and $\lambda = \hat{\psi}(t, z_2(t,x)) := \psi^2(t,x)$, where (see $z_2(t,x) = G(-w(t))[x]$, $t \in [0,\hat{T}]$, $x \in \mathbb{R}^n$), is a solution of the flow equation $x^2_\varphi(t,\lambda) = x \in \mathbb{R}^n$, $t \in [0,\hat{T}]$.

(P4) Under the hypothesis (a)-(b), show (if possible) that $\{\lambda = \psi^1(t,x) : t \in [0,\hat{T}], x \in \mathbb{R}^n\}$ satisfies the following quasilinear (H-J) equations with bounded variation perturbations

$$\begin{cases} d_t\psi^1(t,x) + [\partial_x\psi^1(t,x)]f(x,\varphi(\psi^1(t,x)))dt + \displaystyle\sum_{k=1}^m \partial_x\psi^1(t-,x)g_k(x)d\mu_k(t) = 0, \\ \psi^1(0,x) = x \in \mathbb{R}^n,\ t \in [0,\hat{T}]. \end{cases} \tag{9}$$

In addition, $\lambda = \psi^2(t,x) \in \mathbb{R}^n : t \in [0,\hat{T}], x \in \mathbb{R}^n$ satisfies the following nonlinear SPDEs

$$\begin{cases} d_t\psi^2(t,x) + [\partial_x\psi^2(t,x)]f(x,\varphi(\psi^2(t,x)))dt + \displaystyle\sum_{k=1}^m \partial_x\psi^2(t,x)g_k(x)\hat{\circ}dwk(t) = 0, \\ \psi^2(0,x) = x \in \mathbb{R}^n,\ t \in [0,\hat{T}], \end{cases} \tag{10}$$

where, the Stratonovich integral "$\hat{\circ}$" is computed by

$$h_k(t,x)\hat{\circ}dw_k(t) = h_k(t,x)\cdot dw_k(t) - \frac{1}{2}[\partial_x h_k(t,x)]g_k(x)dt,\ k \in \{1,\ldots,m\},$$

using Itô integral "$\cdot$".

(P5) Under the hypothesis (a)-(b), describe the evolution of the functionals $u^1(t,x) = h(\psi^1(t,x))$ and $u^2(t,x) = h(\psi^2(t,x))$, $t \in [0,\hat{T}]$, $x \in \mathbb{R}^n$, for $f \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^N)$. In particular, $u^1(t,x) := \varphi(\psi^1(t,x))$, $t \in [0,\hat{T}]$, $x \in \mathbb{R}^n$, is a solution of the quasilinear (H-J) equations with bounded variation perturbations

$$\begin{cases} d_t u_j(t,x) + \langle \partial_x u_j(t,x), f(x,u(t,x))\rangle\, dt + \displaystyle\sum_{k=1}^m \langle \partial_x u_j(t-,x), g_k(x)\rangle\, d\mu_j(t) = 0, \\ u_j(0,x) = \varphi_j(x),\ j \in \{1,\ldots,N\},\ x \in \mathbb{R}^n,\ t \in [0,\hat{T}]. \end{cases} \tag{11}$$

and $u^2(t,x) := \varphi(\psi^2(t,x))$, $t \in [0,\hat{T}]$, $x \in \mathbb{R}^n$, is a classical solution of SPDEs

$$\begin{cases} d_t u_j(t,x) + \langle \partial_x u_j(t,x), f(x,u(t,x))\rangle\, dt + \displaystyle\sum_{k=1}^m \langle \partial_x u_j(t,x), g_k(x)\rangle\, \hat{\circ}dw_k(t) = 0, \\ u_j(0,x) = \varphi_j(x),\ j \in \{1,\ldots,N\},\ x \in \mathbb{R}^n,\ t \in [0,\hat{T}]. \end{cases} \tag{12}$$

Here, the Stratonovich integral "$\hat{\circ}$" is computed by

$$h_k(t,x)\hat{\circ}dw_k(t) = h_k(t,x)\cdot dw_k(t) - \frac{1}{2}\left\langle \partial_x h_k(t,x), g_k(x)\right\rangle dt,\ k\in\{1,\dots,m\},$$

using Itô integral "$\cdot$".

3 Solutions for (P1)-(P5)

Lemma 1. (solution for (P1)) *Under the hypothesis* (a)-(b)*, consider the smooth deterministic mapping $S_\varphi(t,\sigma;\lambda)\in\mathbb{R}^n$, $t\in[0,T]$, $\sigma\in\mathbb{R}^m$, $\lambda\in\mathbb{R}^n$ defined in* (5) *and associate the flows $x^1_\varphi(t;\lambda) = S_\varphi(t,\mu(t);\lambda)\in\mathbb{R}^n$ and $x^2_\varphi(t;\lambda) = S_\varphi(t,w(t);\lambda)$, $t\in[0,T]$ (for each $\lambda\in\mathbb{R}^n$) as in* (6) *and* (7)*. Then $\{x^1_\varphi(t;\lambda)\in\mathbb{R}^n : t\in[0,T],\lambda\in\mathbb{R}^n\}$ satisfies the bounded variation characteristic system* (1) *and $\{x^2_\varphi(t;\lambda)\in\mathbb{R}^n : t\in[0,T],\lambda\in\mathbb{R}^n\}$ is the unique continuous and $\mathcal{F}^t$-adapted solution of SDEs* (2)*.*

Proof. By definition (see (5)),

$$S_\varphi(t,\sigma;\lambda) = G(\sigma)[F_\varphi(t;\lambda)],\ t\in[0,T],\ \sigma\in\mathbb{R}^m \quad (\lambda\in\mathbb{R}^n \text{ fixed}) \tag{13}$$

is a continuously differentiable mapping of second order with respect to $t\in[0,T]$ and $\sigma\in\mathbb{R}^m$. As far as the m-orbit $\{G(\sigma)[x]\in\mathbb{R}^n : \sigma\in\mathbb{R}^m\}$ is the unique solution of the gradient system in (3), we get

$$\begin{cases} \partial_{\sigma_i} S_\varphi(t,\sigma;\lambda) = g_i(S_\varphi(t,\sigma);\lambda),\ i\in\{1,\dots,m\}, \\ \partial^2_{\sigma_i} S_\varphi(t,\sigma;\lambda) = [\partial_x g_i(S_\varphi(t,\sigma;\lambda))]g_i(S_\varphi(t,\sigma;\lambda)),\ i\in 1,\dots,m. \end{cases} \tag{14}$$

In addition, using $[f_u, g_k] = 0$, $k\in\{1,\dots,m\}$, $u\in\mathbb{R}^N$ (see the hypotheses (b)), and

$$G(-\sigma)[S_\varphi(t,\sigma;\lambda)] = F_\varphi(t;\lambda),\ t\in[0,T], \tag{15}$$

we compute $\partial_t[S_\varphi(t,\sigma;\lambda)]$ by the following formula

$$\partial_t[S_\varphi(t,\sigma;\lambda)] = [M(\sigma;x)]^{-1} f(G(-\sigma)[x],\varphi(\lambda)),\ \text{for } x = S_\varphi(t,\sigma;\lambda), \tag{16}$$

where $M(\sigma; x) = \partial_x(G(-\sigma)[x])$. Here $H(\sigma; x) := [M(\sigma; x)]^{-1}$ and $M(\sigma; x)$ satisfy the following linear gradient systems

$$\begin{cases} \partial_{\sigma_i} H(\sigma; x) = H(\sigma; x)[\partial_y g_i(G(-\sigma)[x])],\ i \in \{1, \dots, m\}, \\ \partial_{\sigma_i} M(\sigma; x) = -[\partial_y g_i(G(-\sigma)[x])]M(\sigma; x),\ i \in \{1, \dots, m\}, \end{cases} \tag{17}$$

and $H(0; x) = M(0; x) = I_n$ (identity $(n \times n)$ matrix). Notice that by taking straight derivations and using (17) we get

$$\begin{aligned} H(\sigma; x) f(G(-\sigma)[x], \varphi(\lambda)) &= f(x, \varphi(\lambda)) + \\ &\sum_{i=1}^{m} \sigma_i \int_0^1 H(\theta\sigma; x)[f_u, g_i](G(-\theta\sigma)[x]) d\theta = f(x, \varphi(\lambda)), \end{aligned} \tag{18}$$

for any $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$.

In particular, for $x = S_\varphi(t, \sigma; \lambda)$, the equalities in (18) allow us to rewrite (16) as follows

$$\partial_t[S_\varphi(t, \sigma; \lambda)] = f(S_\varphi(t, \sigma; \lambda), \varphi(\lambda)),\ t \in [0, T],\ \sigma \in \mathbb{R}^m,\ \lambda \in \mathbb{R}^n. \tag{19}$$

The dynamic satisfied by $x_\varphi^1(t; \lambda)$, $t \in [0, T]$, follows directly from (14) and (19), i.e.

$$\begin{cases} d_t x_\varphi^1(t; \lambda) = f(x_\varphi^1(t; \lambda), \varphi(\lambda)) dt + \displaystyle\sum_{k=1}^{m} g_k(x_\varphi^1(t-; \lambda)) d\mu_k(t), \\ x_\varphi^1(0; \lambda) = \lambda \in \mathbb{R}^n,\ t \in [0, T],\ \text{where } x(t-; \lambda) = \lim\limits_{t_n \uparrow t} x(t_n; \lambda). \end{cases} \tag{20}$$

As far as $x_\varphi^2 := S_\varphi(t, w(t); \lambda)$, $t \in [0, T]$, is concerned, applying the standard rule of stochastic derivation and using (14) and (19), we obtain SDEs

$$\begin{cases} d_t x_\varphi^2(t; \lambda) = f(x_\varphi^2(t; \lambda), \varphi(\lambda)) dt + \displaystyle\sum_{k=1}^{m} g_k(x_\varphi^2(t; \lambda)) \circ dw_k(t), \\ x_\varphi^2(0; \lambda) = \lambda \in \mathbb{R}^n,\ t \in [0, T], \end{cases} \tag{21}$$

where the Stratonovich integral "$\circ$"is computed by

$$g_i(x) \circ dw_i(t) = g_i(x) \cdot dw_i(t) + \frac{1}{2}[\partial_x g_i(x) g_i(x) dt,\ 1 \le i \le m,$$

using Itô integral "$\cdot$". The proof is complete. □

Lemma 2. (solution for (P2)) *Under the hypothesis (a)-(b), there exist* $0 < \hat{T} \leq T$ *and a unique smooth deterministic mapping* $\{\lambda = \hat{\psi}(t,z) \in \mathbb{R}^n : t \in [0,\hat{T}], z \in \mathbb{R}^n\}$ *satisfying functional equations*

$$F_\varphi(t;\hat{\psi}(t,z)) = z \in \mathbb{R}^n,\ t \in [0,\hat{T}], z \in \mathbb{R}^n.$$

In addition, $\hat{\psi} \in \mathcal{C}^{1,2}([0,\hat{T}] \times \mathbb{R}^n;\mathbb{R}^n)$ *satisfies (H-J) equations* (8)

Proof. By definition, $\{F_\varphi(t;\lambda) \in \mathbb{R}^n : t \in [0,T], \lambda \in \mathbb{R}^n\}$ satisfies ODEs (4) which can be written as integral equations

$$F_\varphi(t;\lambda) = \lambda + \int_0^t f(F_\varphi(s;\lambda),\varphi(\lambda))ds,\ t \in [0,T], \lambda \in \mathbb{R}^n \tag{22}$$

and functional equations $F_\varphi(t;\lambda) = z \in \mathbb{R}^n$ can be written as

$$F_\varphi(t;\lambda) = z \Leftrightarrow \lambda = z - \int_0^t f(F_\varphi(s;\lambda),\varphi(\lambda))ds := \hat{V}(t,z;\lambda),\ t \in [0,T], \lambda \in \mathbb{R}^n. \tag{23}$$

Here the smooth mapping $\hat{V}(t,z;\lambda) : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\hat{V}(t,z;\lambda) = z - \int_0^t f(F_\varphi(s;\lambda),\varphi(\lambda))ds \tag{24}$$

and the functional equations

$$\lambda = \hat{V}(t,z;\lambda) \tag{25}$$

will be solved by applying Banach fixed point theorem. In this respect, we are looking for $0 < \hat{T} \leq T$ sufficiently small such that

$$|\hat{V}(t,z;\lambda'') - \hat{V}(t,z;\lambda')| \leq \rho|\lambda'' - \lambda'|,\ t \in [0,\hat{T}],\ z \in \mathbb{R}^n, \lambda', \lambda'' \in \mathbb{R}^n, \tag{26}$$

for some constant $\rho \in [0,1)$ which shows that the nonlinear mapping $\hat{V}(t,z;\lambda)$ is a contractive one with respect to $\lambda \in \mathbb{R}^n$ and uniformly of $t \in [0,\hat{T}]$, $z \in \mathbb{R}$. For proving (26), define (see $f \in (\mathcal{C}_b^1 \cap \mathbb{C}^2)(\mathbb{R}^n \times \mathbb{R}^N;\mathbb{R}^n)$)

$$K_1 = \sup\{|\partial_u f(x,\varphi(\lambda))||\partial_\lambda \varphi(\lambda)| : \lambda \in \mathbb{R}^n, x \in \mathbb{R}^n\},$$
$$K_2 = \sup\{|\partial_x f(x,\varphi(\lambda))| : \lambda \in \mathbb{R}^n, x \in \mathbb{R}^n\},$$

and $K_3(T) = (1 + K_1 T) \exp K_2 T$. Then using (22) and (24), by a direct computation, we get

$$|F_\varphi(t;\lambda'') - F_\varphi(t;\lambda')| \leq K_3(T)|\lambda'' - \lambda'|,\ t \in [0,T],\ \lambda', \lambda'' \in \mathbb{R}^n, \tag{27}$$

$$|\hat{V}(t,z;\lambda'') - \hat{V}(t,z;\lambda')| \leq T[K_1 + K_2 K_3(T)]|\lambda'' - \lambda'|,\ t \in [0,T], z \in \mathbb{R}^n, \lambda', \lambda'' \in \mathbb{R}^n. \tag{28}$$

Take $0 < \hat{T} \leq T$ sufficiently small such that

$$\rho = \hat{T}[K_1 + K_2 K_3(T)] \in [0,1) \tag{29}$$

and it yelds (see (28))

$$|\hat{V}(t,z;\lambda'') - \hat{V}(t,z;\lambda')| \leq \rho|\lambda'' - \lambda'|,\ t \in [0,\hat{T}],\ z \in \mathbb{R}^n, \lambda'', \lambda' \in \mathbb{R}^n. \tag{30}$$

The contractive property of the mapping $\{\hat{V}(t,z;\lambda) : \lambda \in \mathbb{R}^n\}$ found in (30) allow us to get the fixed point $\lambda = \hat{\psi}(t,z) \in \mathbb{R}^n$ (satisfying (25)) as a limit of a convergent sequence

$$\hat{\psi}(t,z) = \lim_{k\to\infty} \lambda_k(t,z),\ \hat{\psi}(0,z) = z,\ t \in [0,\hat{T}],\ z \in \mathbb{R}^n \tag{31}$$

where

$$\lambda_0(t,z) = z,\ \lambda_{k+1}(t,z) = \hat{V}(t,z;\lambda_k(t,z)),\ k \geq 0. \tag{32}$$

The smooth mapping $\hat{\psi} \in \mathcal{C}^{1,2}([0,\hat{T}] \times \mathbb{R}^n; \mathbb{R}^n)$ is the unique solution of the functional equations (25) and it lead us to

$$\hat{\psi}(t, F_\varphi(t;\lambda)) = \lambda,\ t \in [0,\hat{T}], \tag{33}$$

showing that $\{\hat{\psi}(t,z) \in \mathbb{R}^n : t \in [0,\hat{T}], z \in \mathbb{R}^n\}$ is a fundamental system of first integral associated with the flow $\{F_\varphi(t;\lambda)\}$. Taking derivative with respect to the variable t, from (33), we get

$$\partial_t\hat{\psi}(t, F_\varphi(t;\lambda)) + \partial_z\hat{\psi}(t, F_\varphi(t;\lambda)) f(F_\varphi(t;\lambda), \varphi(\lambda)) = 0,\ t \in [0,\hat{T}] \tag{34}$$

for each $\lambda \in \mathbb{R}^n$ and, in particular, for $\lambda = \hat{\psi}(t,z)$ from (34) and $\hat{\psi}(0,z) = z$ we obtain (H-J) equations (8). The proof is complete. $\square$

Theorem 1. (solution for P3) *Under the same hypotheses as in Lemmas 1 and 2, consider the flows* $\{x^i_\varphi(t,\lambda) \in \mathbb{R}^n : t \in [0,\hat{T}], \lambda \in \mathbb{R}^n\}$, $i \in \{1,2\}$, *defined in Lemma 1 and satisfying the characteristic systems* (1) *and* (2) *correspondingly. Let the smooth deterministic mapping* $\hat{\psi} \in \mathcal{C}^{1,2}([0,\hat{T}]\times\mathbb{R}^n;\mathbb{R}^n)$ *be defined as in Lemma 2. Define* $z^1(t,x) = G(-\mu(t))[x] \in \mathbb{R}^n$ $z^2(t,x) = G(-w(t))[x] \in \mathbb{R}^n$, $t \in [0,\hat{T}]$, $x \in \mathbb{R}^n$, *and*

$$\lambda = \psi^1(t,x) := \hat{\psi}(t, z^1(t,x)) \in \mathbb{R}^n,\ \lambda = \psi^2(t,x) := \hat{\psi}(t, z^2(t,x)) \in \mathbb{R}^n,\ t \in [0,\hat{T}],\ x \in \mathbb{R}^n. \tag{35}$$

Then the bounded variation mapping $\{\psi^1(t,x) : t \in [0,\hat{T}]\}$ *satisfies functional equations*

$$x^1_\varphi(t, \psi^1(t,x)) = x,\ \psi^1(t, x^1_\varphi(t,\lambda)) = \lambda,\ t \in [0,\hat{T}],\ \lambda \in \mathbb{R}^n,\ x \in \mathbb{R}^n, \tag{36}$$

and the continuous and $\mathcal{F}^t$*-adapted process* $\{\lambda = \psi^2(t,x) : t \in [0,\hat{T}]\}$ *satisfies pathwisely the following equations*

$$x^2_\varphi(t, \psi^2(t,x)) = x,\ \psi^2(t, x^2_\varphi(t,\lambda)) = \lambda,\ t \in [0,\hat{T}],\ \lambda \in \mathbb{R}^n,\ x \in \mathbb{R}^n. \tag{37}$$

Proof. By definition, $\lambda = \hat{\psi} \in \mathcal{C}^{1,2}([0,\hat{T}] \times \mathbb{R}^n;\mathbb{R}^n)$ is the unique smooth deterministic mapping satisfying functional equations $F_\varphi(t;\hat{\psi}(t,z)) = z$, $t \in [0,\hat{T}]$, $z \in \mathbb{R}^n$. Replacing $z = z^1(t,x) \in \mathbb{R}^n$ into the above equations, we get $F_\varphi(t,\psi^1(t,x)) = G(-\mu(t))[x]$ and $x^1_\varphi(t;\psi^1(t,x)) = x$, $t \in [0,\hat{T}]$, for each $x \in \mathbb{R}^n$. As far as $z^1(t, x^1_\varphi(t;\lambda)) = F_\varphi(t;\lambda)$, $t \in [0,\hat{T}]$, it yelds $\psi^1(t, x^1_\varphi(t;\lambda)) = \lambda$, $t \in [0,\hat{T}]$ and the conclusion (36) is proved. Similar arguments will be used to get (37) and noticing that $\psi^2(t, x^2_\varphi(t;\lambda)) = \lambda$, $x^2_\varphi(t,\psi^2(t,x)) = x$, $t \in [0,\hat{T}]$, we obtain (37). The proof is complete. □

Theorem 2. (solution for P4) *Under the hypotheses of Theorem 1, consider the mappings* $\{\lambda = \psi^i(t,x) \in \mathbb{R}^n : t \in [0,\hat{T}], x \in \mathbb{R}^n\}$, $i \in \{1,2\}$, *as defined in* (35). *Then* $\{\psi^1(t,x)\}$ *satisfies the following nonlinear (H-J) equations with bounded variation perturbations (see* (9)*)*

$$\begin{cases} d_t\psi^1(t,x) + [\partial_x\psi^1(t,x)]f(x,\varphi(\psi^1(t,x)))dt + \sum_{j=1}^m [\partial_x\psi^1(t-,x)]g_j(x)d\mu_j(t) = 0, \\ \psi^1(0,x) = x \in \mathbb{R}^n,\ t \in [0,\hat{T}],\ \textit{where } h(t-,x) = \lim_{t_n \uparrow t} h(t_n,x). \end{cases} \tag{38}$$

In addition, the continuous and $\mathcal{F}^t$-adapted process $\{\psi^2(t,x)\}$ satisfies the nonlinear SPDEs (see (10)*)*

$$\begin{cases} d_t\psi^2(t,x) + [\partial_x\psi^2(t,x)]f(x,\varphi(\psi^2(t,x)))dt + \sum_{j=1}^{m}([\partial_x\psi^2(t,x)]g_j(x))\hat{\circ}dw_j(t) = 0, \\ \psi^2(0,x) = x \in \mathbb{R}^n,\ t \in [0,\hat{T}]. \end{cases} \tag{39}$$

Proof. Applying the standard rule of deterministic derivation associated with the test functions $\hat{\psi} \in \mathcal{C}^{1,2}([0,\hat{T}] \times \mathbb{R}^n; \mathbb{R}^n)$ and bounded variation process $z^1(t,x) = G(-\mu(t))[x]$, $t \in [0,T]$, we get

$$\begin{cases} d_t\psi^1(t,x) = \partial_t\hat{\psi}(t,z^1(t,x))dt + [\partial_z\hat{\psi}(t,z^1(t-,x))]d_tz^1(t,x) \\ \qquad\qquad\quad = \partial_t\hat{\psi}(t,z^1(t,x))dt - \sum_{j=1}^{m}[\partial_z\hat{\psi}(t,z^1(t-,x))]g_j(z^1(t-,x))d\mu_j(t) \\ \qquad\qquad\quad = E_1(t,x) + E_2(t,x), \\ \psi^1(0,x) = x \in \mathbb{R}^n,\ t \in [0,\hat{T}]. \end{cases} \tag{40}$$

Here, $E_1(t,x) := \partial_t\hat{\psi}(t,z^1(t,x))$ where $\hat{\psi} \in \mathcal{C}^{1,2}([0,\hat{T}] \times \mathbb{R}^n; \mathbb{R}^n)$ satisfies (H-J) equations (8) and yelds

$$\begin{aligned} E_1(t,x) &= -\partial_z\hat{\psi}(t,z^1(t,x))f(z^1(t,x),\varphi(\psi^1(t,x)))dt \\ &= -\partial_x\psi^1(t,x)[\partial_xz^1(t,x)]^{-1}f(z^1(t,x),\varphi(\psi^1(t,x)))dt. \end{aligned} \tag{41}$$

Using the hypothesis $[f_u, g_k] = 0$, $k \in \{1,\ldots,m\}$, $u \in \mathbb{R}^N$, we notice that the product $\{\partial_xG(-\sigma)[x]\}^{-1}f(G(-\sigma)[x],u)$ equals $f(x,u)$ (see (18) of Lemma 1) and, in particular for $\sigma = -\mu(t) \in \mathbb{R}^m$, we may and do write

$$E_1(t,x) = -[\partial_x\psi^1(t,,x)]f(x,\varphi(\psi^1(t,x)))dt,\ t \in [0,\hat{T}]. \tag{42}$$

In addition, using similar arguments, we rewrite $E_2(t,x)$ as follows,

$$E_2(t,x) = -\sum_{j=1}^{m}[\partial_x\psi^1(t-,x)]g_j(x)d\mu_j(t),\ t \in [0,\hat{T}],\ x \in \mathbb{R}^n \tag{43}$$

and (40) coincides with the conclusion (38), i.e.

$$\begin{cases} d_t\psi^1(t,x) + [\partial_x\psi^1(t,x)]f(x,\varphi(\psi^1(t,x)))dt + \sum_{j=1}^{m}[\partial_x\psi^1(t-,x)]g_j(x)d\mu_j(t) = 0, \\ \psi^1(0,x) = x \in \mathbb{R}^n,\ t \in [0,\hat{T}]. \end{cases} \tag{44}$$

Regarding the conclusion (39), we get the result by applying the standard rule of stochastic derivation associated with the test function $\hat{\psi} \in \mathcal{C}^{1,2}([0,\hat{T}] \times \mathbb{R}^n; \mathbb{R}^n)$ and continuous $\mathcal{F}^t$-adapted process $z^2(t,x) = G(-w(t))[x]$, $t \in [0,\hat{T}]$, $x \in \mathbb{R}^n$. The following SPDEs of parabolic type

$$\begin{cases} d_t z^2(t,x) + \sum_{j=1}^{m}([\partial_x z^2(t,x)]g_j(x))\hat{\circ} dw_j(t) = 0, \\ z^2(0,x) = x \in \mathbb{R}^n,\, t \in [0,\hat{T}] \end{cases} \tag{45}$$

is valid for $\{z^2(t,x) : t \in [0,\hat{T}], x \in \mathbb{R}^n\}$ which can be deduced by applying the standard rule of stochastic derivation associated with the smooth deterministic mapping $S(\sigma)[x] = G(-\sigma)[x]$ when $\sigma = w(t) \in \mathbb{R}^m$, $t \in [0,\hat{T}]$. In this respect, notice that

$$\begin{cases} \partial_{\sigma_k} S(\sigma)[x] = -\partial_x\{S(\sigma)[x]\}g_k(x), \\ \partial^2_{\sigma_k}[x] = \partial_{\sigma_k}\{\partial_{\sigma_k}\{S(\sigma)[x]\}\} = \{\partial_x[\partial_x\{S(\sigma)[x]\}g_k(x)]\}g_k(x) \end{cases} \tag{46}$$

for any $\sigma \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$ which allow us to rewrite

$$d_t z^2(t,x) = \sum_{j=1}^{m} \partial_{\sigma_k}\{S(\sigma)[x]\}_{\sigma=w(t)} \cdot dw(t) + \frac{1}{2}\sum_{k=1}^{m} \partial^2_{\sigma_k}\{S(\sigma)[x]\}_{\sigma=w(t)} dt \tag{47}$$

as in (45). Here the Stratonovich integral "$\hat{\circ}$" is computed by

$$h_j(t,x)\hat{\circ} dw_j(t) = h_j(t,x) \cdot dw_j(t) - \frac{1}{2}\partial_x h_j(t,x) g_j(x) dt,\, j \in \{1,\ldots,m\}, \tag{48}$$

using Itô integral "$\cdot$". By definition, $\psi^2(t,x) = \hat{\psi}(t, z^2(t,x))$, $t \in [0,\hat{T}]$, and applying standard rule of stochastic derivation, we get

$$\begin{cases} d_t \psi^2_j(t,x) = \partial_t \hat{\psi}_j(t, z^2(t,x))dt + \partial_z \hat{\psi}_j(t, z^2(t,x)) d_t z^2(t,x) + \\ \qquad + \frac{1}{2}[\sum_{k=1}^{m} \langle \partial^2_z \hat{\psi}_j(t, z^2(t,x)) h_k(t,x), h_k(t,x)\rangle] dt,\, j \in \{1,\ldots,n\}, \\ \psi^2(0,x) = (\psi^2_1(0,x),\ldots,\psi^2_n(0,x)) = x \in \mathbb{R}^n,\, t \in [0,\hat{T}], \end{cases} \tag{49}$$

where $h_k(t,x) = -\partial_x z^2(t,x) g_k(x)$, $k \in \{1,\ldots,m\}$, and Itô equations (45), (48) are used.

The last two terms in (49) can be rewritten as follows

$$\begin{aligned}
&\sum_{k=1}^{m}(\langle\partial_z\hat{\psi}_j(t,z^2(t,x)),h_k(t,x)\rangle\cdot dw_k(t)-\frac{1}{2}\langle\partial_z\hat{\psi}_j(t,z^2(t,x)),\partial_x h_k(t,x)g_k(x)\rangle dt)+\\
&\qquad+\frac{1}{2}[\sum_{k=1}^{m}\langle\partial_z^2\hat{\psi}_j(t,z^2(t,x))h_k(t,x),h_k(t,x)\rangle]dt=\\
&\qquad\quad-\sum_{k=1}^{m}\langle\partial_x\psi_j^2(t,x),g_k(x)\rangle\hat{\circ}dw_k(t),\ t\in[0,\hat{T}],x\in\mathbb{R}^n.
\end{aligned}\tag{50}$$

In addition, using similar arguments as for writting $E_1(t,x)$ (see (42)) we get that the first term in the right-hand side of (49) has the following form

$$\begin{aligned}
\partial_t\hat{\psi}_j(t,z^2(t,x))dt&=-\langle\partial_z\hat{\psi}_j(t,z^2(t,x)),f(z^2(t,x),\varphi(\psi^2(t,x)))\rangle dt\\
&\quad-\langle\partial_x\psi_j^2(t,x),f(x,\varphi(\psi^2(t,x)))\rangle dt,\ t\in[0,\hat{T}],\ x\in\mathbb{R}^n.
\end{aligned}\tag{51}$$

Using (50) and (49) we get the conclusion (39) satisfied and the proof is complete. □

Theorem 3. (Solution for P5) *Under the same hypotheses as in Theorem 2, consider the mappings* $\{\lambda=\psi^i(t,x)\in\mathbb{R}^n : t\in[0,\hat{T}],x\in\mathbb{R}^n\}$, $i\in\{1,2\}$, *satisfying nonlinear (H-J) equations with irregular perturbations* (38) *and* (39). *Let* $h\in(\mathcal{C}_b^1\cap\mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^N)$ *be fixed and define* $u^i(t,x):=h(\psi^i(t,x))$, $t\in[0,\hat{T}]$, $x\in\mathbb{R}^n$. *Then the following (H-J) equations are satisfied*

$$\begin{cases}
d_tu^1(t,x)+[\partial_xu^1(t,x)]f(x,\varphi(\psi^1(t,x)))dt+\sum_{k=1}^{m}[\partial_xu^1(t-,x)]g_k(x)d\mu_k(t)=0,\\
u^1(0,x)=h(x),\ x\in\mathbb{R}^n,\ t\in[0,\hat{T}],
\end{cases}\tag{52}$$

$$\begin{cases}
d_tu^2(t,x)+[\partial_xu^2(t,x)]f(x,\varphi(\psi^2(t,x)))dt+\sum_{k=1}^{m}[\partial_xu^2(t,x)]g_k(x)\hat{\circ}dw_k(t)=0,\\
u^2(0,x)=h(x),\ x\in\mathbb{R}^n,\ t\in[0,\hat{T}],
\end{cases}\tag{53}$$

In particular, $u^i(t,x)=\varphi(\psi^i(t,x))\in\mathbb{R}^n$, $i\in\{1,2\}$, $t\in[0,\hat{T}]$, $x\in\mathbb{R}^n$ *(see* $\varphi\in(\mathcal{C}_b^1\cap\mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^N)$ *and (H-J) equations* (11), (12)*) satisfies (H-J) equations* (11) *and* (12) *correspondingly.*

Proof. By definition, $u^i(t,x) = h(\hat{\psi}(t, z^i(t,x)))$, $t \in [0,\hat{T}]$, $x \in \mathbb{R}^n$ and rewrite it as

$$u^i(t,x) = \hat{h}(t, z^i(t,x)),\ t \in [0,\hat{T}],\ x \in \mathbb{R}^n,\ i \in \{1,2\}, \tag{54}$$

where the smooth deterministic mapping $\hat{h} \in \mathcal{C}^{1,2}([0,\hat{T}] \times \mathbb{R}^n; \mathbb{R}^N)$ is defined by

$$\hat{h}(t,x) = h(\hat{\psi}(t,z)) \in \mathbb{R}^n,\ t \in [0,\hat{T}],\ z \in \mathbb{R}^n. \tag{55}$$

Using similar arguments as in the proof of Theorem 2 (see conclusions (38) and (39)) we get the conclusions (52) and (53) as consequences of the standard deterministic and stochastic rule of derivations. As far as $\hat{u}^i(t,x) = \varphi(\psi^i(t,x))$, $i \in \{1,2\}$, (see $\varphi \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n; \mathbb{R}^N)$) enters the nonlinear vector field $f(x, \varphi(\psi^i(t,x)))$, the conclusions (52) and (53) become (H-J) equations (11) and (12) for $\{\hat{u}^i(t,x)\}$. The proof is complete. $\square$

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