Quasiidentities of Finitely Generated Nilpotent A-loop

by

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Abstract

The paper proves that a finitely generated nilpotent A-loop has a finite basis of quasiidentities, if and only if it is a finite Abelian group.

Key words: loop, nilpotent, basis, quasiidentities, associator, commutator.

Recently special attention has been paid to the theory of quasiidentities developed by A. I. Mal'cev in the '60s of the past century, due to the fact that it has various relations with the mathematical logic, theory of lattices and computer programming. For the latter areas the following problem is of particular importance: when has a given algebraic system with a finite signature a finite basis of quasiidentities? The importance of this problem is also mentioned for theoretical programming by R. McKenzie in the summary of his work [1]. Currently the problem of the finite basis is solved for finite groups (A. Iu. Olishanski [2]), finite associative rings (V. P. Belkin [3]), finitely generated Moufang commutative loop and finitely generated nilpotent Moufang loop (V. I. Ursu [4], [5]). W. Dziobiak [6] extended the condition of the need for V. P. Belkin's result in the case of finite non-associative rings.

This paper proves that a finitely generated nilpotent A-loop has a finite basis of quasiidentities if and only if it is a finite Abelian group. Moreover, if a finite nilpotent A-loop is not commutative or associative, it has no basis of quasiidentities from a finite number of variables. On the basis of this result we found that the lattice of all subquasivarieties of the quasivariety generated by a finitely generated nilpotent A-loop is finite or continuum.

By *loop* we mean an algebra L with an operation of multiplication $\cdot$ and two operations of division $/$, $\backslash$, where there is an element $e \in L$, that for any elements $x, y \in L$ the following equalities hold true

$$ex = xe = x;\ (xy)/y = y\backslash(yx) = (x/y)y = y(y\backslash x) = x.$$

Further in this paper the element e will represent a *unit element* of the loop and $x^{-1} = e/x$.

Let L be a certain loop and H its subloop. H is called a *normal subloop* in L, if for any $x, y \in L$ the following equalities hold true

$$x \cdot H = H \cdot x,\ x \cdot yH = xy \cdot H,\ Hx \cdot y = H \cdot xy.$$

The set

$$Z(L) = \{a \in L \mid ax \cdot y = a \cdot xy,\ x \cdot ya = xy \cdot a,\ ax = xa \text{ for any } x, y \in L\}$$

is called the *centre* of the loop L. It is easy to note that the centre $Z(L)$ is a normal subloop in the loop L.

Let a be a certain element of the loop L. The left and right translations L_a and R_a are defined by the equalities

$$xL_a = ax,\ xR_a = xa;\ x \in L.$$

The group $M(L)$, generated by all translations with the form L_a and R_a, $a \in L$, is called *multiplicative group* of the loop L. The element $\alpha \in M(L)$ is called an *internal substitution*, if $e\alpha = e$, i.e. α applies the unit e of the loop L on itself. All internal substitutions of the loop L form a subgroup in $M(L)$, called the group of internal substitutions of the loop L, which will be denoted by J(L). It is known (see [7]) that the group $J(L)$ is generated by all substitutions of the form

$$R_{x,y} = R_x R_y R_{xy}^{-1},\ L_{x,y} = L_y L_x L_{xy}^{-1},\ T_x = R_x L_x^{-1}\ (x, y \in L).$$

The subloop H of the loop L is normal when and only when $H\alpha = H$ for any $\alpha \in J(L)$ (see [7]-[8]). If all internal substitutions of the loop L are automorphisms, then L is called *$\acute{R}$-loop* [7]-[9].

Let

$$L_0 \supseteq L_1 \supseteq \ldots \supseteq L_n \supseteq \ldots$$

be the central descendant range of the loop L, i.e. $L_0 = L$, and for any $n > 0$, L_n/L_{n+1} is the subloop from the centre of the factor loop L/L_{n+1}. The loop L is called *nilpotent* (or *centrally nilpotent*) if there is such a natural number n that $L_n = \{e\}$. The least natural number n for which $L_n = \{e\}$ is called the *class of nilpotence* of L.

Let L be a nilpotent A-loop of class 2 and x, y and z elements from L. The element $(x,\ y,\ z)x\backslash((xy \cdot z)/(yz))$ is called the associator of the elements $x,\ y,\ z \in L$; the element $[x,\ y] = x/(y/xy)$ is called the commutator of elements $x,\ y \in L$. The subloop of the loop L, generated by the set

$$\{[x,\, y],\, (x,\, y,\, z) \mid x,\, y,\, z \in L\}$$

is called the *commutator* of the A-loop L and will be denoted by L'.

For any loop L with the symbol $\Phi(L)$ we will mark the Frattini subloop of the loop L, i.e. the intersection of all maximal subloops of the loop L [8].

We also remind that if L is a di-associative (i.e. any of its two elements generate a group) and commutative A-loop, then L is a Moufang loop. [10].

According to [11] in any nilpotent A-loop of class 2 the following identities hold true

$$[x \cdot y,\, z] = [x,\, z] \cdot [y,\, z],$$

$$[x,\, y \cdot z] = [x,\, y] \cdot [x,\, z],$$

$$(x,\, y,\, z) = (y,\, x,\, z) \cdot (x,\, z,\, y),$$

$$(x,\, y,\, x) = e,$$

$$(x \cdot y,\, z,\, t) = (x,\, z,\, t) \cdot (y,\, z,\, t),$$

$$(x,\, y \cdot z,\, t) = (x,\, y,\, t) \cdot (x,\, z,\, t),$$

$$(x,\, y,\, z \cdot t) = (x,\, y,\, z) \cdot (x,\, y,\, t)$$

which will be used further by default.

For any class K of A-loops, we denote by $P(K)$ the class of all Cartesian products of loops from K, by $S(K)$ we denote the class of all subloops of the loops from K and by $H(K)$ - the class of all homomorphic images of the loops from the class K.

The smallest quasivariety (respectively, variety) generated by A-loop L is called quasivariety (respectively, variety) generated by A-loop L and is denoted by $Q(L)$ (respectively, $V(L)$).

It is said that the quasiidentities (respectively, identities) of the loop L have a finite basis if there is a finite subset Σ of quasiidentities (respectively, identities) of the signature of the loop so that if a in a loop B, any formula from Σ, holds true, then B belongs to the quasivariety (respectively, variety) generated by loop L.

Let A and B be two A-loops, $a \in Z(A)$ and $b \in Z(B)$ - elements of the same order. We will denote the factor-loop $A \times B/lp(a/b)$ by $A \times B(a = b)$. It is easy to note that there are canonical isomorphisms of inclusion $\varphi : A \to A \times B(a = b)$ and $\psi : B \to A \times B(a = b)$ so that

$$A \times B(a = b) = A^{\varphi} \cdot B^{\psi},\ A^{\varphi} \cap B^{\psi} = lp(a^{\varphi}),$$

In this case we will consider A and A^{φ}, B and B^{ψ} the same loops and instead of $A \times B(a = b)$ we will write AB.

Theorem. *The quasiidentities of the finitely generated nilpotent A-loop have a finite basis of quasiidentities if and only if it is a finite Abelian group.*

Proof. Indeed, if L is a finite Abelian group, then according to Birkhoff Theorem [12] the variety $V(L) = HSP(I(L))$, where $I(L)$ is the class of all loops isomorphic to L. Any cyclic group $C \in SP(I(L))$ is finite and therefore any of its homomorphic images is isomorphic to one of its subgroups. Therefore, all cyclic groups from $V(L)$ belong to the quasivariety $Q(L)$. Now if L is a finitely generated group from $V(L)$, then L is finite and (according to Theorem 8.1.2 from [13]) $L = L_1 \cdot \ldots \cdot L_m$ is a direct product of cyclic subgroups L_i, $i = 1, \ldots, m$. As $L_i \in Q(L)$, $i = 1, \ldots, m$, we have $L \in Q(L)$. Therefore $Q(L) = V(L)$. Hence, the quasiidentities of L have a finite basis and it coincides with the finite basis of the true quasiidentities in the finite abelian group L.

Conversely, let L be a finitely generated nilpotent loop, whose quasiidentities have a finite basis. According to [11], the nilpotent A-loop is monoassociative and the periodical nilpotent A-loop is locally finite. Therefore, if we suppose that L is not finite, then L contains a cyclically infinite group.

According to Theorem 2.6 [11], the loop L satisfies the condition of maximality for subloops and, thus, the total number of prime numbers p for which L contains a p-cyclical group is finite. This, in line with [14], means that the quasiidentities, which hold true in L, have an infinite and independent basis of quasiidentities - contradiction with the hypothesis. Hence, the nilpotent A-loop L is finite. Suppose that L is non-associative or non-commutative. Then L contains a non-associative or non-commutative p-subloop H. Suppose the A-loop H is di-associative. If H is commutative, then, according to [10], H is a Moufang loop; if, though, it is not commutative, then we can consider H as generated by two elements, hence a non-commutative p-group. In this case, according to [4] and [2] respectively, it results that the quasiidentities of the finite loop L do not have a finite basis - contradiction. Suppose now that the subloop H is not di-associative and $exp(H) = p^{\alpha}$. Then we can consider H finitely represented as follows:

$$H = lp(a, b \mid\mid a^{p^{\alpha}} = e,\ b^{p^{\alpha}} = e,\ (a,\ a,\ b)^{p^{\beta}} = e,\ (a,\ b,\ b)^{p^{\gamma}} = e,\ [a,\ b]^{p^{\delta}} = e),$$

where the positive integers α, β, γ and δ satisfy the conditions $\alpha \geq \beta$, $\alpha \geq \gamma$, $\alpha \geq \delta$. Suppose $\beta \geq \gamma$ (the case $\beta \leq \gamma$ is verified by analogy). If

$\delta > \beta$, then for the elements $a_1 = a$, $a_2 = ab^{p^{\delta-1}}$ the following equalities hold true

$$(a_1,\, a_1,\, a_2) = (a,\, a,\, ab^{p^{\delta-1}}) = (a,\, a,\, b)^{p^{\delta-1}} = e,$$

$$(a_1,\, a_2,\, a_2) = (a,\, ab^{p^{\delta-1}},\, ab^{p^{\delta-1}}) = (a,\, a,\, b^{p^{\delta-1}})(a,\, b^{p^{\delta-1}},\, b^{p^{\delta-1}}) = (a,\, a,\, b)^{p^{\delta-1}} \cdot (a,\, b,\, b)^{p^{2(\delta-1)}} = e,$$

$$[a_1,\, a_2]^p = [a,\, ab^{p^{\delta-1}}]^p = [a,\, b]^{p^{\delta}} = e.$$

Hence, the subloop $N = lp(a_1,\, a_2)$ of the loop L is a nilpotent non-commutative p-group and as in the previous case we come to a contradiction. Let $\beta \geq \delta$. In this case for elements $a_1 = a$, $a_2 = ab^{p^{\beta-1}}$ we have

$$(a_1,\, a_1,\, a_2)^p = (a,\, a,\, a \cdot b^{p^{\beta-1}})^p = (a,\, a,\, b^{p^{\beta-1}})^p = (a,\, a,\, b)^{p^{\beta}} = e,$$

$$(a_1,\, a_2,\, a_2)^p = (a,\, a \cdot b^{p^{\beta-1}},\, a \cdot b^{p^{\beta-1}})^p = (a,\, a,\, b^{p^{\beta-1}})^p \cdot (a,\, b^{p^{\beta-1}},\, b^{p^{\beta-1}})^p = (a,\, a,\, b)^{p^{\beta}} \cdot (a,\, b,\, b)^{p^{2\beta-1}} = e$$

and

$$[a_1,\, a_2]^p = [a,\, a \cdot b^{p^{\beta-1}}]^p = [a,\, b]^{p^{\beta}} = e.$$

Hence, the subloop $N = lp(a_1,\, a_2)$ of the loop L is a non-associative A-loop, whose generators a_1, a_2 satifsy the conditions $(a_1,\, a_1,\, a_2) \neq e$, $a_1^{p^{\alpha}} = e$, $a_2^{p^{\alpha}} = e$, $(a_1,\, a_1,\, a_2)^p = e$, $(a_1,\, a_2,\, a_2)^p = e$, $[a_1,\, a_2]^p = e$. Obviously, the non-associative A-loop N is commutative if $[a_1,\, a_2] = e$, which holds true if $\beta > \delta$.

Further we will mark as F_n (or $F_n(x_1,\, \ldots,\, x_n)$) the free loop in the quasivariety $Q(N)$, of the rank m (with free generators $x_1, \ldots, x_n$).

Lemma 1. *The commutator F_n' of the $Q(N)$-free loop $F_n(x_1, \ldots, x_n)$) is a free Abelian group with the exponent p, with the following free generators:*

a)$(x_i,\, x_i,\, x_j)$, $(x_i,\, x_j,\, x_j)$, $1 \leq i < j \leq n$; $(x_l,\, x_j,\, x_k)$, $1 \leq i < j < k \leq n$; $[x_i,\, x_j]$, $1 \leq i < j \leq n$,

if F_n is non-commutative;

b)$(x_i,\, x_i,\, x_j)$, $(x_i,\, x_j,\, x_j)$, $1 \leq i < j \leq n$; $(x_l,\, x_j,\, x_k)$, $1 \leq i < j < k \leq n$,

if F_n is commutative.

Proof. To prove the lemma it is sufficient to show that any relation of equality between the generators of the group F_n' shown in a) (for b) the procedure is similar) is a trivial identity in the variety $V(N)$. Indeed, let

$$\prod_{1\le i<j\le n}(x_i,x_j,x_k)^{\alpha_{ij}}\cdot\prod_{1\le i<j\le n}(x_i,x_i,x_j)^{\beta_{ij}}\cdot\prod_{1\le i<j<k\le n}(x_i,x_j,x_j)^{\gamma_{ijk}}\cdot\prod_{1\le i<j\le n}[x_i,x_j]^{\delta_{ij}}=e \tag{1}$$

be a such a relation of equality

As $x_1,\ \ldots,\ x_n$ are free generators of the $Q(N)$-free loop $F_n(x_1,\ \ldots,\ x_n)$, (1) is a true identity in any loop from the variety $V(N)$. We will consider two cases depending on the prime number p.

Case $p=2$. We will show first that all exponents a_{ij} from (1) are equal to zero, i.e. $\alpha_{ij}=0mod2$, $1\le i<j\le n$. For simplicity, suppose $\alpha_{12}\ne 0mod2$. From the identity (1) for $x_i=e$, $i=3,\ 4,\ \ldots,\ n$ we obtain the identity

$$(x_i,x_i,x_2)\cdot(x_i,x_2,x_2)^{\gamma_{12}}\cdot[x_1,x_2]^{\delta_{12}}=e, \tag{2}$$

true in the A-loop N. If we suppose $\beta_{12}=0mod2$ and $\delta_{12}=0mod2$, then, according to (2), in the A-loop N the identity $(x_1,\ x_1,\ x_2)=e$ is true, which means that N is di-associative - contradiction. Suppose that $\beta_{12}\ne 0mod2$ and $\delta_{12}=0mod2$. Then, from (2), we obtain the identity

$$(x_1,x_1,x_2)\cdot(x_1,x_2,x_2)=e. \tag{3}$$

If we make the substitution $x_1\to x_1\cdot x_2$ in (3), we obtain

$$(x_1x_2,\ x_1x_2,\ x_2)\cdot(x_1x_2,\ x_2,\ x_2)=(x_1,\ x_1,\ x_2)\cdot(x_1,\ x_2,\ x_2)\cdot(x_1,\ x_2,\ x_2)=(x_1,\ x_1,\ x_2)=e,$$

and hence, again we obtain that in the A-loop N the identity $(x_1,\ x_1,\ x_2)=e$ holds true, which is impossible. Now suppose $\beta_{12}=0mod2$ and $\delta_{12}\ne 0mod2$. Then from (2) we obtain the identity

$$(x_1,x_1,x_2)\cdot[x_1,x_2]=e, \tag{4}$$

true in the A-loop N. In (4) we make the substitution $x_1\to x_1\cdot x_2$ and obtain

$(x_1x_2,\ x_1x_2,\ x_2)\cdot[x_1x_2,\ x_2]=(x_1,\ x_1,\ x_2)(x_1,\ x_2,\ x_2)\cdot[x_1,\ x_2]=(x_1,\ x_2,\ x_2)\cdot(x_1,\ x_1,\ x_2)[x_1,\ x_2]=(x_1,\ x_2,\ x_2)=e$,

i.e. $(x_1,\ x_2,\ x_2)=e$. By changing the variable in the latter identity, again we find that in N the identity $(x_1,\ x_1,\ x_2)=e$, holds true, which is impossible. Finally, let there be $\beta_{12}\ne 0mod2$ and $\delta_{12}\ne 0mod2$. Then (2) has the form

$$(x_1, x_1, x_2)(x_1, x_2, x_2)[x_1, x_2] = e, \tag{5}$$

If in (5) we make the substitution $x_1 \to x_1 \cdot x_2$, then we have

$(x_1x_2,\ x_1x_2,\ x_2)(x_1x_2,\ x_2,\ x_2)[x_1x_2,\ x_2] = (x_1,\ x_1,\ x_2)(x_1,\ x_2,\ x_2)(x_1,\ x_2,\ x_2)[x_1,\ x_2] = (x_1,\ x_2,\ x_2) \cdot (x_1,\ x_1,\ x_2)(x_1,\ x_2,\ x_2)[x_1,\ x_2] = (x_1,\ x_2,\ x_2) = e$,

i.e. $(x_1,\ x_2,\ x_2) = e$. From here, as previously, we come to a contradiction. Hence, the assumption $\alpha_{12} \neq 0 mod 2$ is wrong. So we can conclude that $\alpha_{ij} = 0 mod 2$, $1 \leq i < j \leq n$. Similarly we can deduce $\beta_{ij} = 0 mod 2$, $1 \leq i < j \leq n$. Then the identity (1) takes the form

$$\prod_{1 \leq i < j < k \leq n} (x_i, x_j, x_k)^{\gamma_{ijk}} \cdot \prod_{1 \leq i < j \leq n} [x_i, x_j]^{\delta_{ij}} = e. \tag{6}$$

We show that $\delta_{ij} = 0 mod 2$, $1 \leq i < j \leq n$. For simplicity we assume that $\delta_{12} \neq 0 mod 2$. For $x_i = e$, $i = 3,\ 4,\ \ldots,\ n$, from (6) we obtain that in the A-loop N the identity $[x_1,\ x_2] = e$ holds true, which is impossible since N is a non-commutative loop. Hence, the assumption that $\delta_{12} \neq 0 mod 2$ is wrong. Therefore, we can conclude that $\delta_{ij} = 0 mod 2$, $1 \leq i < j \leq n$. Then the identity (6), equivalent to (1), takes the form

$$\prod_{1 \leq i < j < k \leq n} (x_i, x_j, x_k)^{\gamma_{ijk}} = e. \tag{7}$$

Finally, we show that $\gamma_{ijk} = 0 mod 2$, $1 \leq i < j < k \leq n$. For simplicity let us assume $\gamma_{123} \neq 0 mod 2$. For $x_i = e$, $i = 4,\ 5,\ \ldots,\ n$, from (7) we obtain that in the A-loop N the identity $(x_1,\ x_2,\ x_3) = e$, holds true, i.e. N is associative, which is impossible. Therefore, the assumption that $\gamma_{123} \neq 0 mod 2$ is wrong and, hence, we can conclude that $\gamma_{ijk} = 0 mod 2$, $1 \leq i < j < k \leq n$.

Case $p > 2$. As in previous case we will show first that all exponents α_{ij} from (1) are equal to zero. For simplicity, let us assume that $\alpha_{12} \neq 0 mod p$. From the identity (1) for $x_i = e$, $i = 3,\ 4,\ \ldots,\ n$ we obtain the identity

$$(x_1, x_1, x_2)^{\alpha_{12}} (x_1, x_2, x_2)^{\beta_{12}} \cdot [x_1, x_2]^{\delta_{12}} = e, \tag{8}$$

true in the A-loop N. If we assume that $\beta_{12} = 0 mod p$ and $\delta_{12} = 0 mod p$, then, according to (8) in the A-loop N the identity $(x_1,\ x_1,\ x_2)^{\alpha_{12}} = e$, holds true. As p does not divide α_{12}, the latter identity implies the identity $(x_1,\ x_1,\ x_2) = e$, which means that N is di-associative - contradiction. Let us assume that $\beta_{12} \neq 0 mod p$ and $\delta_{12} = 0 mod p$. Then, from (8), we obtain the identity

$$(x_1, x_1, x_2)^{\alpha_{12}} \cdot (x_1, x_2, x_2)^{\beta_{12}} = e, \tag{9}$$

If in (9) we make the substitution $x_1 \to x_1^{-1}$, we obtain

$$(x_1,\, x_1,\, x_2)^{\alpha_{12}} \cdot (x_1,\, x_2,\, x_2)^{-\beta_{12}} = e.$$

Multiplying the latter identity by identity (9), we obtain

$$(x_1,\, x_1,\, x_2)^{2\alpha_{12}} = e.$$

Since p does not divide $2\alpha_{12}$, the latter identity implies $(x_1,\, x_1,\, x_2) = e$. Hence we obtained that in the non-di-associative A-loop N the identity $(x_1,\, x_1,\, x_2) = e$ holds true, which is impossible. Let us assume that $\beta_{12} = 0 mod p$ and $\delta_{12} \neq 0 mod p$. Then from (8) we obtain the identity

$$(x_1, x_1, x_2)^{\alpha_{12}} \cdot [x_1, x_2]^{\delta_{12}} = e. \tag{10}$$

If in (10) we make the substitution $x_1 \to x_1^{-1}$, we obtain

$$(x_1,\, x_1,\, x_2)^{\alpha_{12}} \cdot [x_1,\, x_2]^{-\delta_{12}} = e.$$

Multiplying the latter identity by identity (10), we obtain

$$(x_1,\, x_1,\, x_2)^{2\alpha_{12}} = e$$

which, in its turn, implies $(x_1,\, x_1,\, x_2) = e$. Hence in this case we also obtain that in N the identity $(x_1,\, x_1,\, x_2) = e$ holds true, which is impossible. Finaly, we assume that $\beta_{12} \neq 0 mod p$ and $\delta_{12} \neq 0 mod p$. If in (8) we make the substitution $x_1 \to x_1^{-1}$, we obtain

$$(x_1,\, x_1,\, x_2)^{\alpha_{12}}(x_1,\, x_2,\, x_2)^{-\beta_{12}} \cdot [x_1,\, x_2]^{-\delta_{12}} = e.$$

Multiplying the latter identity by identity (8), we obtain

$$(x_1,\, x_1,\, x_2)^{2\alpha_{12}} = e$$

which again means that $(x_1,\, x_1,\, x_2) = e$. Hence, in the non-di-associative A-loop N the identity $(x_1,\, x_1,\, x_2) = e$ holds true. We have again a contradiction. Therefore, the assumption that $\alpha_{12} \neq 0 mod p$ is wrong. Thus, we can conclude that $\alpha_{ij} = 0 mod p$, $1 \leq i < j \leq n$. Similarly we can deduce $\beta_{ij} = 0 mod p$, $1 \leq i < j \leq n$. Then the identity (1) has the form (6). Now we show that $\delta_{ij} = 0 mod p$, $1 \leq i < j \leq n$. For simplicity we assume $\delta_{12} \neq 0 mod p$. For $x_i = e$, $i = 3,\, 4,\, \ldots,\, n$, from (6) we obtain that in the A-loop N the identity $[x_1,\, x_2]^{\delta_{12}} = e$ holds true. Since p does not divide δ_{12}, the latter identity implies that $[x_1,\, x_2] = e$. Hence, we obtained that in the non-commutative A-loop N the identity $[x_1,\, x_2] = e$ holds true, which is a contradiction. Therefore, the assumption that $\delta_{12} \neq 0 mod 2$ is wrong. Hence, we can conclude that $\delta_{ij} = 0 mod p$, $1 \leq i < j \leq n$. Then the identity (6), equivalent to (1), obtains the form (7). Finally, we show that $\alpha_{ijk} = 0 mod p$, $1 \leq i < j < k \leq n$. For simplicity we assume $\alpha_{123} \neq 0 mod p$. For $x_i = e$, $i = 4,\, 5,\, \ldots,\, n$, from (8) we obtain that in the A-loop N the identity $(x_1,\, x_2,\, x_3)^{\alpha_{123}} = e$ holds true. Since p does not divide α_{123}, the latter identity implies $(x_1,\, x_2,\, x_3) = e$. Therefore, we have that in the non-associative A-loop N the identity $(x_1,\, x_2,\, x_3) = e$ holds true, which is a contradiction. Hence, the assumption that $\alpha_{123} \neq 0 mod p$ is wrong and thus we can conclude that $\alpha_{ijk} = 0 mod p$, $1 \leq i < j < k \leq n$. In this way we obtained that (1) is a trivial identity, i.e. true in any loop from the variety $V(N)$. □

Lemma 2. *The element* $x = (x_1,\, x_1,\, x_2)(x_3,\, x_3,\, x_4) \ldots (x_{2n-1},\, x_{2n-1},\, x_{2n}) \in F_{2n}(x_1,\, \ldots,\, x_{2n})$ *cannot be represented as a product by a smaller number of* n *associators.*

Proof. We assume that x can be written as a product by a smaller number of n associators, i.e. for certain terms $t_1,\, \ldots,\, t_{3k}$ of the loop signature, the following holds true

$$(x_1,\, x_1,\, x_2) \ldots (x_{2n-1},\, x_{2n-1},\, x_{2n}) = \prod_{i=1}^{k} (t_i(x_1,\, \ldots,\, x_{2n}),\, t_{k+i}(x_1,\, \ldots,\, x_{2n}),\, t_{2k+i}(a_1,\, \ldots,\, a_{2n})),$$

where $k < n$. Any relation of equality between the generators of A-loop F_{2n} is a true identity in the variety $V(N)$, i.e. the obtained equality can be considered as identity in $V(N)$. The fact that this identity holds true in any loop $B \in V(N)$, means that any element $g \in B'$ with the following form

$$s = (u_1, u_1, u_2) \cdot (u_3, u_3, u_4) \ldots (u_{2l-1}, u_{2l-1}, u_{2l}) \qquad (11)$$

can be written as a product of associators, whose number does not exceed the number k. Since the number of elements with the form $(u,\, v,\, w)$ from the loop F_s is at most p^{3s}, then the number of elements with the form $(u_1, v_1, w_1, \ldots, (u_k, v_k, w_k$ in the A-loop F_s is not higher than p^{3ks}. This implies that the total number of elements that can be written in the form (11) does not exceed the number $p^{3k\cdot s}$. On the other hand, according to Lemma 1, F'_s is free Abelian group with the exponent p and has $s(s-1)/2$ free generators with the form (x_i, x_i, x_j), $1 \leq i < j \leq s$. Therefore, the total number of elements with the form (11) cannot be smaller than $p^{s(s-1)/2}$. Therefore, for $s > 6k + 1$ we have a contradiction. □

Let's establish a family of $\{B_m | m = 1, 2, \ldots\}$ A-loops, isomorphic to the $Q(N)$-free A-loop F_{2n}. Let A-loop B_m be generated by the elements

$$x_{1,m},\ x_{2,m},\ \ldots,\ x_{2n,m} \text{ and } c_m = \prod_{i=1}^{n}(x_{i,m},\ x_{i,m},\ x_{n+i,m}) \in Z(B_m),$$

then, by definition, we have

$$C_{1,n} = B_1,\ C_{2,n} = C_{1,n} \times B_2(c_1 = c_2),\ \ldots,$$
$$C_{m,n} = C_{m-1,n} \times B_m(c_{m-1} = c_m).$$

For more simplicity, further on we will write C_m instead of $C_{m,n}$.

Lemma 3. *The element*

$$c = c_m = \prod_{i=1}^{n}(x_{i,m},\ x_{i,m},\ x_{n+i,m}) \in C_m (m = 1, 2, \ldots)$$

cannot be represented as a product of fewer than n associators.

Proof. We will apply induction by m. If $m = 1$, then according to Lemma 1, the statement holds true. Let $m > 1$ and let us assume that the statement of the lemma is not true. Then for a $l < n$ and certain elements $d_1, d_2, \ldots, d_{3l} \in C_m$, the following equality holds true

$$c = \prod_{i=1}^{l}(d_i,\ d_{l+i},\ d_{2l+i}).$$

Then for certain elements $a_1, \ldots, a_{3t} \in C_{m-1}$, and $b_1, \ldots, b_{3t} \in B_m$, we have

$$c = \prod_{i=1}^{l}(a_i,\ a_{l+i},\ a_{2l+i}) \cdot \prod_{i=1}^{l}(b_i,\ b_{l+i},\ b_{2l+i}).$$

From this equality it results that the elements

$$\prod_{i=1}^{l}(a_i,\ a_{l+i},\ a_{2l+i}) \text{ and } \prod_{i=1}^{l}(b_i,\ b_{l+i},\ b_{2l+i})$$

belong to the intersection $C_{m-1} \cap B_m = lp(c)$. By the hypothesis of induction this is possible only if both indicated elements are equal to unit, and thus c is also equal to unit, which is not true. $\square$

Lemma 4. *The Frattini subloop $\Phi(C_m)$ of the A-loop $C_m(m = 1, 2, \ldots)$ is the direct product of the subloops C'_m and $C^p_m = lp(x^p|x \in C_m)$, i.e.*
$\Phi(C_m) = C'_m \cdot C^p_m$ *and* $C'_m \cap C^p_m = lp(e)$.

Proof. Let H be a maximal subloop of the A-loop C_m. Since the commutator C'_m is in the centre of $Z(C_m)$, then $(H \cdot C'_m)' = H'$. From here it follows that $C'_m \subseteq H$. Therefore, the subloop H is normal in the A-loop C_m and, obviously, has the index p. But then $x^p \in H$ for any $x \in C_m$, i.e. $C^p_m \subseteq H$. But $\Phi(C_m)$, by definition, is the intersection of all maximal subloops of the loop C_m, that is why $C'_m \cdot C^p_m \subseteq \Phi(C_m)$.

The factor loop $C_m/C'_m C^p_m$ is an elementary abelian group, and thus, it can decompose into a product of cyclic groups of p order. Therefore, it is obvious that the Frattini subloop of the loop $C_m/C'_m C^p_m$ coincides with the unit of the loop. Then, according to Lemma 2.1 ([8], p.97)

$$\Phi(C_m)/C'_p C^p_m \subseteq \Phi(C_b/C'_m C^p_m),$$

and, thus, $C'_b \cdot C^3_b \supseteq \Phi(C_m)$. Therefore, $\Phi(C_b) = C'_b \cdot C^3_b$.

Now we show that $C'_m \cap C^p_m = lp(e)$. First we not that if a p-loop L satisfies the condition $L' \cap L^p = lp(e)$, then this condition is also satisfied by any finite Cartesian power L^m of L and by any factor-loop L/H, where $H \subseteq L'$. Indeed, the equalities $(L^m)' = (L')^m$ and $(L^m)^p = (L^p)^m$ lead to the equality $(L^m)' \cap (L^m)^p = lp(e)$. Now let H be a normal subloop of the loop L contained in L'. Then we have $(L/H)' = L'/H$ and $(L/H)^p = L^p H/H$. Therefore, if we assume that $(L/H)' \cap (L/H)^p \neq lp(e)$, then for certain elements $e \neq a \in L^p$ and $h \in L'$, we have $aH = hH$. Which results in $a \in hH \subseteq L'$, which contradicts the quality $L' \cap L^p = lp(e)$.

Hence, to show that $C'_m \cap C^p_m = lp(e)$, it is sufficient to show that the $Q(N)$-free loop $F = F(x_1, x_2, \ldots)$ of any rank s (finite or infinite) the relation $F' \cap F^p = lp(e)$ holds. Indeed, let $u \in F' \cap F^p$, hence $u = u(x_1, \ldots, x_s)$ can be written in a canonical form, as follows

$$u(x_1, \ldots, x_s) = x_1^{pm_1} x_2^{pm_2} \ldots x_s^{pm_s}.$$

Since the latter expression is an identity in the $Q(N)$-free loop F, after the successive substitution of x_i in it by the unit element e, we obtain the equalities $x_i^{pm_i} = e$, $i = 1, \ldots, s$. Therefore $u = e$. $\square$

Continuation of the proof of the theorem. Further we will show that the finite A-loop L does not have a basis of quasiidentities from a finite number of variables and, particularly, it does not have a finite basis of quasiidentities. For this it is sufficient to have a A-loop A_t for any natural number t, which does not belong to the quasivariety qL, but all its t-generated subloops to belong to the quasivariety qL.

Indeed, let $t > 1$ be a natural number and F_t - a free A-loop of the rank t of the quasivariety $Q(N)$. We denote by $g(t)$ the total number of generators with the form (x, y, z) of the subloop F_t'. Then it is clear that for any loop $K \in V(N)$, generated by t elements, any element from the subloop generated by all associators from K are expressed by a product of $g(t)$ associators with the form $[x, y, z]$. Now consider that $n = g(t) + 1$ and let M be a subloop of the loop $C_m (\equiv C_{m,n})$, generated by t elements. We denote by φ the natural homomorphism of loop $B = B_1 \times \ldots \times B_m$ on loop C_m, whose kern $Ker\varphi = lp(c_1 c_2^{-1}, \ldots, c_{m-1} c_m^{-1})$. Let K be a minimal pro-image through φ of the subloop M. It is clear that K is generated by t generators. Let us show that $K \cap Ker\varphi \subseteq K'$. Indeed, let there be such an element $d \in K \cap Ker\varphi$ and $d \notin K'$. If we admit that $d \in \Phi(K)$, then according to Lemma 4, $d = a \cdot b$, where $a \in K'$ and $b \in K^p \subseteq B^p$. This results in $b = a \backslash d \in K' \subseteq B'$. Hence, $b \in B' \cap B^p$ and, according to Lemma 4, $b = e$. Hence $d = a$ and, thus, $d \in K'$ - contradiction. Therefore, $d \notin \Phi(K)$. Then in K there is a maximal subloop H so as $d \notin H$. Then $K = (K \cap Ker\varphi) \cdot H$, which results in $K^\varphi = H^\varphi$. Thus we obtained that K is not a minimal pro-image through φ of M. Hence, $K \cap Ker\varphi \subseteq K'$ and, since $g(t) < n$ it follows that any element from the intersection $K \cap Ker\varphi$ is expressed as a product of $n - 1$ simple associators with the form (x, y, z). On the other hand, according to Lemmas 2 and 3, the elements that are different from the unit from $Ker\varphi$ cannot be expressed as a product of fewer than n associators. Therefore, $K \cap Ker\varphi = lp(e)$ and $K \equiv M$, but then $M \in Q(N)$. Hence, any subloop generated by t elements of the loop $C_m (m = 1, 2, \ldots)$ belongs to the quasivariety $Q(N)$. Particularly, if we consider $s = |L| + 1$ and $A_t = C_s$ then any t-generated subloop of the loop A_t belongs to the quasivariety $Q(N)$.

Let us show that $A_t \notin qL$. Since the loop A_t is finite, then according to Theorem 8 ([13], p.294], it is sufficient to show that A_t is not included isomorphically in any Cartesian power of the loop L.

Let us assume that the loop A_t is included isomorphically in a Cartesian power of loop L. Then there is such an isomorphism $\psi : A_t \to L$ so that $c_s^\psi \neq e$. If we suppose that for any $j \in \{1, 2, \ldots, m\}$ the set $\{x_{i,j}^\psi | i = 1, 2, \ldots, 2n\}$ contains an element a_j, which does not belong to the subloop $(B_1 \ldots B_{j-1} B_{j+1} \ldots B_s)^\psi$, then L contains s different elements, which is impossible.

Therefore, there is such a $j \in \{1,\, 2,\, \ldots,\, m\}$ so that any element of the set $\{x^{\psi}_{i,j} | i = 1,\, 2,\, \ldots,\, 2n\}$ belongs to the subloop $(B_1 \ldots B_{j-1} B_{j+1} \ldots B_s)^{\psi}$. Then we will have

$$(x_{i,j},\, x_{i,j},\, x_{n+i,j})^{\psi} = (x^{\psi}_{i,j},\, x^{\psi}_{i,j},\, x^{\psi}_{n+1} \in ((B_1 \ldots B_{j-1} B_{j+1} \ldots B_s)^{\psi},\, x^{\psi}_{i,j},\\ x^{\psi}_{n+i,j}) = (B_1 \ldots B_{j-1} B_{j+1} \ldots B_s,\, x_{i,j},\, x_{n+i,j})^{\psi} = lp(e)$$

for each $i = 1,\, 2,\, \ldots,\, n$. From here we obtain

$$c^{\psi}_s = \prod_{i=1}^{n} (x_{i,j},\, x_{i,j},\, x_{n+i,j})^{\psi} = e,$$

which contradicts $c^{\psi}_s \neq e$. Thus the assumption $A_t \in qL$ is not true. Therefore, $A_t \notin Q(L)$. This proves the theorem.

Directly from the Theorem two corollaries follow.

Corollary 1. *If a finite nilpotent A-loop is not commutative or associative, then it has no basis of quasiidentities from a finite number of variables.*

Corollary 2. *If L is a finitely generated nilpotent A-loop, then $Q(L) = V(L)$ if and only if L is a finite Abelian group.*

Similarly to [5] we will prove the following.

Corollary 3. *The lattice of subquasivarieties of the variety generated by a finitely generated nilpotent and non-associative or non-commutative A-loop has the power of the continuum.*

Proof. Indeed, let L be a finitely generated nilpotent A-loop and M the variety generated by the A-loop L. If A-loop L is not finite, it contains an infinite cyclic group and a finite number of p-subloops. Then, according to the proved Theorem, the quasivariety $Q(L)$ is defined in the variety $V(L)$ by a infinite and independent system of quasivarieties. Therefore the number of quasivarieties from $V(L)$ that contain $Q(L)$ is continuum.

Let now the A-loop L be finite. The proof of the above Theorem showed that for any natural number t there is such a finite loop L_t from the variety $V(L)$ that any t-generated subloop from L_t is contained in the quasivariety $Q(L)$, but the loop L_t itself is not contained in the quasivariety generated by all loops from $V(L)$ of a strictly smaller order than t. We construct such an infinite range of natural numbers $\{t_i | i \in N = \{1,\, 2,\, \ldots\}\}$ that: $t_1 = |L|$, $t_{i+1} = |L_i| + |L|$ (here and further on instead of L_{t_i} we will write L_i).

Let us prove that $L_i \notin Q(\{L_j | j \in N \backslash \{i\}\})$. Indeed, if it is not true, then, according to Theorem 8 [13], for any $a \in L_i$, $a \neq 1$, there is an homomorphism φ_a from L_i in a certain loop L_j, $j \neq i$, so as $a\varphi_a \neq 1$. If $i < j$, then $|L^{\varphi}_i| < t_j$ and, since $L^{\varphi}_i \subseteq L_j$, we have $L^{\varphi}_i \in Q(L)$. This means that the element a verges towards the loop L. Therefore, according to the same Theorem 8, $L_i \in Q(\{L, L_j | 1 \leq j < i\})$. But $|L_j| < t_i$, $|L| < t_i$, which contradicts the definition of L_i.

For each i we establish the quasiidentity Φ_i identically true in the quasivariety $Q(\{L_j | j \in N \backslash \{i\}\})$ and false in the loop L_i. Then the system $\{\Phi_i,\ i \in N\}$ of quasiidentities is infinite, independent and, in particular, the lattice of subquasivarieties from the variety M has the power of the continuum.□

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