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1 Introduction

In the classical papers [5] (resp. [3]), Matsusaka and Hoyt gived a necessary and sufficient criterion for an abelian variety A to be a jacobian, respectively a product of jacobians. In [10], Ran reconsiders the subject and gives a more general and probably more natural criterion for this. His method, seems however unsatisfactory in positive characteristic.

The aim of this paper is to reprove Ran's criterion, using results from [1] on the ring of numerical algebraic cycles on A. For the particular case of the Ran-Matsusaka criterion, another proof appeared in [2]. Both proofs are characteristic independent.

In the sequel, for an abelian variety A, we denote by $Z(A)$ the $\mathbb{Q}$-vector space of algebraic cycles with $\mathbb{Q}$-coefficients on A, and by $N(A)$ the quotient by numerical equivalence. It is well known, (cf. [4] or [1]) that on $N(A)$ there are two $\mathbb{Q}$-algebra structures gived by the usual product and by the Pontrjagin product.

The last one, will be very useful through this paper, thanks to his geometric definition and to the fact that it gives a ring structure not only on $N(A)$ but also on $Z(A)$. Below, for $x, y \in N(A)$, we shall denote the usual product by $x \cdot y$ and the Pontrjagin product by $x \star y$.

Also, for two sub-varieties X_1 and X_2 in A, we denote by $X_1 \dot{+} X_2$ and $X_1 \dot{-} X_2$ their sum and difference in the group low of A, for avoiding confusion with the corresponding operations on cycles. Through the paper, the algebraic cycles will often be divisors and 1-cycles, (the last ones being formal sums of curves) and they always have integer coefficients. The therm curve, is reserved for integral, and all 1-cycles will be considered effectives ones. Finally, a prime cycle is an irreducible sub-variety of A of corresponding co-dimension.

2 Generating curves on an abelian variety

Let A be an abelian variety of dimension n, and E be a curve on it which contains the origin 0_A of A. We consider a sequence of closed subsets in A, defined as follows:

$$E_0 = \{0_A\}$$

$$E_i = E \dot{+} E \dot{+} ... \dot{+} E \text{ (i terms, with } 1 \leq i \leq n \text{)}$$

$$\text{and } E_{n+1} = A.$$

It is clear that this sequence is increasing, $dim\ E_i \leq i$ and $dim\ E_{i+1} \leq dim\ E_i + 1$ for every i. As far as E is a curve, E_i is irreducible and there is a first index j such that $E_j = E_{j+1}$. Also, we have $E_i = E_j$ for all $i \geq j$ and $dim\ E_j = j$. It follows that E_j is stable for the group law on A and the induced operation has 0_A as unity. Using a result of Ramanujan ([9] cap. II, §4), E_j is an abelian sub-variety of A and E is clearly a generating curve for it. We denote by $< E >$ the sub-variety E_j. If $j = n$, then $< E >= A$ and E is a generating curve for A.

Remark 2.1. *Matsusaka proves in [6] that every abelian variety has a generating curve. Moreover, from his proof, for a projective embedding of A, every linear section with a convenient linear subspace of appropriate dimension which contain 0_A is a generating curve for A.*

Using the Pontrjagin product (for cycles, not for numerical classes) it is easy to deduce the following useful fact:

Lemma 2.2. *Let E a curve in A with $0_A \in E$, $< E >$ the sub-variety of A generated by E and $j = dim\ < E >$. Then, j is the maximal number i such that E^{*i} ($= E * E * ... * E$ with i terms) is nonzero and $< E >$ is the support of the cycle E^{*j}.*

Consider now a curve $E \subset A$ which does not necessarily contains the origin. It follows easily that for $x \in E$ the abelian variety generated by $E \dot{-} \{x\}$ does not depend on x; it is in fact the sub-group of A generated by $E \dot{-} E$. This abelian variety will also be denoted by $< E >$. If $Z = m_1 E_1 + ... + m_k E_k$ is an effective 1-cycle, we denote by $< Z >$ the abelian sub-variety given by $< E_1 > \dot{+} < E_2 > \dot{+} ... \dot{+} < E_k >$ and we will say that Z is a generating 1-cycle for $< Z >$.

Remark 2.3. *From the definition above, we see that the construction of $< Z >$ is independent of the numbers m_i. In particular, Z and Z_{red} generates the same sub-variety and also for Z and mZ.*

The next lemma will be useful in the sequel:

Lemma 2.4. *a) For $Y, E \subset A$ with Y sub-variety and E a curve, both containing the origin, if $Y * E = 0$ then $E \subset Y$.*
b) For $Y_1, Y_2 \subset A$ abelian sub-varieties and m, n non-zero integers, if mY_1 and nY_2 are numerically equivalents, then $Y_1 = Y_2$.

Proof: a) We have $dim\, Y \leq dim\, (Y \dot{+} E) \leq 1 + dim\, Y$ and from $Y * E = 0$ we deduce that $dim\, (Y \dot{+} E) < 1 + dim\, Y$. So $Y = Y \dot{+} E$ and because $0_A \in Y$ it follows $E \subset Y$.
b) Let E_1 a generating curve for Y_1 (it exists cf. remark 2.1). Then $mY_1 * E_1 = m(Y_1 * E_1) = 0$ and so $nY_1 * E_1 = n(Y_1 * E_1) = 0$. It follows that $Y_2 * E_1 = 0$, and from the first point $E_1 \subset Y_2$. The last inclusion imply that $Y_1 =< E_1 > \subset Y_2$. In the same way is proved the reversed inclusion. **QED**

Remark 2.5. *The point b) above, in the case $m = n = 1$ is a result of Matsusaka in [6].*

Proposition 2.6. *a) For two curves E_1, $E_2 \subset A$ which are numerically equivalent, we have $< E_1 >=< E_2 >$.*
b) For a curve E and Z an 1-cycle which is numerically equivalent with E, we have $< Z >=< E >$.
c) Let D an ample divisor and Z a 1-cycle which is numerically equivalent with D^{n-1}. Then Z is a generating 1-cycle for A.

Proof: a) Using convenient translations we can suppose that E_1 and E_2 contains 0_A. Let r, s be the dimensions of $Y_i =< E_i >$ for $i = 1, 2$. Using lemma 2.2 and the fact that ${E_1}^{*a}$ is numerically equivalent with ${E_2}^{*a}$ for all

positive integers a, we find $r = s$. From lemma 2.2 again, ${E_1}^{*r}$ is a multiple of both Y_1 and Y_2, and the conclusion follows from lemma 2.4b.
b) As in a), denoting $Z = m_1E_1 + ... + m_kE_k$ we can suppose that E and all E_i contains 0_A. By lemma 2.2, for $r = dim \ < E >$, we have $E^{*(r+1)} = 0$. Now, Z^{*r} being numeric equivalent with E^{*r}, we find $Z^{*r} * E = 0$. But again from lemma 2.2 we find a non-zero term in the development of Z^{*r}. With lemma 2.4 a), this term which is in fact a sub-variety contains E, because all terms in Z^{*r} are vanished by Pontrjagin product with E. On the other hand, this term is contained in $< Z >$ and so $< E > \subset < Z >$.

For the reverse inclusion, we consider the development of the left side of: $Z * E^{*r} = 0$. From lemma 2.2 $E^{*r} = n < E >$ with $n \geq 1$ an integer. We find $m_1nE_1* < E > +... + m_knE_k* < E >= 0$, therefore $E_i* < E >= 0$ for all i and then from lemma 2.4a, $E_i \subset < E >$, so $< Z > \subset < E >$.
c) Let m a positive integer with $\mid mD \mid$ very ample. The cycles $m^{n-1}D^{n-1}$ and $m^{n-1}Z$ are therefore numeric equivalents and it will exists an integer curve E in the same numeric class with $m^{n-1}D^{n-1}$ and so with $m^{n-1}Z$. From b) we have $< E >=< m^{n-1}Z >=< Z >$. But from remark 2.1 $< E >= A$, so Z is a generating 1-cycle. **QED**

The point c) above, is a slight generalization of the result from remark 2.1 and will be used to deduce the Matsusaka-Hoyt criterion from the Ran's one.

3 Algebraic cycles constructed from generating curves

We recall a result from [1] which will be the main tool in the proof of Ran's theorem. Let E a generating curve of the n-dimensional abelian variety A. We consider on A, the following cycles: $W_n(E) = \{0_A\}$ and $W_i(E) = \frac{1}{(n-i)!}E^{*(n-i)}$ for $0 \leq i \leq n-1$. From the definition of the Pontrjagin product, $W_i(E)$ is a cycle with irreducible support of co-dimension i on A. In particular $W_1(E)$ is a divisor and there exists $\alpha_E \in \mathbb{Q}$ such that $W_0(E) = \alpha_E \cdot 1_A$, where 1_A is the fundamental cycle on A.

The result we need from [1] is the following:

Proposition 3.1. *All cycles $W_i(E)$ have integer coefficients and in particular $\alpha_E \in \mathbb{Z}$, being evidently positive. Also, $W_1(E)^i = i!\ {\alpha_E}^{i-1}W_i(E)$ for $1 \leq i \leq n$. In particular, $W_1(E)^n > 0$ and so, $W_1(E)$ is ample.*

Remark 3.2. *For E a smooth curve and A his jacobian, these divisors are well-known.*

A first application of the proposition above is the point b) in the following:

Proposition 3.3. *a) Let D an effective divisor and Z a generating 1-cycle. Then $D \cdot Z > 0$.*
b) If moreover D is ample, then $D \cdot Z \geq n = dim\ A$.

Proof : a) We can suppose that D is a prime divisor. Let $E_1, ..., E_k$ the components of Z. We have $D \cdot E_i \geq 0$ for all i, because the general translate of E_i cut properly D. It is therefore sufficient to find an i such that $D \cdot E_i > 0$. Suppose there is no such i. Then cf. a result from [9], cap. 2, §6 translations with elements of the form $\{x\} \dot{-} \{y\}$ with $x, y \in E_i$ leaves D invariant. But Z is generating 1-cycle, and therefore every element in A is of this form. So D is invariant with respect to any tranlation and then numerically equivalent with 0, in contradiction with his effectiveness.

b) Consider a first case where $Z = E$ is a prime cycle (i.e. E is a curve) and without loss of generality $0_A \in E$. Let t a variable and the polynomial $P(t) = (t \cdot W_1(E) + D)^n = n!{\alpha_E}^{n-1}t^n + n!{\alpha_E}^{n-2}(D \cdot E)t^{n-1} + ... + D^n$. Because $W_1(D)$ is ample and D is non-degenerate, the index theorem for abelian varieties cf. [9] asserts that all roots of P are real and negatives. So the means inequality gives $D \cdot E \geq n(\chi(\mathcal{O}_A(D)) \cdot \alpha_E)^{\frac{1}{n}} \geq n$.

For the general case, let $Z = m_1E_1 + ... + m_kE_k$ with all $m_i > 0$, $X_i = < E_i >$ and D_i the restriction of D to X_i. The projection formula gives $D \cdot E_i = D_i \cdot E_i \geq dim\ X_i$ from the particular case above. So, $D \cdot Z = \sum_i m_i D \cdot E_i \geq \sum_i D \cdot E_i \geq \sum_i dim\ X_i \geq dim\ A = n$, because Z is a generating 1-cycle. **QED**

The following consequence of the above proposition, will be useful in the last part of the paper:

Corollary 3.4. *Let A an abelian variety, $D = \sum_{i=1}^{r} m_i D_i$ an ample effective divisor, and $Z = \sum_{i=1}^{s} n_j E_j$ a generating 1-cycle of A (the coefficients are supposed non-zero). If $D \cdot Z = n = dim\ A$ then $m_i = n_j = 1$ for all i, j.*

Proof : We have $n = D \cdot Z = \sum_{i=1}^{r} m_i D_i \cdot Z \geq \sum_{i=1}^{r} D_i \cdot Z \geq n$ because $\sum_{i=1}^{r} D_i$ is ample and one can apply proposition 3.3 b). So $m_i D_i \cdot Z = D_i \cdot Z$ and because the last term is non-zero by 3.3 a), we find $m_i = 1$ for all i.

In the same way, $n = D \cdot Z = \sum_{i=1}^{s} n_j D \cdot E_j \geq \sum_{i=1}^{s} D \cdot E_j = D \cdot \sum_{i=1}^{s} E_j \geq n$ because $\sum_{i=1}^{s} E_j$ remains a generator 1-cycle by remark 2.3. So $n_j D \cdot E_j = D \cdot E_j$ and D being ample, the last term is positive. It results that $n_j = 1$ for all j. **QED**

We can now to proves the following result, which is nothing else than Ran's version of the Matsusaka theorem:

Theorem 3.5. *Let D be an ample divisor on the abelian variety A, and E a generating curve such that $D \cdot E = n = dim\ A$. Then E is smooth, A is its jacobian and D is a translation of $W_1(E)$.*

Proof : In the proof of the point b) from proposition 3.3 we obtained the inequality $D \cdot E \geq n(\chi(\mathcal{O}_A(D)) \cdot \alpha_E)^{\frac{1}{n}} \geq n$. If $D \cdot E = n$ we will have $\chi(\mathcal{O}_A(D)) = \alpha_E = 1$ and so $D^n = n!$. In this case the polynomial $P(t)$ from the same proposition become $P(t) = n!t^n + n! \cdot n \cdot t^{n-1} + ... + n!$. It follows that the arithmetic and geometric means of the roots coincides and so all the roots have the form $-\lambda$ for a positive value of λ. So $P(t) = n!(t + \lambda)^n$ and by identification, $\lambda^n = 1$. It follows that $\lambda = 1$ and then $W_1(E)^{n-1} \cdot D = W_1(E)^{n-2} \cdot D^2 = n!$. These relations imply that $(D - W_1(E)) \cdot W_1(E)^{n-1} = (D - W_1(E))^2 \cdot W_1(E)^{n-2} = 0$. The Hodge index theorem asserts that D is numeric equivalent with $W_1(E)$, and because $W_1(E)$ is a principal polarization (from proposition 3.1 and the equality $\alpha_E = 1$), one deduce that D is a translation of $W_1(E)$.

Consider the normalization $f_0 : T \to E$ for E, and let $f : J \to A$ a prolongation of f_0, where J is a jacobian of T. If we choose as base point in the construction of J, one on T which sits above $0_A \in E$, f will be a morphism of abelian varieties, sending origin to origin. Also, f is surjectif because E is generating for A and for $g =$ genus of T we have $g \geq n$.

Denote by $W_i = W_i(T)$ the canonical cycles on the jacobian J. Therefore $f_*(W_{g-i}) = W_{n-i}(E)$ for $1 \leq i \leq n$: for $i = 1$ this is clear because $W_{g-1} = T$ and for $i > 1$ it is a consequence of the definitions for W_{g-i} and $W_{g-i}(E)$ and also from the fact that f_* commute with the Pontrjagin product. In particular $f_*(W_{g-n}) = W_0(E) = \alpha_E \cdot 1_A$ and so $\alpha_E = 1$ is the degree of the restriction of f to W_{g-n}. Therefore this restriction is a birational morphism and has an inverse: $A - -- \to W_{g-n}$. This inverse, considered as a rational map from A to J can be extended over all the A giving a morphism $A \to J$ cf. [9]. As consequence, the restriction g of f to W_{g-n} will be an isomorphism and W_{g-n} will be an abelian sub-variety of J. But W_{g-n} contains $W_{g-1} = T$ which generates J and so $W_{g-n} = J$. In this case we have $g = n$ and f is bi-rational from J to A hence an isomorphism. **QED**

4 Proof of Ran theorem

The purpose of this section is to give a proof for the full Ran's theorem. Some points are as in [10] and are included only for the sake of completeness. The modifications appears from the replacement of lemma 4 from [10] with the result below whose proof is very simple:

Lemma 4.1. *Let D a prime divisor on an abelian variety A. Then, there exists an abelian variety B, a surjectif morphism of abelian varieties $f : A \to B$ and an ample divisor F on B such that $f^{-1}(F) = D$ as schemes.*

Proof: We consider the closed sub-group K of A defined by $K := \{x \in A \mid \{x\}\dot{+}D = D\}$ and the abelian sub-variety K_0 of A which is the connected component of 0_A in K. We denote by B the quotient A/K_0 and $f : A \to B$ the quotient morphism. Finally denote by F the closed irreducible sub-set $f(D)$ with the reduced structure. We find easily $dim\ F = dim\ B - 1$, so F is a divisor on B and set-theoretically $f^{-1}(F) = D\dot{+}K_0 = D$ because $K_0 \subset K$. Let $x \in A$ such that $\{f(x)\}\dot{+}F = F$. Applying f^{-1} we find $\{x\}\dot{+}D\dot{+}K_0 = D\dot{+}K_0$, and because $D\dot{+}K_0 = D$ we find $\{x\}\dot{+}D = D$ and so $x \in K$. Therefore, the elements in B which leaves F invariant by translations are from $f(K)$. They are then in a finite number, because the index $[K : K_0]$ is finite. So F is an ample divisor on B. Finally the equality $f^{-1}(F) = D$ holds also at the schemes level, because f is smooth from its construction. **QED**

The result we are interested in is the following:

Theorem (Ran) 4.2. *Let A an abelian variety of dimension n, $D = \sum_{i=1}^{n} m_i D_i$ an ample effectif divisor, and $Z = \sum_{j=1}^{s} n_j E_j$ a generating 1-cycle such that $D \cdot Z = n$. Then: $m_i = n_j = 1$ for all i, j, $r = s$ and there are r smooth curves $T_1, ..., T_r$ with jacobians $J_1, ..., J_r$ with a morphism of abelian varieties $h : J_1 \times ... \times J_r \to A$ such that for every i, E_i is a translation of*

$\{0\} \times ... \times \{0\} \times T_i \times \{0\} \times ... \times \{0\}$ (T_i on the i^{th} place) and D_i is a translation of $J_1 \times ... \times J_{i-1} \times W_i \times J_{i+1} \times ... \times J_r$, where W_i is the canonical divisor $W_i(T_I)$ on J_i.

Proof: The fact that $m_i = n_j = 1$ for all i, j is the corollary 3.4. For the other points, the proof follow closely that from [10] with some modifications of the arguments. We begun with three preliminary steps.

Step 1: We prove that for every j there is an unique i such that $D_i \cdot E_j \neq 0$. We translate the curves E_j such that they contain the origin, and denote the result with the same letter. Let $A_j =< E_j >$ and $d_j = dim\ A_j$, so that E_j is a generating curve for A_j. Denote by e_j the inclusion $A_j \subset A$ and by the same letter D a translation of it which has proper intersection with every A_j. Therefore, $e_j{}^*(D) := D_j{}'$ is defined as a cycle and is an ample divisor on A_j. The projection formula gives

$$D \cdot E_j = D_j{}' \cdot E_j$$

and so

$$n = D \cdot E = \sum_{j=1}^{s} D \cdot E_j = \sum_{j=1}^{s} D_j{}' \cdot E_j \geq \sum_{j=1}^{s} d_j \geq n.$$

(the first inequality comes from the fact that on A_j one has $D_j{}' \cdot E_j \geq d_j$ according to proposition 3.3b, and the last one is due to the fact that Z is a generating 1-cycle). So $D_j{}' \cdot E_j = d_j$, and E_j being a generating curve for A_j, from theorem 3.5 one find that E_j is smooth, A_j is its jacobian and $D_j{}'$ is a translation of the canonical divisor on A_j; so $D_j{}'$ is prime as any divisor numeric equivalent with it (it's a principal polarization).

Let's fix an j, and consider for any i a translation of D_i which cuts proper A_j. Every such translation, denoted also by D_i, restricted to A_j is either an effective divisor, or has empty intersection with A_j, in which case $D_i \cdot E_j = 0$. But the sum of these restrictions is numeric equivalent with D_j' and so there cannot exists two indexes i with $D_i \cdot E_j \neq 0$, because in such a case D_j' which is prime, would be the sum of two effective divisors. The existence of an i with $D_i \cdot E_j \neq 0$ comes from the fact that D is ample.

Step 2: This part consists in the proof of the following fact: for an n-dimensional abelian variety A, a prime ample divisor D and a generating

1-cycle $Z = T_1 + ... + T_r$ with $D \cdot Z = n$ one has $r = 1$ (i.e. Z is ireducible and reduced).

The proof is due to Ran cf. lemma III.2 from [10]. Denote by $A_1 = < T_1 >$. From the first step, we know that A_1 is in fact the jacobian of the smooth curve T_1; in particular is principally polarized and isomorphic with its dual. It will suffice to prove that $A_1 = A$, because in this case T_1 will be a generating curve, and the fact that D is ample together with the inequalities $n \leq D \cdot T_1 \leq D \cdot Z = n$ imply that $r = 1$ as desired.

For the time being, we replace D with a translate whose restriction $D_{|A_1} := D_1$ is well defined as divisor on A_1. As in the proof of the step 1, D_1 is numerically equivalent with $W_1(T_1)$. Let $s : A \times A_1 \to A$ the morphism given by $s(r, y) = r + y$, and p, p_1 the projections. Consider on $A \times A_1$ the line bundle $\mathcal{M} = s^*(\mathcal{O}_A(D)) \otimes {p_1}^*(\mathcal{O}_{A_1}(-D_1))$ and on $A_1 \times A_1$ the line bundle $\mathcal{P} = (s_{|A_1 \times A_1})^*(\mathcal{O}_{A_1}(D_1)) \otimes {q_1}^*(\mathcal{O}_{A_1}(-D_1)) \otimes {q_2}^*(\mathcal{O}_{A_1}(-D_1))$, where q_1, q_2 are the projections on the factors of $A_1 \times A_1$. Using the fact that A_1 is a jacobian (and therefore is its Picard variety whith the Poincare bundle equal to $\mathcal{P}$), we deduce the existence of a morphism $f : A \to A_1$ and of a line bundle $\mathcal{N}$ on A such that:

$$\mathcal{M} \otimes p^*(\mathcal{N}) \simeq (f \times Id_{A_1})^*[\mathcal{P}]. \tag{1}$$

Restricting (1) on the fiber$\{x\} \times A_1$, for $x \in A$, one finds an isomorphism

$$e^*({t_x}^*(\mathcal{O}_A(D))) \simeq {t_{f(x)}}^*(\mathcal{O}_{A_1}(D_1)),$$

where e is the embedding $A_1 \hookrightarrow A$. Because D_1 is a principal polarisation, the point $f(x)$ is uniquelly defined by the above property, which can be written in divisorial terms as $(\{-x\}\dot{+}D)_{|A_1} = \{-f(x)\}\dot{+}D_1$, at least for x general such that the divisor $(\{-x\} + D)_{|A_1}$ is well defined. From this one deduces that points in A_1 are fixed by f and so f is surjective with $K \cap A_1 = \{0_A\}$, where K is the kernel of f.

Because K cuts A_1 only in 0_A, the sum morphism $K \times A_1 \to A$ is injectif and so we will have $dim\ (K\dot{+}D_1) = n - dim\ A_1 + dim\ A_1 - 1 = n - 1$. Now, for a general $x \in A$, we have $\{-f(x)\}\dot{+}D_1 = (\{-x\}\dot{+}D)_{|A_1} \subset \{-x\}\dot{+}D$. So, $x - f(x) \in Tran(D_1, D) := \{y \in A \mid \{y\}\dot{+}D_1 \subset D\}$. But $Tran(D_1, D)$ is closed and so for any $x \in A$ we have $\{x - f(x)\}\dot{+}D_1 \subset D$.

Then for $x \in K$, $\{x\}\dot{+}D_1 \subset D$ and therefore $K\dot{+}D_1 \subset D$. For K_0 the connected component of the origin in K, we have $K_0\dot{+}D_1 \subset D$. But $K_0\dot{+}D_1$ is a divisor and D is prime, so the previous inclusion is an equality. Now,

$$K_0\dot{+}D = K_0\dot{+}(K_0\dot{+}D_1) = K_0\dot{+}D_1 = D.$$

But D ample imply K_0 is finite and D prime imply $K_0 = \{0_A\}$ which is equivalent with $A = A_1$.

Step 3: Within this step we prove that for any i there is an unique j such that $D_i \cdot E_j \neq 0$. For this, we consider for all i, an abelian variety B_i, an ample divisor F_i on B_i and a surjectiv morphism $f_i : A \to B_i$ such that ${f_i}^{-1}(F_i) = D_i$. Their existence follow from lemma 4.1.

We have

$$n = D \cdot Z = \sum_{i=1}^{r} {f_i}^{-1}(F_i) \cdot Z = \sum_{i=1}^{r} F_i \cdot (f_i)_* Z \geq \sum_{i=1}^{r} l_i,$$

where $l_i = dim\ B_i$ and the last inequality is from proposition 3.3b. We examine the last sum using the effective construction of the B_i's from lemma 4.1. There, B_i is of the form A/K_i where K_i is an abelian subvariety of A. As consequence, $l_i = codim\ K_i$ and so

$$\sum_{i=1}^{r} l_i = \sum_{i=1}^{r} codim\ K_i \geq codim\ (\cap K_i) = n$$

(by definition of K_i and the ampleness of D, the intersection $\cap K_i$ is finite).

It results that

$$n \geq \sum_{i=1}^{r} F_i \cdot (f_i)_* Z \geq \sum_{i=1}^{r} l_i \geq n,$$

and so $F_i \cdot (f_i)_* Z = l_i$. But F_i is a prim divisor and from step 2 there is an unique j_i with $(f_i)_* E_{j_i}$ a curve. All other curves from the support of Z will be therefore contracted. We fix now i and compute $D_i \cdot E_j = {f_i}^{-1}(F_i) \cdot E_j = F_i \cdot f_{i*} E_j$. This last number is 0 if $j \neq j_i$ and non-zero for $j = j_i$ because F_i is ample. This conclude the third step.

From the first and third steps we find that $i \to j_i$ is a bijection and so $r = s$. Also one can reorder the curves E_j (such that E_{j_i} will be numbered by E_i) and so we can suppose that for all i, j we have $D_i \cdot E_j \neq 0 \Leftrightarrow i = j$. For ending the proof, we consider all the requirements supposed above.

In first place we review the B_i's. Let T_i the cycle $(f_i)_* Z$. From the third step, T_i is in fact a curve, namely $f_i(E_i)$. Also we have seen that $F_i \cdot T_i = l_i = dim\ B_i$ and therefore theorem 3.5 imply that B_i is the jacobian of T_i. To see this, we need only to prove that T_i is a generating curve of B_i and this is implied by the fact that, as we saw, f_i contracts all the curves E_j

for $j \neq i$ and as far as these contains 0_A, the contraction will be to 0_{B_i}. So $f_i(A) = f_i(A_i) = B_i$ and because E_i generates A_i, T_i generates B_i. So, by theorem 3.5, F_i is a translation of the canonical divisor on B_i.

Recall, that in the first step, we supposed (using apropriate translations) that all D_i's cuts proper the sub-varieties A_j's, which means that either ${e_j}^*(D_i)$ is an effectiv divisor on A_j, or $D_i \cap A_j$ is empty, in which case ${e_j}^*(D_i) = 0$. The first case can happen only for $j = i$, because in this situation ${e_j}^*(D_i) \cdot E_j \neq 0$ (more precisely, the projection formula gives ${e_j}^*(D_i) \cdot E_j = D_i \cdot {e_j}_* E_j = D_i \cdot E_j$). So ${e_j}^*(D_i) \neq 0 \Leftrightarrow j = i$ and we have

$$D'_j = {e_j}^*(D) = {e_j}^*(D_j) = {e_j}^*{f_j}^*(F_j) = (f_j \circ e_j)^*(F_j).$$

Let's consider the morphism $f_j \circ e_j : A_j \to B_j$. It sends the generating curve E_j of A_j to the generating curve T_j of B_j, and we saw it's surjective; so $d_j \geq l_j$. But, from the first and third steps, $n = \sum_{j=1}^{r} d_j = \sum_{j=1}^{r} l_j$; this imply that $f_j \circ e_j$ has a finite non-zero degree. On the other hand $f_j \circ e_j$ pull-back the principal polarization F_j from B_j to the principal polarization D'_j on A_j. So its degree is 1 and it is an isomorphism with inverse denoted g_j.

Let $h : B_1 \times ... \times B_r \to A$ defined by $h(b_1, ..., b_r) = \sum_{i=1}^{r} g_i(b_i)$ and $g : A \to B_1 \times ... \times B_r$ defined by $g(a) = (f_1(a), ..., f_r(a))$. Then $h \circ g$ is the identity, being the identity on every A_i. Also, $g \circ h$ is the identity, being the identity on every $\{0_{B_1}\} \times ... \times B_i \times ... \times \{0_{B_r}\}$.

So h is an isomorphism, B_i is the jacobian of T_i and the last part of the theorem concerning the form of the divisors D_i and curves T_i is obvious due to the fact that the transformations of D_i and T_i were translations. **QED**

Finally, we formulate the following corollary which is the result of Hoyt from [3].

Corollary 4.3. *Let A an abelian variety, D an ample divisor with $D^n = n!$ and Z a 1-cycle such that D^{n-1} is numeric equivalent with $(n-1)!Z$. The the conclusion of theorem 4.2 holds.*

Proof: We have $D^n = (n-1)!D \cdot Z \cdot$, so $D \cdot Z = n$. On the other hand, from proposition 2.6c, $(n-1)!Z$ is a generating 1-cycle and therefore Z is a generating 1-cycle. Now all is a consequence of theorem 4.2. **QED**

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