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Boundary behaviour of the mappings satisfying generalized inverse modular inequalities

Mihai Cristea

Abstract: We study the geometric properties of the mappings for which generalized inverse modular inequalities hold. We generalize in this way known theorems from the theory of analytic mappings and the theory of quasiregular mappings, like the theorems of Fatou, M. and F. Riesz, Beurling and Lindelöf and their extensions given for quasiregular mappings by Martio, Rickman and Vuorinen.

Keywords: boundary behaviour of the mappings for which some modular inequalities hold.

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1 Introduction.

Given a domain $D \subset \mathbb{R}^n$, we denote by $A(D)$ the set of all path families from D and for $\Gamma \in A(D)$, we put $F(\Gamma) = \{\rho : \mathbb{R}^n \to [0,\infty]$ Borel maps $| \int_\gamma \rho ds \geq 1$ for every $\gamma \in \Gamma$ locally rectifiable$\}$. We set for $p \geq 1$, $\Gamma \in A(D)$ and $\omega : D \to \mathbb{R}^n$ measurable and finite a.e. the p modulus of weight ω, $M^p_\omega(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{\mathbb{R}^n} \omega(x)\rho(x)^p dx$ and for $\omega = 1$ we obtain the classical p modulus $M_p(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{\mathbb{R}^n} \rho(x)^p dx$.

A known class of continuous, open, discrete mappings $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is the class of K-quasiregular mappings, which are ACL^n, $K \geq 1$ and $|f'(x)|^n \leq K J_f(x)$ a.e. For such mappings the important modular inequality of Poleckii says that $M_n(f(\Gamma)) \leq K M_n(\Gamma)$ for every $\Gamma \in A(D)$. This modular inequality is the key for proving most of the geometric properties of this class of mappings. We recommend the reader the books [33], [34], [50], [57] for basic facts of this theory.

A map $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is of finite distortion if $f \in W^{1,1}_{loc}(D, \mathbb{R}^n) \cap C(D, \mathbb{R}^n)$, $J_f \in L^1_{loc}(D)$ and there exists $K : D \to [0,\infty]$ measurable and finite a.e. such that $|f'(x)|^n \leq K(x) J_f(x)$ a.e. If in addition $f \in W^{1,n}_{loc}(D, \mathbb{R}^n)$, we say that f is of finite dilatation. General classes of such mappings were intensively studied using the modulus method in the last 20 years in [4-7], [14], [17-19], [24-27], [30-32], [35-44], [46-49] and several conditions were imposed to the dilatation K or to the map f, like $K \in BMO(D)$, or such that $exp(A \circ K) \in L^1_{loc}(D)$ for some Orlicz map A, or such that f has locally ACL^n inverses. All of them are open, discrete functions f, and the modular inequality "$M_n(f(\Gamma)) \leq M^n_{K^{n-1}}(\Gamma)$" holds for every $\Gamma \in A(D)$, and this is the main instrument used in studying this functions.

In some recent paper [8-12], we studied classes of continuous, open, discrete mappings $f : D \to \mathbb{R}^n$ satisfying modular inequalities of type "$M_q(f(\Gamma)) \leq \gamma(M^p_\omega(\Gamma))$ for every $\Gamma \in A(D)$", where $q > n-1$, $p > 1$, $\omega \in L^1_{loc}(D)$ and $\gamma : [0,\infty) \to [0,\infty)$ is increasing with $\lim_{t \to 0} \gamma(t) = 0$. We extended partially basic theorems from the theory of quasiregular mappings and from the classes of open, discrete mappings considered in [4-7], [14], [17-19], [24-27], [30-32], [35-44], [46-49] and in some cases we showed that our results were stronger even in the class of quasiregular mappings. We gave Liouville, Montel, Picard type theorems, equicontinuity and eliminability results and we gave estimates of the modulus of continuity. The basic tool for proving this results was the modular inequality "$M_q(f(\Gamma)) \le \gamma(M^p_\omega(\Gamma))$" together with the fact that $M^p_\omega(x) = 0$ in some points $x \in \overline{D}$. Using the modulus method, we developed an unified theory which contains all the classes of mappings of finite distortion mentioned before.

On the other hand, if $f : D \to \mathbb{R}^n$ is K-quasiregular and $N(f, D) < \infty$, the inverse modular inequality "$M_n(\Gamma) \le KN(f, D)M_n(f(\Gamma))$" holds for every $\Gamma \in A(D)$. It is a natural question if a function $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ satisfying a modular inequality of type "$M_q(\Gamma) \le \gamma(M^p_\omega(f(\Gamma)))$" has interesting geometric properties. We show that the answer is positive, using entirely the modulus method. Our methods can be applied to ACL^q mappings, $q > n-1$, having variable Jacobian sign, which are not open or mappings with finite distortion, and have no monotone components, as we can see from Example 1. We show that even for such mappings some results from classical complex analysis still hold and we remark that until now most of the generalizations of complex functions were open functions, or functions having monotone components.

A classical theorem of Fatou states that a bounded analytic function $f : B(0, 1) \to \mathbb{C}$ has a.e. radial limits. It is not known whether a bounded quasiregular mapping $f : B(0, 1) \to \mathbb{R}^n$ has at least one radial limit. K. Rajala showed in [32] that if $f : B(0, 1) \to \mathbb{R}^n$ is a locally injective quasiregular mapping, then f has at least one radial limit. In [25] it is proved that if $f : B(0, 1) \to B(0, 1)$ is a locally injective quasiregular mapping and there exists $C > 0$ and $0 < a < n-1$ such that $N(f, B(0, r)) \le C(1-r)^{-a}$ for $0 < r < 1$, then the Hausdorff dimension of $E(f)$ is less or equal to a, where $E(f) = \{y \in S(0, 1) | f$ has no radial limit in $y\}$. The same result is obtained for bounded quasiregular mappings $f \in B(0, 1) \to \mathbb{R}^n$, as we can see from [20]. Miklyukov proved in [28] that if $f : B(0, 1) \to \mathbb{R}^n$ is a bounded quasiregular map and $\int\limits_{B(0,1)} |f'(x)|^n dx < \infty$, then f has a.e. nontangential limits. In Theorem 5.15 in [23] it is proved that if $f : B(0, 1) \to \mathbb{R}^n$ is quasiregular and there exists $c_1 > 0$ and $0 < \beta < n-1$ such that $\int\limits_{\mathbb{R}^n} N(y, f, B(0, r))dy \le \frac{c_1}{(1-r)^\beta}$ for every $0 < r < 1$, then f has a.e. radial limits. We extend this result in Theorem 5, showing:

Theorem 5. Let $n \ge 2$, $1 < q < p$, let $f : B(0, 1) \to \mathbb{R}^n$ be ACL^q such that there exists $K : B(0, 1) \to [0, \infty]$ measurable and finite a.e. such that $L(x, f)^p \le K(x)|J_f(x)|$ a.e. and suppose that there exists c_1, c_2, α, $\beta > 0$ such that

$$\int\limits_{\mathbb{R}^n} N(y, f, B(0, r))dy \le \frac{c_1}{(1-r)^\beta} \text{ for every } 0 < r < 1 \qquad (1)$$

$$\int\limits_{B(0,r)} K(x)^{q/(p-q)}dx \le \frac{c_2}{(1-r)^\alpha} \text{ for every } 0 < r < 1 \qquad (2)$$

$$\frac{\alpha(p-q)}{p} + \frac{\beta q}{p} < q-1 \qquad (3)$$

Then f has a.e. radial limits.

If we take $p = n$, $\alpha = 0$ and $K(x) \le K$ for every $x \in B(0, 1)$, then condition 3) is "$0 < \beta < \frac{n(q-1)}{q}$" and taking q as close to n, this condition may be "$0 < \beta < n-1$" and this shows that our result improves Theorem 5.15 from [23] given for quasiregular mappings. We can also take the function f to be locally quasiregular or with finite dilatation and satisfying condition 1), 2), 3) from Thereom 5 and we still have a Fatou type result in this class of functions, and our proof is based entirely on the modulus method. A similar result also based on the modulus method is Theorem 1 in [21] which says that if $f \in W^{1,p}(B(0, 1))$ is monotone and $n-1<p\leq n$, then f has nontangential limits at every point $y\in S(0,1)$ with the possible exception of a set of zero p-capacity. When $p>q>n-1$ and the mapping f has monotone components, a related result to Theorem 5 was obtained by T. Äkkinen in [1].

We also give a Fatou type result for ACL^q mappings $f:D\to\mathbb{R}^n$ such that there exists $0<\alpha<q-1$ such that $\int\limits_{B(0,1)}|f^{'}(x)|^q(1-|x|)^\alpha dx<\infty$, in connection with Theorem 2' in [29], given for polyharmonic functions. We show:

Theorem 4. Let $n\geq 2$, $q>1$, $f:B(0,1)\to\mathbb{R}^n$ be ACL^q such that $\mu_n(B_f)=0$, there exists $0<\alpha<q-1$ such that $\int\limits_{B(0,1)}|f^{'}(x)|^q(1-|x|)^\alpha dx<\infty$ and suppose that one of the following conditions hold:

a) f has locally inverses on $f(D\setminus B_f)$ which satisfies condition (N).

b) f satisfies condition (N), there exists $p>0$ and $K:D\to[0,\infty]$ measurable and finite a.e. such that $L(x,f)^p\leq K(x)|J_f(x)|$ a.e. and either $J_f\in L^1_{loc}(D)$ and $q>n-1$, or f is a.e. differentiable.

Then f has a.e. radial limits.

A Fatou type theorem is given in Theorem 2 for mappings satisfying generalized inverse modular inequalities. A local version of it is given in Theorem 1 from which immediately results Theorem 3,4 and 5.

Theorem 2. Let $n\geq 2$, p, $q>1$, $\omega\in L^1(B(0,1))$, $\gamma:[0,\infty)\to[0,\infty)$ increasing such that there exists λ, $M>0$ such that $\gamma(t)\leq Mt^\lambda$ for $t\geq 0$ and let $f:B(0,1)\to\mathbb{R}^n$ be continuous such that $M_q(\Gamma)\leq\gamma(M^p_\omega(f(\Gamma)))$ for every $\Gamma\in A(B(0,1))$. Then f has a.e. radial limits.

We shall give in Lemma 2 and 3 enough consistent conditions in order that some ACL^q mappings to satisfy generalized inverse modular inequalities.

In Theorem 5.17 in [23] Martio and Rickman generalized a theorem of F. and M. Riesz given for bounded analytic functions. They showed that if $f:B(0,1)\to\mathbb{R}^n$ is quasiregular, $c\in\mathbb{R}^n$, $E=\{y\in S(0,1)|\lim\limits_{t\to 1}f(ty)=c\}$ and there exists $c_1>0$, $0<\beta<n-1$ such that $\int\limits_{B(0,r)}N(y,f,B(0,r))dy\leq\frac{c_1}{(1-r)^\beta}$ for every $0<r<1$, then it results that $\mu_{n-1}(E)=0$. We extend this result in Theorem 9, showing that:

Theorem 9. Let $n\geq 2$, $1<q<p$, $f:B(0,1)\to\mathbb{R}^n$ be ACL^q such that there exists $K:B(0,1)\to\mathbb{R}^n$ measurable and finite a.e. such that $L(x,f)^p\leq K(x)|J_f(x)|$ a.e., let $c\in\mathbb{R}^n$ such that $m_{n-2}(f^{-1}(c))=0$, let $f_y:[0,1)\to\mathbb{R}^n$ be defined by $f_y(t)=f(ty)$ for $t\in[0,1)$ and $y\in S(0,1)$ and let $E=\{y\in S(0,1)|c$ is a limit point of $f_y:[0,1)\to\mathbb{R}^n\}$. Suppose that there exists c_1, c_2, α, $\beta>0$ such that

$$\int\limits_{\mathbb{R}^n}N(y,f,B(0,r))dy\leq\frac{c_1}{(1-r)^\beta}\text{ for every }0<r<1\tag{1}$$

$$\int\limits_{B(0,r)}K(x)^{q/(p-q)}dx\leq\frac{c_2}{(1-r)^\alpha}\text{ for every }0<r<1\tag{2}$$

$$\frac{\alpha(p-q)}{p}+\frac{\beta q}{p}<q-1\tag{3}$$

Then $\mu_{n-1}(E)=0$.

Another extension of the theorem of F. and M. Riesz is given for ACL^q mappings for which we impose a boundedness condition of the modulus of the derivative.

Theorem 8. Let $n\geq 2$, $q>1$, $f:B(0,1)\to\mathbb{R}^n$ be ACL^q such that $\mu_n(B_f)=0$, there exists $0<\alpha<q-1$ such that $\int\limits_{B(0,1)}|f^{'}(x)|^q(1-|x|)^\alpha<\infty$, let $c\in\mathbb{R}^n$ such that $m_{n-2}(f^{-1}(c))=0$, let $f_y:[0,1)\to\mathbb{R}^n$ be defined by $f_y(t)=f(ty)$ for $t\in[0,1)$ and $y\in S(0,1)$ and let $E=\{y\in S(0,1)|c$ is a limit point of $f_y:[0,1)\to\mathbb{R}^n\}$. Suppose that one of the following conditions hold:

a) f has locally inverses on $f(D \setminus B_f)$ which satisfies condition (N).

b) f satisfies condition (N), there exists $p > 0$ and $K : D \to [0, \infty]$ measurable and finite a.e. such that $L(x, f)^p \leq K(x)|J_f(x)|$ a.e. and either $J_f \in L^1_{loc}(D)$ and $q > n - 1$ or f is a.e. differentiable.

Then $\mu_{n-1}(E) = 0$.

We also give in Theorem 7 a version of the theorem of F. and M. Riesz for mappings satisfying inverse modular inequalities. A local version of it is Theorem 5 from which immediately results Theorem 8 and 9.

Theorem 7. Let $n \geq 2$, p, $q > 1$, $\omega \in L^1(\mathbb{R}^n)$, $\gamma : [0, \infty) \to [0, \infty)$ be increasing such that there exists $\lambda, M > 0$ such that $\gamma(t) \leq Mt^\lambda$ for $t \geq 0$, let $f : B(0, 1) \to \mathbb{R}^n$ be continuous such that $M_q(\Gamma) \leq \gamma(M^p_\omega(f(\Gamma)))$ for every $\Gamma \in A(B(0, 1))$, let $c \in \mathbb{R}^n$ such that $m_{n-2}(f^{-1}(c)) = 0$, let $f_y : [0, 1) \to \mathbb{R}^n$ be defined by $f_y(t) = f(ty)$ for $t \in [0, 1)$ and $y \in S(0, 1)$ and let $E = \{y \in S(0, 1)|c$ is a limit point of $f_y : [0, 1) \to \mathbb{R}^n\}$. Then $\mu_{n-1}(E) = 0$.

Theorem 5 and 9 in the case $p = q = n$ correspond to the results in 5.15 and 5.17 in [23]. Vuorinen extended in Theorem 14.7 in [57] a result of Beurling, showing that if $f : B(0, 1) \to \mathbb{R}^n$ is quasiconformal and $E = \{y \in S(0, 1)|f$ has no asymptotic value at $y\}$, then it results that $cap_n(F) = 0$ for every compact $F \subset E$. The following theorem generalizes Vuorinen's result.

Theorem 10. Let $n \geq 2$, $n - 1 < q \leq n$, $p \geq 2$, $\omega : D \to [0, \infty]$ measurable and finite a.e. $\gamma : [0, \infty) \to [0, \infty)$ be strictly increasing with $\lim\limits_{t \to 0} \gamma(t) = 0$, let $f : B(0, 1) \to \mathbb{R}^n$ be continuous such that $M_q(\Gamma) \leq \gamma(M^p_\omega(f(\Gamma)))$ for every $\Gamma \in A(B(0, 1))$ and let $E = \{y \in S(0, 1)|f$ has no asymptotic value at $y\}$. Then $cap_q(F) = 0$ for every compact $F \subset E$.

We also give versions of Beurling's theorem for ACL^q mappings, extending partially Theorem 1 in [21] given for monotone functions. Theorem 11 and 12 results immediately from Theorem 10.

Theorem 11. Let $n \geq 2$, $n - 1 < q \leq n$, $f : B(0, 1) \to \mathbb{R}^n$ be ACL^q such that $\int\limits_{B(0,1)} |f'(x)|^q dx < \infty$ and $\mu_n(B_f) = 0$ and let $E = \{y \in S(0, 1)|f$ has no asymptotic values at $y\}$. Suppose that f satisfies conditions $a)$ and $b)$ from Theorem 4. Then $cap_q(F) = 0$ for every compact $F \subset E$.

Theorem 12. Let $n \geq 2$, $n - 1 < q < p$, $q \leq n$, let $f : B(0, 1) \to \mathbb{R}^n$ be ACL^q and suppose that there exists $K \in L^{q/(p-q)}(B(0, 1))$ such that $L(x, f)^p \leq K(x)|J_f(x)|$ a.e. Then, if $E = \{y \in S(0, 1)|f$ has no asymptotic value at $y\}$, it results that $cap_q(F) = 0$ for every compact $F \subset E$.

Vuorinen generalized in Theorem 4.2 in [51] a known result from the theory of quasiconformal mappings in the class of closed quasiregular mappings, showing that if D, G are domains in $\mathbb{R}^n$ and $f : D \to G$ is a closed quasiregular mapping and D is quasiconformally flat at a point $b \in \partial D$, then $C(f, b)$ contains at most one point at which G is finitely connected. We extend partially Vuorinen's result in Theorems 13 and Corollary 2.

Theorem 13. Let $n \geq 2$, D, G be domains in $\mathbb{R}^n$, let $b \in \partial D$ such that D is quasiconformally flat at b, let $f : D \to G$ be ACL^n, open, discrete and closed such that $\mu_n(B_f) = 0$ and $\int\limits_D |f'(x)|^n dx < \infty$ and suppose that one of the following conditions hold:

a) f has locally inverses on $f(D \setminus B_f)$ which satisfies condition (N).

b) f satisfies condition (N), there exists $p > 0$ and $K : D \to [0, \infty]$ measurable and finite a.e. such that $L(x, f)^p \leq K(x)|J_f(x)|$ a.e. and either $J_f \in L^1_{loc}(D)$ and $q > n - 1$ or f is a.e. differentiable.

Then $C(f,b)$ contains at most one point at which G is finitely connected.

Corollary 2. Let $n \geq 2$, D, G be domains in $\mathbb{R}^n$, G finitely connected at the boundary, let $b \in \partial D$ be such that D is quasiconformally flat at b and let $f : D \to G$ be ACL^n, open, discrete and closed such that $\mu_n(B_f) = 0$ and $\int\limits_D |f^{'}(x)|^n dx < \infty$ and suppose that one of the following conditions hold:

a) $J_f(x) \neq 0$ a.e.

b) $\mu_n(f(B_f)) = 0$ and there exists $p > 0$ and $K : D \to [0, \infty]$ measurable and finite a.e. such that $L(x,f)^p \leq K(x)|J_f(x)|$ a.e.

Then there exists $F : D \cup \{b\} \to \overline{\mathbb{R}^n}$ continuous such that $F|D = f$ and we take on $\overline{\mathbb{R}^n}$ the chordal metric.

If in the preceeding theorem $f : D \to G$ is a homeomorphism, we have:

Corollary 3. Let $n \geq 2$, D, G be domains in $\mathbb{R}^n$, G finitely connected at the boundary, $b \in \partial D$ such that D is quasiconformally flat at b and let $f : D \to G$ be an ACL^n homeomorphism such that $\int\limits_D |f^{'}(x)|^n dx < \infty$ and suppose that one of the following conditions hold:

a) $J_f(x) \neq 0$ a.e.

b) there exists $p > 0$ and $K : D \to [0, \infty]$ measurable and finite a.e. such that $L(x,f)^p \leq K(x)|J_f(x)|$ a.e.

Then there exists $F : D \cup \{b\} \to \mathbb{R}^n$ continuous such that $F|D = f$, and we take on $\overline{\mathbb{R}^n}$ the chordal metric.

Corollary 3 is proved by Iwaniecz and Onninen in Theorem 1.3 in [16] without using conditions $a)$ and $b)$, but with some supplementary requirements on the domains D and G and their proof does not use the modulus method.

We prove the following Lindelöf type theorem for mappings satisfying inverse generalized modular inequalities:

Theorem 14. Let $n \geq 2$, $n-1 < q \leq n$, $p > 1$, $\lambda > 0$, $0 < \psi < \varphi \leq \frac{\pi}{2}$, $a = \frac{1}{2}\sin(\varphi - \psi)$, let $D \subset \mathbb{R}^n$ be a domain, $x \in \partial D$, d a half line ending in x such that there exists $\rho > 0$ such that $C_{x,d,\varphi} \cap B(x,\rho) \subset D$, let $C = C_{x,d,\varphi}$, $C_1 = C_{x,d,\psi}$ and $E \subset C_1$ such that $cap_{q,a}\underline{dens}(C,E,x) = \delta > 0$. Let $f : D \to \mathbb{R}^n$ be continuous having monotone components, let $K_r > 0$ and $w_r : D \cap B(x,r) \to [0,\infty]$ be measurable for $0 < r < \rho$ such that $M_q(\Gamma) \leq K_r(M^p_{\omega_r}(f(\Gamma)))^\lambda$ for every $\Gamma \in A(D \cap B(x,r))$ and every $0 < r < \rho$ and suppose that $\lim\limits_{r \to \infty} K_r(\int\limits_{f(B(x,r)\cap D)} \omega_r(z)dz)^\lambda / r^{n-q} = 0$ and that $\lim\limits_{\substack{z \to x \\ z \in E}} f(z) = c$. Then $\lim\limits_{\substack{z \to x \\ z \in C_1}} f(z) = c$.

Using this result, we can give the following partial extensions to Theorem 16.8 in [57]:

Corollary 4. Let $n \geq 2$, $n-1 < q \leq n$, $0 < \psi < \varphi \leq \frac{\pi}{2}$, $a = \frac{1}{2} sin(\varphi - \psi)$, let $D \subset \mathbb{R}^n$ be domain, $x \in \partial D$, d a half line ending in x such that there exists $\rho > 0$ such that $C_{x,d\varphi} \cap B(x,\rho) \subset D$, let $C = C_{x,d,\varphi}$, $C_1 = C_{x,d,\psi}$, $E \subset C_1$ such that $cap_{q,a}\underline{dens}(C,E,x) > 0$, let $f : D \to \mathbb{R}^n$ be ACL^q on D having monotone components such that $\mu_n(B_f) = 0$ and such that $\lim\limits_{r \to 0} \int\limits_{B(x,r)\cap D} |f^{'}(z)|^q dz / r^{n-q} = 0$.

Suppose that one the following conditions hold:

a) f has locally inverses on $f(D \setminus B_f)$ which satisfies condition (N).

b) f satisfies condition (N) and there exists $p \geq 0$, $K : D \to [0,\infty]$ measurable and finite a.e. such that $L(z,f)^p \leq K(z)|J_f(z)|$ a.e. and either f is a.e. differentiable or $J_f \in L^1(D \cap B(x,\rho))$.

Then, if $\lim\limits_{\substack{z \to x \\ z \in E}} f(z) = c$, it results that $\lim\limits_{\substack{z \to x \\ z \in C_1}} f(z) = c$.

Corollary 5. Let $n \geq 2$, $n-1 < q < p$, $q \leq n$, $0 < \varphi < \psi \leq \frac{\pi}{2}$, $a = \frac{1}{2}\sin(\varphi - \psi)$, let

$D \subset \mathbb{R}^n$ be a domain, $x \in \partial D$, d a half line ending in x such that there exists $\rho > 0$ such that $C_{x,d,\varphi} \cap B(x,\rho) \subset D$, let $C = C_{x,d,\varphi}$, $C_1 = C_{x,d,\psi}$, $E \subset C_1$ such that $cap_{q,a}\underline{dens}(C, E, x) > 0$. Suppose that $f : D \to \mathbb{R}^n$ is ACL^q on D, satisfies condition (N) and has monotone components, let $K : D \to [0, \infty]$ measurable and finite a.e. such that $K \in L^{q/(p-q)}(D \cap B(x,\rho))$ be such that $L(z,f)^p \leq K(z)$. $|J_f(z)|$ a.e. and suppose that

$$\lim_{r\to 0}(\int\limits_{B(x,r)\cap D} K(z)^{q/(p-q)}dz)^{\frac{p-q}{q}}(\int\limits_{D\cap B(x,r)} J_f(z)dz)^{q/p}/r^{n-q} = 0.$$

Then, if $\lim\limits_{\substack{z\to x\\ z\in E}} f(z) = c$, it results that $\lim\limits_{\substack{z\to x\\ z\in C_1}} f(z) = c$.

The following theorem is a Lindelöf type theorem without assuming that the components of the function $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ are monotone:

Theorem 15. Let $n \geq 2$, $0 < \varphi \leq \frac{\pi}{2}$, $D \subset \mathbb{R}^n$ a domain, $x \in \partial D$, d a half line ending in x, $0 < \rho < 1$ such that $C_{x,d,\varphi} \cap B(x,\rho) \subset D$, let $C = C_{x,d,\varphi}$, let $f : D \to \mathbb{R}^n$ be ACL^n on D such that $\mu_n(B_f) = 0$ and $\int\limits_{B(x,\rho)\cap D} |f^{'}(z)^n dz < \infty$ and let $\gamma : [0,1] \to C$ be a nonconstant path such that $\gamma(0) = x$ and there exists $\lim\limits_{t\to 0} f(\gamma(t)) = c$. Suppose that one of the following conditions is satisfied:

a) f has locally inverses on $f(D \setminus B_f)$ which satisfies condition (N).

b) f satisfies condition (N) and there exists $p > 0$ and $K : D \to [0, \infty]$ measurable and finite a.e. such that $L(z,f)^p \leq K(z)|J_f(z)|$ a.e. and either f is a.e. differentiable, or $J_f \in L^1(D \cap B(x,\rho))$.

Then, if $S = \{y \in S(x,1) | [x,y] \cap C \cap S(x,\rho) \neq \phi\}$, it results that $\lim\limits_{t\to 1} f(\gamma_y(t)) = c$ for a.e. $y \in S$, where $\gamma_y : [0,1] \to \mathbb{R}^n$ is given by $\gamma_y(t) = tx + (1-t)y$ for $t \in [0,1]$ and $y \in S$. If f has also monotone components and $0 < \psi < \varphi$, then $\lim\limits_{\substack{z\to x\\ z\in C_{x,d,\psi}}} f(z) = c$.

The following two theorems are in connection with Theorem 15.10 in [57]:

Theorem 16. Let $n \geq 2$, $0 < \psi < \varphi \leq \frac{\pi}{2}$, $a = \frac{1}{2}\sin(\varphi - \psi)$, $c \in \mathbb{R}^n$, $D \subset \mathbb{R}^n$ a domain, $x \in \partial D$, d a half line ending in x such that there exists $\rho > 0$ such that $C_{x,d,\varphi} \cap B(x,\rho) \subset D$, let $C = C_{x,d,\varphi}$, $C_1 = C_{x,d,\psi}$, let $f : D \to \mathbb{R}^n$ be ACL^n on D having monotone components such that $\mu_n(B_f) = 0$ and $\int\limits_{B(x,\rho)\cap D} |f^{'}(z)|^n dz < \infty$. Let $E_\epsilon = f^{-1}(B(c,\epsilon)) \cap B(x,\rho) \cap C_1$ for $\epsilon > 0$ be such that $\limsup\limits_{\epsilon\to 0} \int\limits_{E_\epsilon} |f^{'}(z)|^n dz/\epsilon^n = M < \infty$, let $\delta_\epsilon = cap_{n,a}\underline{dens}(C, E_\epsilon, x)$ for $\epsilon > 0$ and suppose that one of the following condition shold:

a) f has locally inverses on $f(D \setminus B_f)$ which satisfies condition (N).

b) f satisfies condition (N) and there exists $p > 0$ and $K : D \to [0, \infty]$ measurable and finite a.e such that $L(z,f)^p \leq K(z)|J_f(z)|$ a.e. and either f is a.e. differentiable, or $f \in L^1(D \cap B(x,\rho))$.

Then, if $\lim\limits_{\epsilon\to 0} \delta_\epsilon(\ln\ln(\frac{1}{\epsilon}))^n = \infty$, it results that $\lim\limits_{\substack{z\to x\\ z\in C_{x,d,\eta}}} f(z) = c$ for every $\psi < \eta < \varphi$.

Theorem 17. Let $n \geq 2$, $0 < \psi < \varphi \leq \frac{\pi}{2}$, $a = \frac{1}{2}\sin(\varphi - \psi)$, $c \in \mathbb{R}^n$, $D \subset \mathbb{R}^n$ a domain $x \in \partial D$, d a half line ending in x such that there exists $\rho > 0$ such that $C_{x,d,\varphi} \cap B(x,\rho) \subset D$, let $C = C_{x,d,\varphi}$, $C_1 = C_{x,d,\psi}$, let $f : D \to \mathbb{R}^n$ be K-quasiregular such that $J_f \in L^1(D \cap B(x,\rho))$ and $N(f, D \cap B(x,\rho)) < \infty$. Let $E_\epsilon = f^{-1}(B(c,\epsilon)) \cap B(x,\rho) \cap C_1$ for $\epsilon > 0$, let $\delta_\epsilon = cap_{n,a}\underline{deus}(C, E, x)$ for $\epsilon > 0$ and suppose that $\lim\limits_{\epsilon\to 0} \delta_\epsilon(\ln\ln(\frac{1}{\epsilon}))^n = \infty$. Then it results that $\lim\limits_{\substack{z\to x\\ z\in C_{x,d,\eta}}} f(z) = c$ for every $\psi < \eta < \varphi$.

2 Preliminaries.

We say that a function $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is ACL if f is continuous and for every cube $Q \subset\subset D$ with the sides parallel to coordinate axes and for every face S of Q it results that $f|P_S^{-1}(y) \cap Q : P_S^{-1}(y) \cap Q \to \mathbb{R}^n$ is absolutely continuous for a.e. $y \in S$, where $P_S : \mathbb{R}^n \to S$ is the projection on S. An ACL map has a.e. first partial derivatives and if $p > 1$, we say that f is ACL^p if f is ACL and the first partial derivatives are locally in L^p. If $p > 1$, we denote by $W^{1,p}_{loc}(D, \mathbb{R}^n)$ the Sobolev space of all functions $f : D \to \mathbb{R}^n$ which are locally in L^p together with their first order distributional derivatives. We see from Proposition 1.2, page 6 in [34] that if $f \in C(D, \mathbb{R}^n)$, then f is ACL^p if and only if $f \in W^{1,p}_{loc}(D, \mathbb{R}^n)$. If $x \in \mathbb{R}^n$, we set $|x| = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$. We set $B(x, r) = \{y \in \mathbb{R}^n | |y - x| < r\}$ and $S(x, r) = \{y \in \mathbb{R}^n | |y - x| = r\}$ for $x \in \mathbb{R}^n$ and $r > 0$. We denote by μ_n the Lebesgue measure in $\mathbb{R}^n$, by μ_{n-1} the spherical measure on $S(0, 1)$, and if $p > 0$, we denote by m_p the p-Hausdorff measure in $\mathbb{R}^n$.

If $D \subset \mathbb{R}^n$ is open, $E, F \subset \overline{D}$, we set $\Delta(E, F, D) = \{\gamma : [a, b] \to \overline{D}$ paths $|\gamma(a) \in E, \gamma(b) \in F$ and $\gamma((a, b)) \subset D\}$ and if $b \in \partial D$, we say that D is quasiconformally flat at b if $M_n(\Delta(E, F)) = \infty$ for every connected sets $E, F \subset \overline{D}$ such that $b \in \overline{E} \cap \overline{F}$. We say that D is finitely connected at the point $b \in \partial D$ if for every $V \in \mathcal{V}(b)$ there exists $U \in \mathcal{V}(b)$ such that $U \subset V$ and $U \cap D$ has a finite number of components. If $\gamma : [a, b] \to \mathbb{R}^n$ is rectifiable, we set $s_\gamma(t) = l(\gamma|[a, t])$ for $t \in [a, b]$ and we define the reparametrisation $\gamma^0 : [0, l(\gamma)] \to \mathbb{R}^n$ of γ by setting $\gamma(t) = \gamma^0(s_\gamma(t))$ for every $t \in [a, b]$. We say that $x \in \overline{\mathbb{R}^n}$ is a limit point of the path $\gamma : [a, b) \to \mathbb{R}^n$ if there exists $a < t_p < b$, $t_p \to b$ such that $\gamma(t_p) \to x$. If $x \in \mathbb{R}^n$, $0 < a < b$, we set $\Gamma_{x,a,b} = \Delta(\overline{B}(x, a), S(x, b), B(x, b) \setminus \overline{B}(x, a))$.

We say that $E = (A, C)$ is a condenser if $C \subset A \subset \mathbb{R}^n$, C is compact and A is open, and if $p > 1$,we define $cap_p(E) = \inf \int_{\mathbb{R}^n} |\nabla u(x)|^p dx$, the p capacity of E, where the infimum is taken over all $u \in C_0^\infty(A)$ such that $u \geq 1$ on C. We set $\Gamma_E = \Delta(C, \partial A, A)$ and we see from Proposition II.10.2, page 54 in [34] that if $p > 1$, then $cap_p(E) = M_p(\Gamma_E)$. We say that a compact set $C \subset \mathbb{R}^n$ is of zero p capacity, $p > 1$, if $cap_p(E) = 0$ for every condenser $E = (A, C)$ with A open and bounded, and the definion does not depend on the open, bounded set A such that $C \subset A$. If $C \subset \mathbb{R}^n$ is compact, we write $cap_p(C) > 0$ if C is not of zero p-capacity.

Let $D \subset \mathbb{R}^n$ be open and $\varphi : \mathcal{B}(D) \to [0, \infty]$. We say that φ is a set function on D if $\varphi(A) < \infty$ for every compact $A \subset D$ and $\varphi(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \varphi(A_i)$ if $A_1, ..., A_i, ...$ are disjoint Borel sets. We say that φ is absolutely continuous if for every $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that $\varphi(A) < \epsilon$ for every $A \in \mathcal{B}(D)$ such that $\mu_n(A) < \delta_\epsilon$, and if φ assumes only finite values, then φ is absolutely continuous if and only if $\varphi(A) = 0$ whenever $\mu_n(A) = 0$. We say that φ has a derivative $\varphi'(x)$ in x if $\varphi'(x) = \lim_{r \to 0} \frac{\varphi(B(x,r))}{\mu_n(B(x,r))}$. A set function φ has a.e. a finite derivative φ' which is a Borel function and if φ is absolutely continuous, then $\varphi(A) = \int_A \varphi'(x) dx$ for every $A \in \mathcal{B}(D)$ (see [50], page 81-83).

Let $D \subset \mathbb{R}^n$ a domain and $f : D \to \mathbb{R}^n$ a map. We denote by $B_f = \{x \in D | f$ is not a local homeomorphism at $x\}$ and if $x \in D$, we set $L(x, f) = \limsup_{h \to 0} \frac{|f(x+h) - f(x)|}{|h|}$. We say that f satisfies condition (N) if $\mu_n(f(A)) = 0$ for every $A \subset D$ such that $\mu_n(A) = 0$. We say that f is open if f carries open sets into open sets, we say that f is closed if f carries closed sets into closed sets and we say that f is discrete if $f^{-1}(y)$ is empty or discrete in D for every $y \in \mathbb{R}^n$. If $A \subset D$, we set $N(y, f, A) = Card\ (f^{-1}(y) \cap A)$ for every $y \in \mathbb{R}^n$ and we set

$N(f,A)=\sup\limits_{y\in\mathbb{R}^n} N(y,f,A)$. If $b\in\partial D$, we set $C(f,b)=\{z\in\overline{\mathbb{R}^n}|$ there exists $b_p\in D, b_p\to b$ such that $f(b_p)\to z\}$. If $f:D\to\mathbb{R}^n$ is continuous, $x\in D$ and f is discrete at x, we set $i(f,x)$ the topological index of f in x, and if f is differentiable in x and $J_f(x)\neq 0$, then $i(f,x)=sgnJ_f(x)$ (see [13] for some basic facts concerning the topological degree).

If $f:B(0,1)\to\mathbb{R}^n$ is a map and $y\in S(0,1)$, we say that f has a radial limit at x if there exists $\lim\limits_{t\to 1} f(ty)\in\mathbb{R}^n$, and we say that f has an asymptotic value at y if there exists $\gamma:[0,1)\to B(0,1)$ a path such that $\lim\limits_{t\to 1}\gamma(t)=y$ and there exists $\lim\limits_{t\to 1} f(\gamma(t))\in\mathbb{R}^n$.

Let $a,b,c\in\mathbb{R}^n$. We set $a(b-a.c-a)$ the angle between $b-a$ and $c-a$ if this angle is less then π. If $x\in\mathbb{R}^n$, $0<\varphi\leq\frac{\pi}{2}$, d is a half line ending in x, we set $C_{x,d,\varphi}=\{z\in\mathbb{R}^n|a(z-x,w-x)<\varphi$, where $w\in d\}$, the cone of center x, direction d and angle φ. Let $q>1$, $0<a\leq 1$, $0<\varphi\leq\frac{\pi}{2}$, $x\in\mathbb{R}^n$, d a half line ending in x, $C=C_{x,d,\varphi}$ and let $E\subset C$. We set the (q,a)-lower capacity density of E in C at the point x by $cap_{q,a}\underline{deus}(C,E,x)=\liminf\limits_{r\to 0} M_q(\Delta(\overline{B}(x,r)\cap E, S(x,(1+a)r)\cap C, B(x,(1+a)r)\cap C)/r^{n-q}$ and if $q=n$ and $a=1$ we have the definition of the lower capacity density of E in x given by M. Vuorinen in Definition 14.9 in [57]. We set the lower radial density of E in x by $rad\ \underline{deus}(E,x)=\liminf\limits_{r\to 0}\frac{m_1\{r\geq 0|S(x,r)\cap E\neq\phi\}}{r}$ and this is Definition 14.10 in [57].

Let $D\subset\mathbb{R}^n$ be open and $f:D\to\mathbb{R}$ be continuous. We say that f is monotone if $\max\limits_{x\in\bar{G}} f(x)=\max\limits_{x\in\partial G} f(x)$ and $\min\limits_{x\in\bar{G}} f(x)=\min\limits_{x\in\partial G} f(x)$ whenever G is a domain such that $\bar{G}\subset D$.

Let $D\subset\mathbb{R}^n$ be open and $f:D\to\mathbb{R}^n$ be ACL^q, $q>1$ such that $J_f(x)\neq 0$ a.e. We can defined a.e. the function $K_{0,q}(f):D\to[0,\infty]$ by $K_{0,q}(f)(x)=\frac{|f'(x)|^q}{|J_f(x)|}$.

We shall use Theorem 4 in [2] which says that if $x\in\mathbb{R}^n$, $0<a<b$, $D=B(x,b)\setminus\overline{B}(x,a)$, $n-1<q\leq n$, $E,F\subset D$ are such that $S(x,t)\cap E\neq\phi$, $S(x,t)\cap F\neq\phi$ for every $a<t<b$, then $M_q(\Delta(E,F,D))\geq C(n,q)(b^{n-q}-a^{n-q})$ if $n-1<q<n$, $M_n(\Delta(E,F,D))\geq C(n)\ln\frac{b}{a}$, where $C(n,q)$ is a constant depending only on n and q and $C(n)$ is a constant depending only on n, and $C(n,q)>0$, $C(n)>0$.

3 Some conditions in order that an ACL^q function to satisfy a generalized inverse modular inequality.

Lemma 1. Let $n\geq 2$, $U,V\subset\mathbb{R}^n$ be open sets, U bounded, $A\subset U$, $B\subset V$ such that $\mu_n(V\setminus B)=0$, let $f:U\to V$ be continuous such that $f|A:A\to B$ is a homeomorphism and let $g:V\to\mathbb{R}^n$, $g(y)=f^{-1}(y)$ if $y\in B$, $g(y)=0$ if $y\in V\setminus B$. Then g is measurable and let $\mu_g:\mathcal{B}(V)\to[0,\infty]$ given by $\mu_g(F)=\mu_n(g(F\cap B))$ for $F\in\mathcal{B}(V)$. Then μ_g is a set function, μ_g' exists a.e. and is a Borel function and if μ_g is absolutely continuous, then $\int\limits_{g(V)} h(z)dz=\int\limits_V h(g(x))\mu_g'(x)dx$ for every Borel function $h:\mathbb{R}^n\to[0,\infty]$.

Proof. Using Lebesgue's theorem, we see that $\int\limits_V\mu_g'(y)dy=\mu_g(V)=\mu_n(g(V))$ and we apply then a standard argument.

Lemma 2. Let $n\geq 2$, $q>1$, $D\subset\mathbb{R}^n$ a domain, $f:D\to\mathbb{R}^n$ be ACL^q such that $\mu_n(B_f)=0$ and $\int\limits_D|f'(x)|^q dx<\infty$ and suppose that one of the following conditions are satisfied:

a) f has locally inverses on $f(D\setminus B_f)$ which satisfies condition (N).

b) f satisfies condition (N), there exists $p>0$ and $K:D\to[0,\infty]$ measurable and finite a.e. such that $L(x,f)^p \le K(x)|J_f(x)|$ a.e. and either $J_f \in L^1_{loc}(D)$ and $q > n-1$, or f is a.e. differentiable.

Then there exists $\omega \in L^1(\mathbb{R}^n)$ such that $M_q(\Gamma) \le C(n)^2 M^q_\omega(f(\Gamma))$ for every $\Gamma \in A(D)$.

Proof. Let $\omega : \mathbb{R}^n \to [0,\infty]$ be given by $\omega(y) = \sum\limits_{x \in f^{-1}(y)\cap(D\setminus B_f)} \mu^{'}_{g_x}(y) L(g_x(y), f)^q$ for a.e. $y \in f(D \setminus B_f)$, $\omega(y) = 0$ otherwise, where g_x is a local inverse of f around the point x such that $g_x(f(x)) = x$. Let D_k be open, bounded, $\overline{D}_k \subset D_{k+1}$, $k \in \mathbb{N}$ be such that $D \setminus B_f = \bigcup\limits_{k=1}^{\infty} \overline{D}_k$. Let $k \in \mathbb{N}$ be fixed and let $\omega_k : \mathbb{R}^n \to [0,\infty]$ be defined by $\omega_k(y) = \sum\limits_{x \in f^{-1}(y)\cap D_k} \mu^{'}_{g_x}(y) L(g_x(y), f)^q$ for a.e. $y \in f(D_k)$, $\omega_k(y) = 0$ otherwise, where g_x is a local inverse of f around the point x such that $g_x(f(x)) = x$. Let $x \in D_k$ and $y = f(x)$. Then $f^{-1}(y) \cap \overline{D}_k$ is a finite set $\{a_1, ..., a_m\} \subset D \setminus B_f$ and since f is a local homeomorphism around each point $a_1, ..., a_m$, we find $V \in \mathcal{V}(y)$ and $U_i \in \mathcal{V}(a_i)$ disjoint such that $\overline{U}_i \subset D_{k+1}$, $f|U_i : U_i \to V$ is a homeomorphism for $i = 1, ..., m$ and $f^{-1}(V) \cap \overline{D}_k \subset \bigcup\limits_{i=1}^{m} U_i$. Let $g_i : V \to U_i$ be the inverse of $f|U_i : U_i \to V$ for $i = 1, ..., m$. We see that $(\omega_k|V)(y) = \sum\limits_{i=1}^{m} \mu^{'}_{g_i}(y) L(g_i(y), f)^q$ for a.e. $y \in V$, hence ω_k is a Borel function for every $k \in \mathbb{N}$ and since $\omega_k \nearrow \omega$ a.e., we see that ω is a Borel function.

Using Besicovitch's covering theorem, we can find a constant $C(n)$ depending only on n and balls V_i, $i \in \mathbb{N}$ such that every point from $f(D_k)$ belongs to at most $C(n)$ balls V_i, $f(D_k) = \bigcup\limits_{i=1}^{\infty} V_i$ and for every $i \in \mathbb{N}$ there exists open, bounded and disjoint sets U_{ij} such that $\overline{U}_{ij} \subset D_{k+1}$, $f|U_{ij} : U_{ij} \to V_i$ is a homeomorphism for $j = 1, ..., j(i)$ and $f^{-1}(V_i) \cap \overline{D}_k \subset \bigcup\limits_{j=1}^{j(i)} U_{ij}$. It also results that every point from D_k belongs to at most $C(n)$ sets U_{ij}, $i \in \mathbb{N}$, $j = 1, ..., j(i)$. We shall denote from now on by $C(n)$ the constant from the theorem of Besicovitch.

Let $g_{ij} : V_i \to U_{ij}$ be the inverse of $f|U_{ij} : U_{ij} \to V_i$ for $i \in \mathbb{N}$, $j = 1, ..., j(i)$ and let $\mu_{g_{ij}} : \mathcal{B}(V_i) \to [0,\infty]$ be the set functions defined by $\mu_{g_{ij}}(F) = \mu_n(g_{ij}(F))$ for $F \in \mathcal{B}(V_i)$, $i \in \mathbb{N}$, $j = 1, ..., j(i)$. Since f is ACL, the first partial derivatives of f exists a.e. and in such a point x we set $f^{'}(x)$ to be the linear map given by the matrix

$$\left(\frac{\partial f_i}{\partial x_j}(x)\right)_{i,j=1,...,n}.$$

Let us show that $L(x,f) \le Q(n,q)|f^{'}(x)|$ a.e. in D, where $Q(n,q)$ is a constant depending only on n and q and $Q(n,q) = 1$ if $q > n$. (1)

Suppose first that f is differentiable in x. We can easy see that $L(x,f) = |f^{'}(x)|$ and suppose now that $q > n$. Since f is ACL^q, we see from Thereom 5.21, page 129 in [13] that f is a.e. differentiable and we can take the constant $Q(n,q) = 1$ if $q > n$.

Suppose now that $n-1 < q \le n$. We see from relation (4.3) in [21] that there exists a constant $C(n,q)$ depending only on n and q such that if $u \in W^{1,q}(D)$ is monotone, $x \in D$ and $B(x,2r) \subset D$, to have that

$$\left(\frac{osc(u, B(x,r))}{r}\right)^q \le C(n,q) \fint_{B(x,2r)} |\nabla u|^q(y)dy \qquad (2)$$

Let $x \in D \setminus B_f$ such that the first partial derivatives of f in x exist and $\lim\limits_{r\to 0} \fint_{B(x,r)} |f^{'}(y)|^q dy = |f^{'}(x)|^q$. Since $x \in D \setminus B_f$, we can find $r_x > 0$ such that $\overline{B}(x, r_x) \subset D$ and $f|B(x, r_x)$:

$B(x, r_x) \to f(B(x, r_x))$ is a homeomorphism, hence the components $f_1, ..., f_n$ of f are monotone functions on $B(x, r_x)$. Using (2), we have

$$|\frac{f_i(y) - f_i(x)}{|y-x|}|^q \le (\frac{osc(f_i, B(x,r))}{r})^q \le C(n,q) \fint_{B(x,2r)} |\nabla f_i|(y)^q dy =$$

$$= C(n,q) \fint_{B(x,2r)} (\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(y)^2)^{\frac{q}{2}} dy = C(n,q) \fint_{B(x,2r)} |f^{'}(y)^T(e_i)|^q dy \le$$

$$\le C(n,q) \fint_{B(x,2r)} |f^{'}(y)^T|^q dy = C(n,q) \fint_{B(x,2r)} |f^{'}(y)|^q dy$$

for $i = 1, ..., n$, $y \in S(x,r)$ and $0 < 2r < r_x$.

Then $\frac{|f(y)-f(x)|^2}{|y-x|^2} = \sum_{i=1}^{n} \frac{|f_i(y)-f_i(x)|^2}{|y-x|^2} \le nC(n,q)^{\frac{2}{q}} (\fint_{B(x,2r)} |f^{'}(y)|^q dy)^{\frac{2}{q}}$ for $y \in S(x,r)$ and $0 < 2r < r_x$ and for $0 < 2r < r_x$ fixed, we see that $L(x,f) \le \sqrt{n} C(n,q)^{\frac{1}{q}} (\fint_{B(x,2r)} |f^{'}(y)|^q dy)^{\frac{1}{q}}$. Letting now $r \to 0$, we find that $L(x,f) \le Q(n,q)|f^{'}(x)|$, where $Q(n,q) = \sqrt{n}C(n,q)^{\frac{1}{q}}$ if $n-1 < q \le n$ and (1) is proved.

Using Theorem 24.5 in [50], we see that

$$\int_{f(D_k)} \omega_k(y) dy \le C(n) \sum_{i=1}^{\infty} \int_{V_i} \omega_k(y) dy = C(n) \sum_{i=1}^{\infty} \int_{V_i} \sum_{j=1}^{j(i)} \mu^{'}_{g_{ij}}(y) L(g_{ij}(y), f)^q dy \le$$

$$\le C(n) \sum_{i=1}^{\infty} \sum_{j=1}^{j(i)} \int_{U_{ij}} L(x,f)^q dx \le C(n)^2 \int_{D_k} L(x,f)^q dx \le$$

$$\le Q(n,q)^q C(n)^2 \int_{D_k} |f^{'}(x)|^q dx = C(n)^2 n^{\frac{q}{2}} C(n,q) \int_{D_k} |f^{'}(x)|^q dx < \infty$$

for every $k \in \mathbb{N}$, hence

$$\int_{f(D_k)} \omega_k(y) dy \le C(n)^2 n^{\frac{q}{2}} C(n,q) \int_{D_k} |f^{'}(x)|^q dx \; for \; k \in \mathbb{N} \qquad (3)$$

Let $\Gamma \in A(D)$, let $\Delta = \{\gamma \in \Gamma | \gamma$ is rectifiable and $f \circ \gamma^0$ is absolutely continuous$\}$ and let $\eta \in F(f(\Gamma))$. Let $\rho : \mathbb{R}^n \to [0, \infty]$ be defined by $\rho(x) = \eta(f(x))L(x,f)$ if $x \in D$, $\rho(x) = 0$ otherwise. We see from Theorem 5.3, page 12 in [50] that $\rho \in F(\Delta)$ and using Fuglede's theorem (see Theorem 28.2, page 95 in [50]), we see that $M_q(\Gamma) = M_q(\Delta)$.

Suppose first that condition a) holds. Since every function g_{ij} satisfies condition (N), we see that every set function $\mu_{g_{ij}}$ is absolutely continuous for $i \in \mathbb{N}$, $j = 1, ..., j(i)$. Using Lemma 1, we have:

$$\int_{D_k} \rho(x)^q dx = \int_{D_k} \eta(f(x))^q L(x,f)^q dx \le C(n) \sum_{i=1}^{\infty} \sum_{j=1}^{j(i)} \int_{U_{ij}} \eta(f(x))^q L(x,f)^q dx =$$

$$= C(n)\sum_{i=1}^{\infty}\sum_{j=1}^{j(i)}\int_{V_i}\eta(f(g_{ij}(y)))^q L(g_{ij}(y),f)^q \mu'_{g_{ij}}(y)dy =$$

$$= C(n)\sum_{i=1}^{\infty}\int_{V_i}\eta(y)^q\sum_{j=1}^{j(i)} L(g_{ij}(y),f)^q \mu'_{g_{ij}}(y)dy =$$

$$= C(n)\sum_{i=1}^{\infty}\int_{V_i}\eta(y)^q\omega_k(y)dy \leq C(n)^2\int_{\mathbb{R}^n}\eta(y)^q\omega_k dy \leq C(n)^2\int_{\mathbb{R}^n}\eta(y)^q\omega(y)dy.$$

Suppose now that condition b) holds and that $q > n-1$ and $J_f \in L^1_{loc}(D)$. We see from Theorem 5.21, page 129 in [13] that f has a.e. a weak differential and let $E_f = \{x \in D | f$ has not a weak differential in $x\}$. Since f satisfies condition (N), we see that $\mu_n(f(E_f)) = 0$ and let $Z_f = \{x \in D \setminus E_f | J_f(x) = 0\}$. Using Theorem 5.6, page 110 in [13], we find that $\mu_n(f(Z_f)) = 0$. Let $h_{ij} : V_i \to \mathbb{R}^n$ be defined by $h_{ij}(y) = g_{ij}(y)$ for $y \in V_i \setminus f(E_f \cup Z_f)$, $h_{ij}(y) = 0$ for $y \in V_i \cap f(E_f \cup Z_f)$, $i \in \mathbb{N}$, $j = 1, ..., j(i)$ and let $\mu_{h_{ij}} : \mathcal{B}(V_i) \to [0,\infty]$ be given by $\mu_{h_{ij}}(F) = \mu_n(h_{ij}(F \cap (V_i \setminus f(E_f \cup Z_f))))$ for $F \in \mathcal{B}(V_i)$, $i \in \mathbb{N}$, $j = 1, ..., j(i)$. Let $i \in \mathbb{N}$ and $j \in \{1, ..., j(i)\}$ be fixed and let $B \subset V_i \setminus f(E_f \cup Z_f)$ be a Borel set such that $\mu_n(B) = 0$ and $A = h_{ij}(B)$. Since f satisfies condition (N) and $f \in W^{1,1}_{loc}(D,\mathbb{R}^n)$, we can use the change of variable formulae (3) from [15] to see that $\int_A J_f(x)dx = \mu_n(f(A)) = \mu_n(B) = 0$, and since $J_f(x) \neq 0$ for every $x \in A$, we see that $\mu_n(A) = 0$. We showed that $\mu_{h_{ij}}(B) = \mu_n(h_{ij}(B)) = \mu_n(A) = 0$ for every $B \in \mathcal{B}(V_i \setminus f(E_f \cup Z_f))$ with $\mu_n(B) = 0$, and this shows that all the set functions $\mu_{h_{ij}}$ are absolutely continuous for $i \in \mathbb{N}$ and $j = 1, ..., j(i)$. Also, $\mu'_{h_{ij}}$ exists a.e. in V_i, $\mu'_{h_{ij}}(y) \leq \mu'_{g_{ij}}(y)$ a.e. in V_i for $i \in \mathbb{N}$, $j = 1, ..., j(i)$ and $L(x,f) = 0$ a.e. in Z_f. We have

$$\int_{D_k}\rho(x)^q dx = \int_{D_k}\eta(f(x))^q L(x,f)^q dx \leq C(n)\sum_{i=1}^{\infty}\sum_{j=1}^{j(i)}\int_{U_{ij}}\eta(f(x))^q L(x,f)^q dx =$$

$$= C(n)\sum_{i=1}^{\infty}\sum_{j=1}^{j(i)}\int_{U_{ij}\setminus(E_f\cup Z_f)}\eta(f(x))^q L(x,f)^q dx = (using\ Lemma 1) =$$

$$C(n)\sum_{i=1}^{\infty}\sum_{j=1}^{j(i)}\int_{V_i}\eta(f(h_{ij}(y)))^q L(h_{ij}(y),f)^q \mu'_{h_{ij}}(y)dy \leq$$

$$\leq C(n)\sum_{i=1}^{\infty}\int_{V_i}\eta(y)^q\sum_{j=1}^{j(i)} L(g_{ij}(y),f)^q \mu'_{g_{ij}}(y)dy =$$

$$= C(n)\sum_{i=1}^{\infty}\int_{V_i}\eta(y)^q\omega_k(y)dy \leq C(n)^2\int_{\mathbb{R}^n}\eta(y)^q\omega_k(y)dy \leq C(n)^2\int_{\mathbb{R}^n}\eta(y)^q\omega(y)dy.$$

If condition b) holds and f is a.e. differentiable, we replace the sets E_f and Z_f by $E_f = \{x \in D | f \text{ is not differentiable in } x\}$ and $Z_f = \{x \in D \setminus E_f | J_f(x) = 0\}$. Then $\mu_n(f(E_f)) = 0$ and using Sard's lemma from [3], we see that $\mu_n(f(Z_f)) = 0$, and we use now that same argument as in the second step of the proof. In all this cases we proved that

$$\int\limits_{D_k} \rho(x)^q dx \le C(n)^2 \int\limits_{\mathbb{R}^n} \eta(y)^q \omega(y) dy \; for \; every \; k \in \mathbb{N} \qquad (4)$$

Letting $k \to \infty$ in (4), we see that

$$M_q(\Gamma) = M_q(\Delta) \le \int\limits_{\mathbb{R}^n} \rho(x)^q dx \le C(n)^2 \int\limits_{\mathbb{R}^n} \eta(y)^q \omega(y) dy \qquad (5)$$

Letting $k \to \infty$ in (3), we see that $\int\limits_{\mathbb{R}^n} \omega(y)dy \le C(n)^2 n^{\frac{q}{2}} C(n,q) \int\limits_D |f^{`}(x)|^q dx < \infty$, and since $\eta \in F(f(\Gamma))$ was arbitrarily chosen, we see from (5) that

$$M_q(\Gamma) \le C(n)^2 M_\omega^q(f(\Gamma))$$

for every $\Gamma \in A(D)$.

Corollary 1. Let $n \ge 2$, $D \subset \mathbb{R}^n$ a domain, $f : D \to \mathbb{R}^n$ be ACL^n such that $\int\limits_D |f^{`}(x)|^n dx < \infty$ and $\mu_n(B_f) = 0$ and suppose that one of the following conditions hold:

a) $J_f(x) \ne 0$ a.e. in D.

b) $\mu_n(f(B_f)) = 0$ and there exists $p > 0$ and $K : D \to [0, \infty]$ measurable and finite a.e. such that $L(x,f)^p \le K(x)|J_f(x)|$ a.e.

Then there exists $\omega \in L^1(\mathbb{R}^n)$ such that $M_n(\Gamma) \le C(n)^2 M_\omega^n(f(\Gamma))$ for every $\Gamma \in A(D)$.

Proof. Suppose first that condition a) holds. Let $U \subset D \setminus B_f$ be a domain such that $f|U : U \to f(U)$ is a homeomorphism. Then $i(f,x)$ is constant on U and if f is differentiable in x and $J_f(x) \ne 0$, then $i(f,x) = sgn J_f(x)$ and this implies that either $J_f(x) > 0$ a.e. in U, or $J_f(x) < 0$ a.e. in U. Let $g = f^{-1} : f(U) \to U$. We see now from Theorem 5.32, page 142 in [13] that g satisfies condition (N) and we apply Lemma 2.

Suppose that condition b) holds. We see from [33], page 190 that f satisfies condition (N) and is a.e. differentiable on $D \setminus B_f$ and since $\mu_n(f(B_f)) = 0$, we see that f satisfies condition (N) and is a.e. differentiable on D. We apply now Lemma 2.

Lemma 3. Let $n \ge 2$, $1 < q < p$, $D \subset \mathbb{R}^n$ be open, let $f : D \to \mathbb{R}^n$ be ACL^q such that there exists $K \in L^{q/(p-q)}(D)$ such that $L(x,f)^p \le K(x)|J_f(x)|$ a.e. and let $\omega : \mathbb{R}^n \to [0,\infty]$ be defined by $\omega(y) = N(y,f,D)$ if $y \in f(D)$, $\omega(y) = 0$ otherwise. Then ω is measurable and if ω is finite a.e. and $C = (\int\limits_D K(x)^{q/(p-q)} dx)^{\frac{p-q}{p}}$ it results that $M_q(\Gamma) \le C M_\omega^p(f(\Gamma))^{q/p}$ for every $\Gamma \in A(D)$. Also, if f satisfies condition (N) and $J_f \in L^1(D)$, then $\int\limits_{\mathbb{R}^n} \omega(y)dy < \infty$.

Proof. Let $\Gamma \in A(D)$, let $\Delta = \{\gamma \in \Gamma | \gamma$ is rectifiable and $f \circ \gamma^0$ is absolutely continuous$\}$ and let $\eta \in F(f(\Gamma))$. Let $\rho : \mathbb{R}^n \to [0,\infty]$ be defined by $\rho(x) = \eta(f(x))L(x,f)$ if $x \in D$, $\rho(x) = 0$ otherwise. Then $\rho \in F(\Delta)$ and using Fuglede's theorem, the change of variable formuale (3) from [15] and Hölder's inequality, we have:

$$M_q(\Gamma) = M_q(\Delta) \le \int\limits_{\mathbb{R}^n} \rho(x)^q dx = \int\limits_D \eta(f(x))^q L(x,f)^q dx \le$$

$$\le \int\limits_D \eta(f(x))^q K(x)^{q/p} |J_f(x)|^{q/p} dx \le C(\int\limits_D \eta(f(x))^p |J_f(x)| dx)^{q/p} \le$$

$$\leq C(\int_{\mathbb{R}^n} N(y,f,D)\eta(y)^p dy)^{q/p} = C(\int_{\mathbb{R}^n} \omega(y)\eta(y)^p dy)^{q/p}.$$

Since $\eta \in F(f(\Gamma))$ was arbitrarily chosen, we find that $M_q(\Gamma) \leq CM_\omega^p(f(\Gamma))^{q/p}$ for every $\Gamma \in A(D)$. If f satisfies condition (N) and $J_f \in L^1(D)$, then $\int_{\mathbb{R}^n} \omega(y)dy = \int_D J_f(x)dx < \infty$.

Example 1. Let $\alpha > 0$, $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (x^\alpha \cos y, x^\alpha \sin y)$ if $x \geq 0$, $y \in \mathbb{R}$, $f(x,y) = f(-x,y)$ if $x < 0$ and $y \in \mathbb{R}$. Then f is continuous on $\mathbb{R}^2$ and f differentiable in every point (x,y) with $x \neq 0$. We have $f'(x,y) = x^{\alpha-1}\begin{pmatrix} \alpha\cos y & \alpha \sin y \\ -x\sin y & x\cos y \end{pmatrix}$ if $x > 0$ and $y \in \mathbb{R}$ and $J_f(x,y) = \alpha x^{2\alpha-1}$ if $x > 0$, $y \in \mathbb{R}$, $J_f(x,y) = -\alpha(-x)^{2\alpha-1}$ if $x < 0$, $y \in \mathbb{R}$, hence $J_f(x,y) = \alpha sgn x|x|^{2\alpha-1}$ if $x \neq 0$, $y \in \mathbb{R}$. Also, $|f'(x,y)|^2 \leq x^{2(\alpha-1)}(x^2+\alpha^2)$ if $x \neq 0$, $y \in \mathbb{R}$, hence $|f'(x,y)| \leq \sqrt{2}$ if $x \neq 0$, $|x| < 1$ and $0 < \alpha \leq 1$.

Let now $D \subset \mathbb{R}^{n-2}$ be a domain, $k \in \mathbb{N}$, $G = D \times (-1,1) \times (-2k\pi, 2k\pi)$. Let $F : G \to \mathbb{R}^n$ be defined by $F(x_1,...,x_n) = (\sqrt{2}x_1, ..., \sqrt{2}x_{n-2}, f(x_{n-1},x_n))$ if $x = (x_1,...,x_n) \in G$ and let $d = \{x \in G | x_{n-1} = 0\}$. Then $F \in C(G,\mathbb{R}^n)$, $F \in C^\infty(G \setminus d, \mathbb{R}^n)$ and $J_F(x) = 2^{\frac{n-2}{2}} J_f(x_{n-1},x_n) = \alpha 2^{\frac{n-2}{2}} sgn x_{n-1}|x_{n-1}|^{2\alpha-1}$ if $x \in G \setminus d$.

Let now $x \in G \setminus d$, $0 < \alpha \leq 1$ and $a \in \mathbb{R}^n$ such that $|a| = 1$. Then $|F'(x)(a)|^2 = |(\sqrt{2}a_1, ..., \sqrt{2}a_{n-2}, f'(x_{n-1},x_n)(a_{n-1},a_n))|^2 \leq 2(a_1^2 + ... + a_{n-2}^2 + |f'(x_{n-1},x_n)|^2(a_{n-1}^2 + a_n^2) \leq 2(a_1^2 + ... + a_n^2) = 2$ and we see that $|F'(x)| = \sqrt{2}$ and that $F \in ACL^q(G)$.

Suppose now that $n \geq 2$, $q > n-1$, $0 < \alpha < \frac{n}{2q} \leq 1$. We see that $B_F = d$, that $N(f, G \setminus d) = 2k$, that $K_{0,n}(F)(x) = \frac{|F'(x)|^n}{|J_F(x)|} = \frac{2}{\alpha |x_{n-1}|^{2\alpha-1}}$ if $x \in G \setminus d$, that $J_F(x) > 0$ if $x \in G \setminus d$ and $x_{n-1} > 0$ and $J_F(x) < 0$ if $x \in G \setminus d$ and $x_{n-1} < 0$. It results that F is not open and is not of finite distortion. We have:

$\int_G K_{0,n}(F)(x)^{\frac{q}{n-q}} dx = \mu_{n-2}(D)4k\pi \int_{-1}^{1} (\frac{2}{\alpha})^{\frac{q}{n-q}} |x_{n-1}|^{\frac{(1-2\alpha)q}{n-q}} dx_{n-1} = \frac{\mu_{n-2}(D)8k\pi 2^{\frac{q}{n-q}}(n-q)}{\alpha^{\frac{q}{n-q}}(n-2\alpha q)} < \infty.$

Let $C = (\int_G K_{0,n}(F)(x)^{\frac{q}{n-q}} dx)^{\frac{n-q}{n}}$ and let $\omega : \mathbb{R}^n \to [0,\infty]$ be defined by $\omega(y) = N(y,F,G)$ if $y \in \mathbb{R}^n$. We see from Lemma 3 that $M_q(\Gamma) \leq C(M_\omega^n(f(\Gamma))^{q/n}$ for every $\Gamma \in A(G)$ and we find that $M_q(\Gamma) \leq C(2k)^{q/n} M_n(f(\Gamma))^{q/n}$ for every $\Gamma \in A(G)$.

We see that a mapping $F : G \to \mathbb{R}^n$ having alternate Jacobian sign which is not open or with finite distortion, and has no monotone components satisfy a modular inequality similar to those used in this paper. We can take $G = B(0,1)$.

4 Some relations between (q,a)-lower capacity density and the lower radial density.

Lemma 4. Let $q > n-1$, $C = C_0, e_1, \frac{\pi}{2} = \{x \in \mathbb{R}^n | x_1 > 0\}$ and let $E \subset C$ be a F_σ set such that rad $\underline{\text{dens}}$ $(E,0) = \delta > 0$. Then there exists a constant $K(n,q,\delta)$ depending only on n,q and δ such that $cap_{q,1}\underline{dens}(C,E,0) \geq K(n,q,\delta) > 0$.

Proof. We set $A_r = \{t \in [0,r) | S(0,t) \cap E \neq \phi\}$ for $r > 0$. Let $0 < \epsilon < \delta$ be such that $m_1(A_r) \geq (\delta - \frac{\epsilon}{2})r$ for $0 < r < \delta$. Let $0 < r < \delta$ be fixed and let $K_r \subset E$ be compact such that if $B_r = \{t \in A_r | S(0,t) \cap K_r \neq \phi\}$ to have that $m_1(B_r) \geq (\delta - \epsilon)r$. Let $F_r = \{z \in \mathbb{R}^n |$ there exists $t \in B_r$ such that $z = te_1\}$ and let $P_r = \{z \in \mathbb{R}^n |$ there exists $y \in K_r$ such that $z = -y\}$. The sets F_r and K_r are compact sets and we see from Theorem 7.5 in [45] that $cap_q(B(0,2r), K_r \cup P_r) \geq cap_q(B(0,2r), F_r)$. Using Ziemer's result from [58] and Lemma 5.22

in [57], we have:

$$M_q(\Delta(E\cap\bar{B}(0,r)),S(0,2r)\cap C,B(0,2r)\cap C)\geq M_q(K_r\cap\bar{B}(0,r),S(0,2r)\cap C,B(0,2r)\cap C)\geq$$

$$\geq\frac{1}{2}M_q(\Delta((K_r\cup P_r)\cap\bar{B}(0,r),S(0,2r),B(0,2r))=\frac{1}{2}cap_q(B(0,2r),K_r\cup P_r)\geq$$

$$\geq\frac{1}{2}cap_q(B(0,2r),F_r)=\frac{1}{2}M_q(\Delta(F_r,S(0,2r),B(0,2r))\geq\frac{1}{2}M_q)(\Delta(F_r,S(0,2r)\cap(B(2e_1,r,2r)\setminus$$

$$\setminus\bar{B}(2e_1r,r)),B(0,2r))\cap(B(2e_1r,2r)\setminus\bar{B}(2e_1r,r)).$$

Let Q_t be the spherical *cap* $S(2e_1r,t)\cap B(0,2r)$ for $r<t<2r$ and let $C_r=\{s\in(0,2r)|$ there exists $t\in B_r$ such that $s=2r-t\}$. Let $\rho\in F(\Delta(F_r,S(0,2r)\cap(B(2e_1r,2r)\setminus\bar{B}(2e_1r,r)),B(0,2r)\cap(B(2e_1r,2r)\setminus\bar{B}(2e_1r,r)))$ be such that $\rho=0$ on $\complement((B(2e_1r,2r)\setminus B(2e_1r,r)))$. Then $\rho|Q_t\in F(\Delta(F_r\cap Q_t,S(0,2r)\cap Q_t,Q_t)$ for $t\in C_r$.

Suppose that $q\neq n$. Using Theorem 3 in [2], we find a constant $Q(n,q)$ depending only on n and q such that $\int\limits_{S(2e_1r,t)}\rho(z)^q d_{S(2e_1r,t)}\geq\frac{Q(n,q)}{t^{q-n+1}}$ for every $t\in C_r$. We have

$$\int\limits_{B(2e_1r,2r)\setminus\bar{B}(2e_1r,r)}\rho(z)^q dz=\int\limits_{r}^{2r}(\int\limits_{S(2e_1r,t)}\rho(z)^q d_{S(2e_1r,t)})dt\geq$$

$$\geq\int\limits_{C_r}(\int\limits_{S(2e_1r,t)}\rho(z)^q d_{S(2e_1r,t)})dt\geq Q(n,q)\int\limits_{C_r}\frac{dt}{t^{q-n+1}}\geq\frac{m_1(C_r)Q(n,q)}{(2r)^{q-n+1}}=$$

$$=\frac{m_1(B_r)Q(n,q)}{(2r)^{q-n+1}}\geq\frac{(\delta-\epsilon)rQ(n,q)}{2^{q-n+1}r^{q-n+1}}=\frac{(\delta-\epsilon)Q(n,q)}{2^{q-n+1}}r^{n-q}.$$

Letting $\epsilon\to 0$, we find that $M_q(\Delta(E\cap\bar{B}(0,r),S(0,2r)\cap C,B(0,2r)\cap C)\geq K(n,q,\delta)r^{n-q}$ for $0<r<\rho$, where $K(n,q,\delta)=\frac{\delta Q(n,q)}{2^{q-n+1}}$. It results that $cap_{q,1}\underline{dens}(C,E,0)\geq K(n,q,\delta)>0$.

If $q=n$, we apply the proof from Lemma 14.11 in [57].

Lemma 5. Let $n\geq 2$, $q>n-1$, $x\in\mathbb{R}^n$, $0<\psi<\rho\leq\frac{\pi}{2}$, d a half line ending in x, let $C=C_{x,d,\varphi}$, let $\gamma:[0,1]\to C_{x,d,\psi}$ be a non-constant path such that $\lim\limits_{t\to 0}\gamma(t)=x$, let $E=Im\gamma$ and $a=\frac{1}{2}\sin(\varphi-\psi)$. Then there exists a constant $Q(n,q,a)$ depending only on n,q and a such that $cap_{q,a}\underline{dens}(C,E,x)\geq Q(n,q,a)>0$.

Proof: Suppose that $E\cap S(x,r)\neq\phi$ for $0<r<2\rho$. Let $0<r<\rho$ and let $t_r>0$ be such that $\gamma(t_r)\in S(x,r)$ and $\gamma([0,t_r))\subset B(x,r)$, Let $y=\gamma(t_r)$ and let w be a point on the line determined by the points x and y such that $|w-y|=ar$ and $w\notin[x,y]$. We see that $B(w,2ar)\subset C$ and that $S(w,t)\cap E\neq\phi$, $S(w,t)\cap S(x,(1+a)r)\neq\phi$ for $ar<t<2ar$.

Suppose that $q\neq n$. Using Theorem 4 in [2], we see that $M_q(\Delta(\bar{B}(x,r)\cap E,S(x,(1+a)r)\cap C,B(x,(1+a)r)\cap C)\geq M_q(\Delta(E\cap\bar{B}(x,r)\cap(B(w,2ar)\setminus\bar{B}(w,ar)),S(x,(1+a)r)\cap(B(w,2ar)\setminus\bar{B}(w,ar)),B(x,(1+a)r)\cap B(w,2ar)\setminus\bar{B}(w,ar)))=M_q(\Delta(E\cap(B(w,2ar)\setminus\bar{B}(w,ar)),S(x,(1+a)r)\cap(B(w,2ar)\setminus\bar{B}(w,r)),B(w,2ar)\setminus\bar{B}(w,ar))\geq C(n,q)((2ar)^{n-q}-(ar)^{n-q})=C(n,q)a^{n-q}(2^{n-q}-1)r^{n-q}$ for $0<r<\rho$. We take now $Q(n,q,a)=C(n,q)a^{n-q}(2^{n-q}-1)$ and we see that $cap_{q,a}\underline{dens}(C,E,x)\geq Q(n,q,a)>0$.

If $q=n$, we apply a similar argument.

Remark 1. Using the proof of Proposition 18 in [2], we see that if $n-1<q<n$, then $M_q(\Delta(\bar{B}(x,r)\cap C,S(x,(1+a)r)\cap C,B(x,(1+a)r)\cap C)=C(a,n)/(r^{\frac{q-n}{q-1}}-((1+a)r)^{\frac{1-n}{q-1}})^{q-1}=$

$K(n,q,a)r^{n-q}$, where $C(a,n)$ is a constant depending only on n and a and $K(n,q,a) = C(a,n)/(1-(1+a)^{\frac{q-n}{q-1}})^{q-1}$. It results that $cap_{q,a}\underline{dens}(C,C,x) \geq K(n,q,a)$ if $q \neq n$.

Also, $cap_{n,a}\underline{dens}(C,C,x) \geq C(a,n)/(\ln(1+a)^{n-1}$, where $C(a,n)$ is a constant depending only on n and a.

Remark 2. Let $C = \{x \in \mathbb{R}^n | x_1 > 0\}$, $E = \bigcup_{k=0}^{\infty}(S(0,\frac{1}{2^k}) \cap C) \cup \{0\}$. We can easy see that rad $\underline{deus}(E,0) = 0$ and $cap_{q,a}\underline{deus}(C,E,0) > 0$ for every $q > n-1$ and every $0 < a \leq 1$.

5 Proofs of the results.

Theorem 1. Let $n \geq 2$, $p,q > 1$, $a,b,\lambda > 0$, $0 < \delta < 1$, $m > \frac{p\lambda+b}{q-1}$, $f : B(0,1) \to \mathbb{R}^n$ be continuous, let $r_k = 1-\delta^{km}$, $Q_k = B(0,r_{k+1})$, $\omega_k : \mathcal{B}(f(Q_k)) \to [0,\infty]$ Borel functions finite a.e. for $k \in \mathbb{N}$, let $\alpha_k = (\int\limits_{f(Q_k)} \omega_k(z)dz)^{1/p}$ and $M_k > 0$ such that $M_k \leq a/\delta^{kb}$ for every $k \in \mathbb{N}$ and suppose that $M_q(\Gamma) \leq M_k(M^p_{\omega_k}(f(\Gamma)))^\lambda$ for every $\Gamma \in A(Q_k)$ and every $k \in \mathbb{N}$ and that $\sum\limits_{k=0}^{\infty} \alpha_k\delta^k < \infty$. Then f has a.e. radial limits.

Proof. We define for $y \in S(0,1)$ and $k \in \mathbb{N}$ the path $\gamma_{y,k} : [r_k, r_{k+1}] \to \mathbb{R}^n$ given by $\gamma_{y,k}(t) = ty$ for $t \in [r_k, r_{k+1}]$ and let $k_0 \in \mathbb{N}$ be such that $r_k > \frac{1}{2}$ for $k \geq k_0$. Let $A_k = \{y \in S(0,1) | d(Im(f \circ \gamma_{y,k}) \geq \alpha_k\delta^k\}$ and $\Gamma_k = \{\gamma_{y,k} | y \in A_k\}$ for $k \in \mathbb{N}$ and let $\eta_k = \frac{1}{\alpha_k\delta^k}\mathcal{X}_{f(Q_k)}$ for $k \in \mathbb{N}$. Then $\eta_k \in F(f(\Gamma_k))$ for $k \in \mathbb{N}$ and we see that $M^p_{\omega_k}(f(\Gamma_k)) \leq \int\limits_{\mathbb{R}^n} \omega_k(z)\eta_k(z)^p dz = \delta^{-kp}$ for $k \in \mathbb{N}$. Let $\rho_k \in F(\Gamma_k)$ and let us fix $k \geq k_0$. Since $r_k \geq \frac{1}{2}$ for $k \geq k_0$, we see that $\int\limits_{r_k}^{r_{k+1}} t^{\frac{1-n}{q-1}} dt \leq 2^{\frac{n-1}{q-1}}(r_{k+1}-r_k)$ and using Hölder's inequality we find that

$$1 \leq \int\limits_{\gamma_{y,k}} \rho_k ds = \int\limits_{r_k}^{r_{k+1}} \rho_k(ty)dy \leq (\int\limits_{r_k}^{r_{k+1}} \rho_k(ty)^q t^{n-1}dt)^{\frac{1}{q}}(\int\limits_{r_k}^{r_{k+1}} t^{\frac{1-n}{q-1}}dt)^{\frac{q-1}{q}}.$$

Then

$$1 \leq 2^{n-1}(r_{k+1}-r_k)^{q-1} \int\limits_{r_k}^{r_{k+1}} \rho_k(ty)^q t^{n-1}dt \; for\ every\ y \in A_k, k \geq k_0 \qquad (1)$$

Let $D = (0,\pi)^{n-2} \times (0,2\pi) \subset \mathbb{R}^{n-1}$ and let $\theta : (0,\infty) \times D \to \mathbb{R}^n$ be the polar coordinates in $\mathbb{R}^n$ and let $g : S(0,1) \to [0,\infty]$ be a Borel map. Then $\int\limits_D g(\theta(1,x))|J_\theta(1,x)|dx = \int\limits_{S(0,1)} g(y)d_{S(0,1)}$ and let us take $g_k : S(0,1) \to [0,\infty]$ given by $g_k(y) = \int\limits_{r_k}^{r_{k+1}} \rho_k(ty)^q t^{n-1}dt$ for $y \in S(0,1)$ and $k \geq k_0$. Then

$$\int\limits_{A_k} g_k(y)d_{S(0,1)} \leq \int\limits_{S(0,1)} g_k(y)d_{S(0,1)} = \int\limits_D g_k(\theta(1,x))|J_\theta(1,x)|dx =$$

$$= \int\limits_D (\int\limits_{r_k}^{r_{k+1}} \rho_k(t\theta(1,x))^q t^{n-1}dt)|J_\theta(1,x)|dx = \int\limits_D \int\limits_{r_k}^{r_{k+1}} \rho_k(\theta(t,x))^q |J_\theta(t,x)|dtdx \leq$$

$$\leq \int_{\mathbb{R}^n} \rho_k(z)^q dz$$

and integrating (1) over $y \in A_k$, we find that

$$\mu_{n-1}(A_k) \leq 2^{n-1}(r_{k+1} - r_k)^{q-1} \int_{A_k} g_k(z) d_{S(0,1)} \leq 2^{n-1}\delta^{km(q-1)} \int_{\mathbb{R}^n} \rho_k(z)^q dz.$$

Since $\rho_k \in F(\Gamma_k)$ was arbitrarily chosen, we find that

$$\mu_{n-1}(A_k) \leq 2^{n-1}\delta^{km(q-1)} M_q(\Gamma_k) \leq 2^{n-1} M_k \delta^{km(q-1)} (M^p_{\omega_k}(f(\Gamma_k)))^\lambda \leq 2^{n-1} a \delta^{k(m(q-1)-b-p\lambda)}$$

for $k \geq k_0$. Let $t = \delta^{m(q-1)-b-p\lambda}$. We proved that

$$\mu_{n-1}(A_k) \leq 2^{n-1} a t^k \; for \; k \geq k_0 \qquad (2)$$

Let $l \geq k_0$. Then $\mu_{n-1}(\bigcup_{k=l}^{\infty} A_k) \leq \sum_{k=l}^{\infty} \mu_{n-1}(A_k) \leq 2^{n-1} a \sum_{k=l}^{\infty} t^k = \frac{2^{n-1}at^l}{1-t} \to 0$ if $l \to \infty$. Let $A = \{y \in S(0,1) |$ there exists $\lim_{t \to 1} f(ty) \in \mathbb{R}^n\}$. Since $\sum_{k=l}^{\infty} \alpha_k \delta^k < \infty$, we see that if $l \geq k_0$ and $y \in \complement(\bigcup_{k=l}^{\infty} A_k)$, it results that $y \in A$, and this implies that $\complement A \subset \bigcup_{k=l}^{\infty} A_k$ for every $l \geq k_0$. Then $\mu_{n-1}(\complement A) \leq \mu_{n-1}(\bigcup_{k=l}^{\infty} A_k) \leq \frac{2^{n-1}at^l}{1-t} \to 0$ if $l \to \infty$, hence $\mu_{n-1}(\complement A) = 0$.

The proof of Theorem 2 results immediately from Theorem 1.

Theorem 3. Let $n \geq 2$, $q > n-1$, $f : B(0,1) \to \mathbb{R}^n$ be ACL^q such that $\mu_n(B_f) = 0$, let $0 < \delta < 1$, $m > \frac{q}{q-1}$, $r_k = 1 - \delta^{km}$, $Q_k = B(0, r_{k+1})$, $\beta_k = (\int_{Q_k} |f'(z)|^q dz)^{\frac{1}{q}}$ for $k \in \mathbb{N}$ be such that $\sum_{k=0}^{\infty} \beta_k \delta^k < \infty$ and suppose that one of the following conditions hold:

a) f has locally inverses on $f(D \setminus B_f)$ which satisfies condition (N).

b) f satisfies condition (N), there exists $p > 0$ and $K : D \to [0, \infty]$ measurable and finite a.e. such that $L(x,f)^p \leq K(x)|J_f(x)|$ a.e. and either $J_f \in L^1_{loc}(D)$ and $q > n-1$, or f is a.e. differentiable.

Then f has a.e. radial limits.

Proof. We see from Lemma 2 that there exists Borel functions $\omega_k : \mathbb{R}^n \to [0, \infty]$ finite a.e. such that $M_q(\Gamma) \leq M^q_{\omega_k}(f(\Gamma))$ for every $\Gamma \in A(Q_k)$ and every $k \in \mathbb{N}$ and we also see from the proof of Lemma 2 that $\int_{f(Q_k)} \omega_k(z) dz \leq n^{\frac{q}{2}} C(n)^2 C(n,q) \int_{Q_k} |f'(z)|^q dz$ for every $k \in \mathbb{N}$. Let $\alpha_k = (\int_{f(Q_k)} \omega_k(z) dz)^{\frac{1}{q}}$ for $k \in \mathbb{N}$. We see that $\sum_{k=0}^{\infty} \alpha_k \delta^k < \infty$ and we apply now Theorem 1, with $p = q$, $\lambda = 1$, $b = 0$, $a = 1$.

Proof of Theorem 4. Let $\lambda > 1$ be such that $\alpha = \frac{q-1}{\lambda}$ and let $\frac{q}{q-1} < m < \frac{\lambda q}{q-1}$. Let $r_k = 1 - \delta^{km}$, $Q_k = B(0, r_{k+1})$ and $\alpha_k = (\int_{Q_k} |f'(z)|^q dz)^{\frac{1}{q}}$ for $k \in \mathbb{N}$. Then $\alpha < \frac{q}{m}$ and let $M > 0$ be such that $\int_{B(0,1)} |f'(z)|^q (1 - |z|)^\alpha dz < M < \infty$. Then $\delta^{(k+1)m\alpha} \int_{Q_k} |f'(z)|^q dz \leq \int_{Q_k} |f'(z)|^q (1-|z|)^\alpha dz \leq M$ for every $k \in \mathbb{N}$, hence $(\int_{Q_k} |f'(z)|^q dz)^{\frac{1}{q}} \leq (\frac{M}{\delta^{m\alpha}})^{\frac{1}{q}} \frac{1}{\delta^{\frac{km\alpha}{q}}}$ for every

$k \in \mathbb{N}$. We see that $\sum\limits_{k=0}^{\infty} \alpha_k \delta^k \leq (\frac{M}{\delta^{m\alpha}})^{\frac{1}{q}} \sum\limits_{k=0}^{\infty} \delta^{kc} < \infty$, where $c = 1 - \frac{m\alpha}{q} > 0$. We apply now Theorem 3.

Proof of Theorem 5. Since $\frac{\alpha(p-q)}{p} + \frac{\beta q}{p} < q-1$, we see that $\frac{q}{(q-1)(1-\frac{\alpha(p-q)}{p(q-1)})} < \frac{p}{\beta}$ and we choose $m \in (\frac{q}{q-1}\frac{1}{(1-\frac{\alpha(p-q)}{p(q-1)})}, \frac{p}{\beta})$. Let $0 < \delta < 1$, $r_k = 1 - \delta^{km}$, $Q_k = B(0, r_{k+1})$, $M_k = (\int\limits_{Q_k} K(x)^{q/(p-q)} dx)^{\frac{p-q}{p}}$ for $k \in \mathbb{N}$ and let $\omega_k : \mathbb{R}^n \to [0,\infty]$, $\omega_k(y) = N(y, f, Q_k)$ for $y \in \mathbb{N}$, $k \in \mathbb{N}$. Let $\alpha_k = (\int\limits_{f(Q_k)} \omega_k(y) dy)^{\frac{1}{p}}$ for $k \in \mathbb{N}$. We see from Lemma 3 that $M_q(\Gamma) \leq M_k(M^p_{\omega_k}(f(\Gamma)))^{\frac{q}{p}}$ for every $\Gamma \in A(Q_k)$ and every $k \in \mathbb{N}$ and using the hypothesis, we see that $M_k \leq (\frac{c_2}{\delta^{m\alpha}})^{\frac{p-q}{p}} \delta^{\frac{-km(p-q)\alpha}{p}}$ for every $k \in \mathbb{N}$. We also see that $\sum\limits_{k=0}^{\infty} \alpha_k \delta^k \leq \frac{(c_1)^{\frac{1}{p}}}{\delta^{\frac{m\beta}{p}}} \sum\limits_{k=0}^{\infty} \delta^{k(1-\frac{m\beta}{p})} < \infty$, since $0 < \frac{m\beta}{p} < 1$. We take $\lambda = \frac{q}{p}$, $b = \frac{m(p-q)\alpha}{p}$, $a = c_2^{\frac{p-q}{p}} / \delta^{\frac{m(p-q)\alpha}{p}}$ in Theorem 1 and since $\frac{p\lambda + b}{q-1} = \frac{q}{q-1} + \frac{m(p-q)\alpha}{p(q-1)} < m$, we can apply Theorem 1 to see that f has a.e. radial limits.

Theorem 6. Let $n \geq 2$, $p, q > 1$, let $f : B(0,1) \to \mathbb{R}^n$ be continuous, let $c \in \mathbb{R}^n$ be such that $m_{n-2}(f^{-1}(c)) = 0$, let $\gamma_y : [0,1) \to \mathbb{R}^n$ be defined by $\gamma_y(t) = ty$ for $t \in [0,1)$ and $y \in S(0,1)$, let $E = \{y \in S(0,1) | c$ is a limit point of $f \circ \gamma_y : [0,1) \to \mathbb{R}^n\}$ and let $\omega_r : \mathcal{B}(f(B(0,r))) \to [0,\infty]$ be measurable and finite a.e. for $0 < r < 1$. Suppose that

1) there exists $c_1 > 0$ and $\beta > 0$ such that $\int\limits_{f(B(0,r))} \omega_r(z) dz \leq \frac{c_1}{(1-r)^\beta}$ for $0 < r < 1$.

2) there exists $\lambda, \gamma, c_2 > 0$, $Q_r > 0$ such that $Q_r \leq \frac{c_2}{(1-r)^\gamma}$ for $0 < r < 1$ and $M_q(\Gamma) \leq Q_r(M^p_{\omega_r}(f(\Gamma)))^\lambda$ for every $\Gamma \in A(B(0,r))$ and every $0 < r < 1$.

3) $\gamma + \lambda\beta < q - 1$.

Then $\mu_{n-1}(E) = 0$.

Proof. Suppose that $\mu_{n-1}(E) > 0$, let $m = q - 1 - \gamma - \beta\lambda > 0$ and $\frac{1}{2} \leq a < 1$. Let $A = \{z \in \overline{B}(0,a) \setminus B(0,\frac{a}{2}) |$ there exists $y \in S(0,1)$ such that $Im(\gamma_y|[\frac{a}{2}, a]) \cap f^{-1}(c) \neq \phi\}$. Then, since $m_{n-2}(f^{-1}(c)) = 0$, we see that $m_{n-1}(A) = 0$. Let $B = \{y \in S(0,1) |$ there exists $z \in A$ and $t \in [0,1]$ such that $z = ty\}$. Then also $m_{n-1}(B) = 0$ and since B is compact, we can find $\rho_0 > 0$ small enough such that if $B_0 = \{y \in S(0,1) | Im(\gamma_y|[\frac{a}{2}, a]) \cap f^{-1}(\overline{B}(c, \rho_0)) \neq \phi\}$, to have that $\mu_{n-1}(B_0) < \frac{\mu_{n-1}(E)}{4}$. Let $E_{-1} = E \setminus B_0$. Then $\frac{3}{4}\mu_{n-1}(E) < \mu_{n-1}(E_{-1})$. Let $r_0 = a$, let $\alpha = (\frac{2^{n-1} c_1^\lambda c_2}{\mu_{n-1}(E)})^{\frac{1}{p\lambda}}$ and we chose $\delta > 0$ small enough such that if $d = (3\delta^m)^{\frac{1}{p\lambda}}$, to have that $\frac{d\alpha}{1-d} < \rho_0$ and $a < 1 - \delta^2$. Let $r_k = 1 - \delta^{k+1}$ for $k \geq 1$ and let $b = \rho_0 - \frac{d\alpha}{1-d} > 0$. Let $\gamma_{y,k} = \gamma_y | [r_k, r_{k+1}]$ for $k \geq 0$ and $y \in S(0,1)$. We chose $\rho_0 > \rho_1 >, ..., > \rho_k > \rho_{k+1} >, ...,$ such that $\rho_k - \rho_{k+1} = \alpha d^{k+1}$ for $k \geq 0$. Then $\rho_0 - \rho_k = d\alpha(1 + d+, ..., +d^{k-1})$ for $k \geq 1$, hence $\rho_k \to b$ and $\rho_k > \frac{b}{2}$ for $k \geq 0$.

Let $E_0 = \{y \in E \setminus B_0 | Im f \circ \gamma_{y,0} \cap \overline{B}(c, \rho_1) \neq \phi\}$ and let $E_k = \{y \in E_{-1} \setminus \bigcup\limits_{i=0}^{k-1} E_i | I_m f \circ \gamma_{y,k} \cap \overline{B}(c, \rho_{k+1}) \neq \phi\}$ for $k \geq 1$. There exists $i_0 \in \mathbb{N}$ such that $E_{i_0} \neq \phi$. Indeed, otherwise $Im f \circ \gamma_{y,l} \cap \overline{B}(c, \rho_{l+1}) = \phi$ for every $y \in E_{-1}$ and every $l \geq 0$, hence $f(\gamma(ty)) \notin B(c, \frac{b}{2}))$ for every $t \geq r_0$ and $y \in E_{-1}$. We reached a contradiction, since $y \in E_{-1}$ and c is a limit point of $f \circ \gamma_y : [0,1) \to \mathbb{R}^n$. If $i_0 = 0$, then $f \circ \gamma_y(r_0) \notin B(c, \rho_0)$. If $i_o > 0$ and $y \in E_{i_0}$, since $E_l = \phi$ for every $l \in \{0, 1, ..., i_0 - 1\}$, we see that $Im f \circ \gamma_{y,l} \cap \overline{B}(c, \rho_{l+1}) = \phi$ for $l \in \{0, 1, ..., i_0 - 1\}$. Taking $l = i_0 - 1$, we see that $f \circ \gamma_y(r_{i_0}) \notin \overline{B}(c, \rho_{i_0})$.

Suppose that we have $i_0 < i_1 <, ..., < i_k$ such that $E_{i_l} \neq \phi$ for $l = 1, ..., k$. Let $y \in E_{i_k}$. Then $y \notin E_{i_{k-1}}$, hence $Im f \circ \gamma_{y, i_{k-1}} \cap \overline{B}(c, \rho_{i_{k-1}+1}) = \phi$. If $i_k = i_{k-1} + 1$, then $f \circ \gamma_y(r_{i_k}) \notin \overline{B}(c, \rho_{i_k})$.

If not, since $E_l = \phi$ for $l \in \{i_{k-1}+1, ..., i_k - 1\}$, we see that $Imf \circ \gamma_{y,l} \cap \overline{B}(c, \rho_{l+1}) = \phi$ for $l \in \{i_{k-1}+1, ..., i_k - 1\}$ and taking $l = i_k - 1$, we find that $Imf \circ \gamma_y(r_{i_k}) \notin \overline{B}(c, \rho_{i_k})$.

We proved that if $E_k \neq \phi$ and $y \in E_k$, then $f \circ \gamma_y(r_k) \notin \overline{B}(c, \rho_k)$ and since also $Imf \circ \gamma_{y,k} \cap \overline{B}(c, \rho_{k+1}) \neq \phi$, we see that for such $k \in \mathbb{N}$, $f \circ \gamma_{y,k}$ meets $\overline{B}(c, \rho_{k+1})$ and $\complement\overline{B}(c, \rho_k)$. Let $\Gamma_k = \{\gamma_{y,k} | y \in E_k\}$ for $k \geq 0$.

Let $k \in \mathbb{N}$ be such that $E_k \neq \phi$. Then $\eta_k = \frac{1}{\rho_k - \rho_{k+1}} \mathcal{X}_{f(B(0,r_{k+1}))} \in F(f(\Gamma_k))$, and we see from Theorem 1 that $\mu_{n-1}(E_k) \leq 2^{n-1}(r_{k+1} - r_k)^{q-1} M_q(\Gamma_k)$. We have

$$\mu_{n-1}(E_k) \leq 2^{n-1}(r_{k+1} - r_k)^{q-1} M_q(\Gamma_k) \leq 2^{n-1}\delta^{(k+1)(q-1)} Q_{r_{k+1}}(M^p_{\omega_{r_{k+1}}}(f(\Gamma_k)))^\lambda \leq$$

$$\leq 2^{n-1}c_2\delta^{(k+1)(q-1-\gamma)}(\int\limits_{\mathbb{R}^n} \omega_{r_{k+1}}(z)\eta_k(z)^p dz)^\lambda = \frac{2^{n-1}c_2\delta^{(k+1)(q-1-\gamma)}}{(\rho_k - \rho_{k+1})^{p\lambda}}(\int\limits_{f(B(0,r_{k+1}))} \omega_{r_{k+1}}(z)dz)^\lambda =$$

$$= 2^{n-1}c_1^\lambda c_2 \delta^{(k+1)m} \frac{\mu_{n-1}(E)}{2^{n-1}c_1^\lambda c_2} \frac{1}{(3\delta^m)^{(k+1)}} = \frac{\mu_{n-1}(E)}{3^{k+1}}$$

for $k \geq 0$.

Since $\mu_{n-1}(\bigcup\limits_{k=0}^{\infty} E_k) \leq \mu_{n-1}(E) \sum\limits_{k=0}^{\infty} \frac{1}{3^{k+1}} = \frac{\mu_{n-1}(E)}{2} < \mu_{n-1}(E_{-1})$, we can find a point $y \in E_{-1} \setminus \sum\limits_{k=0}^{\infty} E_k$. Then $Imf \circ \gamma_{y,k} \cap \overline{B}(c, \rho_{k+1}) = \phi$ for $k \geq 0$, hence $Imf \circ \gamma_{y,k} \cap B(c, \frac{b}{2}) = \phi$ for every $k \geq 0$ and this implies that $f(\gamma(ty)) \notin B(c, \frac{b}{2})$ for $t \geq r_0$. We reached a contradiction, since $y \in E$ and c is a limit point of $f \circ \gamma_y : [0,1) \to \mathbb{R}^n$. We therefore proved that $\mu_{n-1}(E) = 0$.

The proof of Theorem 7 results immediately from Theorem 6.

Proof of Theorem 8. Let $M = \int\limits_{B(0,1)} |f'(z)|^q (1-|z|)^\alpha dz$. We see from Lemma 2 that there exists Borel functions $\omega_r : \mathbb{R}^n \to [0, \infty]$ such that $M_q(\Gamma) \leq M^q_{\omega_r}(f(\Gamma))$ for every $\Gamma \in A(B(0,r))$ and every $0 < r < 1$ and we also see that $\int\limits_{f(B(0,r))} \omega_r(z)dz \leq n^{\frac{q}{2}} C(n)^2 C(n,q) \int\limits_{B(0,r)} |f'(z)|^q dz$ for every $0 < r < 1$. We see that $(1-r)^\alpha \int\limits_{B(0,r)} |f'(z)|^q dz \leq \int\limits_{B(0,r)} |f'(z)|^q (1-|z|)^\alpha dz \leq M$ for $0 < r < 1$ and we find that $\int\limits_{f(B(0,r))} \omega_r(z)dz \leq \frac{Mn^{\frac{q}{2}} C(n)^2 C(n,q)}{(1-r)^\alpha}$ for every $0 < r < 1$. We apply now Theorem 6 with $p = q$, $\gamma = 0$, $\lambda = 1$.

Proof of Theorem 9. Let $\omega_r : \mathcal{B}(f(B(0,r))) \to [0, \infty]$ be given by $\omega_r(y) = N(y, f, B(0,r))$ for $y \in \mathbb{R}^n$ and $0 < r < 1$ and let $Q_r = (\int\limits_{B(0,r)} K(x)^{q/(p-q)} dx)^{\frac{p-q}{p}}$ for $0 < r < 1$. We see from Lemma 3 that $M_q(\Gamma) \leq Q_r(M^p_{\omega_r}(f(\Gamma))^{q/p}$ for every $\Gamma \in A(B(0,r))$ and every $0 < r < 1$ and using the hypothesis, we see that $Q_r \leq \frac{(c_2)^{\frac{p-q}{p}}}{(1-r)^{\frac{\alpha(p-q)}{p}}}$ and $\int\limits_{f(B(0,r))} \omega_r(z)dz \leq \frac{c_1}{(1-r)^\beta}$ for $0 < r < 1$. We take $\lambda = \frac{q}{p}$, $\gamma = \frac{\alpha(p-q)}{p}$ and we see that $\gamma + \beta\lambda = \frac{\alpha(p-q)}{p} + \beta\frac{q}{p} < q - 1$ and we apply now Theorem 6 and we see that $\mu_{n-1}(E) = 0$.

Remark 2. The condition "$m_{n-2}(f^{-1}(c)) = 0$" from the preceding theorems holds if $f^{-1}(c)$ is discrete.

Remark 3. Suppose that we take in Theorem 5 or 9 $p = 2q$. Then, condition 2) is "there exists $c_2 > 0$ and $\alpha > 0$ such that $\int\limits_{B(0,r)} K(x)dx \leq \frac{c_2}{(1-r)^\alpha}$ for every $0 < r < 1$" and condition 3) is "$\alpha + \beta < 2(q-1)$". Also, if we take $p = n$, $q = n-1$, then condition 2) is "there exists

$c_2 > 0$ and $\alpha > 0$ such that $\int\limits_{B(0,r)} K(x)^{n-1}dx \leq \frac{c_2}{(1-r)^\alpha}$ for every $0 < r < 1$" and condition 3) is "$\alpha + (n-1)\beta < n(n-2)$".

Proof of Theorem 10. We can replace $B(0,1)$ with $B(\frac{e_n}{2}, \frac{1}{2})$. Let $E = \{y \in S(\frac{e_n}{2}, \frac{1}{2}) | f$ has no asymptotic value at $y\}$ and suppose that there exists $F \subset E$ compact such that $cap_q(F) > 0$. Let $K = \overline{B}(\frac{e_n}{2}, \frac{1}{4})$ and let $\Gamma_0 = \Delta(K, F, B(\frac{e_n}{2}, 2))$. We show that $M_q(\Gamma_0) > 0$. Let $\Gamma_1 = \Delta(F, S(\frac{e_n}{2}, 2), B(\frac{e_n}{2}, 2))$, $\Gamma_2 = \Delta(K, S(\frac{e_n}{2}, 2), B(\frac{e_n}{2}, 2))$ and let $\delta_i = M_q(\Gamma_i)$ for $i = 1, 2$. Since $cap_q(F) > 0$, we see that $\delta_1 = cap_q(B(\frac{e_n}{2}, 2), F) > 0$ and we see from Proposition 18 in [2] that also $\delta_2 > 0$. Let $\rho \in F(\Gamma_0)$. If $3\rho \in F(\Gamma_1)$ or $3\rho \in F(\Gamma_2)$, then $\int\limits_{\mathbb{R}^n} \rho(x)^q dx \geq \frac{1}{3^q} \min\{\delta_1, \delta_2\}$. In the remaining case, there exists $\gamma_i \in \Gamma_i$ such that $\int\limits_{\gamma_i} \rho ds < \frac{1}{3}$ for $i = 1, 2$. Let $\Gamma_3 = \Delta(Im\gamma_1, Im\gamma_2, B(\frac{e_n}{2}, 2) \setminus \overline{B}(\frac{e_n}{2}, \frac{1}{2}))$. Then $3\rho \in F(\Gamma_3)$ and using Caraman's result from Theorem 4 in [1], we see that $\delta_3 = M_q(\Gamma_3) > 0$. We proved that $M_q(\Gamma_0) \geq \frac{1}{3^q} min\{\delta_1, \delta_2, \delta_3\} > 0$.

Let $\Gamma = \Delta(K, F, B(\frac{e_n}{2}, \frac{1}{2}))$. We want to show that $M_q(\Gamma) > 0$, We see from Theorem 19 in [2] that the q modulus of all paths passing through some points in $\mathbb{R}^n$ is zero, hence we can take $r > 0$ small enough such that if $\Gamma_4 = \Delta(K, F \setminus B(e_n, r), B(e_n, 2) \setminus \overline{B}(e_n, r))$, to have that $M_q(\Gamma_4) > 0$. Let $\Gamma_5 = \Delta(K, F \setminus B(e_n, r), B(\frac{e_n}{2}, \frac{1}{2}) \setminus B(e_n, r))$. Let us show first that $M_q(\Gamma_5) > 0$. Let $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ be the inversion given by $f(x) = e_n + \frac{x - e_n}{|x - e_n|^2}$ if $x \neq e_n$, $f(e_n) = \infty$, $f(\infty) = e_n$. Then f is a homeomorphism, $f = f^{-1}$, $f(S(\frac{e_n}{2}, \frac{1}{2})) = \mathbb{R}^{n-1}$, and if $H = \{x \in \mathbb{R}^n | x_n < 0\}$, we see that $f(B(\frac{e_n}{2}, \frac{1}{2})) \subset H$. Let $D = \{x \in \mathbb{R}^n | r < |x - e_n| < 2\}$ and let $G = \{y \in \mathbb{R}^n | \frac{1}{2} < |y - e_n| < \frac{1}{r}\}$. Then $f(D) = G$ and $|f^{'}(x)| = |x - e_n|^{-2}$ if $x \neq e_n$.

Let $\eta \in F(f(\Gamma_4))$ and let $\rho : \mathbb{R}^n \to [0, \infty]$, $\rho(x) = \eta(f(x))L(x, f)$ if $x \in D$, $\rho(x) = 0$ if $x \notin D$. Then $\rho \in F(\Gamma_4)$ and

$$M_q(\Gamma_4) \leq \int\limits_D \rho(x)^q dx = \int\limits_D \eta(f(x))^q L(x, f)^q dx = \int\limits_D \eta(f(x))^q |f^{'}(x)|^n \frac{1}{|f^{'}(x)|^{n-q}} dx \leq$$

$$\leq 2^{2(n-q)} \int\limits_D \eta(f(x))^q |J_f(x)| dx = 2^{2(n-q)} \int\limits_G \eta(y)^q dy \leq 2^{2(n-q)} \int\limits_{\mathbb{R}^n} \eta(y)^q dy.$$

Since $\eta \in F(f(\Gamma_4))$ was arbitrarily chosen, we proved that $M_q(\Gamma_4) \leq 2^{2(n-q)} M_q(f(\Gamma_4))$.

In the same way we see that $M_q(f(\Gamma_5)) \leq r^{-2(n-q)} M_q(\Gamma_5)$. Let $\Gamma^{'} = \Delta(f(K), f(F \setminus B(e_n, r)), \mathbb{R}^n)$. Then $f(\Gamma_4) \subset \Gamma^{'}$ and we can take $r > 0$ small enough such that $M_q(f(\Gamma_5)) \geq \frac{1}{2} M_q(f(K), f(F \setminus B(e_n, r)), H)$. Using Lemma 5.22 in [57], we have:

$$M_q(\Gamma_5) \geq r^{2(n-q)} M_q(f(\Gamma_5)) = \frac{r^{2(n-q)}}{2} M_q(f(K), f(F \setminus B(e_n, r)), H) \geq$$

$$\geq \frac{r^{2(n-q)}}{4} M_q(\Gamma^{'}) \geq \frac{r^{2(n-q)}}{4} M_q(f(\Gamma_4)) \geq \frac{1}{4} (\frac{r}{2})^{2(n-q)} M_q(\Gamma_4) > 0.$$

We proved that $M_q(\Gamma_5) > 0$ and since $\Gamma_5 \subset \Gamma$, it results that $M_q(\Gamma) > 0$. If $q = n$, we can prove that $M_n(\Gamma) > 0$ by the method from Lemma 14.7 in [57].

Let $\rho_0 : \mathbb{R}^n \to [0, \infty]$, $\rho_0(x) = 0$ for every $x \in \mathbb{R}^n$. Then $\rho_0 \in F(\phi)$ and since $\omega < \infty$ a.e., we see that $0 \leq M_\omega^q(\phi) \leq \int\limits_{\mathbb{R}^n} \omega(x) \rho_0(x)^q dx = 0$, hence $M_\omega^q(\phi) = 0$.

Let $\Gamma_r = \{\gamma \in \Gamma | \gamma$ is rectifiable$\}$ and let $\Gamma_r^{'} = \{\beta \in f(\Gamma_r) | \beta$ is rectifiable$\}$. We see from Lemma 3 in [7] that $M_\omega^p(\Gamma_r^{'}) = M_\omega^p(f(\Gamma_r))$ and we see that $0 < M_q(\Gamma) = M_q(\Gamma_r) \leq \gamma(M_\omega^p(f(\Gamma_r))) = \gamma(M_\omega^p(\Gamma_r^{'}))$.

It results that $\Gamma_r^{'} \neq \phi$, hence there exists $\gamma \in \Gamma_r$ such that $f \circ \gamma$ is rectifiable and this contradicts the choice of F. We therefore proved that $cap_q(F) = 0$.

Proof of Theorem 11. We see from Lemma 2 that there exists $\omega \in L^1(\mathbb{R}^n)$ such that $M_q(\Gamma) \leq M_\omega^q(f(\Gamma))$ for every $\Gamma \in A(B(0,1))$ and we apply Theorem 10.

Proof of Theorem 12. Let $C = (\int\limits_{B(0,1)} K(x)^{q/(p-q)}dx)^{\frac{p-q}{p}}$. We see from Lemma 3 that $M_q(\Gamma) \leq CM_p(f(\Gamma))^{q/p}$ for every $\Gamma \in A(B(0,1))$ and we apply Theorem 10.

Proof of Theorem 13. Suppose that G is finitely connected at two distinct points $b_1, b_2 \in C(f,b)$. Let $V_1 \in \mathcal{V}(b_1)$, $V_2 \in \mathcal{V}(b_2)$ be such that $\overline{V_1}, \overline{V_2}$ are compact, disjoint sets and let $x_j \to b$, $y_j \to b$ be such that $f(x_j) \to b_1$, $f(y_j) \to b_2$. Taking if necessarily a subsequence, we can find a component F_1 of $V_1 \cap G$ such that $f(x_j) \in F_1$ for $j \in \mathbb{N}$. Using Lemma 3.2 and Lemma 3.6 from [51], we see that $f^{-1}(F_1)$ has a finite number of components which are mapped by f onto F_1, and taking if necessarily a subsequence, we can find a component E_1 of $f^{-1}(F_1)$ such that $f(E_1) = F_1$ and $x_j \in E_1$ for every $j \in \mathbb{N}$. In the same way we can find a component F_2 of $V_2 \cap G$ and a component E_2 of $f^{-1}(F_2)$ such that $f(E_2) = F_2$ and $y_j \in E_2$ for every $j \in \mathbb{N}$.

Using Lemma 2, we can find $\omega \in L^1(\mathbb{R}^n)$ such that $M_n(\Gamma) \leq C(n)^2 M_\omega^n(f(\Gamma))$ for every $\Gamma \in A(D)$. Let $\Gamma_0 = \Delta(E_1, E_2, D)$. Since $b \in \overline{E}_1 \cap \overline{E}_2$ and D is quasiconformally flat at b, we see that $M_n(\Gamma_0) = \infty$. Let $h = d(\overline{V}_1, \overline{V}_2) > 0$ and let $\rho = \frac{1}{h}$. Then $\rho \in F(f(\Gamma_0))$ and we find that $\infty = M_n(\Gamma_0) \leq C(n)^2 M_\omega^n(f(\Gamma_0)) \leq C(n)^2 \int\limits_{\mathbb{R}^n} \rho(x)^n \omega(x)dx = \frac{C(n)^2}{h^n} \int\limits_{\mathbb{R}^n} \omega(x)dx < \infty$ and we reached a contradiction. It results that $C(f,b)$ contains at most one point at which G is finitely connected.

Proof of Corollary 2. We see from Corollary 1 that there exists $\omega \in L^1(\mathbb{R}^n)$ such that $M_n(\Gamma) \leq C(n)^2 M_\omega^n(f(\Gamma))$ for every $\Gamma \in A(D)$ and we apply the same arguments as in Theorem 13.

Proposition 1. Let $n-1 < q \leq n$, $p > 1$, $\lambda > 0$, $\rho > 0$, $0 < \psi < \rho \leq \frac{\pi}{2}$, $a = \frac{1}{2}\sin(\varphi - \psi)$, let $D \subset \mathbb{R}^n$ be a domain, d a half line ending in x such that $C_{x,d,\varphi} \cap B(x,\rho) \subset D$, let $K_r > 0$ and $w_r : D \cap B(x,r) \to [0,\infty]$ be measurable for $0 < r < \rho$, let $f : D \to \mathbb{R}^n$ be continuous having monotone components such that $M_q(\Gamma) \leq K_r M_{w_r}^p(f(\Gamma))^\lambda$ for every $\Gamma \in A(D \cap B(x,r))$ and every $0 < r < \rho$ and suppose that $\lim\limits_{r \to 0} K_r(\int\limits_{f(D \cap B(x,r))} w_r(z)dz)^\lambda / r^{n-q} = 0$. Let $\epsilon > 0$ and $b_k \in C_{x,d,\psi}, b_k \to 0$. Then there exists $k_\epsilon \in \mathbb{N}$ such that $|f(y) - f(b_k) \leq \epsilon$ for every $y \in B(b_k, a|b_k - x|)$ and every $k \geq k_\epsilon$.

Proof: Let us fix $i \in \{1,...,n\}$ and let $r_k = |b_k - x|$ for $k \in \mathbb{N}$. We can suppose that $B(b_k, 2ar_k) \subset C_{x,d,\varphi} \cap B(b,\rho) \subset D$ for every $k \in \mathbb{N}$ and let us fix $k \in \mathbb{N}$. Let $y \in B(b_k, ar_k)$ and let $\alpha_k = \omega_{(1+2a)r_k}$ and $C_k = K_{(1+2a)r_k}$. Suppose first that $f_i(y) > f_i(b_k)$ and let $A = \{z \in B(b_k, 2ar_k) | f_i(z) \leq f_i(b_k)\}$ and $B = \{z \in B(b_k, 2ar_k) | f_i(z) > f_i(y)\}$. Since f_i is monotone, we see that $S(b_k,t) \cap A \neq \phi$, $S(b_k,t) \cap B \neq \phi$ for $|y - b_k| < t < 2ar_k$. Let $\Gamma_k \equiv \Delta(A, B, B(b_k, 2ar_k) \setminus \bar{B}(b_k, |y - b_k|))$ and let $\rho_k = \frac{1}{f_i(y) - f_i(b_k)} \chi_{f(B(b_k, 2ar_k))}$. Then $\rho_k \in F(f(\Gamma_k))$ and $B(b_k, 2ar_k) \subset B(x, (1+2a)r_k)$.

Suppose first that $q \neq n$. Using Theorem 4 in [2], we have:

$$C(n,q)((2ar_k)^{n-q} - (ar_k)^{n-q}) \leq C(n,q)((2ar_k)^{n-q} - |y - b_k|^{n-q}) \leq M_q(\Gamma_k) \leq$$

$$\leq C_k M_{\alpha_k}^p(f(\Gamma_k))^\lambda \leq C_k(\int\limits_{\mathbb{R}^n} \alpha_k(z)\rho_k(z)^p dz)^\lambda =$$

$$\frac{C_k}{|f_i(y)-f_i(b_k)|^{p\lambda}}(\int\limits_{f(B(b_k,2ar_k))}\alpha_k(z)dz)^\lambda \leq$$

$$\leq \frac{C_k}{|f_i(y)-f_i(b_k)|^{p\lambda}}(\int\limits_{f(B(x,(1+2a)r_k))}\alpha_k(z)dz)^\lambda .$$

It results that $|f_i(y)-f_i(b_k)|^{p\lambda} \leq \frac{(1+2a)^{n-q}C_k(\int\limits_{f(B(x,(1+2a)r_k)}\alpha_k(z)dz)^\lambda}{C(n,q)a^{n-q}(2^{n-q}-1)((1+2a)r_k)^{n-q}}$.

This inequality is valid for $i=1,...,n$ and also if $f_i(b_k)>f_i(y)$, hence we can find $k_\epsilon \in \mathbb{N}$ such that $|f(y)-f(b_k)| \leq \epsilon$ for every $y \in B(b_k, ar_k)$ and every $k \geq k_\epsilon$.

Suppose now that $q=n$. Then

$$C(n)\ln 2 \leq C(n)\ln(\frac{2ar_k}{|y-b_k|}) \leq M_n(\Gamma_k) \leq C_k M^p_{\alpha_k}(f(\Gamma))^\lambda \leq$$

$$\leq \frac{C_k}{|f_i(y)-f_i(b_k)|^{p\lambda}}(\int\limits_{f(B(b_k,2ar_k))}\alpha_k(z)dz)^\lambda \leq$$

$$\leq \frac{C_k}{|f_i(y)-f_i(b_k)|^{p\lambda}}(\int\limits_{f(B(x,(1+2a)r_k))}\alpha_k(z)dz)^\lambda$$

and we see that

$$|f_i(y)-f_i(b_k)|^{p\lambda} \leq \frac{C_k}{C(n)\ln 2}(\int\limits_{f(B(x,(1+2a)r_k))}\alpha_k(z)dz)$$

for $i=1,...,n$.

We see that also in this case we can find $k_\epsilon \in \mathbb{N}$ such that $|f(y)-f(b_k)| \leq \epsilon$ for every $y \in B(b_k, ar_k)$ and every $k \geq k_\epsilon$.

Proof of Theorem 14. Suppose that $q \neq n$. Suppose that there exists $\epsilon > 0$ and $b_k \in C_1$, $b_k \to 0$ such that $|f(b_k)-c| \geq 3\epsilon$ for every $k \in \mathbb{N}$ and let $r_k = |b_k - x|$ for $k \in \mathbb{N}$. Using Proposition 1, we can suppose that $|f(y)-c| > 2\epsilon$ for every $k \in \mathbb{R}$ and every $y \in B(b_k, ar_k)$ and we can suppose that $|f(y)-c| \leq \epsilon$ for every $y \in B(x,r_k) \cap E$ and every $k \in \mathbb{N}$.

Let us fix $k \in \mathbb{N}$. Let $B_k = B(b_k, \frac{a}{2}r_k)$ and $\Gamma_{1k} = \Delta(B_k, S(x,(1+a)r_k) \cap C, B(x,(1+a)r_k) \cap C)))$. Using Proposition 18 in [2], we see that there exists a constant $C_1(n,q,\varphi,\psi)$ depending only on n,q,φ and ψ such that $M_q(\Gamma_{1k}) \geq C_1 r_k^{n-q}$. Let $\Gamma_{2k} = \Delta(\bar{B}(x,r_k) \cap E, S(x,(1+a)r_k) \cap C, B(x,(1+a)r_k) \cap C)$. We can suppose that $M_q(\Gamma_{2k}) \geq \delta r_k^{n-q}$. Let $\Gamma_k = \Delta(B_k, \bar{B}(x,r_k) \cap E, B(x,(1+a)r_k) \cap C)$ and let $\rho_k \in F(\Gamma_k)$. If $3\rho_k \notin F(\Gamma_{1k})$, $3\rho_k \notin F(\Gamma_{2k})$, we can find $\alpha_k \in \Gamma_{1k}$, $\beta_k \in \Gamma_{2k}$ locally rectifiable such that $\int\limits_{\alpha_k}\rho_k ds < \frac{1}{3}$ and $\int\limits_{\beta_k}\rho_k ds < \frac{1}{3}$. Let $\Gamma_{3k} = \Delta(Im\alpha_k, Im\beta_k, (B(x,(1+a)r_k) \setminus \bar{B}(x,(1+\frac{1}{a})r_k) \cap C))$. Then $3\rho_k \in F(\Gamma_{3k})$ and using Theorem 4 in [2], we see that $C(n,q)r_k^{n-q}((1+a)^{n-q} - (1+\frac{a}{2})^{n-q} \leq M_q(\Gamma_{3k}) \leq 3^q \int\limits_{\mathbb{R}^n}\rho_k(z)^p dz$. We find that

$$\int\limits_{\mathbb{R}^n}\rho_k(z)^p dz \geq \frac{1}{3^q}\min\{M_q(\Gamma_{1k}), M_q(\Gamma_{2k}), M_q(\Gamma_{3k})\}$$

and since $\rho_k \in F(\Gamma_k)$ was arbitrarily chosen, we find that there exists $C_2(n,q,\varphi,\psi,\delta)$, a constant depending only on n,q,φ,ψ and δ such that $M_q(\Gamma_k) \geq C_2(n,q,\varphi,\psi,\delta)r_k^{n-q}$. Let

$\eta_k = \frac{1}{\epsilon}\chi_{f(B(x,(1+a)r_k)\cap D)}$. Then $\eta_k \in F(f(\Gamma_k))$ and let $\alpha_k = \omega_{(1+a)r_k}$ and $Q_k = K_{(1+a)r_k}$. It results that

$$C_2 r_k^{n-q} \leq M_q(\Gamma_k) \leq Q_k M^p_{\alpha_k}(f(\Gamma_k))^\lambda \leq Q_k(\int\limits_{\mathbb{R}^n} \alpha_k(z)\eta_k(z)^p dz)^\lambda \leq$$

$$\leq \frac{Q_k}{\epsilon^{p\lambda}}(\int\limits_{f(B(x,(1+a)r_k)\cap D)} \alpha_k(z)dz)^\lambda.$$

We find that $0 < \frac{C_2\epsilon^{p\lambda}}{(1+a)^{n-q}} \leq Q_k(\int\limits_{f(B(x,(1+a)r_k)\cap D)} \alpha_k(z)dz)^\lambda/((1+a)r_k)^{n-q} \to 0$ if $k \to \infty$ and we reached a contradiction.

We proved that $\lim\limits_{\substack{z\to x\\ z\in C_1}} f(z) = c$. The proof in the case $q = n$ is done in a similar manner.

Proof of Corollary 4. We see from Lemma 2 that there exists $\epsilon > 0$ and $\omega : D \cap B(x,\epsilon) \to [0,\infty]$ measurable and finite a.e. and constants $Q(n)$ and $C(n)$ depending only on n such that $M_q(\Gamma) \leq C(n)M^q_\omega(f(\Gamma))$ for every $\Gamma \in A(D \cap B(x,\epsilon))$ and $\int\limits_{f(B(x,r)\cap D)} \omega(z)dz \leq Q(n)(\int\limits_{D\cap B(x,r)} |f^{\text{`}}(z)^q dz)$ for $0 < r < \epsilon$. We apply now Theorem 1.

Remark 4. If $D = B(0,1) \subset \mathbb{R}^n$ and $f \in W^{1,q}(B(0,1))$, $n-1 < q < n$, we see from Lemma 3.2 in [21] that for every $\epsilon > 0$ there exists $U \subset \mathbb{R}^n$ open with $B_{1,q}(U) < \epsilon$ and such that $\lim\limits_{r\to 0}(\int\limits_{D\cap B(x,r)} |\nabla f(z)|^q dz/r^{n-q} = 0$ for every $x \notin U \cap S(0,1)$. Here, we denote by $B_{1,q}(U)$ the Bessel capacity of U.

Proof of Corollary 5. Let $\omega_r : \mathbb{R}^n \to [0,\infty]$ be given by $\omega_r(y) = N(y, f, D\cap B(x,r))$ if $y \in f(D \cap B(x,r))$, $\omega_r(y) = 0$ otherwise for $0 < r < \rho$ and let $Q_r = (\int\limits_{B(x,r)\cap D} K(z)^{q/(p-q)}dz)^{\frac{p-q}{p}}$ for $0 < r < \rho$. We can suppose that $J_f \in L^1(D \cap B(x,\rho))$ and using relation (3) from [15], we see that $\omega_r \in L^1(\mathbb{R}^n)$ and we see from Lemma 3 that $M_q(\Gamma) \leq Q_r M^p_{\omega_r}(f(\Gamma))^{q/p}$ for every $\Gamma \in A(D \cap B(x,r))$ and every $0 < r < \rho$. We also see that $Q_r(\int\limits_{f(D\cap B(x,r))} \omega_r(z)dz)^{q/p}/r^{n-q} = Q_r(\int\limits_{D\cap B(x,r)} J_f(z)dz)^{q/p}/r^{n-q} \to 0$ if $r \to 0$. We apply now Theorem 1.

We have the following consequences:

Corollary 6. Let $n \geq 2$, $n-1 < q \leq n$, $0 < \psi < \varphi \leq \frac{\pi}{2}$, $a = \frac{1}{2}\sin(\varphi-\psi)$, $D \subset \mathbb{R}^n$ a domain, $x \in \partial D$, d a half line ending in x such that there exists $\rho > 0$ such that $C_{x,d,\varphi} \cap B(x,\rho) \subset D$, let $C = C_{x,d,\varphi}$, $C_1, C_{x,d,\psi}$ and let $E \subset C_1$ be such that $cap_{q,a}\underline{dens}(C,E,x) > 0$. Let $f : D \to \mathbb{R}^n$ be ACL^n on D be such that there exists $K : D \to [0,\infty]$ measurable and finite a.e. such that $|f^{\text{`}}(z)|^n \leq K(z)J_f(z)$ a.e., $K \in L^{q/(n-q)}(D \cap B(x,\rho)) \cap L^{q/(n-q)}_{loc}(D)$ and that $\lim\limits_{r\to 0}(\int\limits_{B(x,r)\cap D} K(z)^{q/(n-q)}dz)^{(n-q)/n}(\int\limits_{B(x,r)\cap D} J_f(z)dz)^{q/n}/r^{n-q} = 0$. Then, if $\lim\limits_{\substack{z\to x\\ z\in E}} f(z) = c$, it results that $\lim\limits_{\substack{z\to x\\ z\in C_1}} f(z) = c$.

Proof: We see from [22] that f is open, discrete and hence has monotone components and we also see from [22] that f satisfies condition (N). We apply now Corollary 5.

Corollary 7. Let $n \geq 2$, $n-1 < q \leq n$, $0 < \psi < \varphi \leq \frac{\pi}{2}$, $a = \frac{1}{2}\sin(\varphi-\psi)$, $D \subset \mathbb{R}^n$ a domain, $x \in \partial D$, d a half line ending in x such that there exists $\rho > 0$ such that $C_{x,d,\varphi} \cap B(x,\rho) \subset D$, let $C = C_{x,d,\varphi}$, $C_1 = C_{x,d,\psi}$ and let $E \subset C_1$ be such that $cap_{q,a}\underline{dens}(C,E,x) > 0$. Let $f : D \to \mathbb{R}^n$ be ACL^n on D and $K : D \to [0,\infty]$ measurable and finite a.e. such that $|f^{'}(z)|^n \leq K(z)J_f(z)$ a.e., $K \in L^{q/(n-q)}(D \cap B(x,\rho)) \cap L_{loc}^{q/(n-q)}(D)$ and suppose that $\lim\limits_{r\to 0}(\int\limits_{B(x,r)\cap D} K(z)^{q/(n-q)}dz)^{\frac{n-q}{n}}(\int\limits_{\mathbb{R}^n} N(y,f,D\cap B(x,r))dy)^{q/n}/r^{n-q} = 0$. Then, if $\lim\limits_{\substack{z\to x\\ z\in E}} f(z) = c$, it results that $\lim\limits_{\substack{z\to x\\ z\in C_1}} f(z) = c$.

Remark 5. The set E from Theorem 1, 2, 3 and Corollary 4, 5, 6, 7 may be a nonconstant path $\gamma : [0,1] \to C_{x,d,\psi}$ such that $\gamma(0) = x$. If $C = C_{x,d,\frac{\pi}{2}}$ is a half plane. the set E may be such that rad $\underline{\text{dens}}(E,x) > 0$, and this thing is proved in Lemma 4 and Lemma 5.

Proof of Theorem 15. Let $\epsilon > 0$ and let $r_\epsilon > 0$ ne such that $f(\gamma(t)) \in B(c,\frac{\epsilon}{3})$ for $t \in (0,r_\epsilon)$. Let $D_\epsilon = f^{-1}(\complement\bar{B}(c,\epsilon)) \cap B(x,\rho) \cap C$ and let $A_\epsilon = \{t \in (0,r_\epsilon) | S(x,t) \cap D_\epsilon \neq \phi\}$. Then A_ϵ is an open subset of $(0,r_\epsilon)$, hence $A_\epsilon = \bigcup\limits_{k=0}^{\infty} (\alpha_{k,\epsilon},\beta_{k,\epsilon})$, where $(\alpha_{k\epsilon},\beta_{k\epsilon}) \cap (\alpha_{j\epsilon},\beta_{j\epsilon}) = \phi$ for $k \neq j$, $k,j \in \mathbb{N}$ and let $C_{k,\epsilon} = (B(x,\beta_{k\epsilon}) \setminus \bar{B}(x,\alpha_{k\epsilon})) \cap C$ for $k \in \mathbb{N}$. Let $\Delta_{k\epsilon} = \Delta(D_\epsilon, Im\alpha, C_{k,\epsilon})$ and let $\eta_k = \frac{3}{2\epsilon}\chi_{f(C_{k,\epsilon})}$ for $k \in \mathbb{N}$. Then $\eta_k \in F(f(\Delta_{k\epsilon}))$ for $k \in \mathbb{N}$. Using Lemma 4, we can find constants $C_1(n)$, $C(n)$ and $Q(n)$ depending only on n and $\omega \in L^1(\mathbb{R}^n)$ such that $C_1(n)\ln\frac{\beta_{k,\epsilon}}{\alpha_{k,\epsilon}} \leq M_n(\Delta_{k\epsilon}) \leq C(n)M_\omega^n(f(\Delta_{k\epsilon})) \leq C(n)\int\limits_{\mathbb{R}^n}\omega(z)\eta_k(z)^n dz = \frac{3^nC(n)}{2^n\epsilon^n}\int\limits_{f(C_{k,\epsilon})}\omega(z)dz \leq \frac{3^nC(n)Q(n)}{2^n\epsilon^n}\int\limits_{C_{k,\epsilon}}|f^{'}(z)|^n dz$ for every $k \in \mathbb{N}$. It results that $\ln\frac{\beta_{k,\epsilon}}{\alpha_{k,\epsilon}} \to 0$ and let $k_\epsilon \in \mathbb{N}$ be such that $\ln\frac{\beta_{k,\epsilon}}{\alpha_{k,\epsilon}} < 1$ for $k \geq k_\epsilon$.

Let $C(f,x,y) = \{z \in \overline{\mathbb{R}^n} |$ there exists $t_p \to 1$ such that $f(\gamma_y(t_p)) \to z\}$ for $y \in S$. Let $B_\epsilon = \{y \in S | C(f,x,y) \cap \complement B(c,3\epsilon) \neq \phi\}$. Suppose that $\mu_{n-1}(B_\epsilon > 0$. We can chose $k_\epsilon \in \mathbb{N}$ such that $\frac{C(n)Q(n)}{(2\epsilon)^n}\int\limits_{B(x,r_{k\epsilon})\cap D}|f^{'}(z)|^n dz < \frac{\mu_{n-1}(B_\epsilon)}{2}$.

Let $t_{k,\epsilon} = 1 - \frac{\alpha_{k,\epsilon}}{|y-x|}$, $s_{k,\epsilon} = 1 - \frac{\beta_{k,\epsilon}}{|y-x|}$ and let $\gamma_{y,k,\epsilon} = \gamma_y|[s_{k,\epsilon},t_{k,\epsilon}]$ for $y \in B_\epsilon$, $k \geq k_\epsilon$ and let $B_{k,e} = \{y \in S | Im\gamma_{y,k,\epsilon} \cap f^{-1}(\complement B(c,3\epsilon)) \neq \phi\}$ for $k \geq k_\epsilon$, $k \in I_\epsilon$. We see that if $y \in B_{k,\epsilon}$ and $k \in I_\epsilon$, then $f\circ\gamma_{y,k,\epsilon}$ meets both $\complement B(c,3\epsilon)$ and $\bar{B}(c,\epsilon)$, hence $d(f\circ\gamma_{y,k,\epsilon}) \geq 2\epsilon$ for $y \in B_{k,\epsilon}$ and $k \in I_\epsilon$. Let us fix $k \in I_\epsilon$ and let $\Gamma_{k,\epsilon} = \{\gamma_{y,k,\epsilon} | y \in B_{k,\epsilon}\}$. Then $\rho_k = \frac{1}{2\epsilon}\chi_{f(C_{k,\epsilon})} \in F(f(\Gamma_{k,\epsilon}))$ and we have $\frac{\mu_{n-1}(B_{k,\epsilon})}{(\ln(\frac{\beta_{k,\epsilon}}{\alpha_{k,\epsilon}}))^{n-1}} \leq M_n(\Gamma_{k,\epsilon}) \leq C(n)M_\omega^n(f(\Gamma_{k,\epsilon}))) \leq C(n)\int\limits_{\mathbb{R}^n}\omega(z)\rho_k(z)^n dz \leq \frac{C(n)}{(2\epsilon)^n}\int\limits_{f(C_{k,\epsilon})}\omega(z)dz \leq \frac{C(n)Q(n)}{(2\epsilon)^n}\int\limits_{C_{k,\epsilon}}|f^{'}(z)|^n dz$.

We find that

$$\sum_{k\in I_\epsilon}\mu_{n-1}(B_{k,\epsilon}) \leq \sum_{k\in I_\epsilon}\frac{C(n)Q(n)}{(2\epsilon)^n}(\ln(\frac{\beta_{k,\epsilon}}{\alpha_{k,\epsilon}}))^{n-1}\int\limits_{C_{k,\epsilon}}|f^{'}(z)|^n dz \leq$$

$$\leq \sum_{k\in I_\epsilon}\frac{C(n)Q(n)}{(2\epsilon)^n}\int\limits_{C_{k,\epsilon}}|f^{'}(z)|^n dz \leq \frac{C(n)Q(n)}{(2\epsilon)^n}\int\limits_{B(x,r_{k_\epsilon})\cap D}|f^{'}(z)|^n dz \leq \frac{\mu_{n-1}(B_\epsilon)}{2}.$$

Since $B_\epsilon \subset \bigcup\limits_{k\in I_\epsilon} B_{k,\epsilon}$, we obtain that $\mu_{n-1}(B_\epsilon \leq \mu_{n-1}(\bigcup\limits_{k\in I_\epsilon} B_{k,\epsilon}) \leq \sum\limits_{k\in I_\epsilon}\mu_{n-1}(B_{k,\epsilon}) \leq \frac{\mu_{n-1}(B_\epsilon)}{2}$ and we reached a contradiction.

It results that $\mu_{n-1}(B_\epsilon) = 0$ for every $\epsilon > 0$. Let $B = \bigcup\limits_{\epsilon>0} B_\epsilon$. Then $\mu_{n-1}(B) = 0$ and if $y \in S \setminus B$, then $Card\ C(f,x,y) = 1$ and hence $\lim\limits_{t\to 1} f(\gamma_y(t)) = c$ for every $y \in S \setminus B$. If f has monotone components and $0 < \psi < \varphi$, we see from Theorem 14 that $\lim\limits_{\substack{z\to x\\ z\in C_{x,d,\psi}}} f(z) = c$.

We also have:

Corollary 8. Let $n \geq 2$, $0 < \psi < \varphi \leq \frac{\pi}{2}$, $D \subset \mathbb{R}^n$ a domain, $x \in \partial D$, d a half line ending in x such tat there exists $\rho > 0$ such that $C_{x,d,\varphi} \cap B(x,\rho) \subset D$, let $C = C_{x,d,\varphi}$, $C_1 = C_{x,d,\psi}$, let $\gamma : [0,1] \to C$ be a nonconstant path such that $\gamma(0) = x$, let f be ACL^n on D such that $\mu_n(B_f) = 0$ and there exists $p > n-1$ and $K \in L^p_{loc}(D)$ such that $|f^{`}(z)|^n \leq K(z)J_f(z)$ a.e. Suppose that $\int\limits_{D\cap B(x,\rho)} |f^{`}(z)|^n dz < \infty$ and $\lim\limits_{t\to 0} f(\gamma(t)) = c$ and either $\mu_n(f(B_f)) = 0$, or f has locally inverses on $f(D \setminus B_f)$ which satisfies condition (N). Then $\lim\limits_{\substack{z\to x\\ z\in C_1}} f(z) = c$.

Proof: Since f is a local homeomorphism which is ACL^n on $D \setminus B_f$, we see from Lemma 6.7, page 190 in [33] that f is a.e. differentiable and satisfies condition (N) on $D \setminus B_f$. It results that if $\mu_n(f(B_f)) = 0$, then f is a.e. differentiable and satisfies condition (N) on D. Using Theorem 15, we can find a nonconstant path $\gamma : [0,1] \to C_1$ such that $\gamma(0) = x$ and $\lim\limits_{t\to 0} f(\gamma(t)) = c$. We see from [22] that f is open, discrete and hence has monotone components. We apply now Corollary 4 and Lemma 5.

Proof of Theorem 16. We see from Lemma 2 that there exists $\omega \in L^1(\mathbb{R}^n)$ such that $\omega = 0$ on $\complement B(x,\rho) \cap C_1$ and constants $C(n), Q(n)$ depending only on n such that $M_n(\Gamma) \leq C(n)M^n_\omega(f(\Gamma))$ for every $\Gamma \in A(B(x,\rho) \cap C_1)$ and $\int\limits_{f(A)} \omega(z)dz \leq Q(n)\int\limits_A |f^{`}(z)|^n dz$ for every measurable $A \subset B(x,\rho) \cap C_1$.

Suppose that there exists $\alpha > 0$ and $b_k \in C_1$, $b_k \to 0$ such that $|f(b_k) - c| \geq 2\alpha$ for every $k \in \mathbb{N}$ and let $0 < \epsilon < \alpha$. Let $r_\epsilon > 0$ be such that $M_n(\Delta(B(x,r) \cap E_\epsilon, S(x,(1+a)r) \cap C, B(x,(1+a)r) \cap C) \geq \delta_\epsilon$ for $0 < r \leq r_\epsilon$, let $k_\epsilon \in \mathbb{N}$ be such that $|f(y) - f(b_k)| \leq \epsilon$ for every $y \in B(b_k, a|b_k - x|)$ and every $k \geq k_\epsilon$ and suppose that $|b_{k_\epsilon} - x| \leq r_\epsilon$. Let $r_{k_\epsilon} = |b_{k_\epsilon} - x|$, $\Gamma_{1\epsilon} = \Delta(B(x,r_{k_\epsilon}) \cap E_\epsilon, S(x,(1+a)r_{k_\epsilon}) \cap C, B(x,(1+a)r_{k_\epsilon}) \cap C)$, $\Gamma_{2\epsilon} = \Delta(B(b_{k_\epsilon}, \frac{a}{2}r_{k_\epsilon}), S(x,(1+a)r_{k_\epsilon}) \cap C, B(x,(1+a)r_{k_\epsilon} \cap C)$ and let $\Gamma_\epsilon = \Delta(B(x,r_{k_\epsilon}) \cap E_\epsilon, B(b_{k_\epsilon}, \frac{a}{2}r_{k_\epsilon}), B(x,(1+a)r_{k_\epsilon}) \cap C)$.

Let $\rho_\epsilon \in F(\Gamma_\epsilon)$. If $3\rho_\epsilon \notin F(\Gamma_{1\epsilon})$, $3\rho_\epsilon \notin F(\Gamma_{2\epsilon})$, we can find $\alpha_\epsilon \in \Gamma_{1\epsilon}$, $\beta_\epsilon \in \Gamma_{2\epsilon}$ locally rectifiable such that $\int\limits_{\alpha_\epsilon} \rho_\epsilon ds < \frac{1}{3}$, $\int\limits_{\beta_\epsilon} \rho_\epsilon ds < \frac{1}{3}$. Let $\Gamma_{3\epsilon} = \Delta(Im\alpha_\epsilon, Im\beta_\epsilon, (B(x,(1+a)r_{k_\epsilon}) \setminus \bar{B}(x,(1+\frac{a}{2})r_{k_\epsilon})) \cap C)$. Then $M_n(\Gamma_{1\epsilon}) \geq \delta_\epsilon$ and there exists constants $C_1(n,\varphi,\psi)$, $C_2(n,\varphi,\psi)$ depending only on n, φ and ψ such that $M_n(\Gamma_{2\epsilon}) \geq C_1(n,\varphi,\psi)$ and $M_n(\Gamma_{3\epsilon}) \geq C_2(n,\varphi,\psi)\ln(\frac{1+a}{1+\frac{a}{2}})$. We see that $3\rho_\epsilon \in F(\Gamma_{3\epsilon})$, hence $\int\limits_{\mathbb{R}^n} \rho_\epsilon(z)^n dz \geq \frac{1}{3^n}\min\{M_n(\Gamma_{1\epsilon}), M_n(\Gamma_{2\epsilon}), M_n(\Gamma_{3\epsilon})\}$ and since $\rho_\epsilon \in F(\Gamma_\epsilon)$ was arbitrarily chosen, we see that $M_n(\Gamma_\epsilon) \geq \frac{\delta_\epsilon}{3^n}$ for $\epsilon > 0$ small enough. We see that $f(\Gamma_\epsilon) > \Gamma_{c,\epsilon,\alpha}$ and that $\int\limits_{B(c,r)} \omega(z)dz = \int\limits_{f(E_r)} \omega(z)dz \leq Q(n)\int\limits_{E_r} |f^{`}(z)|^n dz$ for $0 < r \leq r_\epsilon$, hence $\int\limits_{B(c,r)} \omega(z)dz/r^n \leq MQ(n)$ for $0 < r \leq r_\epsilon$. Using Theorem 2 in [5], we find that

$$\delta_\epsilon \leq 3^n M_n(\Gamma_\epsilon) \leq 3^n C(n) M^n_\omega(f(\Gamma_\epsilon)) \leq 3^n C(n) M^n_\omega(\Gamma_{c,\epsilon,\alpha}) \leq \frac{3^n M C(n) e^n Q(n) \sum\limits_{k=1}^{\infty} \frac{1}{k^n}}{(\ln\ln(\frac{\alpha e}{\epsilon}))^n}.$$

It results that $\delta_\epsilon(\ln\ln(\frac{\alpha e}{\epsilon}))^n \leq 3^n C(n)Q(n)Me^n \sum\limits_{k=1}^{\infty} \frac{1}{k^n} < \infty$, and we reached a contradiction, since $\lim\limits_{\epsilon\to 0} \delta_\epsilon(\ln\ln(\frac{\alpha e}{\epsilon}))^n = \infty$.

It results that $\lim\limits_{\substack{z\to x\\ z\in C_1}} f(z) = c$. We apply now Theorem 15, Lemma 5 and Corollary 4 to see that $\lim\limits_{\substack{z\to x\\ z\in C_{x,d,\eta}}} f(z) = c$ for every $\psi < \eta < \varphi$.

Proof of Theorem 17. Let V_n be the volume of the unit ball in $\mathbb{R}^n$. We see that $\int\limits_{E_\epsilon} |f^{'}(z)|^n dz \le K \int\limits_{E_\epsilon} J_f(z)dz \le K \int\limits_{f(E_\epsilon)} N(y,f,D\cap B(x,\rho))dy \le KV_nN(f,D\cap B(x,\rho))\epsilon^n$ for every $\epsilon > 0$ and we see that $\lim\limits_{\epsilon\to 0} \int\limits_{E_\epsilon} |f^{'}(z)|^n dz/\epsilon^n < \infty$. Also, $\int\limits_{D\cap B(x,\rho)} |f^{'}(z)|^n dz \le K \int\limits_{D\cap B(x,\rho)} J_f(z)dz < \infty$. We apply Theorem 16 to see that $\lim\limits_{\substack{z\to x \\ z\in C_{x,d,\eta}}} f(z) = c$ for every $\psi < \eta < \varphi$.

Remark 6. If the function f from Theorem 17 is bounded on $B(x,\rho)\cap D$, then the condition "$\int\limits_{B(x,\rho)\cap D} J_f(z)dz < \infty$" is satisfied. Indeed, $\int\limits_{B(x,\rho)\cap D} J_f(z)dz \le \int\limits_{f(B(x,\rho)\cap D)} N(y,f,B(x,\rho)\cap D))dy \le \mu_n(f(D\cap B(x,\rho))N(f,D\cap B(x,\rho)) < \infty$.

A recent research concerning the properties of the mappings satisfying generalized inverse modular inequalities may be found in [49].

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