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Heat transfer with first-order interfacial jump in a biconnected structure

by
Dan Poliševski , Renata Schiltz-Bunoiu and Alina Stănescu

Abstract

The paper deals with the asymptotic behaviour of heat conduction in a bounded domain having the ε-periodic structure introduced in [13], formed by two interwoven connected components separated by an interface on which the heat flux is continuous and the temperature subjects to a first-order jump condition. Considering that the conductivities of the two components and the transfer coefficient of the jump condition have various orders of magnitude with respect to ε, we derive the macroscopic laws and the effective coefficients in all regular cases by means of the two-scale convergence technique of the periodic homogenization theory.

Key Words: Homogenization, heat conduction, first-order jump interface, two-scale convergence,
2010 Mathematics Subject Classification: 35B27, 80M40, 76M50.

1 Introduction

In this paper we study the asymptotic behaviour, when $\varepsilon \to 0$, of the temperature governed by the heat transfer problem in the ε-periodic structure introduced by [13], which is a realistic periodic structure composed of two connected components and having many convenient properties.

We set the reference conductor (where the conductivity is of unity order with respect to ε) in the ambient component, the only one which is reaching the boundary of the domain. The second component contains the core material of the structure, where the conductivity is set of $\varepsilon^{2\beta}$-order, with $\beta \in [0,1]$. Let us remark here that for $\beta > 1$ the temperature becomes singular with respect to ε.

On the interface between the reference conductor and the core material we set ε^r to be the order of the transmission coefficient in the jump condition. A counterexample of [8] shows that the temperature cannot be asymptotically finite for $r > 1$; furthermore, we restrain to $r \in (-1, 1]$.

In order to derive the macroscopic laws and the effective coefficients in all regular cases we apply the two-scale convergence technique of the periodic homogenization theory (see [1], [11] and [5]). In the present framework, it turns out that there are exactly six distinct cases, given by $\beta = 0$, $\beta \in (0,1)$ or $\beta = 1$ and $r = 1$ or $r \in (-1, 1)$. We determine in each case the specific local-periodic problems. The solutions of these specific problems define the effective coefficients which allow the identification of the homogenized systems which uniquely define the asymptotic behaviour of the temperature.

We have to mention that besides heat conduction there are many other phenomena which lead to asymptotic problems similar to the one studied here; for instance, the pressure distribution in a partially fractured porous medium, the dispersion of a concentration of solute in a domain with highly different diffusivities or the diffusion of a dissolved chemical in a fluid flowing through a porous medium with highly different permeabilities. Thus, in such different frameworks, this problem has already been treated when the core material is composed of isolated grains for $\beta = 0$ and $r = 0$ in [9] and for $\beta = 0$ and various values of r, especially $r = 1$, which corresponds to the case when the transmission coefficient balance the total measure of the interface, in [2], [12], [3], [8], [10] and [6]. For our geometry, only the case $\beta = 0$ and $r = 1$ have already been studied in [7].

2 The heat conduction problem

Let Ω be an open connected bounded set in $\mathbb{R}^N$ $(N \geq 3)$, locally located on one side of the boundary $\partial\Omega$, a Lipschitz manifold composed of a finite number of connected components.

Let Y_a be a Lipschitz open connected subset of the unit cube $Y = (0,1)^N$. We assume that $Y_b = Y \setminus \overline{Y}_a$ has a locally Lipschitz boundary and that the intersections of ∂Y_b with ∂Y are reproduced identically on the opposite faces of the cube, denoted for every $i \in \{1, 2, \ldots, N\}$ by

$$\Sigma^{+i} = \{y \in \partial Y : \; y_i = 1\} \text{ and } \Sigma^{-i} = \{y \in \partial Y : \; y_i = 0\}, \tag{2.1}$$

with the property that

$$\overline{Y}_b \cap \Sigma^{\pm i} \subset\subset \Sigma^{\pm i}, \quad \forall i \in \{1, 2, ..., N\}. \tag{2.2}$$

We assume that repeating Y by periodicity, the reunion of all the $\overline{Y}_a$ parts is a connected domain in $\mathbb{R}^N$ with a locally C^2 boundary; we denote it by $\mathbb{R}^N_a$ and further $\mathbb{R}^N_b = \mathbb{R}^N \setminus \mathbb{R}^N_a$. Obviously, the origin of the coordinate system can be set such that there exists $R > 0$ with the property $B(0, R) \subseteq \mathbb{R}^N_a$.

For any $\varepsilon \in (0, 1)$ we denote

$$\mathbb{Z}_\varepsilon = \{k \in \mathbb{Z}^N : \; \varepsilon k + \varepsilon Y \subseteq \Omega\}, \tag{2.3}$$

$$I_\varepsilon = \{k \in \mathbb{Z}_\varepsilon : \; \varepsilon k \pm \varepsilon e_i + \varepsilon Y \subseteq \Omega, \; \forall i \in \{1, ..., N\}\}, \tag{2.4}$$

where e_i are the unit vectors of the canonical basis in $\mathbb{R}^N$.

The core component of our structure is defined by

$$\Omega_{\varepsilon b} = \mathrm{int}\left(\bigcup_{k\in I_\varepsilon}(\varepsilon k + \varepsilon \overline{Y}_b)\right) \tag{2.5}$$

and the reference conductor by

$$\Omega_{\varepsilon a} = \Omega \setminus \overline{\Omega}_{\varepsilon b}. \tag{2.6}$$

The interface between the two components is denoted by

$$\Gamma_\varepsilon = \partial\Omega_{\varepsilon a} \cap \partial\Omega_{\varepsilon b} = \partial\Omega_{\varepsilon b}. \tag{2.7}$$

Finally, let us remark that all the boundaries are at least locally Lipschitz, $\Omega_{\varepsilon a}$ is connected and $\Omega_{\varepsilon b}$ can be, in particular, connected too.

We introduce the Hilbert space

$$H_\varepsilon = \left\{ v \in L^2(\Omega) : v\Big|_{\Omega_{\varepsilon a}} \in H^1(\Omega_{\varepsilon a}),\ v\Big|_{\Omega_{\varepsilon b}} \in H^1(\Omega_{\varepsilon b}),\ v = 0 \text{ on } \partial\Omega \right\} \tag{2.8}$$

endowed with the scalar product

$$(u, v)_{H_\varepsilon} = \int_{\Omega_{\varepsilon a}} \nabla u \nabla v + \varepsilon^2 \int_{\Omega_{\varepsilon b}} \nabla u \nabla v + \varepsilon \int_{\Gamma_\varepsilon} [u][v], \tag{2.9}$$

where $[u] = \gamma_{\varepsilon b} u - \gamma_{\varepsilon a} u$ and $\gamma_{\varepsilon a} u,\ \gamma_{\varepsilon b} u$ are the traces of u on Γ_ε defined in $H^1(\Omega_{\varepsilon a})$ and $H^1(\Omega_{\varepsilon b})$, respectively.

From now on, let us denote $\Gamma := \partial Y_a \cap \partial Y_b$. Obviously,

$$\bigcup_{k\in\mathbb{Z}_\varepsilon} (\varepsilon k + \varepsilon\Gamma) \subseteq \Gamma_\varepsilon \tag{2.10}$$

and if ν is the normal on Γ(exterior to Y_a) and $x \in (\varepsilon k + \varepsilon\Gamma)$ for some $k \in \mathbb{Z}_\varepsilon$ then

$$\nu^\varepsilon(x) = \nu\left(\left\{\frac{x}{\varepsilon}\right\}\right) \tag{2.11}$$

where $\left\{\frac{x}{\varepsilon}\right\}$ is formed by the fractional parts of the components of $\varepsilon^{-1}x$.

Our domain has the following well-known properties [4], [7]:

Lemma 1. *There exists an extension operator $P_\varepsilon \in \mathcal{L}\left(H^1(\Omega_{\varepsilon a}); H^1_0(\Omega)\right)$ such that*

$$P_\varepsilon v = v \text{ in } \Omega_{\varepsilon a}, \tag{2.12}$$

$$|\nabla P_\varepsilon v|_{L^2(\Omega)} \le C\, |\nabla v|_{L^2(\Omega_{\varepsilon a})},\ \forall v \in H^1(\Omega_{\varepsilon a}) \tag{2.13}$$

where $C > 0$ is a constant independent of ε.

Lemma 2. *For any $v \in H_\varepsilon$ there exists $C > 0$, independent of ε, such that*

$$|v|_{L^2(\Omega_{\varepsilon a})} \leq C \, |\nabla v|_{L^2(\Omega_{\varepsilon a})}, \tag{2.14}$$

$$\varepsilon^{1/2} \, |\gamma_{\varepsilon a} v|_{L^2(\Gamma_\varepsilon)} \leq C \left(|v|_{L^2(\Omega_{\varepsilon a})} + \varepsilon \, |\nabla v|_{L^2(\Omega_{\varepsilon a})} \right), \tag{2.15}$$

$$|v|_{L^2(\Omega_{\varepsilon b})} \leq C \left(\varepsilon^{1/2} \, |\gamma_{\varepsilon b} v|_{L^2(\Gamma_\varepsilon)} + \varepsilon \, |\nabla v|_{L^2(\Omega_{\varepsilon b})} \right). \tag{2.16}$$

Remark 1. *Taking in account the L^2-norm of the jump on Γ_ε the results of the previous Lemma have an important consequence:*

$$|v|_{L^2(\Omega_{\varepsilon b})} \leq C \, |v|_{H_\varepsilon}, \forall v \in H_\varepsilon. \tag{2.17}$$

For any $\varepsilon \in (0,1)$ we introduce the transmission factor $h^\varepsilon(x) = h(x/\varepsilon)$ and the symmetric conductivities $a^\varepsilon_{ij}(x) = a_{ij}(x/\varepsilon)$ and $b^\varepsilon_{ij}(x) = b_{ij}(x/\varepsilon)$, where h, a_{ij} and b_{ij} belong to $L^\infty_{per}(Y)$ and have the property that there exists $\delta > 0$ such that

$$h \geq \delta, \;\; a.e. \;\; on \;\; Y \tag{2.18}$$

$$a_{ij}\xi_i\xi_j \geq \delta\xi_i\xi_i \;\; \text{and} \;\; b_{ij}\xi_i\xi_j \geq \delta\xi_i\xi_i, \;\; \forall \xi \in \mathbb{R}^N, \;\; a.e. \;\; on \;\; Y. \tag{2.19}$$

Considering that $\beta \in [0,1]$, $r \leq 1$ and $f \in L^2(\Omega)$ are also given, we look for the temperature u^ε which satisfies the heat conduction equations

$$-\frac{\partial}{\partial x_i}\left(a^\varepsilon_{ij} \frac{\partial u^\varepsilon}{\partial x_j} \right) = f \text{ in } \Omega_{\varepsilon a}, \tag{2.20}$$

$$-\varepsilon^{2\beta} \frac{\partial}{\partial x_i}\left(b^\varepsilon_{ij} \frac{\partial u^\varepsilon}{\partial x_j} \right) = f \text{ in } \Omega_{\varepsilon b}, \tag{2.21}$$

with the following transmission and boundary conditions

$$a^\varepsilon_{ij} \frac{\partial u^\varepsilon}{\partial x_j} \nu^\varepsilon_i = \varepsilon^{2\beta} b^\varepsilon_{ij} \frac{\partial u^\varepsilon}{\partial x_j} \nu^\varepsilon_i = \varepsilon^r h_\varepsilon \left(\gamma_{\varepsilon b} u^\varepsilon - \gamma_{\varepsilon a} u^\varepsilon \right) \text{ on } \Gamma_\varepsilon, \tag{2.22}$$

$$u_\varepsilon = 0 \text{ on } \partial\Omega. \tag{2.23}$$

The variational formulation of the problem (2.20)-(2.23) is the following:
To find $u^\varepsilon \in H_\varepsilon$ such that

$$a_\varepsilon(u^\varepsilon, v) := \int_{\Omega_{\varepsilon a}} a^\varepsilon_{ij} \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial v}{\partial x_j} + \varepsilon^{2\beta} \int_{\Omega_{\varepsilon b}} b^\varepsilon_{ij} \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial v}{\partial x_j} + \varepsilon^r \int_{\Gamma_\varepsilon} h^\varepsilon [u^\varepsilon][v] = \int_\Omega f v,$$

$$\forall v \in H_\varepsilon. \tag{2.24}$$

The variational problem (2.24) it is well-posed:

Theorem 1. *For any $\varepsilon \in (0,1)$ there exists a unique $u^\varepsilon \in H_\varepsilon$, solution of the problem (2.24).*

Proof: Due the fact that the form $a_\varepsilon(\cdot,\cdot)$ is coercive and continous, the theorem is proved by applying the Lax-Milgram Theorem. The coerciveness of the form $a_\varepsilon(\cdot,\cdot)$ can be easily shown using (2.18)-(2.19) and the continuity of the right-hand side of (2.24) by applying the inequalities (2.14)-(2.16).

□

3 A priori estimates of the temperature

We begin by giving the a priori estimates of u^ε, solution of (2.24), for any $\beta \in [0,1]$ and $r \in (-1,1]$.

Setting $v = u^\varepsilon$ in (2.24) and using the coerciveness of $a_\varepsilon(\cdot,\cdot)$ we obtain

$$\{u^\varepsilon\}_\varepsilon \text{ bounded in } H_\varepsilon \tag{3.1}$$

and we find some $C > 0$, independent of ε, such that

$$|\nabla u^\varepsilon|_{L^2(\Omega_{\varepsilon a})} \le C, \quad \varepsilon^\beta |\nabla u^\varepsilon|_{L^2(\Omega_{\varepsilon b})} \le C, \quad \varepsilon^{r/2} |[u^\varepsilon]|_{L^2(\Gamma_\varepsilon)} \le C. \tag{3.2}$$

Using (2.14)-(2.16) we get

$$|u^\varepsilon|_{L^2(\Omega_{\varepsilon a})} \le C, \;\; |u^\varepsilon|_{L^2(\Omega_{\varepsilon b})} \le C, \;\; |\nabla u^\varepsilon|_{L^2(\Omega_{\varepsilon a})} \le C, \;\; \varepsilon^\beta |\nabla u^\varepsilon|_{L^2(\Omega_{\varepsilon b})} \le C. \tag{3.3}$$

Let us introduce the following Hilbert spaces

$$H^1_{per}(Y_a) = \left\{\varphi \in H^1_{loc}\left(\mathbb{R}^N_a\right) : \ \varphi \text{ is } Y\text{-periodic}\right\} \tag{3.4}$$

and

$$\widetilde{H}^1_{per}(Y_a) = \left\{\varphi \in H^1_{loc}\left(\mathbb{R}^N_a\right) : \int_{Y_a} \varphi = 0 \text{ and } \varphi \text{ is } Y\text{-periodic}\right\}. \tag{3.5}$$

Hereafter, for any $u \in H^1(\Omega_{\varepsilon\alpha})$, $\alpha \in \{a,b\}$, we use the notations

$$\widehat{u}^\varepsilon_\alpha = \begin{cases} u & \text{in } \Omega_{\varepsilon\alpha} \\ 0 & \text{in } \Omega - \Omega_{\varepsilon\alpha} \end{cases} \qquad \widehat{\nabla u}^\varepsilon_\alpha = \begin{cases} \nabla u & \text{in } \Omega_{\varepsilon\alpha} \\ 0 & \text{in } \Omega - \Omega_{\varepsilon\alpha}. \end{cases} \tag{3.6}$$

Now, we can present the main compactness result.

Theorem 2. *For every $\beta \in [0,1]$ and $r \in (-1,1]$ there exists $u_a \in H^1_0(\Omega)$, $\eta_a \in L^2\left(\Omega; \widetilde{H}^1_{per}(Y_a)\right)$ and $u_b \in L^2(\Omega, L^2_{per}(Y_b))$ such that the following convergences hold on some subsequence*

$$\widehat{u}^\varepsilon_a \overset{2s}{\rightharpoonup} \chi_a u_a, \tag{3.7}$$

$$\widehat{\nabla u}^\varepsilon_a \overset{2s}{\rightharpoonup} \chi_a\left(\nabla_x u_a + \nabla_y \eta_a(\cdot,y)\right), \tag{3.8}$$

$$\widehat{u}_b^\varepsilon \stackrel{2s}{\rightharpoonup} \chi_b u_b, \tag{3.9}$$

where $\chi_\alpha : L^2(\Omega\times Y_\alpha) \to L^2(\Omega\times Y)$, $\alpha \in \{a,b\}$, denotes the straight prolongation with zero; sometimes it can be identified with the characteristic value of Y_α.

When $\beta = 0$ we find that u_b is independent of y, with $u_b \in H^1(\Omega)$. Moreover, there exists $\eta_b \in L^2\left(\Omega; \widetilde{H}^1_{per}(Y_b)\right)$ such that it holds

$$\widehat{\nabla u_b}^\varepsilon \stackrel{2s}{\rightharpoonup} \chi_b \left(\nabla_x u_b + \nabla_y \eta_b(\cdot, y)\right). \tag{3.10}$$

When $\beta \in (0,1)$ we find that u_b is independent of y, with $u_b \in L^2(\Omega)$.

When $\beta = 1$ it holds

$$\varepsilon\widehat{\nabla u_b}^\varepsilon \stackrel{2s}{\rightharpoonup} \chi_b \nabla_y u_b. \tag{3.11}$$

Proof: Using the a priori estimates obtained previously we deduce that $\{u_a^\varepsilon\}_\varepsilon$ and $\{\nabla u_a^\varepsilon\}_\varepsilon$ are bounded in $L^2(\Omega_{\varepsilon a})$ and $\left[L^2(\Omega_{\varepsilon a})\right]^N$. Obviously, the sequences corresponding to $\widehat{u}_a^\varepsilon$ and $\widehat{\nabla u_a}^\varepsilon$ are bounded in $L^2(\Omega)$ and $\left[L^2(\Omega)\right]^N$. Using the main compactness theorem of the two-scale theory we find that $\exists g(x,y) \in L^2(\Omega\times Y)$ and $\exists G(x,y) \in \left[L^2(\Omega\times Y)\right]^N$ such that, on some subsequence, we have

$$\widehat{u}_a^\varepsilon \stackrel{2s}{\rightharpoonup} g \tag{3.12}$$

and

$$\widehat{\nabla u_a}^\varepsilon \stackrel{2s}{\rightharpoonup} G. \tag{3.13}$$

Therefore, for every $\psi(x,y) \in L^2(\Omega; C_{per}(Y))$ we have

$$\int_\Omega \widehat{u}_a^\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right) dx \longrightarrow \int_{\Omega\times Y} g(x,y)\psi(x,y)dxdy. \tag{3.14}$$

Moreover, if we set $\psi = 0$ on $\Omega \times Y_b$, we obtain

$$\int_{\Omega\times Y_b} g(x,y)\psi(x,y)dxdy = 0,$$

which obviously imply

$$g(x,y) = 0, \quad \text{for a.a. } (x,y) \in \Omega \times Y_b. \tag{3.15}$$

In conclusion, there exists $u_a(x,y) \in L^2(\Omega\times Y)$ such that

$$g = \chi_a\, u_a \quad \text{in } \Omega \times Y.$$

Further, we prove that u_a is independent of y. Let $\Psi \in \left[\mathcal{D}(\Omega; C^\infty_{per}(Y))\right]^N$. According to (3.13) we have

$$\int_{\Omega} \widehat{\nabla u_a^{\varepsilon}}(x)\Psi\left(x, \frac{x}{\varepsilon}\right) dx \longrightarrow \int_{\Omega\times Y} G(x,y)\Psi(x,y)dxdy, \tag{3.16}$$

and hence

$$\varepsilon \int_{\Omega} \widehat{\nabla u_a^{\varepsilon}}(x)\Psi\left(x, \frac{x}{\varepsilon}\right) dx \longrightarrow 0. \tag{3.17}$$

Integrating by parts, the left-hand side becomes

$$\varepsilon \int_{\Omega} \widehat{\nabla u_a^{\varepsilon}}\Psi dx = -\int_{\Omega_{\varepsilon a}} u_a^{\varepsilon}(\mathrm{div}_y \Psi)dx - \varepsilon \int_{\Omega_{\varepsilon a}} u_a^{\varepsilon}(\mathrm{div}_x \Psi)dx.$$

Using the definition of $\widehat{u}_a^{\varepsilon}$ and the convergence (3.12) it follows

$$-\int_{\Omega} \widehat{u}_a^{\varepsilon}(x)(\mathrm{div}_y \Psi)\left(x, \frac{x}{\varepsilon}\right) dx \longrightarrow -\int_{\Omega\times Y} \chi_{\alpha}(y)u_a(x,y)(\mathrm{div}_y \Psi)(x,y)dxdy.$$

Then

$$\int_{\Omega\times Y_a} u_a(x,y)\mathrm{div}_y \Psi(x,y)dxdy = 0,\ \forall \Psi \in \left[\mathcal{D}\left(\Omega; C_{per}^{\infty}(Y)\right)\right]^N.$$

We choose $\Psi(x,y) = \varphi(x)\Phi(y)$ where $\varphi \in \mathcal{D}(\Omega)$ and $\Phi \in \left[C_{per}^{\infty}(Y)\right]^N$; then, for a.a. $x \in \Omega$ we have

$$\int_{Y_a} u_a(x,y)(\mathrm{div}_y \Phi)(y)dy = 0.$$

Because there exists $v \in L^2(Y_a)$ with $\int_{Y_a} v dy = 0$, such that $\mathrm{div}_y \Phi = v$, it follows that $u_a(x,y)$ is constant with respect to $y \in Y_a$.

Next, we look for the form of $G(x,y)$. Choosing $\Psi \in \left[\mathcal{D}(\Omega; C_{per}^{\infty}(Y))\right]^N$ in (3.16) with $\mathrm{div}_y \Psi = 0$ and integrating by parts the left-hand side, we get

$$-\int_{\Omega} \widehat{u}_a^{\varepsilon}(x)(\mathrm{div}_x \Psi)\left(x, \frac{x}{\varepsilon}\right) dx \longrightarrow \int_{\Omega\times Y} G(x,y)\Psi(x,y)dxdy.$$

As $\widehat{u}_a^{\varepsilon} \xrightarrow{2s} \chi_{\alpha}(y)u_a$, we have

$$\int_{\Omega\times Y} G(x,y)\Psi(x,y)dxdy = -\int_{\Omega\times Y_a} u_a(x)(\mathrm{div}_x \Psi)(x,y)dxdy. \tag{3.18}$$

For $\Psi(x,y) = 0,\ \forall y \in Y_b$, we obtain

$$\int_{\Omega\times Y_b} G(x,y)\Psi(x,y)dxdy = 0.$$

Then $G(x,y) = 0$, for any $x \in \Omega$ and $y \in Y_b$ i.e. there exists $F(x,y) \in L^2(\Omega \times Y)$ such that

$$G(x,y) = \chi_a(y)F(x,y). \tag{3.19}$$

Thus (3.18) becomes

$$\int_{\Omega\times Y_a} F(x,y)\Psi(x,y)dxdy = -\int_{\Omega\times Y_a} \nabla u_a(x)\Psi(x,y)dxdy. \tag{3.20}$$

We choose again $\Psi(x,y) = \varphi(x)\Phi(y)$ where $\varphi \in \mathcal{D}(\Omega)$ and $\Phi \in \left[C^\infty_{per}(Y)\right]^N$ with $\mathrm{div}\Phi = 0$ in Y_a. From (3.20) we obtain

$$\int_{Y_a}\left[F(x,y) - \nabla u_a(x)\right]\Phi(y)dy = 0, \text{ for a.a. } x \in \Omega. \tag{3.21}$$

Hence, there exists $\eta_a \in L^2\left(\Omega; \widetilde{H}^1(Y_a)\right)$ such that

$$F(x,y) - \nabla u_a(x) = (\nabla_y \eta_a)(x,y). \tag{3.22}$$

Let us remark here that $u_a \in H^1(\Omega)$. Moreover, using (3.22) and recalling (3.21) with $\Phi \cdot \nu = 0$ on Γ, we get

$$\sum_{i=1}^N \left(\int_{\sum_a^{+i}} \eta_a \cdot \Phi dy_i' - \int_{\sum_a^{-i}} \eta_a \cdot \Phi dy_i'\right) = 0,$$

where $\sum_a^{\pm i} = \sum^{\pm i} \bigcap \overline{Y}_a$. Consequently $\eta_a \in L^2\left(\Omega; \widetilde{H}^1_{per}(Y_a)\right)$ and

$$G(x,y) = \chi_a(y)\left(\nabla u_a + \nabla_y \eta_a(x,y)\right).$$

For u_a, it remains to prove that it vanishes on $\partial\Omega$. As the estimations (3.3) imply that $\left\{|\nabla u_a^\varepsilon|_{L^2(\Omega_{\varepsilon a})}\right\}_\varepsilon$ is bounded, then using the Poincaré-Friedrichs inequality and the extension operator (2.12)-(2.13) we obtain

$$|P_\varepsilon u_a^\varepsilon|_{H_0^1(\Omega)} \leq C\,|\nabla P_\varepsilon u_a^\varepsilon|_{L^2(\Omega)} \leq C\,|\nabla u_a^\varepsilon|_{L^2(\Omega_{\varepsilon a})} \leq C,$$

which shows that $\{P_\varepsilon u_a^\varepsilon\}$ is bounded in $H_0^1(\Omega)$. Hence, there exists $u_a' \in H_0^1(\Omega)$ such that $P_\varepsilon u_a^\varepsilon \rightharpoonup u_a'$ in $H_0^1(\Omega)$ and consequently $\chi_a(\{\frac{x}{\varepsilon}\})P_\varepsilon u_a^\varepsilon \xrightharpoonup{2s} \chi_a(y)u_a'$.

On the other hand, as $\chi_a(\{\frac{x}{\varepsilon}\})P_\varepsilon u_a^\varepsilon = \widehat{u}_a^\varepsilon$ and $\widehat{u}_a^\varepsilon \xrightharpoonup{2s} \chi_a(y)u_a$, then, by identifying the limits, we get $u_a = u_a'$ in Ω.

When $\beta = 0$, we find from (3.2) and (2.16) that there exists $C > 0$, independent of ε, with the property that

$$|\nabla u^\varepsilon|_{L^2(\Omega_{\varepsilon b})} \leq C \text{ and } \varepsilon^{1/2}|[u^\varepsilon]|_{L^2(\Gamma_\varepsilon)} \leq C. \tag{3.23}$$

It follows that $\{\widehat{u}_b^\varepsilon\}_\varepsilon$ and $\left\{\widehat{\nabla u_b^\varepsilon}\right\}_\varepsilon$ are bounded in $L^2(\Omega)$ and $\left[L^2(\Omega)\right]^N$ and the rest of the proof is similar to that for u_a.

When $\beta \in (0,1)$, we have to prove that u_b is independent of y. Using the a priori estimate (3.2), for any $\Psi \in \left[\mathcal{D}(\Omega; C_{per}^{\infty}(Y))\right]^N$ it holds

$$\varepsilon \int_{\Omega} \widehat{\nabla u_b^{\varepsilon}}(x)\Psi\left(x, \frac{x}{\varepsilon}\right) dx = \varepsilon^{1-\beta}\varepsilon^{\beta} \int_{\Omega} \widehat{\nabla u_b^{\varepsilon}}(x)\Psi\left(x, \frac{x}{\varepsilon}\right) dx \longrightarrow 0, \tag{3.24}$$

which is identical to (3.17). Hence, the rest of the proof is similar to that for the corresponding property of u_a.

Finally, when $\beta = 1$, the estimations (3.3) imply that $\left\{\varepsilon \widehat{\nabla u_b^{\varepsilon}}\right\}_{\varepsilon}$ is bounded in $L^2(\Omega)$ and hence we can assume that it has a two-scale limit on the same subsequence as $\left\{\varepsilon \widehat{u_b^{\varepsilon}}\right\}_{\varepsilon}$ (see the main compactness theorem of [1] or [11]). The form of this limit, that is (3.11), can be found by using standard methods (see Proposition 1.14 of [1]).

□

Now, for any $k \in \{1,2,...,N\}$, we define $\eta_{ak} \in \widetilde{H}^1_{per}(Y_a)$ as the unique solution of the local-periodic problem

$$-\frac{\partial}{\partial y_i}\left(a_{ij}\frac{\partial\left(\eta_{ak} + y_k\right)}{\partial y_j}\right) = 0 \quad \text{in } Y_a, \tag{3.25}$$

$$a_{ij}\frac{\partial\left(\eta_{ak} + y_k\right)}{\partial y_j}\nu_i = 0 \quad \text{on } \Gamma. \tag{3.26}$$

The effective conductivity A is given by the classical formula

$$A_{ij} = \int_{Y_a} \left(a_{ij} + a_{ik}\frac{\partial \eta_{aj}}{\partial y_k} \right) dy, \qquad \forall i,j \in \{1,2,...,N\}. \tag{3.27}$$

Remark 2. *The homogenized tensor A is symmetric and positively defined.*

Remark 3. *Similarly to (3.25)-(3.26), for any $k \in \{1,2,...,N\}$, we consider the local-periodic problem associated to b_{ij} in Y_b; its solution is denoted by $\eta_{bk} \in \widetilde{H}^1_{per}(Y_b)$. Correspondingly, we define the effective conductivity B_{ij} like in (3.27).*

Next, we introduce the functions w_0 and w_1, which are the only solutions in $H^1_{per}(Y_b)$ of the following problems:

$$-\frac{\partial}{\partial y_i}\left(b_{ij}\frac{\partial w_0}{\partial y_j}\right) = 1 \quad \text{in } Y_b, \tag{3.28}$$

$$w_0 = 0 \quad \text{on } \Gamma \tag{3.29}$$

and

$$-\frac{\partial}{\partial y_i}\left(b_{ij}\frac{\partial w_1}{\partial y_j}\right) = 1 \text{ in } Y_b, \tag{3.30}$$

$$-b_{ij}\frac{\partial w_1}{\partial y_j}\nu_i + h w_1 = 0 \text{ on } \Gamma. \tag{3.31}$$

Due to the existence of the first-order jump interface Γ_ε, there are two effective coefficients describing the microscopic transfer:

$$\widetilde{h} = \int_\Gamma h(y)d\sigma, \tag{3.32}$$

$$\widetilde{w_1 h} = \int_\Gamma w_1(y)h(y)d\sigma. \tag{3.33}$$

4 The homogenization process for $\beta = 0$ and $r = 1$

Passing (2.24) to the limit, we obtain like in [7]:

Theorem 3. *For any $\Phi_a \in \mathcal{D}(\Omega)$, $\Phi_b \in C^\infty(\Omega)$ and $\varphi_\alpha \in \mathcal{D}(\Omega; C^\infty_{per}(Y_\alpha))$, $\alpha \in \{a,b\}$, we have*

$$\sum_{\alpha\in\{a,b\}} \int_{\Omega\times Y_\alpha} \alpha_{ij}\left(\frac{\partial u_\alpha}{\partial x_j} + \frac{\partial \eta_\alpha}{\partial y_j}\right)\left(\frac{\partial \Phi_\alpha}{\partial x_i} + \frac{\varphi_\alpha}{\partial y_i}\right) +$$
$$+\widetilde{h}\int_\Omega (u_b - u_a)(\Phi_b - \Phi_a) = \int_{\Omega\times Y} (\chi_a\Phi_a + \chi_b\Phi_b) f, \tag{4.1}$$

where $\widetilde{h}$ is defined in (3.32).

Remark 4. *Using density arguments it follows that*

$$((u_a,u_b),(\eta_a,\eta_b)) \in V_1 := \left[H^1_0(\Omega)\times H^1(\Omega)\right]\times\left[L^2(\Omega,\widetilde{H}^1_{per}(Y_\alpha))\right]$$

$\alpha \in \{a,b\}$, is solution of the problem:

To find $((u_a,u_b),(\eta_a,\eta_b)) \in V_1$ satisfying

$$\sum_{\alpha\in\{a,b\}} \int_{\Omega\times Y_\alpha} \alpha_{ij}\left(\frac{\partial u_\alpha}{\partial x_j} + \frac{\partial \eta_\alpha}{\partial y_j}\right)\left(\frac{\partial \Phi_\alpha}{\partial x_i} + \frac{\partial\varphi_\alpha}{\partial y_i}\right) + \widetilde{h}\int_\Omega (u_b - u_a)(\Phi_b - \Phi_a) =$$
$$= \int_{\Omega\times Y} (\chi_a\Phi_a + \chi_b\Phi_b) f, \quad \forall\,((\Phi_a,\Phi_b),(\varphi_a,\varphi_b)) \in V_1. \tag{4.2}$$

It easy to verify that (4.2) is a well-posed problem in the Hilbert space V_1, endowed with the scalar product:

$$\langle((u_a,u_b),(\eta_a,\eta_b)),((\Phi_a,\Phi_b),(\varphi_a,\varphi_b))\rangle_{V_1} = \sum_{\alpha\in\{a,b\}}\int_\Omega \nabla u_\alpha \nabla\Phi_\alpha +$$
$$+\int_\Omega (u_b - u_a)(\Phi_b - \Phi_a) + \sum_{\alpha\in\{a,b\}}\int_{\Omega\times Y_\alpha} \nabla_y\eta_\alpha\nabla_y\varphi_\alpha. \tag{4.3}$$

Theorem 4. *If u_ε is the solution of (2.24) then, the convergences (3.7), (3.8), (3.9) and (3.10) hold on the whole sequence and the corresponding limit $(u_a, u_b) \in H_0^1(\Omega) \times H^1(\Omega)$ is the unique solution of the homogenized problem*

$$\int_\Omega A_{ij} \frac{\partial u_a}{\partial x_j} \frac{\partial \Phi_a}{\partial x_i} + \int_\Omega B_{ij} \frac{\partial u_b}{\partial x_j} \frac{\partial \Phi_b}{\partial x_i} + \widetilde{h} \int_\Omega (u_b - u_a)(\Phi_b - \Phi_a) =$$

$$= \int_\Omega \left(|Y_a|\Phi_a + |Y_b|\Phi_b\right) f, \quad \forall\, (\Phi_a, \Phi_b) \in H_0^1(\Omega) \times H^1(\Omega). \tag{4.4}$$

Proof: Let $(v_a, v_b) \in H_0^1(\Omega) \times H^1(\Omega)$ be the unique solution of the well-posed problem (4.4). Recalling the local problem (3.25), we define η_α by

$$\eta_\alpha(x, y) = \eta_{\alpha k}(y) \frac{\partial v_\alpha}{\partial x_k}(x), \quad \alpha \in \{a, b\}. \tag{4.5}$$

Now, we easily verify that $((v_a, v_b), (\eta_a, \eta_b)) \in V_1$ is the only solution of (4.2). This imply that $v_a = u_a$ and $v_b = u_b$ and the proof is completed.

□

The homogenization process can be summarized in this case by:

Theorem 5. *If u^ε is the solution of (2.24) then*

$$u^\varepsilon \overset{2s}{\rightharpoonup} \chi_a u_a + \chi_b u_b \tag{4.6}$$

where $(u_a, u_b) \in H_0^1(\Omega) \times H^1(\Omega)$ is the unique solution of

$$\int_\Omega A_{ij} \frac{\partial u_a}{\partial x_j} \frac{\partial \Phi_a}{\partial x_i} + \int_\Omega B_{ij} \frac{\partial u_b}{\partial x_j} \frac{\partial \Phi_b}{\partial x_i} + \widetilde{h} \int_\Omega (u_b - u_a)(\Phi_b - \Phi_a) =$$

$$= \int_\Omega \left(|Y_a|\Phi_a + |Y_b|\Phi_b\right) f, \quad \forall\, (\Phi_a, \Phi_b) \in H_0^1(\Omega) \times H^1(\Omega). \tag{4.7}$$

5 The homogenization process for $\beta = 0$ and $r \in (-1, 1)$

In this case (2.17) and (3.2) imply:

$$|\nabla u^\varepsilon|_{L^2(\Omega_{\varepsilon b})} \le C \quad \text{and} \quad \varepsilon^{r/2} |[u^\varepsilon]|_{L^2(\Gamma_\varepsilon)} \le C$$

for some $C > 0$, independent of ε. A first result follows, like in the previous section:

Theorem 6. *There exists $u_b \in H_0^1(\Omega)$ and $\eta_b \in L^2\left(\Omega; \widetilde{H}^1_{per}(Y_b)\right)$ such that:*

$$\widehat{\nabla u_b^\varepsilon} \overset{2s}{\rightharpoonup} \chi_b \left(\nabla_x u_b + \nabla_y \eta_b(\cdot, y)\right). \tag{5.1}$$

Next, we can present a preliminary homogenization result:

Theorem 7. *There exists $u \in H_0^1(\Omega)$ such that*

$$u_a = u_b = u \text{ in } \Omega. \tag{5.2}$$

Moreover, for any $\Phi \in \mathcal{D}(\Omega)$ and $\varphi_\alpha \in \mathcal{D}(\Omega; C^\infty_{per}(Y_\alpha))$, $\alpha \in \{a, b\}$ we have

$$\sum_{\alpha \in \{a,b\}} \int_{\Omega \times Y_\alpha} \alpha_{ij} \left(\frac{\partial u}{\partial x_j} + \frac{\partial \eta_\alpha}{\partial y_j} \right) \left(\frac{\partial \Phi}{\partial x_i} + \frac{\partial \varphi_\alpha}{\partial y_i} \right) dxdy = \int_\Omega f \Phi dx. \tag{5.3}$$

Proof: First we want to prove (5.2). We multiply the variational problem (2.24) with ε^{1-r} and we consider the following test function

$$v(x) = \left(\Phi_a(x) + \varepsilon \varphi_a \left(x, \frac{x}{\varepsilon} \right), \Phi_b(x) + \varepsilon \varphi_b \left(x, \frac{x}{\varepsilon} \right) \right), \tag{5.4}$$

where $\Phi_\alpha \in \mathcal{D}(\Omega)$ and $\varphi_\alpha \in \mathcal{D}(\Omega; C^\infty_{per}(Y_\alpha))$. Passing to the limit, we get

$$\widetilde{h} \int_\Omega (u_b - u_a)(\Phi_b - \Phi_a)\, dx = 0, \quad \forall \Phi_a, \Phi_b \in \mathcal{D}(\Omega),$$

which obviously yields (5.2).

Finally, we set in (2.24) the test function (5.4) with $\Phi_a = \Phi_b = \Phi$; passing it to the limit, we obtain (5.3) by usual arguments. □

Remark 5. *Using density arguments it follows that*

$$(u, \eta_a, \eta_b) \in V_2 := H_0^1(\Omega) \times L^2(\Omega, \widetilde{H}^1_{per}(Y_a)) \times L^2(\Omega, \widetilde{H}^1_{per}(Y_b))$$

is solution of the problem:

To find $(u, \eta_a, \eta_b) \in V_2$ satisfying

$$\sum_{\alpha \in \{a,b\}} \int_{\Omega \times Y_\alpha} \alpha_{ij} \left(\frac{\partial u}{\partial x_j} + \frac{\partial \eta_\alpha}{\partial y_j} \right) \left(\frac{\partial \Phi}{\partial x_i} + \frac{\partial \varphi_\alpha}{\partial y_i} \right) dxdy = \int_\Omega f \Phi dx,$$

$$\forall\, (\Phi, \varphi_a, \varphi_b) \in V_2. \tag{5.5}$$

It easy to verify that (5.5) is a well-posed problem in the Hilbert space V_2, endowed with the scalar product:

$$\langle (u, \eta_a, \eta_b), (\Phi, \varphi_a, \varphi_b) \rangle_{V_2} = \int_\Omega \nabla u \nabla \Phi + \sum_{\alpha \in \{a,b\}} \int_{\Omega \times Y_\alpha} \nabla_y \eta_\alpha \nabla_y \varphi_\alpha. \tag{5.6}$$

In this case the homogenization process is summarized by:

Theorem 8. *If u^ε is the solution of the problem (2.24) then*

$$u^\varepsilon \xrightarrow{2s} u \tag{5.7}$$

where $u \in H_0^1(\Omega)$, is the unique solution of the homogenized problem

$$\int_\Omega (A + B) \nabla u \nabla \Phi = \int_\Omega f \Phi, \;\; \forall \Phi \in H_0^1(\Omega), \tag{5.8}$$

and A, B are the effective positive matrices defined by (3.27).

6 The homogenization process for $\beta \in (0,1)$ and $r = 1$

Recalling Theorem 2, we obtain a preliminary two-scale behavior of the limits.

Theorem 9. *For any $\Phi_a, \Phi_b \in \mathcal{D}(\Omega)$ and $\varphi_a \in \mathcal{D}(\Omega; C^\infty_{per}(Y_a))$, we have*

$$\int_{\Omega\times Y_a} a_{ij}\left(\frac{\partial u_a}{\partial x_j} + \frac{\partial \eta_a}{\partial y_j}\right)\left(\frac{\partial \Phi_a}{\partial x_i} + \frac{\partial \varphi_a}{\partial y_i}\right) + \int_\Omega \widetilde{h}(u_b - u_a)(\Phi_b - \Phi_a) =$$
$$= \int_{\Omega\times Y} (\chi_a \Phi_a + \chi_b \Phi_b)\, f, \tag{6.1}$$

where $\widetilde{h}$ is defined in (3.32).

Proof: Setting the test function (5.4) in (2.24) we have

$$\int_{\Omega_{\varepsilon a}} a^\varepsilon_{ij} \frac{\partial u^\varepsilon_a}{\partial x_i}\left(\frac{\partial \Phi_a}{\partial x_j} + \frac{\partial \varphi_a}{\partial y_j}\right) + \varepsilon \int_{\Omega_{\varepsilon a}} a^\varepsilon_{ij} \frac{\partial u^\varepsilon_a}{\partial x_i}\frac{\partial \varphi_a}{\partial x_j} +$$
$$+\varepsilon^{2\beta} \int_{\Omega_{\varepsilon b}} b^\varepsilon_{ij} \frac{\partial u^\varepsilon_b}{\partial x_i}\left(\frac{\partial \Phi_b}{\partial x_j} + \frac{\partial \varphi_b}{\partial y_j}\right) + \varepsilon^{2\beta+1} \int_{\Omega_{\varepsilon b}} b^\varepsilon_{ij} \frac{\partial u^\varepsilon_b}{\partial x_i}\frac{\partial \varphi_b}{\partial x_j} +$$
$$+\varepsilon \int_{\Gamma_\varepsilon} h_\varepsilon (u^\varepsilon_b - u^\varepsilon_a)\left(\Phi_b - \Phi_a + \varepsilon(\varphi_b - \varphi_a)\right) = \sum_{\alpha\in\{a,b\}} \left(\int_{\Omega_{\varepsilon\alpha}} f\Phi_\alpha + \varepsilon \int_{\Omega_{\varepsilon\alpha}} f\varphi_\alpha\right). \tag{6.2}$$

We obtain (6.1) by passing (6.2) to the limit and using the same arguments as in the proof of Theorem 3. □

Remark 6. *Using density arguments it follows that*

$$(u_a, u_b, \eta_a) \in V_3 := H^1_0(\Omega) \times L^2(\Omega) \times L^2(\Omega, \widetilde{H}^1_{per}(Y_a))$$

is solution of the problem:

To find $(u_a, u_b, \eta_a) \in V_3$ satisfying

$$\int_{\Omega\times Y_a} a_{ij}\left(\frac{\partial u_a}{\partial x_j} + \frac{\partial \eta_a}{\partial y_j}\right)\left(\frac{\partial \Phi_a}{\partial x_i} + \frac{\partial \varphi_a}{\partial y_i}\right) + \widetilde{h}\int_\Omega (u_b - u_a)(\Phi_b - \Phi_a) =$$
$$= \int_{\Omega\times Y} (\chi_a \Phi_a + \chi_b \Phi_b)\, f, \quad \forall\, (\Phi_a, \Phi_b, \varphi_a) \in V_3. \tag{6.3}$$

It easy to verify that (6.3) is a well-posed problem in the Hilbert space V_3, endowed with the scalar product:

$$\langle (u_a, u_b, \eta_a), (\Phi_a, \Phi_b, \varphi_a)\rangle_{V_3} = \int_\Omega \nabla u_a \nabla \Phi_a + \int_\Omega (u_b - u_a)(\Phi_b - \Phi_a) +$$
$$+ \int_{\Omega\times Y_a} \nabla_y \eta_a \nabla_y \varphi_a. \tag{6.4}$$

Theorem 10. *If u_ε is the solution of the problem (2.24) the convergences (3.7)-(3.11) hold on the whole sequence and the limit $(u_a, u_b) \in H_0^1(\Omega) \times L^2(\Omega)$ is the unique solution of the homogenized problem*

$$\int_\Omega A_{ij} \frac{\partial u_a}{\partial x_j} \frac{\partial \Phi_a}{\partial x_i} + \int_\Omega \widetilde{h}(u_b - u_a)(\Phi_b - \Phi_a) = \int_\Omega (|Y_a|\Phi_a + |Y_b|\Phi_b) f,$$

$$\forall (\Phi_a, \Phi_b) \in H_0^1(\Omega) \times L^2(\Omega). \tag{6.5}$$

Consequently, the results of the homogenization process are summarized in this case by:

Theorem 11. *If u^ε is the solution of the problem (2.24) then*

$$u^\varepsilon \overset{2s}{\rightharpoonup} u + \frac{|Y_b|}{\widetilde{h}} \chi_b f, \tag{6.6}$$

where $u \in H_0^1(\Omega)$ is the unique solution of the Dirichlet problem

$$\int_\Omega A \nabla u \nabla \Phi = \int_\Omega f \Phi, \quad \forall \Phi \in H_0^1(\Omega). \tag{6.7}$$

7 The homogenization process for $\beta \in (0,1)$ and $r \in (-1,1)$

Here it is the preliminary result specific to this case:

Theorem 12. *There exists $u \in H_0^1(\Omega)$ such that*

$$u_a = u_b = u \text{ in } \Omega. \tag{7.1}$$

Moreover, for any $\Phi \in \mathcal{D}(\Omega)$ and $\varphi_a \in \mathcal{D}(\Omega; C^\infty_{per}(Y_a))$, it holds

$$\int_{\Omega \times Y_a} a_{ij} \left(\frac{\partial u_a}{\partial x_j} + \frac{\partial \eta_a}{\partial y_j} \right) \left(\frac{\partial \Phi}{\partial x_i} + \frac{\partial \varphi_a}{\partial y_i} \right) dxdy = \int_\Omega f \Phi dx. \tag{7.2}$$

Proof: We begin by proving (7.1). We use the same method as in the proof of Theorem 7, multiplying the variational problem (2.24) with ε^{1-r}. Again, we set (5.4) as test function. Using the two-scale convergences of Theorem 2 and calculating separately each integral we get:

$$\varepsilon^{1-r} \int_{\Omega_{\varepsilon a}} a_{ij}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial v}{\partial x_j} dx \longrightarrow 0,$$

$$\varepsilon^{1-r} \varepsilon^{2\beta} \int_{\Omega_{\varepsilon b}} b_{ij}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial v}{\partial x_j} \longrightarrow 0,$$

$$\varepsilon \int_{\Gamma_\varepsilon} h^\varepsilon [u^\varepsilon][v] \longrightarrow \int_\Omega \widetilde{h}(u_b - u_a)(\Phi_b - \Phi_a),$$

$$\varepsilon^{1-r}\int_\Omega fv \longrightarrow 0.$$

By substitution we obtain:

$$\int_\Omega \widetilde{h}\,(u_b-u_a)\,(\Phi_b-\Phi_a)\,dx=0, \quad \forall \Phi_a,\Phi_b\in\mathcal{D}(\Omega),$$

from which (7.1) follows.

In order to prove (7.2) we set (5.4) as test function in (2.24) with $\Phi_a=\Phi_b=\Phi$. Taking (7.1) into account we complete the proof as usual, all the convergences being straightforward. □

Remark 7. *By density arguments we remark that*

$$(u,\eta_a)\in V_4 := H_0^1(\Omega)\times L^2(\Omega,\widetilde{H}^1_{per}(Y_a))$$

is solution of the problem:

To find $(u,\eta_a)\in V_4$ satisfying

$$\int_{\Omega\times Y_a} a_{ij}\left(\frac{\partial u}{\partial x_j}+\frac{\partial \eta_a}{\partial y_j}\right)\left(\frac{\partial \Phi}{\partial x_i}+\frac{\partial \varphi_a}{\partial y_i}\right)dxdy=\int_\Omega f\Phi dx$$
$$\forall\,(\Phi,\varphi_a)\in V_4. \tag{7.3}$$

It easy to verify that (7.3) is a well-posed problem in the Hilbert space V_4, endowed with the scalar product:

$$\langle (u,\eta_a)\,,(\Phi,\varphi_a)\rangle_{V_4}=\int_\Omega \nabla u\nabla\Phi+\int_{\Omega\times Y_a}\nabla_y\eta_a\nabla_y\varphi_a. \tag{7.4}$$

In the present case the asymptotic behavior can be summarized by:

Theorem 13. *If u^ε is the solution of the problem (2.24) then,*

$$u^\varepsilon \stackrel{2s}{\rightharpoonup} u, \tag{7.5}$$

where $u\in H_0^1(\Omega)$ is the unique solution of (6.7).

8 The homogenization process for $\beta=1$ and $r=1$

The preliminary result of this section is given by:

Theorem 14. *If u^ε is the solution of the problem (2.24) then we have*

$$\int_{\Omega\times Y_a} a_{ij}\left(\frac{\partial u_a}{\partial x_i}+\frac{\partial \eta_a}{\partial y_i}\right)\left(\frac{\partial \Phi}{\partial x_j}+\frac{\partial \varphi_a}{\partial y_j}\right)+\int_{\Omega\times Y_b} b_{ij}\frac{\partial u_b}{\partial y_i}\frac{\partial \varphi_b}{\partial y_j}+$$
$$+\int_{\Omega\times\Gamma} h\,(u_b-u_a)\,(\varphi_b-\Phi)=\int_{\Omega\times Y_a} f\Phi+\int_{\Omega\times Y_b} f\varphi_b, \tag{8.1}$$

for any $\Phi\in\mathcal{D}(\Omega)$, $\varphi_\alpha\in\mathcal{D}\left(\Omega;C^\infty_{per}(Y_\alpha)\right)$, $\alpha\in\{a,b\}$.

Proof: In order to prove (8.1), for $\Phi \in \mathcal{D}(\Omega)$ and $\varphi_\alpha \in \mathcal{D}\left(\Omega; C^\infty_{per}(Y_\alpha)\right)$, with $\alpha \in \{a, b\}$, we set $v \in H_\varepsilon$ in (2.24) as follows:

$$v(x) = \left(\Phi(x) + \varepsilon\varphi_a\left(x, \frac{x}{\varepsilon}\right), \varphi_b\left(x, \frac{x}{\varepsilon}\right)\right), \quad x \in \Omega. \tag{8.2}$$

Now, the variational problem (2.24) becomes:

$$\int_{\Omega_{\varepsilon a}} a^\varepsilon_{ij}\frac{\partial u^\varepsilon}{\partial x_i}\left(\frac{\partial \Phi}{\partial x_j} + \frac{\partial \varphi_a}{\partial y_j}\right) + \varepsilon\int_{\Omega_{\varepsilon a}} a^\varepsilon_{ij}\frac{\partial u^\varepsilon}{\partial x_i}\frac{\partial \varphi_a}{\partial x_j} + \varepsilon^2\int_{\Omega_{\varepsilon b}} b^\varepsilon_{ij}\frac{\partial u^\varepsilon}{\partial x_i}\frac{\partial \varphi_b}{\partial x_j} +$$

$$+\varepsilon\int_{\Omega_{\varepsilon b}} b^\varepsilon_{ij}\frac{\partial u^\varepsilon}{\partial x_i}\frac{\partial \varphi_b}{\partial y_j} + \varepsilon\int_{\Gamma_\varepsilon} h^\varepsilon\left(u^\varepsilon_b - u^\varepsilon_a\right)\left(\varphi_b - \Phi - \varepsilon\varphi_a\right) =$$

$$= \int_{\Omega_{\varepsilon a}} f\Phi + \int_{\Omega_{\varepsilon b}} f\varphi_b + \varepsilon\int_{\Omega_{\varepsilon a}} f\varphi_a. \tag{8.3}$$

The proof is completed by passing (8.3) to the limit. In the following we shall present only the convergences of the term involving the integral on Γ_ε, all the other being straightforward. We obviously have

$$I_\varepsilon = \varepsilon\int_{\Gamma_\varepsilon} h^\varepsilon u^\varepsilon_b\left(\varphi_b - \Phi\right) - \varepsilon\int_{\Gamma_\varepsilon} h^\varepsilon u^\varepsilon_a\left(\varphi_b - \Phi\right) + (\text{terms} \to 0). \tag{8.4}$$

As Γ is C^2, there exists $\Psi \in \left(\mathcal{D}(\Omega; C^1_{per}(Y))\right)^N$ such that

$$\Psi(\cdot, y) = \left(\varphi_b(\cdot, y) - \Phi(\cdot)\right)\cdot\nu(y), \quad \forall\, y \in \Gamma. \tag{8.5}$$

Then denoting $\Psi^\varepsilon(x) = \Psi\left(x, \frac{x}{\varepsilon}\right)$, $x \in \Omega$, we obtain

$$\varepsilon\int_{\Gamma_\varepsilon} h^\varepsilon u^\varepsilon_b\left(\varphi_b - \Phi\right) = \varepsilon\int_{\Gamma_\varepsilon} h^\varepsilon u^\varepsilon_b \Psi^\varepsilon\cdot\nu^\varepsilon =$$

$$= -\int_\Omega \varepsilon\widehat{\nabla} u^\varepsilon_b(x) h\Psi - \int_\Omega \widehat{u}^\varepsilon_b(x)\mathrm{div}_y\left(h\Psi\right) + (\text{terms} \to 0).$$

As usual,

$$\varepsilon\int_{\Gamma_\varepsilon} h^\varepsilon u^\varepsilon_b\left(\varphi_b - \Phi\right) \to -\int_{\Omega\times Y_b} \mathrm{div}_y\left(hu_b\Psi\right) = \int_\Omega\left(\int_\Gamma hu_b\Psi\cdot\nu\right)$$

$$= \int_{\Omega\times\Gamma} hu_b\left(\varphi_b - \Phi\right). \tag{8.6}$$

In a similar way, using (3.7)-(3.8), we get

$$\varepsilon\int_{\Gamma_\varepsilon} h^\varepsilon u^\varepsilon_a\left(\varphi_b - \Phi\right) = \int_\Omega \widehat{u}^\varepsilon_a \mathrm{div}_y\left(h\Psi\right) + (\text{terms} \to 0) \longrightarrow$$

$$\longrightarrow \int_{\Omega\times Y_a} u_a\mathrm{div}_y\left(h\Psi\right) = \int_\Omega u_a\left(\int_\Gamma h\Psi\cdot\nu\right) = \int_{\Omega\times\Gamma} hu_a\left(\varphi_b - \Phi\right). \tag{8.7}$$

Finally, (8.4), (8.6) and (8.7) yield

$$I_\varepsilon \longrightarrow \int_{\Omega\times\Gamma} h\,(u_b - u_a)\,(\varphi_b - \Phi)\,.$$

□

Remark 8. *Using density arguments it follows that*

$$(u_a, u_b, \eta_a) \in V_5 := H_0^1(\Omega) \times L^2(\Omega; H^1_{per}(Y_b)) \times L^2(\Omega, \widetilde{H}^1_{per}(Y_a))$$

is solution of the problem:

To find $(u_a, u_b, \eta_a) \in V_5$ satisfying

$$\int_{\Omega\times Y_a} a_{ij}\left(\frac{\partial u_a}{\partial x_i} + \frac{\partial \eta_a}{\partial y_i}\right)\left(\frac{\partial \Phi}{\partial x_j} + \frac{\partial \varphi_a}{\partial y_j}\right) + \int_{\Omega\times Y_b} b_{ij}\frac{\partial u_b}{\partial y_i}\frac{\partial \varphi_b}{\partial y_j} +$$
$$+ \int_{\Omega\times\Gamma} h\,(u_b - u_a)\,(\varphi_b - \Phi) = \int_{\Omega\times Y_a} f\Phi + \int_{\Omega\times Y_b} f\varphi_b,$$
$$\forall\,(\Phi, \varphi_b, \varphi_a) \in V_5. \tag{8.8}$$

It easy to verify that (8.8) is a well-posed problem in the Hilbert space V_5, endowed with the scalar product:

$$\langle (u_a, u_b, \eta_a)\,, (\Phi, \varphi_a, \varphi_b)\rangle_{V_5} = \int_\Omega \nabla u_a \nabla \Phi + \int_{\Omega\times Y_b} \nabla u_b \nabla \varphi_b +$$
$$+ \int_{\Omega\times\Gamma} (u_b - u_a)\,(\varphi_b - \Phi) + \int_{\Omega\times Y_a} \nabla_y \varphi_a \nabla_y \eta_a. \tag{8.9}$$

The results of the homogenization process can be summarized in this case by:

Theorem 15. *If u^ε is the solution of (2.24) then*

$$u^\varepsilon \overset{2s}{\rightharpoonup} \left(|\,Y_a\,| + \widetilde{w_1 h}\right) u + w_1\chi_b f \tag{8.10}$$

where $u \in H_0^1(\Omega)$ is the unique solution of the homogenized problem (6.7).

Proof: If $u \in H_0^1(\Omega)$ is the solution of the homogenized system (6.7) then it is easy to verify that the only solution of the problem (8.8) is given by

$$u_a(x) = \left(|\,Y_a\,| + \widetilde{w_1 h}\right) u(x),\ x \in \Omega \tag{8.11}$$

$$u_b(x, y) = \left(|\,Y_a\,| + \widetilde{w_1 h}\right) u(x) + w_1(y) f(x), \quad (x, y) \in \Omega \times Y_b, \tag{8.12}$$

$$\eta_a(x, y) = \left(|\,Y_a\,| + \widetilde{w_1 h}\right) \eta_{a_k}(y) \frac{\partial u}{\partial x_k}(x), \quad (x, y) \in \Omega \times Y_a, \tag{8.13}$$

and thus the proof is completed. □

9 The homogenization process for $\beta = 1$ and $r \in (-1, 1)$

The preliminary result of this case is the following:

Theorem 16. *For any $\Phi \in \mathcal{D}(\Omega)$ and $\varphi_\alpha \in \mathcal{D}(\Omega; C^\infty_{per}(Y_\alpha))$, $\alpha \in \{a, b\}$ such that*

$$\varphi_b(x, y) = \Phi(x), \ \forall (x, y) \in \Omega \times \Gamma \tag{9.1}$$

we have:

$$\int_{\Omega \times Y_a} a_{ij} \left(\frac{\partial u_a}{\partial x_i} + \frac{\partial \eta_a}{\partial y_i} \right) \left(\frac{\partial \Phi}{\partial x_j} + \frac{\partial \varphi_a}{\partial y_j} \right) + \int_{\Omega \times Y_b} b_{ij} \frac{\partial u_b}{\partial y_i} \frac{\partial \varphi_b}{\partial y_j} =$$

$$= \int_{\Omega \times Y_a} f \Phi + \int_{\Omega \times Y_b} f \varphi_b. \tag{9.2}$$

Moreover,

$$u_a = u_b \ \text{on} \ \Omega \times \Gamma. \tag{9.3}$$

Proof: We begin by proving (9.3). We use the same ideas as in the proof of the Theorem 7. We multiply the variational problem (2.24) with ε^{1-r} and then we take the test function (8.2) with $\Phi \in \mathcal{D}(\Omega)$, $\varphi_a \in \mathcal{D}(\Omega; C^\infty_{per}(Y_a))$ and $\varphi_b \in \mathcal{D}(\Omega; C^\infty_{per}(Y_b))$. We get

$$\varepsilon^{1-r} \int_{\Omega_{\varepsilon a}} a^\varepsilon_{ij} \frac{\partial u^\varepsilon}{\partial x_i} \left(\frac{\partial \Phi}{\partial x_j} + \frac{\partial \varphi_a}{\partial y_j} \right) + \varepsilon^{2-r} \int_{\Omega_{\varepsilon a}} a^\varepsilon_{ij} \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial \varphi_a}{\partial x_j} + \varepsilon^{3-r} \int_{\Omega_{\varepsilon b}} b^\varepsilon_{ij} \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial \varphi_b}{\partial x_j} +$$

$$+ \varepsilon^{2-r} \int_{\Omega_{\varepsilon b}} b^\varepsilon_{ij} \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial \varphi_b}{\partial y_j} + \varepsilon \int_{\Gamma_\varepsilon} h^\varepsilon \left(u^\varepsilon_b - u^\varepsilon_a \right) \left(\varphi_b - \Phi - \varepsilon \varphi_a \right) =$$

$$= \varepsilon^{1-r} \int_{\Omega_{\varepsilon a}} f \Phi + \varepsilon^{1-r} \int_{\Omega_{\varepsilon b}} f \varphi_b + \varepsilon^{2-r} \int_{\Omega_{\varepsilon a}} f \varphi_a. \tag{9.4}$$

As in the proofs of Theorem 7, Theorem 12 and Theorem 14, when passing to the limit with $(\varepsilon \to 0)$ in the (9.4) we obtain

$$\int_{\Omega \times \Gamma} h(y) \left(u_b(x, y) - u_a(x) \right) \left(\varphi_b(x, y) - \Phi(x) \right) = 0,$$

$$\forall \Phi \in \mathcal{D}(\Omega), \ \varphi_b \in \mathcal{D}(\Omega; C^\infty_{per}(Y_b)), \tag{9.5}$$

which obviously imply (9.3).

In order to obtain the homogenized equation (9.2) we take in (2.24) the same test function (8.2) with the supplementary condition (9.1). The proof is completed again in a straightforward manner, the term corresponding to the integral on Γ_ε being of order $\varepsilon^{1+r/2}$. $\square$

In the light of the previous result, we introduce the space

$$V := \left\{ (\Phi, \varphi) \in H^1_0(\Omega) \times L^2(\Omega; H^1_{per}(Y_b)), \ \ \varphi = \Phi \text{ on } \Omega \times \Gamma \right\}. \tag{9.6}$$

Remark 9. *Using density arguments it follows that*

$$((u_a, u_b), \eta_a) \in V_6 := V \times L^2(\Omega; \widetilde{H}^1_{per}(Y_a))$$

is solution of the problem:

To find $((u_a, u_b), \eta_a) \in V_6$ *satisfying*

$$\int_{\Omega\times Y_a} a_{ij}\left(\frac{\partial u_a}{\partial x_i} + \frac{\partial \eta_a}{\partial y_i}\right)\left(\frac{\partial \Phi}{\partial x_j} + \frac{\partial \varphi_a}{\partial y_j}\right) + \int_{\Omega\times Y_b} b_{ij}\frac{\partial u_b}{\partial y_i}\frac{\partial \varphi_b}{\partial y_j} =$$

$$= \int_{\Omega\times Y_a} f\Phi + \int_{\Omega\times Y_b} f\varphi_b, \quad \forall\, ((\Phi, \varphi_b), \varphi_a) \in V \times L^2(\Omega; \widetilde{H}^1_{per}(Y_a)). \tag{9.7}$$

It easy to verify that (9.7) is a well-posed problem in the Hilbert space V_6, *endowed with the scalar product:*

$$\langle ((u_a, u_b), \eta_a), ((\Phi, \varphi_a), \varphi_b)\rangle_{V_6} = \int_\Omega \nabla u_a \nabla \Phi + \int_{\Omega\times Y_b} \nabla_y u_b \nabla_y \varphi_b + \int_{\Omega\times Y_a} \nabla_y \varphi_a \nabla_y \eta_a. \tag{9.8}$$

The results of the homogenization process can be summarized in this case by:

Theorem 17. *If* u^ε *is the solution of the problem (2.24) then,*

$$u^\varepsilon \overset{2s}{\rightharpoonup} |Y_a| u + w_0 \chi_b f, \tag{9.9}$$

where $u \in H^1_0(\Omega)$ *is the unique solution of (6.7).*

Proof: If $u \in H^1_0(\Omega)$ is the unique solution of (6.7) then we verify that the unique solution of (9.7) is the following:

$$u_a = |Y_a| u, \quad u_b = |Y_a| u + w_0 f, \quad \eta_a = |Y_a| \eta_{ak} \frac{\partial u}{\partial x_k},$$

where η_{ak} and w_0 are defined by (3.25)-(3.26) and (3.28)-(3.29). □

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