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ON DIFFERENTIAL CALCULUS ON PRE-LIE GROUPS

MIHAI NICOLAE

Abstract. We fill a gap in the literature on the Lie theoretic investigations on general topological groups by providing the proof of the generalization of Faà di Bruno's formula to general topological groups. We then apply that formula for establishing the differentiability properties of the multiplication mapping in a pre-Lie group.

1. Introduction

There is a recent interest in differential calculus on topological groups, developed after the pattern of Lie groups; see for instance [NS13], where this calculus was used for Fréchet-Lie supergroups. This article fills a gap in the literature devoted to that circle of ideas, namely we give the proof of the generalization of Faà di Bruno's formula to general topological groups and we then study the differentiability properties of the multiplication mapping in a pre-Lie group. Our main results are stated below, in the second part of this introduction, after some necessary preliminaries. Then the proof of the main result is given in Section 2.

Preliminaries. We use differential calculus on topological groups as developed in the book [BCR81]. Unless otherwise mentioned, G is any topological group. (Evey topological group is assumed Hausdorff in the present paper.) Denoting by $\mathcal{C}(\cdot,\cdot)$ the spaces of continuous maps, one defines

$$\Lambda(G) := \{\gamma \in \mathcal{C}(\mathbb{R}, G) \mid (\forall t, s \in \mathbb{R}) \quad \gamma(t+s) = \gamma(t)\gamma(s)\}.$$

This set is endowed with the topology of uniform convergence on compact subsets of $\mathbb{R}$. The *adjoint action* of the topological group G is the mapping

$$\mathrm{Ad}_G \colon G \times \Lambda(G) \to \Lambda(G), \quad (g, \gamma) \mapsto \mathrm{Ad}_G(g)\gamma := g\gamma(\cdot)g^{-1}, \tag{1.1}$$

which is continuous [HM07, Prop. 2.28] and homogeneous in the sense that $\mathrm{Ad}_G(g)(r\cdot\gamma) = r \cdot (\mathrm{Ad}_G(g)\gamma)$ for all $r \in \mathbb{R}$, $g \in G$, and $\gamma \in \Lambda(G)$, where one defines

$$(\forall r, t \in \mathbb{R})(\forall \gamma \in \Lambda(G)) \quad (r \cdot \gamma)(t) := \gamma(rt).$$

For $r = -1$ and $\gamma \in \Lambda(G)$ we denote $-\gamma := (-1) \cdot \gamma \in \Lambda(G)$.

Let an arbitrary open subset $V \subseteq G$ and $\mathcal{Y}$ be any real locally convex space. If $\varphi \colon V \to \mathcal{Y}$, $\gamma \in \Lambda(G)$, and $g \in V$, then we denote

$$(D_\gamma \varphi)(g) := \lim_{t \to 0} \frac{\varphi(g\gamma(t)) - \varphi(g)}{t} \tag{1.2}$$

if the limit in the right-hand side exists. One defines $\mathcal{C}^1(V, \mathcal{Y})$ as the set of all functions $\varphi \in \mathcal{C}(V, \mathcal{Y})$ for which the function

$$D\varphi \colon V \times \Lambda(G) \to \mathcal{Y}, \quad (D\varphi)(g; \gamma) := (D_\gamma \varphi)(g)$$

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is well defined and continuous. One also denotes $D\varphi =: D^1\varphi$.

Now let $n \geq 2$ si spatiul $\mathcal{C}^{n-1}(V,\mathcal{Y})$ and the mapping D^{n-1} have been defined. Then $\mathcal{C}^n(V,\mathcal{Y})$ is defined as the set of all $\varphi \in \mathcal{C}^{n-1}(V,\mathcal{Y})$ for which the function

$$D^n\varphi\colon V \times \Lambda(G) \times \cdots \times \Lambda(G) \to \mathcal{Y},$$
$$(g;\gamma_1,\dots,\gamma_n) \mapsto (D_{\gamma_n}(D_{\gamma_{n-1}}\cdots(D_{\gamma_1}\varphi)\cdots))(g)$$

is well defined and continuous.

Moreover $\mathcal{C}^\infty(V,\mathcal{Y}) := \bigcap_{n\geq 1}\mathcal{C}^n(V,\mathcal{Y})$ and $\mathcal{C}^\infty_0(V,\mathcal{Y})$ is the set of all $\varphi \in \mathcal{C}^\infty(V,\mathcal{Y})$ having compact support. If $\mathcal{Y} = \mathbb{C}$, then we write simply $\mathcal{C}^n(G) := \mathcal{C}^n(V,\mathbb{C})$ etc., for $n = 1,2,\dots,\infty$.

It will be convenient to use the notations

$$D_\gamma\varphi := D_{\gamma_n}(D_{\gamma_{n-1}}\cdots(D_{\gamma_1}\varphi)\cdots)\colon G \to \mathcal{Y}$$

whenever $\gamma := (\gamma_1,\dots,\gamma_n) \in \Lambda(G)\times\cdots\times\Lambda(G)$ and $\varphi \in \mathcal{C}^n(G,\mathcal{Y})$.

We use the notation

$$\gamma^x : \mathbb{R} \to G, \gamma^x(t) = x^{-1}\gamma(t)x.$$

A *pre-Lie group* is any topological group G satisfying the conditions:

(1) The topological space $\Lambda(G)$ has the structure of a locally convex Lie algebra over $\mathbb{R}$, whose scalar multiplication, vector addition and bracket satisfy the following conditions for all $t,s \in \mathbb{R}$ and $\gamma_1,\gamma_2 \in \Lambda(G)$:

$$\begin{aligned}(t\cdot\gamma_1)(s) &= \gamma_1(ts);\\ (\gamma_1+\gamma_2)(t) &= \lim_{n\to\infty}(\gamma_1(t/n)\gamma_2(t/n))^n;\\ [\gamma_1,\gamma_2](t^2) &= \lim_{n\to\infty}(\gamma_1(t/n)\gamma_2(t/n)\gamma_1(-t/n)\gamma_2(-t/n))^{n^2},\end{aligned} \tag{1.3}$$

where the convergence is assumed to be uniform on the compact subsets of $\mathbb{R}$.

(2) For every nontrivial $\gamma \in \Lambda(G)$ there exists a function φ of class $\mathcal{C}^\infty$ on some neighborhood of $\mathbf{1} \in G$ such that $(D^\lambda_\gamma\varphi)(\mathbf{1}) \neq 0$.

Every locally compact group (in particular, every finite-dimensional Lie group) is a pre-Lie group ([BCR81, pag. 41–41]).

We will see below that if G is a pre-Lie group, then the multiplication mapping $\pi\colon G\times G \to G$, $(x,y)\mapsto xy$, is smooth (cf. [BCR81, Th. 1.3.2.2 and subsect. 1.1.2] or alternatively [BR80, Th. and Sect. 1]), where differentiability of maps between open sets of topological groups is understood in the following sense:

Let G_1, G_2 be two pre-Lie groups with some open sets $X_1 \subseteq G_1$ and $X_2 \subseteq G_2$, and $f : X_1 \to G_2$ be any continuous function. We say that f *is of class* C^k if there exist the maps $D^\ell f : X_1 \times \Lambda^\ell(G_1) \to \Lambda(G_2)$, $\ell = 1,\dots,k$, such that for every localy convex space $\mathcal{Y}$ and every function $\varphi \in C^\ell(X_2,\mathcal{Y}), 0 \leq \ell \leq k$ we have $\varphi\circ f \in C^\ell(X_1\cap f^{-1}(X_2),\mathcal{Y})$ and for every $\gamma = (\gamma_1,\dots,\gamma_\ell) \in \Lambda^\ell(G_1)$ the following chain rule holds,

$$D^\ell(\varphi\circ f)(x;\gamma) = \sum_{k=1}^{\ell}\sum_{(A_1,\dots,A_k)} D^{D^{A_1(\gamma)}f(x)}\dots D^{D^{A_j(\gamma)}f(x)}\varphi(f(x)). \tag{1.4}$$

The second summation in the above formula is performed after all partitions $\{1,2,\dots,\ell\} = A_1 \sqcup \dots \sqcup A_k$ into nonempty subsets with $\min A_1 > \dots > \min A_k$. For any fixed $k \in \{1,\dots,\ell\}$, and every $j = 1,\dots,k$, we have denoted $A_j = \{i^j_1,\dots,i^j_{m_j}\} \subseteq \{1,2,\dots,\ell\}$, with $i_1^j < \ldots < i_{m_j}^j$, and moreover

$$A_j(\gamma) := (\gamma_{i_1^j}, \ldots, \gamma_{i_{m_j}^j}) \in \Lambda^{m_j}(G_1) \text{ and } D^{A_j(\gamma)} f(x) := D^{m_j} f(x; A_j(\gamma)) \in \Lambda(G_2).$$

Note that $m_j = |A_j|$ for $j = 1, \ldots, k$, hence $1 \le m_1, \ldots, m_k \le \ell$ with $m_1 + \cdots + m_k = \ell$. We also note that the uniqueness of the above maps $D^\ell f$ follows by using the condition (2) in the definition of a pre-Lie group along with te chain rule.

Main results. The main result of this paper is the following formula:

Theorem 1.1. *Let G be any topological group and $\mathcal{Y}$ be any locally convex space. Define $\pi\colon G \times G \to G$, $\pi(x, y) = xy$. For every $f \in C^k(G, \mathcal{Y})$, and $k \ge 1$ one has*

$$\begin{aligned} & D^k(f \circ \pi)((x, y); (\lambda_{11}, \lambda_{12}), \ldots, (\lambda_{k1}, \lambda_{k2})) \\ & = \sum_{\ell=0}^{k} \sum_{\substack{i_1 < \cdots < i_\ell \\ i_{\ell+1} < \cdots < i_k}} D^k f(xy; \lambda_{i_1 2}, \ldots, \lambda_{i_\ell 2}, \lambda^y_{i_{\ell+1} 1}, \ldots, \lambda^y_{i_k 1}) \end{aligned}$$

where the above sum is performed according to the condition

$$\{i_1, \ldots, i_l\} \cup \{i_{l+1}, \ldots, i_k\} = \{1, \ldots, k\}.$$

Moreover it follows that if $f \in C^\infty(G, \mathcal{Y})$, then $f \circ \pi \in C^\infty(G \times G, \mathcal{Y})$.

We mention that the formula from Theorem 1.1 is the corrected version of a formula that was indicated without any proof on [BCR81, page 46], and is in fact the generalization of the Faa di Bruno formula to topological groups (see [Jo02] for more details on that formula in the classical setting on $\mathbb{R}^n$).

Corollary 1.2. *If the topological group G is abelian and $\mathcal{Y}$ is any locally convex space, then the following assertions hold:*

(1) *The map $\pi\colon G \times G \to G$, $\pi(x, y) = xy$, is a morphism of topological groups.*
(2) *For every $f \in C^k(G, \mathcal{Y})$, and $k \ge 1$ one has*

$$D^k(f \circ \pi)((x, y); (\lambda_{11}, \lambda_{12}), \ldots, (\lambda_{k1}, \lambda_{k2})) = D^k f(xy; \lambda_{11} + \lambda_{12}, \ldots, \lambda_{k1} + \lambda_{k2})$$

where the sums $\lambda_{j1} + \lambda_{j2} \in \Lambda(G)$, for $j = 1, \ldots, k$, are understood in the sense of the equality (1.3).

Proof. Using Theorem 1.1, we obtain

$$\begin{aligned} & D^k(f \circ \pi)((x, y); (\lambda_{11}, \lambda_{12}), \ldots, (\lambda_{k1}, \lambda_{k2})) \\ & = \sum_{\ell=0}^{k} \sum_{\substack{i_1 < \cdots < i_\ell \\ i_{\ell+1} < \cdots < i_k}} D^k f(xy; \lambda_{i_1 2}, \ldots, \lambda_{i_\ell 2}, \lambda^y_{i_{\ell+1} 1}, \ldots, \lambda^y_{i_k 1}) \\ & = \sum_{\ell=0}^{k} \sum_{\substack{i_1 < \cdots < i_\ell \\ i_{\ell+1} < \cdots < i_k}} D^k f(xy; \lambda_{i_1 2}, \ldots, \lambda_{i_\ell 2}, \lambda_{i_{\ell+1} 1}, \ldots, \lambda_{i_k 1}) \\ & = D^k f(xy; \lambda_{11} + \lambda_{12}, \ldots, \lambda_{k1} + \lambda_{k2}) \end{aligned}$$

□

Corollary 1.3. *If the topological group G is a pre-Lie group, then the map $\pi\colon G \times G \to G$, $\pi(x, y) = xy$, is of class C^∞.*

Proof. We put

$$D\pi((x,y);(\alpha,\beta)) = \alpha^y + \beta$$

and

$$D^j\pi((x,y);(\alpha_1,\beta_1),\dots,(\alpha_j,\beta_j)) = [\dots[\alpha_1^y,\beta_j],\dots,\beta_2].$$

For every open set $X \subseteq G$, every locally convex space $\mathcal{Y}$, and every function $\varphi \in C^k(X,\mathcal{Y})$, if $j \leq k$ and $x,y \in G$ with $xy \in X \subseteq G$, then

$$\begin{aligned}
&D^j(\varphi\circ\pi)((x,y);(\alpha_1,\beta_1),\dots,(\alpha_j,\beta_j))\\
&\quad = \sum_{\ell=1}^{j} \sum_{\substack{i_1<\dots<i_\ell\\ i_{\ell+1}<\dots<i_j}} D^j\varphi(xy;\beta_{i_1},\dots,\beta_{i_l},\alpha^y_{i_{l+1}},\dots,\alpha^y_{i_j})\\
&\quad = \sum_{\ell=1}^{j} \sum_{\substack{i_1<\dots<i_\ell\\ i_{\ell+1}<\dots<i_j}} D^{\alpha^y_{i_j}}\dots D^{\alpha^y_{i_{l+1}}} D^{\beta_{i_l}}\dots D^{\beta_{i_1}}\varphi(xy)
\end{aligned}$$

and we thus obtain the chain rule (1.4). This shows that the map π is of class C^k and, since k is arbitrary, it follows that π is of class C^∞. □

2. Proof of Theorem 1.1

The proof of Theorem 1.1 is based on two lemmas that will be first proved and in whose statements we assume he setting of the theorem.

Lemma 2.1. *Let $f \in C^k(G,\mathcal{Y})$, $k \geq 1$, $\lambda_{1i_1},\lambda_{2i_2},\dots,\lambda_{ki_k} \in \Lambda(G)$, $i_1,\dots,i_k \in \{1,2\}$,$m = i_1+i_2+\dots+i_k-k$. The equality $i_l = 2$ has m solutions denoted $a_1 < \dots < a_m$. The equality $i_l = 1$ has $(k-m)$ solutions denoted $a_{m+1} < \dots < a_k$. Then we have*

$$\partial^{\lambda_{1i_1}\lambda_{2i_2}\dots\lambda_{ki_k}}(f\circ\pi)(x,y) = D^k f(xy;\lambda_{a_12},\dots,\lambda_{a_m2},\lambda^y_{a_{m+1}1},\dots,\lambda^y_{a_k1}).$$

Note that in the above statement, m is the number of occurrences of 2 in the set $\{i_1,\dots,i_k\}$.

Proof of Lemma 2.1. The proof will be by induction on $k \geq 1$. In the case $k=1$ we will prove the following two relations:

$$\partial^{\lambda_{11}}(f\circ\pi)(x,y) = Df(xy;\lambda^y_{11}) \text{ si } \partial^{\lambda_{12}}(f\circ\pi)(x,y) = Df(xy;\lambda_{12}).$$

We have

$$\begin{aligned}
\partial^{\lambda_{12}}(f\circ\pi)(x,y) &= \frac{d}{dt}\Big|_{t=0}(f\circ\pi)(x;y\lambda_{12}(t))\\
&= \frac{d}{dt}\Big|_{t=0} f(xy\lambda_{12}(t))\\
&= Df(xy;\lambda_{12}).
\end{aligned}$$

On the other hand

$$\begin{aligned}
\partial^{\lambda_{11}}(f\circ\pi)(x,y) &= \frac{d}{dt}\Big|_{t=0}(f\circ\pi)(x\lambda_{11}(t);y)\\
&= \frac{d}{dt}\Big|_{t=0} f(x\lambda_{11}(t)y)\\
&= \frac{d}{dt}\Big|_{t=0} f(xy\lambda^y_{11}(t))\\
&= Df(xy;\lambda^y_{11})
\end{aligned}$$

and the case $k = 1$ ends.

For passing from k to $k+1$, we calculate

$$\partial^{\lambda_{1i_1}\lambda_{2i_2}\ldots\lambda_{ki_k}\lambda_{k+1,i_{k+1}}}(f\circ\pi)(x,y) = \partial^{\lambda_{k+1,i_{k+1}}}(\partial^{\lambda_{1i_1}\lambda_{2i_2}\ldots\lambda_{ki_k}}(f\circ\pi))(x,y).$$

We have two cases: $i_{k+1} = 1$ or $i_{k+1} = 2$.

• The case $i_{k+1} = 1$. In this case $i_1+i_2+\ldots+i_k+i_{k+1}-(k+1) = i_1+i_2+\ldots+i_k-k = m$ therefore m remains unchanged by passing from k to $k+1$. We have

$$\begin{aligned}
&\partial^{\lambda_{1i_1}\ldots\lambda_{ki_k}\lambda_{k+1,i_{k+1}}}(f\circ\pi)(x,y)\\
&\qquad = \frac{d}{dt}\Big|_{t=0}\partial^{\lambda_{1i_1}\ldots\lambda_{ki_k}}(f\circ\pi)(x\lambda_{k+1,1}(t),y)\\
&\qquad = \frac{d}{dt}\Big|_{t=0}D^k f(x\lambda_{k+1,1}(t)y;\lambda_{a_12},\ldots,\lambda_{a_m2},\lambda^y_{a_{m+1}1},\ldots,\lambda^y_{a_k1})\\
&\qquad = D^{k+1}f(xy;\lambda_{a_12},\ldots,\lambda_{a_m2},\lambda^y_{a_{m+1}1},\ldots,\lambda^y_{a_k1},\lambda^y_{k+1,1})
\end{aligned}$$

and the case $i_{k+1} = 1$ ends.

• The case $i_{k+1} = 2$. In this case we have

$$i_1+i_2+\ldots+i_k+i_{k+1}-(k+1) = i_1+i_2+\ldots+i_k-k+1 = m+1.$$

We have

$$\begin{aligned}
&\partial^{\lambda_{1i_1}\ldots\lambda_{ki_k}\lambda_{k+1,i_{k+1}}}(f\circ\pi)(x,y)\\
&\quad = \frac{d}{dt}\Big|_{t=0}\partial^{\lambda_{1i_1}\ldots\lambda_{ki_k}}(f\circ\pi)(x;y\lambda_{k+1,2}(t))\\
&\quad = \frac{d}{dt}\Big|_{t=0}D^k f(xy\lambda_{k+1,2}(t);\lambda_{a_12},\ldots,\lambda_{a_m2},\lambda^{y\lambda_{k+1,2}(t)}_{a_{m+1}1},\ldots,\lambda^{y\lambda_{k+1,2}(t)}_{a_k1})
\end{aligned}$$

which must be equal with

$$D^{k+1}f(xy;\lambda_{a_12},\ldots,\lambda_{a_m2},\lambda_{k+1,2},\lambda^y_{a_{m+1}1},\ldots,\lambda^y_{a_k1}).$$

It is enough to prove the above relation for $y = \mathbf{1}\in G$. For abitrary $s\in\mathbb{R}$ we define $g_s:\mathbb{R}^k\to\mathcal{Y}$ by

$$\begin{aligned}
&g_s(t_1,\ldots,t_k)\\
&\quad := f(y\lambda_{k+1,2}(s)\lambda^{y\lambda_{k+1,2}(s)}_{a_k1}(t_1)\ldots\lambda^{y\lambda_{k+1,2}(s)}_{a_{m+1}1}(t_{k-m})\lambda_{a_m2}(t_{k+1-m})\ldots\lambda_{a_12}(t_k))\\
&\quad = f(\lambda_{a_k1}(t_1)\ldots\lambda_{a_{m+1}1}(t_{k-m})y\lambda_{k+1,2}(s)\lambda_{a_m2}(t_{k+1-m})\ldots\lambda_{a_12}(t_k)).
\end{aligned}$$

We define $h:\mathbb{R}^{k+1}\to\mathcal{Y}$ by

$$\begin{aligned}
&h(t_1,\ldots,t_k,t_{k+1})\\
&\quad = f(y\lambda^y_{a_k1}(t_1)\ldots\lambda^y_{a_{m+1}1}(t_{k-m})\lambda_{k+1,2}(t_{k+1-m})\lambda_{a_m2}(t_{k+2-m})\ldots\lambda_{a_12}(t_{k+1}))\\
&\quad = f(\lambda_{a_k1}(t_1)\ldots\lambda_{a_{m+1}1}(t_{k-m})y\lambda_{k+1,2}(t_{k+1-m})\lambda_{a_m2}(t_{k+2-m})\ldots\lambda_{a_12}(t_{k+1})).
\end{aligned}$$

We have $h\in C^{k+1}(\mathbb{R}^{k+1},\mathcal{Y})$, $g_s\in C^k(\mathbb{R}^k,\mathcal{Y})$, and the connection between these functions is

$$g_s(t_1,\ldots,t_k) = h(t_1,\ldots,t_{k-m},s,t_{k-m+1},\ldots,t_k).$$

The requested relation is equivalent to

$$\frac{d}{ds}\Big|_{s=0}\frac{\partial^k g_s}{\partial t_1\ldots\partial t_k}(0,0,\ldots,0) = \frac{\partial^{k+1}h}{\partial t_1\ldots\partial t_k\partial t_{k+1}}(0,0,\ldots,0).$$

For $t := (t_1, \dots, t_k)$ we sequentially have the relations

$$\frac{\partial g_s}{\partial t_k}(t) = \frac{\partial h}{\partial t_{k+1}}(t_1, \dots, t_{k-m}, s, t_{k-m+1}, \dots, t_k)$$

$$\frac{\partial^m g_s}{\partial t_{k-m+1} \dots \partial t_k}(t) = \frac{\partial^m h}{\partial t_{k-m+2} \dots \partial t_{k+1}}(t_1, \dots, t_{k-m}, s, t_{k-m+1}, \dots, t_k)$$

$$\frac{\partial^{m+1} g_s}{\partial t_{k-m} \dots \partial t_k}(t) = \frac{\partial^{m+1} h}{\partial t_{k-m}\partial t_{k-m+2} \dots \partial t_{k+1}}(t_1, \dots, t_{k-m}, s, t_{k-m+1}, \dots, t_k)$$

$$\frac{\partial^k g_s}{\partial t_1 \dots \partial t_k}(t) = \frac{\partial^k h}{\partial t_1 \dots \partial t_{k-m}\partial t_{k-m+2} \dots \partial t_{k+1}}(t_1, \dots, t_{k-m}, s, t_{k-m+1}, \dots, t_k)$$

$$\frac{\partial^k g_s}{\partial t_1 \dots \partial t_k}(0, 0, \dots, 0) = \frac{\partial^k h}{\partial t_1 \dots \partial t_{k-m}\partial t_{k-m+2} \dots \partial t_{k+1}}(0, \dots, 0, s, 0, \dots, 0).$$

It follows that

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0} \frac{\partial^k g_s}{\partial t_1 \dots \partial t_k}(0, \dots, 0) &= \frac{d}{ds}\Big|_{s=0} \frac{\partial^k h}{\partial t_1 \dots \partial t_{k-m}\partial t_{k-m+2} \dots \partial t_{k+1}}(0, \dots, 0, s, 0, \dots, 0) \\ &= \frac{\partial^{k+1} h}{\partial t_{k-m+1}\partial t_1 \dots \partial t_{k-m}\partial t_{k-m+2} \dots \partial t_{k+1}}(0, \dots, 0) \\ &= \frac{\partial^{k+1} h}{\partial t_1 \dots \partial t_k \partial t_{k+1}}(0, \dots, 0)\end{aligned}$$

and this completes the proof by induction. □

Lemma 2.2. *Let G_1, G_2 topological groups and X an open set from $G_1 \times G_2$ and $h \in C^k(X, \mathcal{Y})$, $k \geq 1$. Then the partial derivatives of order $\leq k$ of h are continuous from X to $\mathcal{Y}$ and we have the relation*

$$D^k h((x, y); (\lambda_{11}, \lambda_{12}), \dots, (\lambda_{k1}, \lambda_{k2})) = \sum_{i_1, \dots, i_k = 1,2} \partial^{\lambda_{1i_1}\lambda_{2i_2}\dots\lambda_{ki_k}} h(x, y)$$

for every $(x, y) \in X$.

Proof. The case $k = 1$. We must show that

$$Dh((x, y); (\lambda_{11}, \lambda_{12})) = \partial^{\lambda_{11}} h(x, y) + \partial^{\lambda_{12}} h(x, y).$$

We have

$$\partial^{\lambda_{11}} h(x, y) = \frac{d}{dt}\Big|_{t=0} h(x\lambda_{11}(t), y) = Dh((x, y); (\lambda_{11}, 0))$$

therefore the function $\partial^{\lambda_{11}} h : X \to \mathcal{Y}$ is continuous. On the other hand

$$\partial^{\lambda_{12}} h(x, y) = \frac{d}{dt}\Big|_{t=0} h(x, y\lambda_{11}(t)) = Dh((x, y); (0, \lambda_{12}))$$

therefore the function $\partial^{\lambda_{12}} h : X \to \mathcal{Y}$ is continuous as well.

Moreover, $(\lambda_{11}, 0)(t)(0, \lambda_{12})(t) = (\lambda_{11}(t), \lambda_{12}(t)) = (\lambda_{11}, \lambda_{12})(t)$. From

$$Dh((x, y); (\lambda_{11}, \lambda_{12})) = Dh((x, y); (\lambda_{11}, 0)) + Dh((x, y); (0, \lambda_{12}))$$

we get

$$Dh((x, y); (\lambda_{11}, \lambda_{12})) = \partial^{\lambda_{11}} h(x, y) + \partial^{\lambda_{12}} h(x, y)$$

and the case $k = 1$ ends.

The case $k = 2$. We must show that

$$\begin{aligned} D^2h((x,y);(\lambda_{11},\lambda_{12}),(\lambda_{21},\lambda_{22})) =& \partial^{\lambda_{11}\lambda_{21}}h(x,y) + \partial^{\lambda_{11}\lambda_{22}}h(x,y) \\ &+ \partial^{\lambda_{12}\lambda_{21}}h(x,y) + \partial^{\lambda_{12}\lambda_{22}}h(x,y). \end{aligned}$$

We have

$$\begin{aligned} \partial^{\lambda_{11}\lambda_{21}}h(x,y) &= \partial^{\lambda_{11}}(\partial^{\lambda_{21}}h)(x,y) \\ &= \frac{d}{dt}\Big|_{t=0} \partial^{\lambda_{11}}h(x\lambda_{21}(t),y) \\ &= \frac{d}{dt}\Big|_{t=0} Dh(x\lambda_{21}(t),y);(\lambda_{11},0)) \\ &= D^2h((x,y);(\lambda_{11},0),(\lambda_{21},0)). \end{aligned}$$

Similarly we get

$$\begin{aligned} \partial^{\lambda_{11}\lambda_{22}}h(x,y) &= D^2h((x,y);(\lambda_{11},0),(0,\lambda_{22})) \\ \partial^{\lambda_{12}\lambda_{22}}h(x,y) &= D^2h((x,y);(0,\lambda_{12}),(0,\lambda_{22})) \\ \partial^{\lambda_{12}\lambda_{21}}h(x,y) &= D^2h((x,y);(0,\lambda_{12}),(\lambda_{21},0)). \end{aligned}$$

The above relations imply that the 2nd oder partial derivatives of h are continuous.

We have

$$\begin{aligned} D^2h((x,y);&(\lambda_{11},\lambda_{12}),(\lambda_{21},\lambda_{22})) \\ &= D^2h((x,y);(\lambda_{11},\lambda_{12}),(\lambda_{21},0)) + D^2h((x,y);(\lambda_{11},\lambda_{12}),(0,\lambda_{22})) \\ &= D^2h((x,y);(\lambda_{11},0),(\lambda_{21},0)) + D^2h((x,y);(0,\lambda_{12}),(\lambda_{21},0)) \\ &\quad + D^2h((x,y);(\lambda_{11},0),(0,\lambda_{22})) + D^2h((x,y);(0,\lambda_{12}),(0,\lambda_{22})) \\ &= \partial^{\lambda_{11}\lambda_{21}}h(x,y) + \partial^{\lambda_{11}\lambda_{22}}h(x,y) + \partial^{\lambda_{12}\lambda_{21}}h(x,y) + \partial^{\lambda_{12}\lambda_{22}}h(x,y) \end{aligned}$$

and the case $k = 2$ ends.

The case $k \geq 3$. We have

$$\partial^{\lambda_{1i_1}\lambda_{2i_2}\dots\lambda_{ki_k}}h(x,y) = D^kh((x,y);\gamma_1,\dots,\gamma_k)$$

where $\gamma_j = (\lambda_{j1},0)$ if $i_j = 1$, while $\gamma_j = (0,\lambda_{j2})$ if $i_j = 2$. It follows by the above relations that the partial derivatives of order k of the function h are continuous.

As in the case $k = 2$ we have

$$\begin{aligned} D^kh((x,y);&(\lambda_{11},\lambda_{12}),\dots,(\lambda_{k-1,1},\lambda_{k-1,2}),(\lambda_{k1},\lambda_{k2})) \\ &= D^kh((x,y);(\lambda_{11},\lambda_{12}),\dots,(\lambda_{k-1,1},\lambda_{k-1,2}),(\lambda_{k1},0)) \\ &\quad + D^kh((x,y);(\lambda_{11},\lambda_{12}),\dots,(\lambda_{k-1,1},\lambda_{k-1,2}),(0,\lambda_{k2})) \\ &= D^kh((x,y);(\lambda_{11},\lambda_{12}),\dots,(\lambda_{k-1,1},0),(\lambda_{k1},0)) \\ &\quad + D^kh((x,y);(\lambda_{11},\lambda_{12}),\dots,(0,\lambda_{k-1,2}),(\lambda_{k1},0)) \\ &\quad + D^kh((x,y);(\lambda_{11},\lambda_{12}),\dots,(\lambda_{k-1,1},0),(0,\lambda_{k2})) \\ &\quad + D^kh((x,y);(\lambda_{11},\lambda_{12}),\dots,(0,\lambda_{k-1,2}),(0,\lambda_{k2})) \\ &= \sum_{i_1,\dots,i_k=1,2} \partial^{\lambda_{1i_1}\lambda_{2i_2}\dots\lambda_{ki_k}}h(x,y) \end{aligned}$$

and this completes the proof. □

Proof of Theorem 1.1. Using Lema 2.2, we obtain

$$D^k(f\circ\pi)((x,y);(\lambda_{11},\lambda_{12}),\dots,(\lambda_{k1},\lambda_{k2})) \quad = \sum_{i_1,\dots,i_k=1,2} \partial^{\lambda_{1i_1}\lambda_{2i_2}\dots\lambda_{ki_k}}(f\circ\pi)(x,y).$$

B replacing the 2^k partial derivatives from the right hand side by their values provided by Lemma 2.1 we obtain the sum from the requested relation. □

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