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# Axiomatization and local theorem for certain classes of loops

**Vasile I. Ursu**

***Abstract**: in class of loops, it is proved local theorem for properties $RN$, $RI$, $Z$, $\overline{RN}$, $\overline{RI}$, $\overline{Z}$, $\overline{N}$, ordering and free ordering, using the language of mathematical logic.*

***Key words**: loop system, subnormal, classes, local, axiom, quasiuniversal, ordonable.*

The group investigations which have infinite normal or infinite subnormal rows of certain types lead to the appearance of some new groups, called Kurosh-Chernikov groups classes. Normally, analogical classes appear in the theory of loops, which we'll call Kurosh-Chernikov as well. In this work it is indicated that the classes of *RN-, RI-, Z-* loops can be axiomatizated by means of certain universal formulas, but the $\overline{RN}$ -, $\overline{RI}$ -, $\overline{Z}$ -, $\ddot{N}$ - loops classes – by means of certain quasiuniversal formulas and the local theorem is proved for these loops and also for ordinable or ordinable free loops. Combining the properties *RN, RI* and others provided to be connected of the subloops in their deffinition, one can obtain new local theorems for ordonable partial loops.

1. **Deffinitions and notations.** It will be self-understood a $L$ algebra through the loop with a multiplication opperation · and two devision opperations /, \, where the element $e \in L$ is being existed, so that for any elements $x, y \in L$ the next egualities are set:

$$ex = xe = x; \quad (xy)/y = y\backslash(yx) = (x/y)y = y(y\backslash x) = x.$$

The element $e$ will represent the *unit element* of the loop further on.

Let $L$ be a loop. The inner mapping group $J(L)$ of loop $L$ is the group generated by all the substitutions of form

$$R_{x,y} = R_x R_y R_{xy}^{-1}, \quad L_{x,y} = L_y L_x L_{xy}^{-1}, \quad T_x = R_x L_x^{-1}, \text{ where } xR_y = yL_x = x \cdot y \quad (x, y \in L).$$

The set

$$Z(L) = \{a \in L \mid ax \cdot y = a \cdot xy, x \cdot ya = xy \cdot a, ax = xa \text{ for any } x, y \in L\}$$

is being called the *centre* of the loop $L$. A subset $H \subseteq L$ closed in terms with the loop $L$ operations is called the subloop of $L$. Subloop $H$ of loop $L$ is called normal in $L$ if for any $x, y \in L$ are true the equalities

$$x \cdot H = H \cdot x,\ x \cdot yH = xy \cdot H,\ Hx \cdot y = H \cdot xy,$$

which is equivalent with $H^{\varphi} = H$ for any $\varphi \in J(L)$ (see [1]-[3]). The commutator $[x, y]$ of the elements $x, y \in L$, the left asociator $[x, y, z]$ and the right associator $(x, y, z)$ of the elements $x, y, z \in L$ are being defined through the following equalities

$$[x, y] = x/(y/xy),\quad [x,y,z] = ((xy)\backslash(x \cdot yz))/z,\quad (x,y,z) = x\backslash((xy \cdot z)/(yz)).$$

For any subloop $H$ of the loop $L$ through $[H, L]$ we note the subloop in $L$ generated by set

$$\{[x, y],[z, y, x],(x, y, z) \mid x \in X,\ y, z \in L\}.$$

Particularlysubloop, $[L, L]$ contains all the commutators and associators of loop $L$ and is called associant-commutant of the loop $L$. A loop $L$ is called rezoluble if it does exists a natural number $m$ so that $L^{(m)} = \{e\}$, where $L^{(m)}$ is defined in the following way $L^{(0)} = L,\ L^{(m)} = [L^{(m-1)}, L^{(m-1)}]$ for $m=1,2, \ldots$ .

The set of symbols which makes the language of the predicational calculus (CP) is made of objects and variable predicate, brackets, logical connectors ( $\wedge, \vee, \neg, \rightarrow, =$) and existentional ($\exists$) and universal ($\forall$) quantificators. It is known that any formula of CP is equivalent to a formula, of which quantificators are placed before the logical symbols and it is called *prenex*. A prenex formula is called *universal* if it doesn't contain the quantificator $\exists$ and it is called *universal-object* if it doesn't contain the quantificators $\exists$ which is refered to variable objects (see [4]). CP formula is called quasiuniversal if it has form $(\forall r_1)\ldots(\forall r_n)\Phi$, where $\Phi$ is a formula formed by means of the logical connectors $\wedge$, $\vee$, $\neg$, $\rightarrow$ from the universal-objectional formulas which doesn't contain free variable objects but they contain free variable predicate only from the $\{r_1, \ldots, r_n\}$ set (in this form the deffinition is given [5] or [6].

The totality of subsets $A_i, i \in I$ of the set $A$ is called *local covering* if $A = \bigcup_{i \in I} A_i$ and for any $i \in I$ there is a $k \in I$ so that $A_i \cup A_j \subseteq A_k$.

**2. Loop classes.** It is known that a loop is rezoluble (nilpotent) if it has a finite subnormal (normal) row of subloops of wich the factors are abelien groups (centrale). The group researches which have infinite subnormal rows or infinite normal of certain types lead to the appearance of certain new classes of groups called Kourosh-Chernikov group classes. Normally analogical classes appear in the loop theory which we'll call as well Kourosh-Chernikov. Further on we'll show the definitions and notations of these classes.

The base notion of Kourosh-Chernikov classes theory is the subnormal system, not indespensable finite.

System *S* of subloops of the loop *L* is called subnormal if it:

1) contains the unitary subloop $\{e\}$ and the loop *L* itself;

2) is ordonated linear related to the inclusion, namely for any $A, B \in S$ or $A \subseteq B$;

3) is closed related to the reunion and intersection; particularly, together with any subloop $A \neq L$ it contains $\overset{\sqcup}{A}$ intersection of the all $H \in S$ for which $A \subset H$ and $A$ reunion of all $N \in S$ for which $N \subset A$;

4) checks the condition: *A* is a normal subloop of the loop $\overset{\sqcup}{A}$ for any $A \in S$.

Factor-loop $\overset{\sqcup}{A}/A$ is called factor of *S* subnormal system. It is said that the system *S* is *wholly ascendent ordonated* if $\overset{\sqcup}{A} \neq A$ for any $A \neq L$, and *wholly descendent ordonated* if $A \neq A$ for any $A \neq \{e\}$ $(A \in S)$.

*S* subnormal system is called *normal* if all its terms are normal subloops of loop *L*. If a subnormal system contains another subnormal system of the same loop, it means the first is called the *refining* of the second one. A subnormal system is called *rezoluble* if all its factors are abeliene groups. The system *S* is called *central* if all its factors are central, that is

$$\overset{\sqcup}{A}/A \subseteq Z(L) \text{ for any } A \neq L$$

or the equivalent of this one

$$[\overset{\sqcup}{A}, L] \subseteq A \text{ for any } A \neq L \ \ (A \in S).$$

Subloops $A$ and $B$ of the subnormal system $S$ we'll call *neighbours* in $S$ if $A \subset B$ and there is no the third subloop of $S$ between them, that is for any $C \in S$ the double inclusion $A \subseteq C \subseteq B$ involves $A$=$C$ or $C$=$B$.

A class of loops is called *Kourosh-Chernikov* if its loops checks one of the following properties (we'll keep the traditional notations as for groups):

$RN$ : the loop has a rezoluble subnormal system;

$RN^*$ : the loop has a rezoluble subnormal system entirely ascendent ordonable;
$\overline{RN}$ : any subnormal system of the loop can be refined up to a rezoluble subnormal system;
$RI$ : the loop has a rezoluble normal system;
$RI^*$ :the loop has a rezoluble normal system entirely ascendent ordonable;
$\overline{RI}$ : any normal system can be refined to a rezoluble normal system;
$Z$ : the loop has a central system;
$ZA$ : the loop has a central system entirely ascendent ordonable;
$ZD$ : the loop has a central system entirely descendent ordonable;
$\overline{Z}$ : any normal system can be refined up to a central system;
$\widetilde{N}$ : any subloop belongs to a subnormal system;
$N$ : any subloop belongs to a subnormal system entirely ascendent ordonable.

The Kourosh-Chernikov loop classes are enough large. For example, loops-$Z$ class owns all the free loops. It results from Higman theorem [7], according to which any free loop admits a central system. A very important example of loops-$RN$ is that one of the ordonated diasociative $A$-loops of which the internal substitutions are automorphisms. The loop of which every internal substitution is automorphism is called A-*loop* (see [8]).

Really, let $L$ be an ordonated $A$-loop with linear ordination $\leq$. The subset $H$ of $L$ is called *conex* if for any $a, b \in H$ elements and any $c \in L$ from the double inequality $a \leq c \leq b$ results $c \in H$. We show that totality $S$, made from all the connex loops of the $A$-loop $L$, is a rezoluble subnormal subsystem. Really, it is evidently the condition 1) from the subnormal system definition. Now we check the condition 2). Let $H$ and $K$ two normal subloops of the $L$ $A$-loop for which it exists an element $h \in H$ so that $h \notin K$. Let's suppose $h > c$ ($h < c$ case is being investigated analogically). From the property of connexity of $H$ subloop it results that for any $x \in K$ takes place $c \leq x \leq h$ or $c \leq x^{-1} \leq h$. From where it results $x \in H$. Consequently, $K \subseteq H$. The condition 3) is evident. Let's check the condition 4), that is, if $H \subset K$ and there is no any connex subloop between them, that is connex subloop $H$ is normal in connex subloop $K$. For any intern substitution $\varphi$ of $A$-loop $H$ we have $K^{\varphi} \subseteq H^{\varphi}$ and, in the same time, it is clearly that $K^{\varphi}$, $H^{\varphi}$ there are also connex subloops and there are no other connex subloops between them. Because $H^{\varphi}=H$ we have $K^{\varphi}=K$. So, *K* is invariant related to any intern substitution of *H* subloop and therefore *K* is a normal subloop of loop *H*.

Since factor loop $H/K$ is ordonable and doesn't contain proper connex subloops, rezults that $H/K$ is arhimedian (that is for any $x, y \in H/K$ with $x > e$ and $y > e$ there is a natural number *n* so that $x^n > y$). Then factor loop $H/K$ is abelian group. Therefore we can conclude that the normal subloop system of diasociativ *A*-loop, *L* is rezoluble, this means that *L* is a *RN*-loop.

Further on we'll investigate only Kourosh-Cernikov classes of $RN$—, $\overline{RN}$—, $RI$—, $\overline{RI}$—, $Z$—, $\overline{Z}$—, $\widetilde{N}$—loops.

**3. The local theorem.** A.I.Mal'cev's theorem states: *if an universal or quasiuniversal formula is true on any subsystem of a local covering of an algebric system, then it is true and on this algebric system.*

As it is known (see [9], [10]) for groups the $RN$, $RI$, $Z$, $\overline{RN}$, $\overline{RI}$, $\overline{Z}$, $\widetilde{N}$ properties satisfy the local theorem. A more moderne demonstration of this theorem in mathematical logical language is offered in [11] and in topological language is offered in [12]. For loops we'll demonstrate the local theorem in the mathematical logical language.

**Theorem.** *In the loop class* $RN$, $RI$, $Z$ *properties can be stated by universal formulae and* $\overline{RN}$, $\overline{RI}$, $\overline{Z}$, $\widetilde{N}$ *properties can be stated by quasiuniversal formulae, but that's why for all these properties the local theorem is true.*

*Demonstration.* On thebasis of the procedure applicated by A.I.Mal'tsev for groups, we'll pass from the subnormal system language of a loop *L* to the predicational language. For each subnormal system *S* of subloops of *L* loop one distribute a binary predicate $r_s = r$ defined on *L* so that: $xry$ is true if and only if it is subloop $H \in S$ so that $x \in H$ and $y \notin H$. It is clearly that $r$ satisfies the following universal axioms

1) $(\forall x)\, x\bar{r}x$;
2) $(\forall x)(\forall y)(xry \,\&\, yrz \rightarrow xrz)$;
3) $(\forall x)(\forall y)(xrz \,\&\, y\bar{r}z \rightarrow xry)$;
4) $(\forall x)(\forall y)(\forall z)(xrz \,\&\, yrz \rightarrow x \cdot yrz \,\&\, x/yrz \,\&\, y\backslash xrz)$;
5) $(\forall x)(\neg(x = e) \rightarrow erx)$;
6 a) $(\forall x)(\forall y)(xry \rightarrow [x, y]ry)$;
b) $(\forall x)(\forall y)(\forall z)(xry \,\&\, (xrz \vee x\bar{r}z) \rightarrow (x, y, z)ry \,\&\, (x, z, y)ry)$;
c) $(\forall x)(\forall y)(\forall z)(xry \,\&\, (xrz \vee x\bar{r}z) \rightarrow [y, z, x]ry \,\&\, [z, y, x]ry)$.

The condition (6)=(a)&(b)&(c) is equivalent to the fact that for any pair of $A \subset B$ neighbour subloop in *S*, *A* subloop is normal in *B* loop.

Conversely, let on $L$ loop definit $r$ predicate with the properties 1)-6). We're noting with $ry$ the totality of all those elements $x \in L$ for which $xry$ is true. According to (4) condition every $ry$ subset is a subloop of $L$ loop. It is easy to check that

$$ry = \bigsqcup_{y \in H \in S} H, \quad H = \bigcap_{y \in H} ry \quad (H \in S, H \neq L),$$

this means that $ry \in S$ and every $H \in S, H \neq L$ is reprezented as an intersection of certain $ry$ subloops of loop $L$. The set system of the forme $ry(y \in L)$ make an ordonated complete system of subloops related to the inclusion. If we adjust to this system the intersection and reunion of any subset number out of it, so we obtain $S$ subnormal system of subloops. According to 1)-3) it results that $r_s = r$. So we obtained that the totality of subnormal systems of loop $L$ and the totality of binare predicate which are checking the conditions 1)-6) are into a biunivocal correspondation.

We can notice that subnormal system $S_1$ is a rafination of subnormal system $S$ of the same loop, if and only if $r_1$ and $r$ adequate predicates check $xry \to xr_1y$ axiom.

The axiom

7) $\neg(x = x/x) \,\&\, x\bar{r}y \,\&\, x\bar{r}z \,\&\, y\bar{r}x \,\&\, z\bar{r}x \to [x,y]rx \,\&\, (x,y,z)rx \,\&\, [y,z,x]rx$

is equivalent on condition that factor-loop of two neighbours subloops from a subnormal system is abelian group.

The axiom

8) $xry \to [x,z]ry \,\&\, (x,z,t)ry \,\&\, [z,t,x]ry$

is equivalent on condition that all the subloops of the subnormal system to be central subloops of loop $L$.

According to the definition $CP$ formula doesn't contain operational symbols. Further on we'll consider that the equality relations $x \cdot y = z, x/y = z, x \backslash y = z$ are respectively ternary predicates $q(x,y,z)$, $p(x,y,z)$, $t(x,y,z)$. Then all axioms 1)-8) can be writen as $CP$ formulae. For example axiom 6a) has the predicational form

$$(\exists p)(\exists q)(\exists t)(\forall x)(\forall y)(\forall t)(\forall u)(xry \,\&\, p(x,y,z) \,\&\, q(y,t,z) \,\&\, t(x,z,t) \to try).$$

Now the properties of loop $L$ of being *RN-*, *RI-*, *Z-* loop can be stated accordingly through universal-objectual formulae

*RN*: $(\exists r)((1) \,\&\, (2) \,\&\, (3) \,\&\, (4) \,\&\, (5) \,\&\, (6) \,\&\, (7))$;
*RI*: $(\exists r)((1) \,\&\, (2) \,\&\, (3) \,\&\, (4) \,\&\, (5) \,\&\, (6) \,\&\, (7) \,\&\, (8))$;

$$Z: \quad (\exists r)((1) \,\&\, (2) \,\&\, (3) \,\&\, (4) \,\&\, (5) \,\&\, (6) \,\&\, (8)) \ ,$$

where (1)-(8) are expressions from 1)-8). It is clear that *RN, RI* and *Z* axioms are universal-object formulae.

We'll investigate now $\overline{RN}$ -, $\overline{RI}$ -, $\overline{Z}$ -, $\widetilde{N}$ -loops. We respectively note with $\alpha_1(r), \alpha_2(r), \alpha_3(r)$ the part without the quantifier $(\exists r)$ of the *RN, RI, Z* expressions. We consider

$$\alpha(r) = (1) \,\&\, (2) \,\&\, (3) \,\&\, (4) \,\&\, (5) \,\&\, (6),$$

$$\beta(r) = (1) \,\&\, (2) \,\&\, (3) \,\&\, (4) \,\&\, (5) \,\&\, (7) \, .$$

Then the loop property of being $\overline{RN}$ -, $\overline{RI}$ - or $\overline{Z}$ - loop can be stated through the axioms

$$\overline{RN}: \quad (\forall r)(\alpha(r) \rightarrow (\exists q)(\alpha_1(q) \,\&\, (\forall u)(\forall v)(urv \rightarrow uqv)) \ ,$$
$$\overline{RI}: \quad (\forall r)(\beta(r) \rightarrow (\exists q)(\alpha_2(q) \,\&\, (\forall u)(\forall v)(urv \rightarrow uqv)) \ ,$$
$$\overline{Z}: \quad (\forall r)(\beta(r) \rightarrow (\exists q)(\alpha_3(q) \,\&\, (\forall u)(\forall v)(urv \rightarrow uqv)) \ .$$

It is evident that the obtained axioms are quasiuniversal formulae. Consequently, $\overline{RN}$ -, $\overline{RI}$ -, $\overline{Z}$ - loops make up quasiuniversal subclasses of the loop class.

For writing the definition of *N*-loop we must know the satisfied condition of *r* predicate for characterizing in *CP* language the systems of the $\{e\} \subseteq L_1 \subseteq L$ form, where $L_1$ is any subloop of loop *L*. It is evident that it is necessarily to complete the 1)-5) axioms on condition

$$9) \ (\forall x)(\forall y)(\neg(x = x/x) \,\&\, xry \rightarrow y\bar{r}z) \ .$$

We note

$$\gamma(r) = (1) \,\&\, (2) \,\&\, (3) \,\&\, (4) \,\&\, (5) \,\&\, (7) \ .$$

Then the property of being $\widetilde{N}$ —loop can be represented by the following form:

$$\widetilde{N}: \quad (\forall r)(\gamma(r) \rightarrow (\exists q)(\alpha_1(q) \,\&\, (\forall u)(\forall v)(urv \rightarrow uqv)) \ .$$

This axiom is again a quasiuniversal formula. Consequently, $\widetilde{N}$ - loops make up a quasiuniversal subclass in the class of all the loops.

Theorem is proved.

From the theorem demonstration the following results

**Corollary.** *In RN-, RI-, Z -loop class the loops make up universal subclasses and* $\overline{RN}$ -, $\overline{RI}$ -, $\overline{Z}$ *-loops make up quasiuniversal subclasses of the loop class.*

Further on we'll show other class examples of universal and quasiuniversal loops. The binar relation $r$ defined on a set $L$ is called *ordonated partialy* if it takes place:

$(\forall x)(xrx)$ - reflexivity,
$(\forall x)(\forall y)(xry \,\&\, yrx \longrightarrow x = y)$ - antisimetria,
$(\forall x)(\forall y)(\forall z)(xry \,\&\, yrz \longrightarrow xrz)$ - transitivity.
If it occurs $(\forall x)(\forall y)(xry \;\; xyrx)$ linearity then relation $r$ is called *ordonated linear* on $L$. The predicate $r$ defined on loop $L$ is called *stable related to multiplication*, if

$$(\forall x)(\forall y)(\forall z)(xry \longrightarrow x \cdot zry \cdot z) \,\&\, z \cdot xrz \cdot y .$$

The loop is called *ordonable* if it can be defined on it a stable linear order related to multiplication. The loop in which any partial order stable in terms with multiplication, can be continued up to the linear order, stable in terms with multiplication, we'll call *ordonable free*.

Let $\alpha(r)$ be formula

$$xrx \,\&\, (xry \,\&\, yrx \to x = y) \,\&\, (xry \,\&\, yrz \to xrz) \,\&\, (xry \to x \cdot zry \cdot z \,\&\, t \cdot xrt \cdot y),$$

but $\beta(r)$ the conjunction of $u(r)$ and $xry \vee yrx$ formula. Then the property of the loop of being ordonable can be stated by the following axiom

$$(\exists r)(\forall x)(\forall y)(\forall z)(\forall t)\,\beta(r),$$

but the loop property of being ordonable linear free can be represented by the formula

$$(\exists r)(\forall x)(\forall y)(\forall z)(\forall t)(\beta(r) \longrightarrow (\exists q)(\beta(q) \,\&\, xry \longrightarrow xqy)) .$$

So the loop property of being ordonable can be stated through a universal formula but the loop property of being ordonable free can be stated through a quasiuniversal formula. Consequently, *ordonable loops make a universal subclass of the loop class, but ordonable free loops make a quasiuniversal subclass.* According, to as above it results that *loop properties of being ordonable or ordonable free satisfy the local theorem*.

4. **Observations. a)** The above speeches show the connexion that exists between the loops of *RN-, RI-* classes and others, and the class loops of ordonable and ordonable free loops. This connexion can be more pronounced if instead of $r$ predicate we'll investigate the negation $\neg r$ or $\bar{r}$. Really $q = \bar{r}$, we can that 1), 2) and 3) axioms to present by the form

$$(\forall x)(xqx),\ (\forall x)(\forall y)(\forall y)\,(xqy \,\&\, yqz \longrightarrow xqz),\ (\forall x)(\forall y)(xqy \vee yqx),$$

and then $q$ predicate defines a quasiorder. Then the loops from the mentioned classes are quasiordonable loops of type *RN, RI, ...* and the quasiorde of which check certain written properties by the form of universal and quasiuniversal formulae.

**b)** Other properties of the loops for which the local theorem is true can be obtained through the deffinition of the notions of ordonability and free ordonability for the loop of the type $RN\neg$, $\overline{RN}$ etc, for which is natural to suppose that the subloops from the linear complet ordonated system for them to be connex. The above descriptions show that the obtained classes in this way are axiomatizated classes of universal or quasiuniversal formulae.

**c)** An eloquent example that illustrates the importance of the local theorem is its application to the demonstration of the following affirmation:

*If in loop L each generated-finit subloop contains an associative (or commutative) normal subloop of* $\leq n$ *indexes then loop L itself contains an associative (or commutative) normal subloop of* $\leq n$ *indexes.*

Indeed, let $H_i$, $i \in S$ system of generated-finite subloops be a local covering for loop *L*. We fix the family of associative subloops (respective, commutative) normale $A_i \subseteq H_i$, $i \in I$ which satisfy the theorem condition. For each associative (respective, commutative) subloop $A_i$ we annex binar predicate $r_i(x, y)$ defined in a such way:

$$H_i \models r_i(x, y) \Leftrightarrow xA_i = yA_i .$$

We notice that predicate $r_i$ satisfies the properties:

$$\varphi_1(x,y,z;r_i) = (\forall x)(\forall y)(\forall z)(r_i(x,x) \,\&\, (r_i(x,y) \rightarrow r_i(y,x)) \,\&\, (r_i(x,y) \,\&$$

$$(r_i(y,z) \rightarrow r_i(x,z)) \,\&\, (r_i(x,y) \,\&\, r_i(z,t) \rightarrow r_i(x\cdot z, y\cdot t) \,\&\, r_i(x/z, y/t) \,\&\, r_i(z\backslash x, t\backslash y)),$$

$$\varphi_2(x, y, z; r_i) = (\forall x)(\forall y)(\forall z) r_i(x, e) \,\&\, r_i(y, e) \,\&\, r_i(z, e) \Rightarrow xy \cdot z = x \cdot yz)$$

(respectively, $\varphi_3(x, y, z; r_i) = (\forall x)(\forall y)(r_i(x, e) \,\&\, r_i(y, e) \Rightarrow x \cdot y = y \cdot x)$),

$$\varphi_4(x, y, z; r_i) = (\forall x)(\forall y)(\forall z)(r_i(x, e) \longrightarrow r_i([x, y], e) \,\&\, r_i([x, y, z], e) \,\&\, r_i((x, y, z), e)) ,$$

$$\varphi_5(x_1,..., x_n; r_i) = (\forall x_1)(\forall x_2)...(\forall x_n)(\neg r_i(x_1, e) \,\&\, \neg r_i(x_2, e) \,\&\, ...$$

$$\& \neg r_i(x_n, e) \longrightarrow r_i(x_1, x_2) \vee ... \vee r_i(x_1, x_n) \vee r_i(x_2, x_3) \vee ... \vee r_i(x_2, x_n) \vee r_i(x_3, x_4) \vee ... r_i(x_{n-1}, x_n)).$$

So on each generated-finit subloop *H* of loop *L* the universal-object formula is true

$$(\exists r)(\varphi_1(x_1, x_2, x_3; r) \,\&\, \varphi_2(y_1, y_2, y_3; r) \,\&\, \varphi_3(z_1, z_2, z_3; r) \,\&\, \varphi_5(t_1, , , , x_n; r))$$

(respectively, $(\exists r)(\varphi_1(x_1, x_2, x_3; r) \,\&\, \varphi_2(y_1, y_2, y_3; r) \,\&\, \varphi_4(z_1, z_2, z_3; r) \,\&\, \varphi_5(t_1, , , , x_n; r))$).

So, we obtained that the property *any generated-finit subloop contains an associative (or commutative) normal subloop of* $\leq n$ *indexes* it is being uttered through an universal-object formula. But then according to the local theorem and loop *L* contains associative (or commutative) normal subloop of finite indexes.

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