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Lattice preradicals with applications

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Abstract

In this paper, we introduce and investigate the latticial counterpart of the module-theoretical concept of preradical. Applications are given to Grothendieck categories and module categories equipped with hereditary torsion theories.

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Introduction

In this paper we introduce and investigate the latticial counterpart of the module-theoretical concept of preradical.

Section 0 lists some notation, definitions, and results about general lattices, especially from [4] and [18], and linear modular lattices introduced and investigated in [5]. We also relate our concept of a linear morphism of lattices with that of a qframe morphism introduced in [19].

In Section 1 we define the concept of a lattice preradical and prove the latticial counterpart of the property of a preradical on modules to commute with arbitrary internal direct sums.

Section 2 is devoted to lattice preradicals in full subcategories of the category $\mathcal{LM}$ of all linear modular lattices.

In Section 3 we introduce and investigate the latticial counterparts of the well-known preradicals $\mathrm{Tr}\,(\mathcal{U}, M)$ and $\mathrm{Rej}\,(\mathcal{U}, M)$ of an arbitrary class $\mathcal{U}$ of right R-modules in a right R-module M.

Section 4 discusses various forms of the socle and radical of complete modular lattices satisfying additional conditions.

The last two sections give applications to Grothendieck categories and module categories equipped with a hereditary torsion theory; they are obtained at once by specializing the results of the previous sections to the lattice $\mathcal{L}(X)$ of all subobjects of an object X of a Grothendieck category $\mathcal{G}$ and to the lattice $\mathrm{Sat}_\tau(M_R)$ of all τ-saturated submodules of a module M_R with respect to a hereditary torsion theory τ on Mod-R.

0 Preliminaries

All lattices considered in this paper are assumed to have a least element denoted by 0 and a greatest element denoted by 1, in other words they are bounded. By $\mathcal{L}$ we shall denote the class of all (bounded) lattices. Throughout this paper, L will always denote such a lattice and L^o its *opposite* or *dual* lattice. By $\mathcal{M}$ we shall denote the class of all modular lattices (with 0 and 1). We shall use $\mathbb{N}$ to denote the set $\{1, 2, \ldots\}$ of all positive integers.

For a lattice L and elements $a \leqslant b$ in L we write

$$b/a := [a, b] = \{\, x \in L \mid a \leqslant x \leqslant b\,\}.$$

An *initial interval* (respectively, a *quotient interval*) of b/a is any interval c/a (respectively, b/c) for some $c \in b/a$.

A lattice L is said to be *simple* (respectively, *zero*) if it has exactly two elements (respectively, one element); so, L is simple (respectively, zero) if $L = \{0, 1\}$ and $0 \neq 1$ (respectively, $L = \{0\}$).

An element $a \in L$ is said to be an *atom* if $a \neq 0$ and $a/0 = \{0, a\}$, i.e., $a/0$ is a simple lattice. We denote by $A(L)$ the set, possibly empty, of all atoms of L. The *socle* $\mathrm{Soc}(L)$ of a complete lattice L is the join of all atoms of L, i.e., $\mathrm{Soc}\,(L) := \bigvee A(L)$. A *coatom* of L is an element $b \in L$ which is a maximal element of $L \setminus \{1\}$. We denote by $M(L)$ the set, possibly empty, of all coatoms of L, so $M(L) = A(L^o)$.

As in [12], a lattice L is said to be *atomic* (respectively, *strongly atomic*) if for every $0 \neq x \in L$ there exists an atom $a \in L$ such that $a \leqslant x$ (respectively, for every $x < y$ in L the interval y/x contains an atom). As in [15], a lattice L is said to be *semi-atomic* (respectively, *semi-Artinian*) if 1 is a join of atoms of L (respectively, if for every $1 \neq x \in L$, the lattice $1/x$ has at least an atom). An upper continuous modular lattice L is strongly atomic if and only if it is semi-Artinian (see, e.g., [15, Proposition 1.9.3]).

An element $c \in L$ is a *complement* (in L) if there exists an element $a \in L$ such that $a \wedge c = 0$ and $a \vee c = 1$; we say in this case that c is a *complement* of a (in L). The lattice L is said to be *complemented* if every element of L has a complement in L.

An element $b \in L$ is a *pseudo-complement in* L if there exists an element $a \in L$ such that $a \wedge b = 0$ and b is maximal with this property; we say in this case that b is a *pseudo-complement* of a.

For a lattice L and $a, b, c \in L$, the notation $a = b \dot{\vee} c$ will mean that $a = b \vee c$ and $b \wedge c = 0$, and we say that a is a *direct join* of b and c. Also, for a non-empty subset S of L, we use the *direct join* notation $a = \dot{\bigvee}_{b \in S} b$ or $a = \dot{\bigvee} S$ if S is an independent subset of L and $a = \bigvee_{b \in S} b$. Recall that a non-empty subset S of L is called *independent* if $0 \notin S$, and for every $x \in S$, $n \in \mathbb{N}$, and subset $T = \{t_1, \ldots, t_n\}$ of S with $x \notin T$, one has $x \wedge (t_1 \vee \cdots \vee t_n) = 0$. Clearly a subset S of L is independent if and only if every finite subset of S is independent.

An element c of a lattice L is called *compact* in L if whenever $c \leqslant \bigvee_{x \in A} x$ for a subset A of L, there is a finite subset F of A such that $c \leqslant \bigvee_{x \in F} x$. The lattice L is said to be *compact* if 1 is a compact element in L, and *compactly generated* if it is complete and every element of L is a join of compact elements.

An element $e \in L$ is said to be *essential* (in L) if $e \wedge x \neq 0$ for every $x \neq 0$ in L. One denotes by $E(L)$ the set of all essential elements of L. An element $s \in L$ is called *small* or *superfluous* (in L) provided $s \in E(L^o)$. Thus, a small element s of L is characterized by the fact that $1 \neq s \vee a$ for any element $a \in L$ with $a \neq 1$. We shall denote by $S(L)$ the set of small elements of L, so that $S(L) = E(L^o)$. The *radical* $\mathrm{Rad}(L)$ of a complete lattice L is the join of all small elements of L, i.e., $\mathrm{Rad}\,(L) := \bigvee S(L)$.

Notice that a different concept of radical, much closer to its module-theoretical correspondent, has been considered in [15]: for a complete lattice L the *radical* r_L of L is the meet of all coatoms of L, i.e., $r_L = \bigwedge_{m \in M(L)} m$, putting $r_L = 1$ if L has no coatoms. In order to avoid any confusion, we shall denote this r_L by $\mathrm{Jac}\,(L)$, and call it the *Jacobson radical* of L. We shall discuss in Section 4 the connections between $\mathrm{Jac}\,(L)$ and $\mathrm{Rad}\,(L)$; in particular we shall see that they coincide if L is a compactly generated modular lattice.

For all other undefined notation and terminology on lattices, the reader is referred to [4], [12], and [18].

Throughout this paper R will denote an associative ring with non-zero identity element, and Mod-R the category of all unital right R-modules. The notation M_R will be used to designate a unital right R-module M, and $N \leqslant M$ will mean that N is a submodule of M. The lattice of all submodules of a module M_R will be denoted by $\mathcal{L}(M_R)$.

We present now after [5] the concept of a *linear morphism* of lattices that evokes the property of a linear mapping $\varphi : M \longrightarrow N$ between modules M_R and N_R to have a kernel $\mathrm{Ker}\,(\varphi)$ and to verify the Fundamental Theorem of Isomorphism: $M/\mathrm{Ker}\,(\varphi) \simeq \mathrm{Im}\,(\varphi)$.

Definition 0.1. *Let $f : L \longrightarrow L'$ be a mapping between the lattice L with least element 0 and last element 1 and the lattice L' with least element $0'$ and last element $1'$.*

The mapping f is called a linear morphism *if there exist $k \in L$, called a* kernel *of f, and $a' \in L'$ such that the following two conditions are satisfied.*

(1) *$f(x) = f(x \vee k)$, $\forall\, x \in L$.*

(2) *f induces an isomorphism of lattices $\bar{f} : 1/k \xrightarrow{\sim} a'/0'$, $\bar{f}(x) = f(x)$, $\forall\, x \in 1/k$.* □

Examples 0.2. (1) Let $\varphi : M_R \longrightarrow M'_R$ be a morphism of modules, and consider the mapping $f : \mathcal{L}(M_R) \longrightarrow \mathcal{L}(M'_R)$ defined by $f(N) = \varphi(N)$ for every $N \leqslant M$. Then f is a linear morphism with kernel Ker (φ).

(2) For any lattice L and any $a \leqslant b$ in L, the mapping $p : b/0 \longrightarrow b/a$, $p(x) := x \vee a$, is a surjective linear morphism with kernel a, as it can be easily seen. This linear morphism is the latticial counterpart of the canonical surjective mapping from any module M_R to the factor module M/N, where N is any submodule of M.

(3) For any lattice L and any a, b in L, such that $a \wedge b = 0$, the mapping

$$q : (a \vee b)/0 \longrightarrow a/0,\ q(x) := (x \vee b) \wedge a,$$

is a surjective linear morphism with kernel b.

Indeed, q is obtained by composing the linear morphism

$$(a \vee b)/0 \longrightarrow (a \vee b)/b,\ x \mapsto x \vee b,$$

considered in (2) with the lattice isomorphism (and thus linear morphism)

$$(a \vee b)/b \xrightarrow{\sim} a/0,\ x \mapsto x \wedge a.$$

This is the latticial counterpart of the canonical projection $M \oplus M' \longrightarrow M$ for two modules M_R and M'_R. □

We list below from [5] some of the basic properties of linear morphisms we need in the sequel.

Proposition 0.3. ([5, Proposition 1.3]). *The following assertions hold for a linear morphism $f : L \longrightarrow L'$ with a kernel k.*

(1) *For $x, y \in L$, $f(x) = f(y) \iff x \vee k = y \vee k$.*

(2) *$f(k) = 0'$ and k is the greatest element of L having this property, where $0'$ is the least element of L'; so, the kernel of a linear morphism is uniquely determined.*

(3) *If $a \in L$ is such that $f(a) = 0'$, then f induces a linear morphism*

$$h : 1/a \longrightarrow L',\ h(x) = f(x),\ \forall\, x \in 1/a.$$

(4) *$f(x \vee y) = f(x) \vee f(y),\ \forall\, x, y \in L$.* □

Corollary 0.4. ([5, Corollary 1.4]). *Any linear morphism is an increasing mapping.* □

Proposition 0.5. ([5, Proposition 2.2]). *The following statements hold.*

(1) *The class $\mathcal{M}$ of all (bounded) modular lattices becomes a category, denoted by $\mathcal{LM}$, if for any L, $L' \in \mathcal{M}$ one takes as morphisms from L to L' all the linear morphisms from L to L'.*

(2) *The isomorphisms in the category $\mathcal{LM}$ are exactly the isomorphisms in the full category $\mathcal{M}$ of the category $\mathcal{L}$ of all (bounded) lattices.*

(3) *The monomorphisms in the category $\mathcal{LM}$ are exactly the injective linear morphisms.*

(4) *The epimorphisms in the category $\mathcal{LM}$ are exactly the surjective linear morphisms.*

(5) *The subobjects of $L \in \mathcal{LM}$ can be taken as the intervals $a/0$ for any $a \in L$.* □

We end this section with new properties of linear morphisms of arbitrary modular lattices not mentioned in [5], that allow us to relate them to qframe morphisms introduced in [19].

Lemma 0.6. *The following assertions hold for a linear morphism $f : L \longrightarrow L'$.*

(1) *f commutes with arbitrary joins, i.e., $f\left(\bigvee_{i\in I} x_i\right) = \bigvee_{i\in I} f(x_i)$ for any family $(x_i)_{i\in I}$ of elements of L, provided both joins exist.*

(2) *f preserves intervals, i.e., for any $u \leqslant v$ in L, one has $f(v/u) = f(v)/f(u)$.*

Proof. By definition, there exist $k \in L$, the kernel of f, and $a' \in L'$ such that f induces a lattice isomorphism $\bar{f} : 1/k \xrightarrow{\sim} a'/0'$.

(1) The proof follows the same idea of [5, Proposition 1.3 (4)]. We have

$$f\big(\bigvee_{i\in I} x_i\big) \;=\; f\big(\big(\bigvee_{i\in I} x_i\big) \vee k\big) \;=\; f\big(\bigvee_{i\in I} (x_i \vee k)\big) \;=\; \bar{f}\big(\bigvee_{i\in I} (x_i \vee k)\big).$$

Being a lattice isomorphism, $\bar{f}$ commutes with arbitrary joins, so

$$\bar{f}\big(\bigvee_{i\in I} (x_i \vee k)\big) \;=\; \bigvee_{i\in I} \bar{f}(x_i \vee k) \;=\; \bigvee_{i\in I} f(x_i \vee k) \;=\; \bigvee_{i\in I} f(x_i).$$

(2) Let $\varphi : L \longrightarrow 1/k$ be the mapping defined by $\varphi(x) = x \vee k$, $\forall\, x \in L$. We have $f = \bar{f} \circ \varphi \circ \iota$, where $\iota : a'/0' \hookrightarrow L'$ is the canonical inclusion mapping. First, we show that for every $u \leqslant v$ in L we have

$$\varphi(v/u) \;=\; \varphi(v)/\varphi(u).$$

To see this, notice that φ is an increasing mapping, so $\varphi(v/u) \subseteq \varphi(v)/\varphi(u)$. For the other inclusion, let $y \in \varphi(v)/\varphi(u) = (v \vee k)/(u \vee k)$. Then $u \leqslant (u \vee k) \wedge v \leqslant y \wedge v \leqslant (v \vee k) \wedge v = v$, so $y \wedge v \in v/u$. Now, since $k \leqslant y$, using modularity we obtain $\varphi(y \wedge v) = (y \wedge v) \vee k = y \wedge (v \vee k) = y$. which proves that φ preserves intervals.

Now we can show that f preserves intervals. Since $\bar{f}$ is a lattice isomorphism, it follows that

$$f(v)/f(u) \;=\; \bar{f}(\varphi(v))/\bar{f}(\varphi(u)) \;=\; \bar{f}(\varphi(v)/\varphi(u)).$$

Using the fact that φ preserves intervals, we deduce that

$$\bar{f}(\varphi(v)/\varphi(u)) \;=\; \bar{f}(\varphi(v/u)) \;=\; f(v/u),$$

as desired. □

Remark 0.7. According to the terminology of [19], a *quasi-frame,* or shortly, a *qframe* is nothing else than an upper continuous modular lattice, and a *qframe morphism* is by definition any mapping between two qframes that preserves intervals and commutes with arbitrary joins; further, a qframe morphism $f : L \longrightarrow L'$ is said to be *algebraic* if the restriction $h : 1/K(f) \longrightarrow L'$ of f to the interval $1/K(f)$ of L is injective, where $K(f) := \bigvee_{f(x)=0'} x$. If $f : L \longrightarrow L'$ is any linear morphism between two qframes then, by Proposition 0.3(2), $K(f)$ coincides with its kernel k we defined in [5]. □

Proposition 0.8. *The following assertions are equivalent for a a mapping* $f : L \longrightarrow L'$ *between the upper continuous modular lattices* L *and* L'*.*

(1) f *is a linear morphism.*

(2) f *is an algebraic qframe morphism.*

Proof. (1) $\Longrightarrow$ (2): This is exactly Lemma 0.6.

(2) $\Longrightarrow$ (1): Assume that f is an algebraic qframe morphism. Set

$$K := \{\, x \in L \,|\, f(x) = 0' \,\} \text{ and } c := \textstyle\bigvee_{x \in K} x.$$

Then

$$f(c) = f(\bigvee_{x\in K} x) = \bigvee_{x \in K} f(x) = 0'$$

because f commutes with arbitrary joins. Now, since f preserves interval, we have

$$f(1/c) = f(1)/f(c) = a'/0',$$

where $a' := f(1)$. Because the qframe morphism f is algebraic, it induces a lattice isomorphism

$$\bar{f} : 1/c \xrightarrow{\sim} a'/0',\ \bar{f}(x) = f(x),\ \forall\, x \in 1/k.$$

Finally, notice that $f(x \vee c) = f(x) \vee f(c) = f(x) \vee 0' = f(x), \forall\, x \in L$. Consequently, f is a linear morphism with kernel c. □

Example 0.9. Consider the four-element lattice $L = \{\, 0,\, a,\, b,\, 1\}$ with $0 < a,\, b < 1$ and a, b incomparable. Also consider $L' = \{0',\, 1'\}$ with $0' < 1'$. Clearly, L and L' are both modular upper continuous lattices. Let $f : L \longrightarrow L'$ be the map defined by

$$f(0) = 0' \text{ and } f(a) = f(b) = f(1) = 1'.$$

Then f is a qframe morphism with $K(f) = \{0\}$. However, f is not a linear morphism (and so, it is not an algebraic qframe morphism). Indeed, if f would be a linear morphism, then its kernel as a linear morphism would be $k = 0$, so $L = 1/k \simeq f(L) = L'$, which is a contradiction. □

1 Lattice preradicals and direct joins

In this section we define the concept of a preradical on linear modular lattices and prove that any such preradical commutes with arbitrary direct joins.

Definition 1.1. *A* lattice preradical *is any functor* $r : \mathcal{LM} \longrightarrow \mathcal{LM}$ *satisfying the following two conditions:*

(1) $r(L) \leqslant L$ *for any* $L \in \mathcal{LM}$.

(2) *For any morphism* $f : L \longrightarrow L'$ *in* $\mathcal{LM}$, $r(f) : r(L) \longrightarrow r(L')$ *is the restriction and corestriction of* f *to* $r(L)$ *and* $r(L')$, *respectively.*

In other words, a lattice preradical is nothing else than a subfunctor of the identity functor $1_{\mathcal{LM}}$ *of the category* $\mathcal{LM}$. □

Let $r : \mathcal{LM} \longrightarrow \mathcal{LM}$ be a lattice preradical. For any $L \in \mathcal{LM}$ and $a \in L$, the subobject $r(a/0)$ of L in $\mathcal{LM}$ is necessarily an initial interval of $a/0$. We denote

$$r(a/0) := a^r/0.$$

If $a \leqslant b$ in L, the inclusion mapping $\iota : a/0 \hookrightarrow b/0$ is clearly a linear morphism. Applying now r we obtain $r(\iota) : a^r/0 \longrightarrow b^r/0$ as a restriction of ι, and so $a^r \leqslant b^r$.

Definition 1.2. *Let* $r : \mathcal{LM} \longrightarrow \mathcal{LM}$ *be a lattice preradical. We say that* r *is a* radical *if, with notation above,* $r(1/1^r) = 1^r/1^r$ *for all* $L = 1/0 \in \mathcal{LM}$.

Further, r *is said to be an* idempotent (*respectively,* left exact *or* hereditary) *preradical if for all* $L \in \mathcal{LM}$, $r(r(L)) = r(L)$ (*respectively,* $a^r = a \wedge 1^r$ *for every* $a \in L$). □

Proposition 1.3. *For any lattice* $L \in \mathcal{LM}$ *and any finite independent family* $(a_i)_{1\leqslant i\leqslant n}$ *of* L, *with* $n \in \mathbb{N}$, *one has*

$$\Big(\dot{\bigvee_{1\leqslant i\leqslant n}} a_i\Big)^r = \dot{\bigvee_{1\leqslant i\leqslant n}} a_i^r.$$

Proof. We proceed by induction on n. Clearly, it suffices to consider only the case $n = 2$. For simplicity, set $a := a_1$, $b := a_2$, and $c := a_1 \vee a_2$. We have $a^r \leqslant c^r$ and $b^r \leqslant c^r$, so

$$a^r \vee b^r \leqslant c^r.$$

To prove the opposite inequality, consider the linear morphism

$$q : c/0 \longrightarrow a/0,\, q(x) = (x \vee b) \wedge a$$

(see Example 0.2(3)). Applying r, we obtain $r(q) : c^r/0 \longrightarrow a^r/0$, thus

$$r(q)(c^r) \;=\; (c^r \vee b) \wedge a \;\leqslant\; a^r.$$

In a similar way, we have $(c^r \vee a) \wedge b \;\leqslant\; b^r$, so

$$((c^r \vee b) \wedge a) \vee ((c^r \vee a) \wedge b) \leqslant a^r \vee b^r.$$

Now, using modularity and the facts that $(c^r \vee a) \wedge b \leqslant b \leqslant c^r \vee b$ and $c^r \leqslant c = a \vee b$, we deduce that

$$((c^r \vee b) \wedge a) \vee ((c^r \vee a) \wedge b) = (c^r \vee b) \wedge (a \vee ((c^r \vee a) \wedge b)) = (c^r \vee b) \wedge ((a \vee b) \wedge (c^r \vee a)) = (c^r \vee a) \wedge (c^r \vee b),$$

and, consequently

$$c^r \;\leqslant\; ((c^r \vee b) \wedge a) \vee ((c^r \vee a) \wedge b).$$

It follows that

$$c^r \leqslant a^r \vee b^r,$$

which ends the proof. □

Proposition 1.4. *For any upper continuous lattice* $L \in \mathcal{LM}$ *and any independent family* $(a_i)_{i \in I}$ *of* L *one has*

$$(\dot{\bigvee_{i \in I}} a_i)^r = \dot{\bigvee_{i \in I}} a_i^r.$$

Proof. If we set $c := \bigvee_{i \in I} a_i$, then $a_i^r \leqslant c^r$ for each $i \in I$, so

$$\bigvee_{i \in I} a_i^r \;\leqslant\; c^r.$$

To prove the opposite inequality, consider $F \subseteq I$, F finite. Since

$$(\bigvee_{i \in F} a_i) \wedge (\bigvee_{i \in I \setminus F} a_i) \;=\; 0,$$

the canonical mapping

$$q : c/0 \longrightarrow (\bigvee_{i \in F} a_i)/0,\, q(x) = \big(x \vee (\bigvee_{i \in I \setminus F} a_i)\big) \wedge (\bigvee_{i \in F} a_i),$$

is a linear morphism by Example 0.2(3).

Applying r, we obtain the linear mapping

$$r(q) : c^r/0 \longrightarrow (\bigvee_{i \in F} a_i)^r/0.$$

By Proposition 1.3, we have

$$(\dot{\bigvee_{i\in F}} a_i)^r = \dot{\bigvee_{i\in F}} a_i^r,$$

thus

$$r(q)(c^r) = \big(c^r \vee (\bigvee_{i\in I\setminus F} a_i)\big) \wedge (\bigvee_{i\in F} a_i) \leqslant \bigvee_{i\in F} a_i^r,$$

and, consequently

$$c^r \wedge (\bigvee_{i\in F} a_i) \leqslant \bigvee_{i\in F} a_i^r.$$

Denote by $\mathcal{P}_f(I)$ the upward directed set of all finite subsets of I. Using the upper continuity of L, we obtain

$$c^r = c^r \wedge c = c^r \wedge \big(\bigvee_{F\in\mathcal{P}_f(I)} (\bigvee_{i\in F} a_i)\big) = \bigvee_{F\in\mathcal{P}_f(I)} \big(c^r \wedge (\bigvee_{i\in F} a_i)\big) \leqslant \bigvee_{F\in\mathcal{P}_f(I)} (\bigvee_{i\in F} a_i^r) = \bigvee_{i\in I} a_i^r,$$

and we are done. □

2 Preradicals in full subcategories of $\mathcal{LM}$.

In this section we discuss preradicals in full subcategories of the category $\mathcal{LM}$ of all linear modular lattices. First, we recall some definitions from [9] and [10].

Definitions 2.1. *Let $\varnothing \neq \mathcal{X} \subseteq \mathcal{L}$. We say that:*

(1) $\mathcal{X}$ *is an* abstract class *if it is closed under lattice isomorphisms, i.e., if $L, K \in \mathcal{L}$, $K \simeq L$, and $L \in \mathcal{X}$, then $K \in \mathcal{X}$.*

(2) $\mathcal{X}$ *is* hereditary (*respectively,* cohereditary) *if it is an abstract class and for any $L \in \mathcal{L}$ and any $a \leqslant b \leqslant c$ in L such that $c/a \in \mathcal{X}$, it follows that $b/a \in \mathcal{X}$ (respectively, $c/b \in \mathcal{X}$).*

(3) $\mathcal{X}$ *is* closed under joins *if it is an abstract class and for any complete lattice L, any $a \in L$, and any family of elements $(a_i)_{i\in I}$, I arbitrary set, with $a \leqslant a_i$ in L and $a_i/a \in \mathcal{X}$, $\forall\, i \in I$, it follows that $\big(\bigvee_{i\in I} a_i\big)/a \in \mathcal{X}$.*

(4) $\mathcal{X}$ *is* closed under meets *if it is an abstract class and for any complete lattice L, any $a \in L$, and any family of elements $(a_i)_{i\in I}$, I arbitrary set, with $a \geqslant a_i$ in L and $a/a_i \in \mathcal{X}$, $\forall\, i \in I$, it follows that $a/\big(\bigwedge_{i\in I} a_i\big) \in \mathcal{X}$.* □

For any non-empty subclass $\mathcal{C}$ of $\mathcal{M}$ we shall denote by $\mathcal{LC}$ the full subcategory of $\mathcal{LM}$ having $\mathcal{C}$ as the class of its objects.

Lemma 2.2. *Let $\mathcal{C}$ be an abstract subclass of $\mathcal{M}$ such that the monomorphisms in the category $\mathcal{LC}$ are injective linear morphisms. Then, for every $L \in \mathcal{C}$, every subobject of L in the category $\mathcal{LC}$ is represented by an initial interval $a/0$ of $L = 1/0$ for some $a \in L$.*

Proof. Let (S, α) be a subobject of L in $\mathcal{LC}$, i.e., $S \overset{\alpha}{\rightarrowtail} L$ is a monomorphism in $\mathcal{LC}$, so an injective linear morphism by hypothesis. Thus, the kernel of α is 0. By definition of a linear morphism, there exists $a \in L$ such that α induces a lattice isomorphism $\overline{\alpha} : S \overset{\sim}{\longrightarrow} a/0$. Since $S \in \mathcal{C}$, it follows that $a/0 \in \mathcal{C}$ because $\mathcal{C}$ is an abstract class. By Proposition 0.5(2), $\overline{\alpha}$ is an isomorphism in the category $\mathcal{LM}$, so it is also an isomorphism in its full subcategory $\mathcal{LC}$.

Let $a/0 \overset{\iota}{\hookrightarrow} L$ be the inclusion mapping. Clearly, ι is an injective linear morphism, so it is a monomorphism in $\mathcal{LC}$. Hence $(a/0, \iota)$ is a subobject of L in $\mathcal{LC}$. We have $\alpha = \iota \circ \overline{\alpha}$, and, because $\overline{\alpha}$ is an isomorphism, we deduce that the subobjects (S, α) and $(a/0, \iota)$ of L are equivalent in $\mathcal{LC}$, therefore $(a/0, \iota)$ represents (S, α). □

Proposition 2.3. *The following assertions are equivalent for an abstract subclass $\mathcal{C}$ of $\mathcal{M}$.*

(1) *$\mathcal{C}$ is hereditary.*

(2) *For any $L \in \mathcal{C}$, the subobjects of L in the category $\mathcal{LC}$ can be taken as the initial intervals $a/0$ of $L = 1/0$, $a \in L$.*

In this case, the monomorphisms in the category $\mathcal{LC}$ are precisely the injective linear morphisms.

Proof. (1) $\Longrightarrow$ (2): First, observe that any injective linear morphism in $\mathcal{LC}$ is clearly a monomorphism in $\mathcal{LC}$. Let $L \in \mathcal{C}$ and $a \in L$. Then $a/0 \in \mathcal{C}$, because $\mathcal{C}$ is hereditary. The inclusion mapping $a/0 \overset{\iota}{\hookrightarrow} L$ is an injective linear morphism, so it is a monomorphism in $\mathcal{LC}$. Thus $(a/0, \iota)$ is a subobject of L in $\mathcal{LC}$.

Now, we are going to show that every subobject of L is represented by some $a/0$. By Lemma 2.2, it suffices to prove that the monomorphisms in $\mathcal{LC}$ are injective linear morphisms. For that, we simply notice that the argument in the proof of [5, Proposition 2.2 (3)] works in this context. For the reader's convenience, we include it below.

Let $f : L \longrightarrow L'$ be a monomorphism in the category $\mathcal{LC}$. Because f is a linear morphism, there exists a kernel $k \in L$ of f. Since $\mathcal{C}$ is a hereditary class, it follows that $K = k/0$ is a member of $\mathcal{C}$. Consider the linear morphisms $\kappa : K \longrightarrow L$, $\kappa(x) = x, \forall\, x \in K$, and $o : K \longrightarrow L$, $o(x) = 0, \forall\, x \in K$. We have $f \circ \kappa = f \circ o$, and, since f is a monomorphism, we deduce that $\kappa = o$. Thus $k = \kappa(k) = o(k) = 0$, and, consequently, f is injective.

(2) $\Longrightarrow$ (1): For every $a \in L$, $a/0$ is an object of $\mathcal{LC}$, so is a member of $\mathcal{C}$. □

Proposition 2.3 provides the correct setting to define a lattice preradical r on a full subcategory $\mathcal{LC}$ of $\mathcal{LM}$ in such a manner that the image $r(L)$ of a lattice L in $\mathcal{LC}$ to be an initial interval of L.

Definition 2.4. *For any hereditary class $\mathcal{C}$ of $\mathcal{M}$, a* lattice preradical *on $\mathcal{C}$ is a subfunctor $r : \mathcal{LC} \longrightarrow \mathcal{LC}$ of the identity functor $1_{\mathcal{LC}}$.* □

In a dual manner, we obtain the following results.

Lemma 2.5. *Let $\mathcal{C}$ be an abstract subclass of $\mathcal{M}$ such that the epimorphisms in the category $\mathcal{LC}$ are surjective linear morphisms. Then, for every $L \in \mathcal{C}$, every quotient object of L in the category $\mathcal{LC}$ is represented by a quotient interval $1/a$ of $L = 1/0$ for some $a \in L$.* □

Proposition 2.6. *The following assertions are equivalent for an abstract subclass $\mathcal{C}$ of $\mathcal{M}$.*

(1) *$\mathcal{C}$ is cohereditary.*

(2) *For any $L \in \mathcal{C}$, the quotient objects of L in the category $\mathcal{LC}$ can be taken as the quotient intervals $1/a$ of $L = 1/0$, $a \in L$.*

In this case, the epimorphisms in the category $\mathcal{LC}$ are precisely the surjective linear morphisms. □

3 Trace and Reject

In this section we introduce and investigate the latticial counterparts of the well-known preradicals $\mathrm{Tr}\,(\mathcal{U}, M)$ and $\mathrm{Rej}\,(\mathcal{U}, M)$ of an arbitrary class $\mathcal{U}$ of right R-modules in a right R-module M (see [11]). As particular cases we deduce that the socle and radical define two lattice preradicals on the full subcategory $\mathcal{LM}_c$ of $\mathcal{LM}$ consisting of all complete modular lattices.

Definition 3.1. *Let L be a complete lattice, and let $\varnothing \neq \mathcal{X} \subseteq \mathcal{L}$ be an abstract class of lattices. We define the* trace of $\mathcal{X}$ in L *and the* reject of $\mathcal{X}$ in L *by*

$$\mathrm{Tr}\,(\mathcal{X},\, L) := \bigvee \{\, x \in L \,|\, x/0 \in \mathcal{X} \,\} \quad \textit{and} \quad \mathrm{Rej}\,(\mathcal{X},\, L) := \bigwedge \{\, x \in L \,|\, 1/x \in \mathcal{X} \,\}.$$ □

Remarks 3.2. (1) If the abstract class $\mathcal{X}$ contains a zero lattice, hence all zero lattices, in particular, if $\mathcal{X}$ is a hereditary or cohereditary class, then the sets $\{\, x \in L \,|\, x/0 \in \mathcal{X} \,\}$ and $\{\, x \in L \,|\, 1/x \in \mathcal{X} \,\}$ are both non-empty for any lattice L.

(2) For a class $\varnothing \neq \mathcal{X} \subseteq \mathcal{L}$ closed under joins, we have

$$\mathrm{Tr}\,(\mathcal{X},\, L)/0 \,\in\, \mathcal{X},$$

and so, $L \in \mathcal{X} \iff \mathrm{Tr}\,(\mathcal{X},\, L) = 1$. If additionally $\mathcal{X}$ contains a zero lattice, then the trace of $\mathcal{X}$ in L is the greatest element $b \in L$ such that $b/0 \in \mathcal{X}$.

(3) Dually, for a class $\varnothing \neq \mathcal{X} \subseteq \mathcal{L}$ closed under meets, we have

$$1/\mathrm{Rej}\,(\mathcal{X},\, L) \,\in\, \mathcal{X},$$

and so, $L \in \mathcal{X} \iff \mathrm{Rej}\,(\mathcal{X},\, L) = 0$. If additionally $\mathcal{X}$ contains a zero lattice, then the reject of $\mathcal{X}$ in L is the least element $b \in L$ such that $1/b \in \mathcal{X}$. □

For two lattices L and L' we shall denote by 0 (respectively, 1) the least (respectively, greatest) element of L, and by $0'$ (respectively, $1'$) the least (respectively, greatest) element of L'.

Lemma 3.3. *Let $\mathcal{X}$ be an abstract class of lattices, let L, L' be two complete lattices, and let $f : L \xrightarrow{\sim} L'$ be a lattice isomorphism. Then*

$$f(\mathrm{Tr}\,(\mathcal{X},\, L)) \,=\, \mathrm{Tr}\,(\mathcal{X},\, L') \quad \textit{and} \quad f(\mathrm{Rej}\,(\mathcal{X},\, L)) \,=\, \mathrm{Rej}\,(\mathcal{X},\, L').$$

Proof. Since $\mathcal{X}$ is an abstract class, f induces a bijection between the sets $\{\, x \in L \,|\, x/0 \in \mathcal{X}\,\}$ and $\{\, x' \in L' \,|\, x'/0' \in \mathcal{X}\,\}$. Being a lattice isomorphism, f commutes with arbitrary joins, so

$$\begin{aligned} f(\mathrm{Tr}\,(\mathcal{X},\, L)) \;&=\; f\big(\bigvee \{\, x \in L \,|\, x/0 \in \mathcal{X}\,\}\big) \,=\, \bigvee \{\, f(x) \in L \,|\, x/0 \in \mathcal{X}\,\} \\ &=\; \bigvee \{\, x' \in L' \,|\, x'/0' \in \mathcal{X}\,\} \,=\, \mathrm{Tr}\,(\mathcal{X},\, L'). \end{aligned}$$

Similarly, f induces a bijection between the sets $\{\, x \in L \,|\, 1/x \in \mathcal{X}\,\}$ and $\{\, x' \in L' \,|\, 1'/x' \in \mathcal{X}\,\}$, and we obtain the second equality. □

Proposition 3.4. *Let $\mathcal{X}$ be a cohereditary class of lattices, and let L, L' be complete modular lattices. Then, for any linear morphism $f : L \longrightarrow L'$ in $\mathcal{LM}$ we have*

$$f(\mathrm{Tr}\,(\mathcal{X},\, L)) \leqslant \mathrm{Tr}\,(\mathcal{X},\, L').$$

Proof. If k is the kernel of f, then there exists $a' \in L'$ such that f induces a lattice isomorphism $\bar{f} : 1/k \xrightarrow{\sim} a'/0'$, $\bar{f}(x) = f(x), \forall\, x \in 1/k$.

Let $x \in L$ be such that $x/0 \in \mathcal{X}$. Since $\mathcal{X}$ is cohereditary, it follows that $x/(x \wedge k) \in \mathcal{X}$. By modularity, we have $(x \vee k)/k \simeq x/(x \wedge k)$, and since $\mathcal{X}$ is an abstract class, we have $(x \vee k)/k \in \mathcal{X}$.

Set $t := \mathrm{Tr}\,(\mathcal{X},\, L)$. We have

$$t \vee k \,=\, \Big(\bigvee_{x \in L,\ x/0 \in \mathcal{X}} x\Big) \vee k \,=\, \bigvee_{x \in L,\ x/0 \in \mathcal{X}} (x \vee k) \,\leqslant\, \bigvee_{y \in 1/k,\ y/k \in \mathcal{X}} y \,=\, \mathrm{Tr}\,(\mathcal{X},\, 1/k).$$

Using Proposition 0.3 and Corollary 0.4, we deduce that

$$f(t) \,=\, f(t \vee k) \,\leqslant\, f(\mathrm{Tr}\,(\mathcal{X},\, 1/k)) \,=\, \bar{f}(\mathrm{Tr}\,(\mathcal{X},\, 1/k)).$$

Since $\bar{f}$ is a lattice isomorphism, by Lemma 3.3 we have

$$\bar{f}(\mathrm{Tr}\,(\mathcal{X},\, 1/k)) \,=\, \mathrm{Tr}\,(\mathcal{X},\, a'/0') \,=\, \bigvee_{x' \in a'/0',\ x'/0' \in \mathcal{X}} x' \,\leqslant\, \bigvee_{x' \in L',\ x'/0' \in \mathcal{X}} x' \,=\, \mathrm{Tr}\,(\mathcal{X},\, L').$$

Hence $f(t) \leqslant \mathrm{Tr}\,(\mathcal{X},\, L')$, and we are done. □

Corollary 3.5. *Let $\mathcal{X}$ be a cohereditary class of lattices and let $\mathcal{LM}_c$ be the full subcategory of $\mathcal{LM}$ consisting of all complete modular lattices. Then, the assignment $L \mapsto \mathrm{Tr}\,(\mathcal{X},\, L)/0$ defines a preradical on $\mathcal{LM}_c$. Moreover, it is an idempotent preradical.*

Proof. The first part follows from Proposition 3.4. For the second part, we have

$$\{\, a \in L \,|\, a/0 \in \mathcal{X} \,\} \;=\; \{\, a \in L \,|\, a/0 \in \mathcal{X},\, a \leqslant \mathrm{Tr}\,(\mathcal{X},\, L) \,\} \;=\; \{\, a \in \mathrm{Tr}\,(\mathcal{X},\, L)/0 \,|\, a/0 \in \mathcal{X} \,\},$$

so $\mathrm{Tr}\,(\mathcal{X},\, L) \;=\; \mathrm{Tr}\,(\mathcal{X},\, \mathrm{Tr}\,(\mathcal{X},\, L)/0)$. □

Example 3.6. Denote by $\mathcal{S}$ the class of all lattices having at most two elements, i.e., of all zero and simple lattices. Clearly, $\mathcal{S}$ is a cohereditary class. For a complete lattice L, we have $\mathrm{Tr}\,(\mathcal{X},\, L) = \mathrm{Soc}\,(L)$. In particular, Corollary 3.5 implies at once the following result of [6]: the assignment $\sigma : \mathcal{LM}_c \longrightarrow \mathcal{LM}_c\,,\ \sigma(L) = \mathrm{Soc}\,(L)/0$, defines an idempotent preradical on $\mathcal{LM}_c$.

Moreover, if we consider the restriction (and corestriction) σ' of σ to the full subcategory $\mathcal{LM}_u$ of $\mathcal{LM}$ consisting on all upper continuous modular lattices, then σ' is a left exact preradical. To see this, let $L \in \mathcal{LM}_u$ and $y \in L$. Since $\mathrm{Soc}\,(y/0) \leqslant y$ and $\mathrm{Soc}\,(y/0) \leqslant \mathrm{Soc}\,(L)$, it follows that $\mathrm{Soc}\,(y/0) \leqslant y \wedge \mathrm{Soc}\,(L)$. Now, $\mathrm{Soc}\,(L)/0$ is an upper continuous modular semi-atomic lattice, so $(y \wedge \mathrm{Soc}\,(L))/0$ is also semi-atomic, by [15, Theorem 1.8.2 and Corollary 1.8.4]. Hence $y \wedge \mathrm{Soc}\,(L) \leqslant \mathrm{Soc}\,(y/0)$, and consequently, $y \wedge \mathrm{Soc}\,(L) = \mathrm{Soc}\,(y/0)$. □

Lemma 3.7. *Let $\mathcal{X}$ be a hereditary class of lattices, let L be a complete modular lattice, and let $a \in L$. Then*

$$\mathrm{Rej}\,(\mathcal{X},\, a/0) \leqslant \mathrm{Rej}\,(\mathcal{X},\, L) \leqslant \mathrm{Rej}\,(\mathcal{X},\, 1/a).$$

Proof. For the first inequality, let $x \in L$ with $1/x \in \mathcal{X}$. Since $\mathcal{X}$ is a hereditary class, it follows that $(a \vee x)/x \in \mathcal{X}$. By modularity, we have $(a \vee x)/x \simeq a/(a \wedge x)$, and because $\mathcal{X}$ is an abstract class, we deduce that $a/(a \wedge x) \in \mathcal{X}$. Thus, by definition, $a \wedge x \geqslant \mathrm{Rej}\,(\mathcal{X},\, a/0)$, and so, $x \geqslant \mathrm{Rej}\,(\mathcal{X},\, a/0)$. Consequently, $\mathrm{Rej}\,(\mathcal{X},\, L) \;=\; \bigwedge \{\, x \in L \,|\, 1/x \in \mathcal{X} \,\} \;\geqslant\; \mathrm{Rej}\,(\mathcal{X},\, a/0)$.

The second inequality follows from $\{\, x \in L \,|\, 1/x \in \mathcal{X} \,\} \;\supseteq\; \{\, x \in 1/a \,|\, 1/x \in \mathcal{X} \,\}$. □

Proposition 3.8. *Let $\mathcal{X}$ be a hereditary class of lattices, and let $L,\, L'$ be complete modular lattices. Then, for any linear morphism $f : L \longrightarrow L'$ in $\mathcal{LM}$ we have*

$$f(\mathrm{Rej}\,(\mathcal{X},\, L)) \leqslant \mathrm{Rej}\,(\mathcal{X},\, L').$$

Proof. If k is the kernel of f, then there exists $a' \in L'$ such that f induces a lattice isomorphism $\bar{f} \,:\, 1/k \,\simeq\, a'/0'$. By Lemma 3.7, we have $\mathrm{Rej}\,(\mathcal{X},\, L) \,\leqslant\, \mathrm{Rej}\,(\mathcal{X},\, 1/k)$. Since $k \leqslant \mathrm{Rej}\,(\mathcal{X},\, 1/k)$, we deduce that

$$\mathrm{Rej}\,(\mathcal{X},\, L) \vee k \;\leqslant\; \mathrm{Rej}\,(\mathcal{X},\, 1/k).$$

Thus

$$f(\mathrm{Rej}\,(\mathcal{X},\, L)) \;=\; f(\mathrm{Rej}\,(\mathcal{X},\, L) \vee k) \;\leqslant\; f(\mathrm{Rej}\,(\mathcal{X},\, 1/k)) \;=\; \bar{f}(\mathrm{Rej}\,(\mathcal{X},\, 1/k)).$$

By Lemma 3.3, we have

$$\bar{f}(\mathrm{Rej}\,(\mathcal{X},\, 1/k)) \;=\; \mathrm{Rej}\,(\mathcal{X},\, a'/0'),$$

and using again Lemma 3.7, we obtain

$$\mathrm{Rej}\,(\mathcal{X},\, a'/0')) \leqslant \mathrm{Rej}\,(\mathcal{X},\, L').$$

Hence $f(\mathrm{Rej}\,(\mathcal{X},\, L)) \leqslant \mathrm{Rej}\,(\mathcal{X},\, L')$. $\square$

Corollary 3.9. *For any hereditary class $\mathcal{X}$ of lattices, the assignment $L \mapsto \mathrm{Rej}\,(\mathcal{X},\, L)/0$ defines a preradical on the full subcategory $\mathcal{LM}_c$ of $\mathcal{LM}$ consisting of all complete modular lattices. Moreover, it is a radical.*

Proof. The first part follows from Proposition 3.8. For the second part, we have

$$\{\, a \in L \,|\, 1/a \in \mathcal{X} \,\} = \{\, a \in L \,|\, 1/a \in \mathcal{X},\, a \geqslant \mathrm{Rej}\,(\mathcal{X},\, L) \,\} = \{\, a \in 1/\mathrm{Rej}\,(\mathcal{X},\, L) \,|\, 1/a \in \mathcal{X} \,\},$$

so $\mathrm{Rej}\,(\mathcal{X},\, L) = \mathrm{Rej}\,(\mathcal{X},\, 1/\mathrm{Rej}\,(\mathcal{X},\, L))$. $\square$

Example 3.10. Denote by $\mathcal{S}$ the class of all lattices having at most two elements, i.e., of all zero and simple lattices. Clearly, $\mathcal{S}$ is a hereditary class. For a complete lattice L, we have $\mathrm{Rej}\,(\mathcal{X},\, L) = \mathrm{Jac}\,(L)$. Thus, Corollary 3.9 implies that $\varrho : \mathcal{LM}_c \longrightarrow \mathcal{LM}_c\,,\ \varrho(L) = \mathrm{Jac}\,(L)/0$, is a radical on $\mathcal{LM}_c$. $\square$

4 The socle and radical of lattices with additional conditions

In this section we discuss other expressions of the socle and radical of a complete modular lattice satisfying additional conditions.

Recall that for any complete lattice L, we have denoted

$$M(L) := A(L^o) \quad \text{and} \quad \mathrm{Jac}\,(L) := \bigwedge M(L).$$

Proposition 4.1. ([6, Proposition 3.1]). *For any compactly generated modular lattice L we have*

$$\mathrm{Soc}\,(L) = \bigwedge E(L) \quad \textit{and} \quad \mathrm{Rad}\,(L) = \mathrm{Jac}\,(L). \qquad \square$$

In the sequel we shall present other cases when one or both equalities in Proposition 4.1 hold.

For a lattice L we clearly have the following implications:

$$\boxed{\text{semi-atomic} \Longrightarrow \text{semi-Artinian} \Longrightarrow \text{atomic}}$$

The implications above are strict. Indeed, the sublattice $L = \{0\} \cup [1, 2\,]$ of the usually ordered lattice $\mathbb{R}$ of all real numbers is an upper continuous modular lattice which is atomic but not semi-Artinian, and the chain $\{\, 0,\, 1,\, 2 \,\}$ is semi-Artinian but not semi-atomic.

The well-known property of a non-zero finitely generated module M to possess a proper maximal submodule that include a given proper submodule, known as *Krull's Lemma,* is taken below as a definition for the following condition a lattice L may have:

(KL) $\quad\quad \forall\, x \in L \setminus \{1\},\ \exists\, m \in M(L)$ with $x \leqslant m$, i.e., $M(1/x) \neq \varnothing$,

where (KL) is an acronym for "Krull's Lemma".

Remarks 4.2. (1) A lattice L satisfies (KL) $\Longleftrightarrow$ its opposite lattice L^o is atomic.

(2) (Krull's Lemma) Any complete compact lattice L satisfies the condition (KL) (see, e.g., Lemma [4, Lemma 2.1.13] or [15, Lemma 1.8.5]).

(3) Any Noetherian lattice L satisfies the condition (KL) because each non-empty subset $(1/x) \setminus \{1\}$ has a maximal element. This also follows from (2) since a Noetherian (bounded) lattice is necessarily complete and compact (see, e.g., [4, Proposition 2.1.14, Corollary 2.1.15]).

(4) If L is a semi-atomic upper continuous modular lattice then L satisfies the condition (KL). Indeed, let $x < 1$ in L. Since L is semi-atomic, there exists $a \in A(L)$ such that $a \nleqslant x$, and then $a \wedge x = 0$ because a is an atom. Using Zorn's Lemma, there exists a pseudo-complement m of a such that $m \geqslant x$. Then, by [18, Chapter III, Proposition 6.4], we have $m \vee a \in E(L)$. On the other hand, by Theorem [15, Theorem 1.8.2], L is a complemented lattice, so $E(L) = \{1\}$, and then $m \vee a = 1$. By modularity, we have

$$1/m \,=\, (m \vee a)/m \,\simeq\, a/(m \wedge a) \,=\, a/0,$$

so $1/m$ is simple, and consequently, $m \in M(L)$ with $x \leqslant m$, as desired. □

Proposition 4.3. *The following statements hold for a lattice* L.

(1) *If* $a \in A(L)$ *and* $e \in E(L)$, *then* $a \leqslant e$.

(2) *If* L *is complete, then* $\mathrm{Soc}\,(L) \leqslant \bigwedge E(L)$.

(3) *If* L *is complete and atomic, then* $\mathrm{Soc}\,(L) \in E(L)$, *and so,* $\mathrm{Soc}\,(L) = \bigwedge E(L)$.

Proof. (1) If $a \in A(L)$ and $e \in E(L)$, then $a \wedge e \neq 0$, and since a is an atom, it follows that $a \leqslant e$.

(2) follows from (1).

(3) Because L is atomic, for every $0 \neq x \in L$ there exists $a \in A(L)$ such that $a \leqslant x$, and so $0 < a \leqslant x \wedge \mathrm{Soc}\,(L)$. □

By duality, we obtain the following result. The first part of (3) is exactly the latticial counterpart of the well-known *Nakayama's Lemma.*

Proposition 4.4. *The following statements hold for a lattice* L.

(1) *If* $s \in S(L)$ *and* $m \in M(L)$, *then* $s \leqslant m$.

(2) *If* L *is complete, then* $\mathrm{Rad}\,(L) \leqslant \mathrm{Jac}\,(L)$.

(3) *If* L *is complete and satisfies* (KL)*, then* $\mathrm{Jac}\,(L) \in S(L)$*, so* $\mathrm{Rad}\,(L) = \mathrm{Jac}\,(L)$. □

Proof. (1) and (2) are clear. To prove (3), set $j := \mathrm{Jac}\,(L)$, and let $x \in L$ with $x \vee j = 1$. Assume that $x \neq 1$. Then, by (KL), there exists $m \in M(L)$ such that $x \leqslant m$, and then $1 = x \vee j \leqslant x \vee m = m$, which is a contradiction. So, $x = 1$, i.e., $j \in S(L)$, and then $j \leqslant \mathrm{Rad}\,(L)$. Thus $\mathrm{Rad}\,(L) = \mathrm{Jac}\,(L)$ by (2). □

5 Applications to Grothendieck categories

In this section we apply the lattice-theoretical results established in the previous section to Grothendieck categories.

Throughout this section $\mathcal{G}$ will denote a *Grothendieck category*, i.e., an Abelian category with exact direct limits and with a generator. For any object X of $\mathcal{G}$, $\mathcal{L}(X)$ will denote the lattice of all subobjects of X. It is well-known that $\mathcal{L}(X)$ is an upper continuous modular lattice (see, e.g., [18, Chapter 4, Proposition 5.3, and Chapter 5, Section 1]. For any object X of $\mathcal{G}$, and for any subset $\mathcal{X} \subseteq \mathcal{L}(X)$ we denote

$$\bigvee \mathcal{X} := \sum_{A \in \mathcal{X}} A \quad \text{and} \quad \bigwedge \mathcal{X} := \bigcap_{A \in \mathcal{X}} A.$$

Recall that an object $X \in \mathcal{G}$ is said to be *finitely generated* if the lattice $\mathcal{L}(X)$ is compact. The category $\mathcal{G}$ is called *locally finitely generated* if it has a family of finitely generated generators, or equivalently if the lattices $\mathcal{L}(X)$ are compactly generated for all objects X of $\mathcal{G}$ (see Stenström [18, p. 122]). As in [3], we say that an object $X \in \mathcal{G}$ is *locally finitely generated* if the lattice $\mathcal{L}(X)$ of all its subobjects is compactly generated. For all undefined notation and terminology on Abelian categories the reader is referred to [18].

We say that Y is a *small* subobject of an object X of $\mathcal{G}$ if Y is a small element of the lattice $\mathcal{L}(X)$. More generally, if $\mathbb{P}$ is any property on lattices, we say that an object $X \in \mathcal{G}$ is/has $\mathbb{P}$ if the lattice $\mathcal{L}(X)$ is/has $\mathbb{P}$. Similarly, a subobject Y of an object $X \in \mathcal{G}$ is/has $\mathbb{P}$ if the element Y of the lattice $\mathcal{L}(X)$ is/has $\mathbb{P}$. Thus, we obtain the concepts of an *atom, coatom, socle, radical, Jacobson radical,* etc.

Lemma 5.1. *For any Abelian category* $\mathcal{A}$ *and any morphism* $f : X \longrightarrow Y$ *in* $\mathcal{A}$*, the canonical mapping*

$$\varphi : \mathcal{L}(X) \longrightarrow \mathcal{L}(Y),\, \varphi(A) := f(A),\, \forall\, A \leqslant X,$$

is a linear morphism.

Proof. Let $K := \mathrm{Ker}\,(f)$ and $I := \mathrm{Im}\,(f)$. First, observe that $\varphi(A+K) = \varphi(A)$, $\forall\, A \leqslant X$. Since $\mathcal{A}$ is an Abelian category, the morphism f induces an isomorphism $\overline{f} : X/K \xrightarrow{\sim} I$. This produces a lattice isomorphism

$$\psi : \mathcal{L}(X/K) \xrightarrow{\sim} \mathcal{L}(I),\, \psi(U) := f(U),\, \forall\, U \leqslant X/K.$$

On the other hand, the lattice $\mathcal{L}(X/K)$ is canonically isomorphic to the interval $[K, X]$ of the lattice $\mathcal{L}(X)$, the lattice $\mathcal{L}(I)$ is isomorphic to the interval $[0, I]$ of $\mathcal{L}(Y)$, and the composed isomorphism $[K, X] \longrightarrow [0, I]$ is given by $Z \mapsto f(Z)$. This shows that

$$\varphi : \mathcal{L}(X) \longrightarrow \mathcal{L}(Y),\, \varphi(A) := f(A),\, \forall\, A \leqslant X,$$

is a linear morphism, as desired. □

For any object X of $\mathcal{G}$ we set

$$A(X) := A(\mathcal{L}(X)),\, S(X) := S(\mathcal{L}(X)),\, E(X) := E(\mathcal{L}(X)),\, M(X) := M(\mathcal{L}(X)),$$

$$\mathrm{Soc}\,(X) := \mathrm{Soc}\,(\mathcal{L}(X)),\, \mathrm{Rad}\,(X) := \mathrm{Rad}\,(\mathcal{L}(X)),\, \mathrm{Jac}\,(X) := \mathrm{Jac}\,(\mathcal{L}(X)).$$

According to the definition above of objects of $\mathcal{G}$ that are/have a certain property $\mathbb{P}$, we call the elements of $A(X)$ (respectively, $S(X)$, $E(X)$, $M(X)$) atoms (respectively, small subobjects, essential subobjects, coatoms) of X, and $\mathrm{Soc}\,(X)$ (respectively, $\mathrm{Rad}\,(X)$, $\mathrm{Jac}\,(X)$) are known as the socle (respectively, radical, Jacobson radical) of X. In the sequel we shall use the well-established term of a *simple* (respectively, *proper maximal*) subobject of an object X instead of that of an atom (respectively, coatom) of X. Also, as mentioned above, for the term of a *compact object* we shall use that of a *finitely generated object.*

In view of Lemma 5.1, all the notions and results of the previous sections have categorical versions obtained by specializing them from an arbitrary modular lattice L to the upper continuous modular lattice $\mathcal{L}(X)$ of any object X of a Grothendieck category $\mathcal{G}$. No further proofs are required. We shall present below only a few results, and leave the others to the reader.

Proposition 5.2. *For any morphism $f : X \longrightarrow Y$ in a Grothendieck category $\mathcal{G}$, one has*

$$f(\mathrm{Soc}\,(X)) \leqslant \mathrm{Soc}\,(Y),\;\; f(\mathrm{Rad}\,(X)) \leqslant \mathrm{Rad}\,(Y),\;\; \textit{and}\; f(\mathrm{Jac}\,(X)) \leqslant \mathrm{Jac}\,(Y).$$ □

Proposition 5.3. *The following statements are true for an object X of a Grothendieck category $\mathcal{G}$.*

(1) $\mathrm{Jac}\,(X/\mathrm{Jac}\,(X)) = 0$.

(2) $\mathrm{Soc}\,(\mathrm{Soc}\,(X)) = \mathrm{Soc}\,(X)$.

(3) $\mathrm{Soc}\,(Y) = Y \cap \mathrm{Soc}\,(X)$ *for any subobject Y of X.* □

The next result is hard to be proved without invoking its latticial counterpart, where the condition that the considered lattice is compactly generated is essential. We do not have any counter-example showing its failure for arbitrary objects of a Grothendieck category.

Proposition 5.4. *For any locally finitely generated object X of a Grothendieck category $\mathcal{G}$ we have*

$$\mathrm{Rad}\,(X) \;=\; \mathrm{Jac}\,(X) \quad \textit{and} \quad \mathrm{Soc}\,(X) \;=\; \bigcap_{E\in E(X)} E.$$

In particular, the equalities above hold for any object X of a locally finitely generated Grothendieck category $\mathcal{G}$. □

Following our definition above of objects of $\mathcal{G}$ that are/have a property $\mathbb{P}$, we say that an object $X \in \mathcal{G}$ is *atomic* (respectively, satisfies (KL)) if the lattice $\mathcal{L}(X)$ is atomic (respectively, satisfies (KL)).

Proposition 5.5. *If $X \in \mathcal{G}$ is atomic, then* $\mathrm{Soc}\,(X) = \bigcap_{E\in E(X)} E$. □

Proposition 5.6. *If $X \in \mathcal{G}$ satisfies* (KL) *then* $\mathrm{Rad}\,(X) = \mathrm{Jac}\,(X)$. □

Corollary 5.7. *If $X \in \mathcal{G}$ is a finitely generated object, in particular a Noetherian object, then* $\mathrm{Rad}\,(X) = \mathrm{Jac}\,(X)$. □

6 Applications to module categories equipped with a hereditary torsion theory

In this section, we present relative versions with respect to a hereditary torsion theory on Mod-R of some results related to preradicals on modules. Their proofs are immediate applications of the lattice-theoretical results obtained in the previous sections.

Throughout this section R denotes a ring with non-zero identity, Mod-R the category of all unital right R-modules, $\tau = (\mathcal{T}, \mathcal{F})$ a fixed hereditary torsion theory on Mod-R, and $\tau(M)$ the τ-torsion submodule of a right R-module M. We shall use the notation M_R to emphasize that M is a right R-module. For any M_R we shall denote

$$\mathrm{Sat}_\tau(M) \;:=\; \{\; N \mid N \leqslant M \text{ and } M/N \in \mathcal{F} \;\},$$

and for any $N \leqslant M$ we shall denote by $\overline{N}$ the *τ-saturation* of N (in M) defined by $\overline{N}/N = \tau(M/N)$. The submodule N is called *τ-saturated* if $N = \overline{N}$. Note that

$$\mathrm{Sat}_\tau(M) = \{\, N \,|\, N \leqslant M,\; N = \overline{N}\,\},$$

so $\mathrm{Sat}_\tau(M)$ is the set of all τ-*saturated* submodules of M, which explains the notation. It is known that for any M_R, $\mathrm{Sat}_\tau(M)$ is an upper continuous modular lattice with respect to the inclusion $\subseteq$ and the operations $\bigvee$ and $\bigwedge$ defined as follows:

$$\bigvee_{i\in I} N_i := \overline{\sum_{i\in I} N_i} \qquad \text{and} \qquad \bigwedge_{i\in I} N_i := \bigcap_{i\in I} N_i,$$

having least element $\tau(M)$ and greatest element M (see [18, Chapter 9, Proposition 4.1]).

For all undefined notation and terminology on torsion theories the reader is referred to [8] and [18].

If $\mathbb{P}$ is any property on lattices, we say that a module M_R is/has τ-$\mathbb{P}$ if the lattice $\mathrm{Sat}_\tau(M)$ is/has $\mathbb{P}$. Thus, we obtain the concepts of a τ-*Noetherian* module, τ-*compact* module, τ-*compactly generated* module, etc. We say that a submodule N of M_R is/has τ-$\mathbb{P}$ if its τ-saturation $\overline{N}$, which is an element of $\mathrm{Sat}_\tau(M)$, is/has $\mathbb{P}$. Thus, we obtain the concepts of a τ-*essential* submodule of a module, τ-*small* submodule of a module, τ-*independent* set/family of submodules of a module, etc. Since $\overline{N} = \overline{\overline{N}}$, it follows that N is/has τ-$\mathbb{P}$ if and only if $\overline{N}$ is/has τ-$\mathbb{P}$.

Recall that in Torsion Theory a module U_R is said to be τ-*simple* if $U \notin \mathcal{T}$ and $\mathrm{Sat}_\tau(U) = \{\tau(U), U\}$, and τ-*cocritical* if it τ-simple and $U \in \mathcal{F}$. Thus, U_R is τ-*simple* if and only if $\mathrm{Sat}_\tau(U_R)$ is a simple lattice, so this concept agrees with that of a τ-$\mathbb{P}$ module defined above.

We denote by $\mathrm{Max}_\tau(M)$ the set, possibly empty, of all maximal elements of the poset $\mathrm{Sat}_\tau(M) \setminus \{M\}$, by $A_\tau(M)$ the set of all τ-simple submodules of M, by $C_\tau(M)$ the set of all τ-cocritical submodules of M, by $S_\tau(M)$ the set of all τ-small submodules of M, and by $E_\tau(M)$ the set of all τ-essential submodules of M.

Lemma 6.1. ([4, Lemmas 3.4.2 and 3.4.4]). *The following statements hold for a module M_R and submodules $P \subseteq N$ of M_R.*

(1) *The mapping*

$$\alpha : \mathrm{Sat}_\tau(N/P) \longrightarrow \mathrm{Sat}_\tau(\overline{N}/\overline{P}),\ X/P \mapsto \overline{X}/\overline{P},$$

is a lattice isomorphism.

(2) $\mathrm{Sat}_\tau(N) \simeq \mathrm{Sat}_\tau(\overline{N})$.

(3) *If $N \in \mathcal{T}$, then $\mathrm{Sat}_\tau(M) \simeq \mathrm{Sat}_\tau(M/N)$, in particular $\mathrm{Sat}_\tau(M) \simeq \mathrm{Sat}_\tau(M/\tau(M))$.*

(4) *If $M/N \in \mathcal{T}$, then $\mathrm{Sat}_\tau(M) \simeq \mathrm{Sat}_\tau(N)$.*

(5) *If $N,\ P \in \mathrm{Sat}_\tau(M)$, then the assignment $X \mapsto X/P$ defines a lattice isomorphism from the interval $[P, N]$ of the lattice $\mathrm{Sat}_\tau(M)$ onto the lattice $\mathrm{Sat}_\tau(N/P)$.* □

Proposition 6.2. *The following assertions hold for a module M_R and $N \leqslant M$.*

(1) *$N \in \mathrm{Max}_\tau(M)$.*

(2) *N is a maximal element of the set* $\{ P \leqslant M \mid M/P \notin \mathcal{T} \}$.

(3) *M/N is a τ-cocritical module.*

Proof. The equivalence (1) $\Longleftrightarrow$ (2) follow from the same arguments as in the proof of [1, Proposition 0.2] for $M_R = R_R$, or of [16, Proposition 1.2], where the assumption that the module M_R is finitely generated is not necessary.

(1) $\Longleftrightarrow$ (3): If $N \in \mathrm{Max}_\tau(M)$, then, by Lemma 6.1(5) the lattice $\mathrm{Sat}_\tau(M/N)$ is isomorphic to the interval $[N, M]$, so $\mathrm{Sat}_\tau(M/N) = \{N, M\}$. Since $M/N \in \mathcal{F}$, we deduce that M/N is a τ-cocritical module.

(3) $\Longrightarrow$ (1): If M/N is a τ-cocritical module, then, $N \in \mathrm{Sat}_\tau(M)$, and, again by Lemma 6.1(5), we have $[N, M] = \{N, M\}$ and $N \neq M$, i.e., $N \in \mathrm{Max}_\tau(M)$, as desired. □

Lemma 6.3. *The following assertions hold for a module M_R and $N \leqslant M$.*

(1) $N \in E_\tau(M) \Longleftrightarrow (\forall\, P \leqslant M,\, P \cap N \in \mathcal{T} \Longrightarrow P \in \mathcal{T})$.

(2) $N \in S_\tau(M) \Longleftrightarrow (\forall\, P \leqslant M,\, M/(P+N) \in \mathcal{T} \Longrightarrow M/P \in \mathcal{T})$.

Proof. (1) See [3, Proposition 5.3(1)].

(2) Assume that N is τ-small, i.e., $\overline{N}$ is a small element of $\mathrm{Sat}_\tau(M)$, and let $P \leqslant M$ with $M/(P+N) \in \mathcal{T}$. Then $M = \overline{P+N} = \overline{P} \vee \overline{N}$, so $\overline{P} = M$, i.e., $M/P \in \mathcal{T}$, as desired.

Conversely, if N has the stated property, let $X \in \mathrm{Sat}_\tau(M)$ with $\overline{N} \vee X = M$, i.e., $\overline{\overline{N} + X} = M$. By [3, Lemma 5.1(1)], we have $\overline{N + X} = M$, i.e., $M/(N+X) \in \mathcal{T}$. Then $M/X \in \mathcal{T}$ by hypothesis, in other words $X = \overline{X} = M$. This shows that $\overline{N}$ is a small element of $\mathrm{Sat}_\tau(M)$, i.e., N is τ-small. □

We define now the relative version of $\mathrm{Soc}\,(M)$, $\mathrm{Rad}\,(M)$, and $\mathrm{Jac}\,(M)$ as

$$\mathrm{Soc}_\tau(M) := \overline{\sum_{A \in A_\tau(M)} A},$$

$$\mathrm{Rad}_\tau(M) := \overline{\sum_{N \in S_\tau(M)} N},$$

$$\mathrm{Jac}_\tau(M) := \bigcap_{P \in \mathrm{Max}_\tau(M)} P,$$

and call them the *τ-socle, τ-radical,* and *τ-Jacobson radical* of M, respectively. Notice that $\mathrm{Soc}_\tau(M)$, $\mathrm{Rad}_\tau(M)$, and $\mathrm{Jac}_\tau(M)$ are all elements of $\mathrm{Sat}_\tau(M)$. The result following the next lemma shows that they agree with the concepts of τ-$\mathbb{P}$ modules defined above.

Lemma 6.4. *Let $(N_i)_{i\in I}$ be a family of submodules of a module M_R. Then*

$$\overline{\sum_{i \in I} N_i} = \overline{\sum_{i\in I} \overline{N_i}}.$$

Proof. Clearly, $\sum_{i\in I} N_i \subseteq \sum_{i\in I} \overline{N_i}$, so $\overline{\sum_{i\in I} N_i} \subseteq \overline{\sum_{i\in I} \overline{N_i}}$. For the opposite inclusion, we have

$$\overline{\sum_{i\in I} \overline{N_i}} \subseteq \overline{\overline{\sum_{i\in I} N_i}} = \overline{\sum_{i\in I} N_i},$$

and we are done. □

Proposition 6.5. *The following statements hold for a module M_R.*

(1) $\mathrm{Soc}_{\tau}(M) = \mathrm{Soc}\,(\mathrm{Sat}_{\tau}(M)) = \overline{\sum_{C\in C_\tau(M)} C}.$

(2) $\mathrm{Rad}_{\tau}(M) = \mathrm{Rad}\,(\mathrm{Sat}_{\tau}(M)).$

(3) $\mathrm{Jac}_{\tau}(M) = \mathrm{Jac}\,(\mathrm{Sat}_{\tau}(M)).$

Proof. (1) First, observe that for any $N \leqslant P \leqslant M$ the τ-saturation $\overline{P/N}$ of P/N in M/N is exactly $\overline{P}/N$, and

$$N \in A_\tau(M) \iff \overline{N} \in A_\tau(M) \iff \overline{N} \in C_\tau(M) \iff \overline{N} \in A(\mathrm{Sat}_{\tau}(M)).$$

By definition and Lemma 6.4 we have

$$\mathrm{Soc}_{\tau}(M) := \overline{\sum_{A\in A_\tau(M)} A} = \overline{\sum_{A\in A_\tau(M)} \overline{A}} = \overline{\sum_{C\in C_\tau(M)} C} =$$

$$= \overline{\sum_{B\in A(\mathrm{Sat}_{\tau}(M))} B} = \bigvee_{B\in A(\mathrm{Sat}_{\tau}(M))} B = \mathrm{Soc}\,(\mathrm{Sat}_{\tau}(M)).$$

(2) The proof is similar to that of (1).

(3) By definitions, we have

$$\mathrm{Jac}_{\tau}(M) := \bigcap_{P\in \mathrm{Max}_{\tau}(M)} P = \bigcap_{P\in M(\mathrm{Sat}_{\tau}(M))} P = \mathrm{Jac}\,(\mathrm{Sat}_{\tau}(M)).$$

□

Notice that, to the best of our knowledge, the τ-Jacobson radical $\mathrm{Jac}_{\tau}(M)$ of a module M_R has been considered for the first time in the literature in [13], but only for $M_R = R_R$, and then used in [14] to prove in a pretty complicated manner the *Relative Hopkins-Levitzki Theorem,* also called the *Miller-Teply Theorem.* The reader is referred to [2] for more details about various aspects of the *Classical Hopkins-Levitzki Theorem,* including the *Relative Hopkins-Levitzki Theorem,* the *Categorical Hopkins-Levitzki Theorem,* the *Latticial Hopkins-Levitzki Theorem,* as well as the connections between them. Other properties of $\mathrm{Jac}_{\tau}(M)$ are given in [16] and [17].

Proposition 6.6. *For any morphism* $f : M_R \longrightarrow N_R$ *of right* R*-modules, the canonical mapping of lattices*

$$f_\tau : \mathrm{Sat}_\tau(M) \longrightarrow \mathrm{Sat}_\tau(N), \ \ f_\tau(A) = \overline{f(A)},$$

is a linear morphism.

Proof. If we set $K := \mathrm{Ker}\,(f)$ and $I := \mathrm{Im}\,(f)$, then the Fundamental Theorem of Isomorphism in Mod-R gives an isomorphism $M/K \simeq I$ of R-modules, which induces a lattice isomorphism

$$\mathrm{Sat}_{\,\tau}(M/K) \simeq \mathrm{Sat}_{\,\tau}(I).$$

By Lemma 6.1, we obtain the following sequence of canonical lattice isomorphisms

$$\mathrm{Sat}_{\,\tau}(M/K) \simeq \mathrm{Sat}_{\,\tau}(\overline{M}/\overline{K}\,) \simeq \mathrm{Sat}_{\,\tau}(M/\overline{K}\,) \simeq [\,\overline{K}, M\,] \simeq \mathrm{Sat}_{\,\tau}(\overline{I}) \simeq [\,\overline{0}, \overline{I}\,],$$

where the interval $[\,\overline{0}, \overline{I}\,]$ is considered in the lattice $\mathrm{Sat}_{\,\tau}(N)$. It is easily checked that their composition is exactly the isomorphism

$$\mathrm{Sat}_{\,\tau}(M/K) \overset{\sim}{\longrightarrow} [\,\overline{0}, \overline{I}\,],\ X/K \mapsto \overline{f(X)},$$

and we are done. □

Of course, all the results of the previous sections have relative versions obtained by specializing them from an arbitrary modular lattice L for the upper continuous modular lattice $\mathrm{Sat}_\tau(M)$ of any module M_R. No further proofs are required. We shall present below only a few results, and leave the others to the reader.

Proposition 6.7. *For any morphism* $f : M_R \longrightarrow N_R$ *of right modules one has*

$$f(\mathrm{Soc}_{\,\tau}(M)) \leqslant \mathrm{Soc}_{\,\tau}(N),\ \ f(\mathrm{Rad}_{\,\tau}(M)) \leqslant \mathrm{Rad}_{\,\tau}(N),\ \ \textit{and}\ f(\mathrm{Jac}_{\,\tau}(M)) \leqslant \mathrm{Jac}_{\,\tau}(N). \quad \square$$

As in Proposition 5.4, the next result is hard to be proved without invoking its latticial counterpart, where the condition that the considered lattice is compactly generated is essential.

Proposition 6.8. *The following statements hold for a* τ*-compactly generated module* M_R*.*

(1) $\mathrm{Soc}_{\,\tau}(M) = \displaystyle\bigcap_{X \in E(\mathrm{Sat}_{\,\tau}(M))} X = \bigcap_{N \in E_\tau(M)} \overline{N}.$

(2) $\mathrm{Rad}_{\,\tau}(M) = \mathrm{Jac}_{\,\tau}(M) = \displaystyle\bigcap_{P \in \mathrm{Max}_{\,\tau}(M)} P.$ □

Lemma 6.9. ([18, Proposition 1.1, Chap. XXIII]). *The following assertions are equivalent for a module* M_R*.*

(1) M *is* τ*-compact.*

(2) *The filter* $F(M) := \{\, N \leqslant M \,|\, M/N \in \mathcal{T}\,\}$ *has a basis consisting of finitely generated submodules, i.e.,* $\forall\, N \leqslant M$ *with* $M/N \in \mathcal{T}$, $\exists\, N' \leqslant N$ *such that* N' *is finitely generated and* $M/N' \in \mathcal{T}$. □

Proposition 6.10. *Let* M_R *be a module such that the filter* $F(M)$ *has a basis consisting of finitely generated submodules. Then* M *satisfies the relative condition below*

$$(\mathrm{KL}_\tau) \qquad \forall\, N \in \mathrm{Sat}_\tau(M) \setminus \{M\}, \exists\, P \in \mathrm{Max}_\tau(M) \text{ with } N \leqslant P.$$

Proof. Use Lemma 6.9 and apply Remarks 4.2(2) for the lattice $L = \mathrm{Sat}_\tau(M)$. □

Corollary 6.11. *Any* τ*-Noetherian module* M_R *satisfies the condition* (KL_τ).

Proof. By [7, Corollaire 1.6], for any τ-Noetherian module M, the filter $F(M)$ has a basis consisting of finitely generated submodules, so the result follows from Proposition 6.10. □

Corollary 6.12. *For any module* M_R *such that the filter* $F(M)$ *has a basis consisting of finitely generated submodules, in particular, for any* τ*-Noetherian module* M*, we have*

$$\mathrm{Rad}_\tau(M) = \mathrm{Jac}_\tau(M).$$

Proof. The result follows from Proposition 6.10 by specializing Proposition 4.4(3) for the lattice $\mathrm{Sat}_\tau(M)$. □

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