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Simion Stoilow Institute of Mathematics of the Romanian Academy, Research Unit 5, P.O. Box 1-764, RO-010145 Bucharest 1, ROMANIA
E-mail: Toma.Albu@imar.ro
Bucharest Univerisity, Department of Mathematics Academy Str. 14. RO - 010014 Bucharest 1, ROMANIA
E-mail: miosifg@gmail. com

Modular C_{11} lattices and lattice preradicals

TOMA ALBU
Simion Stoilow Institute of Mathematics of the Romanian Academy
Research Unit 5, P.O. Box 1 - 764
RO - 010145 Bucharest 1, ROMANIA
e-mail: Toma.Albu@imar.ro

MIHAI IOSIF
Bucharest University, Department of Mathematics
Academiei Str. 14, RO - 010014 Bucharest 1, ROMANIA
e-mail: miosifg@gmail.com

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Abstract

This paper deals with properties of modular C_{11} lattices involving hereditary preradicals on hereditary classes of modular lattices. Applications are given to Grothendieck categories and module categories equipped with hereditary torsion theories.

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Introduction

In our recent papers [9] and [8] we have introduced and investigated the conditions (C_i), $i = 1, 2, 3, 11, 12,$ in arbitrary bounded lattices and preradicals in bounded modular lattices, respectively. This paper studies the behavior under lattice preradicals of the condition (C_{11}) in modular lattices. The main ingredient in our study is the concept of a linear morphism of lattices introduced in [5]. As in [9] and [6], we shall also illustrate here a general strategy which consists on putting a module-theoretical definition/result into a latticial frame, in order to translate that definition/result to Grothendieck categories and to module categories equipped with a hereditary torsion theory.

In Section 0 we list some general definitions, notation, and results on lattices from [3], [4], and [15], as well as two basic results of [9] on C_{11} lattices, that will be used in the sequel.

Section 1 presents some basic definitions and results of [5] and [8] on linear morphisms of lattices and lattice preradicals, respectively.

In Section 2 we prove the main result of the paper, which is the latticial counterpart of the module-theoretical main result of [14] on C_{11} modules. Our proof is not a simple adaptation of the corresponding one in the module case because not all the involved module-theoretical tools work in a latticial frame.

In Section 3 we describe a weaker version of a lattice preradical, that we used as a first attempt to specialize our main result to objects of Grothendieck categories. Even if this attempt was not successfully, the instrument devised for it is worth mentioning in the general context of lattice preradicals.

In Sections 4 and 5 we do manage to specialize the main result of Section 2 for the lattice $\mathcal{L}(X)$ of all subobjects of an object X of a Grothendieck category $\mathcal{G}$ and for the lattice $\mathrm{Sat}_\tau(M_R)$ of all τ-saturated submodules of a module M_R with respect to a hereditary torsion theory τ on Mod-R, by using another instrument of specialization proposed by us in [7].

0 Preliminaries

All lattices considered in this paper are assumed to have a least element denoted by 0 and a greatest element denoted by 1, in other words they are *bounded.* Throughout this paper, L will always denote such a lattice. We shall denote by $\mathcal{L}$ the class of all (bounded) lattices and by $\mathcal{M}$ the class of all (bounded) modular lattices.

For a lattice L and elements $a \leqslant b$ in L we write

$$b/a := [a, b] = \{\ x \in L \mid a \leqslant x \leqslant b\,\}.$$

An *initial interval* of b/a is any interval c/a for some $c \in b/a$.

An element $c \in L$ is a *complement* (in L) if there exists an element $a \in L$ such that $a \wedge c = 0$ and $a \vee c = 1$; we say in this case that c is a *complement* of a (in L). One denotes by $D(L)$ the set of all complements of L. The lattice L is said to be *complemented* if every element of L has a complement in L.

For a lattice L and $a, b, c \in L$, the notation $a = b \dot{\vee} c$ will mean that $a = b \vee c$ and $b \wedge c = 0$, and we say that a is a *direct join* of b and c. Similarly, as for modules, one defines the concepts of a direct join and an independent family of elements of L. (see, e.g., [3, §1.2] or [4, §1.2]).

An element $a \in L$ is said to be an *atom* of L if $a \neq 0$ and $a/0 = \{0, a\}$. The *socle* $\mathrm{Soc}(L)$ of a complete lattice L is the join of all atoms of L.

An element $b \in L$ is a *pseudo-complement* (in L) if there exists an element $a \in L$ such that $a \wedge b = 0$ and b is maximal in the set of all elements c in L with $a \wedge c = 0$; we say in this case that b is a *pseudo-complement* of a. As in [3] or [4], L is called *pseudo-complemented* if every element of L has a pseudo-complement, and *strongly pseudo-complemented* if for all $a, b \in L$ with $a \wedge b = 0$, there exists a pseudo-complement p of a in L such that $b \leqslant p$. Every upper continuous modular lattice is strongly pseudo-complemented.

An element $e \in L$ is said to be *essential* (in L) if $e \wedge x \neq 0$ for every $x \neq 0$ in L. One denotes by $E(L)$ the set of all essential elements of L.

An element $c \in L$ is said to be *closed in* L if $c \notin E(a/0)$ for all $a \in L$ with $c < a$, and $C(L)$ will denote the set of all closed elements of L. For an element $a \in L$, we say that $c \in L$ is a *closure* of a in L if $a \in E(c/0)$ and $c \in C(L)$. Clearly, c is a closure of a in L if and only if c is a maximal element in the set $\{ x \in L \,|\, a \in E(x/0)\}$. A lattice L is called *essentially closed* if for each element $a \in L$ there exists a closure of a in L. Every strongly pseudo-complemented lattice (hence every upper continuous modular lattice) is essentially closed by [3, Theorem 1.2.24] or Theorem [4, Theorem 1.2.24]. We say that $a \in L$ has a *unique closure* in L if a has exactly one closure in L.

For all other undefined notation and terminology on lattices, the reader is referred to [3], [4], [12], and/or [15].

We summarize the following notation that will be used in this paper.

$E(L) :=$ the set of all *essential* elements of L (E for "*Essential*"),

$C(L) :=$ the set of all *closed* elements of L (C for "*Closed*"),

$D(L) :=$ the set of all *complement* elements of L (D for "*Direct summand*"),

We introduced and studied in [9] and [6] the condition (C_{11}) on lattices as the latticial counterpart of the known corresponding condition on modules (see [13] and [14]).

Definition 0.1. *We say that a lattice L satisfies the condition (C_{11}), or shortly, L is a C_{11} lattice, if for every $x \in L$ there exists a pseudo-complement p of x with $p \in D(L)$.* $\square$

We present below two results that will be used in Section 2.

Proposition 0.2. ([9, Proposition 1.8]). *A lattice $L \in \mathcal{M}$ is C_{11} $\Longleftrightarrow$ $\forall\, x \in L, \exists\, d \in D(L)$ with $d \wedge x = 0$ and $d \vee x \in E(L)$.* $\square$

Proposition 0.3. ([9, Proposition 2.5]). *Let $L \in \mathcal{M}$, and let $(a_i)_{1 \leqslant i \leqslant n}$ be a finite independent family of elements of L such that $1 = \dot{\bigvee}_{1 \leqslant i \leqslant n} a_i$ and $a_i/0$ is C_{11} for all $1 \leqslant i \leqslant n$. Then L is C_{11}.* $\square$

1 Linear morphisms of lattices and lattice preradicals

In this section we recall from [5] and [8] the concepts of a *linear morphism* and of a *lattice preradical,* respectively, and list some of their basic properties.

Definition 1.1. ([5, Definitions 1.1]). *A mapping $f : L \longrightarrow L'$ between the lattices L and L' is called a* linear morphism *if there exist $k \in L$, called a* kernel *of f, and $a' \in L'$ such that the following two conditions are satisfied.*

(1) $f(x) = f(x \vee k), \forall\, x \in L$.

(2) *f induces an isomorphism of lattices $\bar{f} : 1/k \xrightarrow{\sim} a'/0'$, $\bar{f}(x) = f(x), \forall\, x \in 1/k$, where $0'$ is the least element of L'.* $\square$

Proposition 1.2. *The following assertions hold for a linear morphism $f : L \longrightarrow L'$ with a kernel k.*

(1) *For $x, y \in L$, $f(x) = f(y) \iff x \vee k = y \vee k$.*

(2) *$f(k) = 0'$ and k is the greatest element of L having this property, so, the kernel of a linear morphism is uniquely determined.*

(3) *f commutes with arbitrary joins, i.e., $f\left(\bigvee_{i\in I} x_i\right) = \bigvee_{i\in I} f(x_i)$ for any family $(x_i)_{i\in I}$ of elements of L, provided both joins exist.*

(4) *f is an increasing mapping.*

(5) *f preserves intervals, i.e., for any $u \leqslant v$ in L, one has $f(v/u) = f(v)/f(u)$.*

(6) *If $L \in \mathcal{M}$, then for any $a \in L$, the restriction $f_a : a/0 \longrightarrow L'$, $f_a(x) = f(x)$, $\forall\, x \in a/0$, of f to $a/0$ is a linear morphism with kernel $a \wedge k$.*

Proof. (1) and (2) are parts of [5, Proposition 1.3], (3) and (5) are exactly [8, Lemma 0.6], and (4) is exactly [5, Corollary 1.4]).

(6) Set $k_a := a \wedge k$. Since $k_a \leqslant k$ we have $f(k_a) = 0'$. Thus

$$f_a(x) = f(x) = f(x) \vee f(k_a) = f(x \vee k_a) = f_a(x \vee k_a),\ \forall\, x \in a/0.$$

Now, f induces a lattice isomorphism $\overline{f} : 1/k \overset{\sim}{\longrightarrow} f(1)/0'$, $\overline{f}(x) = f(x)$, $\forall\, x \in L$. By a second restriction, we obtain a lattice isomorphism

$$g : (a \vee k)/k \overset{\sim}{\longrightarrow} f(a \vee k)/0',\ g(x) = f(x),\ \forall\, x \in (a \vee k)/k.$$

By modularity, we have the lattice isomorphism

$$\varphi : a/(a \wedge k) \overset{\sim}{\longrightarrow} (a \vee k)/k,\ \varphi(x) = x \vee k,\ \forall\, x \in a/(a \wedge k).$$

If we set $\overline{f_a} := g \circ \varphi : a/(a \wedge k) \overset{\sim}{\longrightarrow} f(a \vee k)/0'$, then

$$\overline{f_a}(x) = g(\varphi(x)) = f(x \vee k) = f(x) = f_a(x), \forall\, x \in a/(a \wedge k),$$

hence f_a induces the lattice isomorphism

$$\overline{f_a} : a/k_a \overset{\sim}{\longrightarrow} f_a(a)/0',\ \overline{f_a}(x) = f_a(x),\ \forall\, x \in a/k_a\,,$$

i.e., f_a is a linear morphism with kernel $k_a = a \wedge k$, as desired. □

Proposition 1.3. ([5, Proposition 2.2]). *The following statements hold.*

(1) *The class $\mathcal{M}$ of all (bounded) modular lattices becomes a category, denoted by $\mathcal{LM}$, if for any $L, L' \in \mathcal{M}$ one takes as morphisms from L to L' all the linear morphisms from L to L'.*

(2) *The isomorphisms in the category $\mathcal{LM}$ are exactly the isomorphisms in the full category $\mathcal{M}$ of the category $\mathcal{L}$ of all (bounded) lattices.*

(3) *The monomorphisms in the category $\mathcal{LM}$ are exactly the injective linear morphisms.*

(4) *The epimorphisms in the category $\mathcal{LM}$ are exactly the surjective linear morphisms.*

(5) *The subobjects of $L \in \mathcal{LM}$ can be viewed as the intervals $a/0$, $a \in L$.* □

Definitions 1.4. ([11, Definitions 2.1]). *Let $\varnothing \neq \mathcal{C} \subseteq \mathcal{L}$. We say that:*

(1) *$\mathcal{C}$ is an* abstract class *if it is closed under lattice isomorphisms, i.e., if $L, K \in \mathcal{L}$, $K \simeq L$, and $L \in \mathcal{C}$, then $K \in \mathcal{C}$.*

(2) *$\mathcal{C}$ is* hereditary *if it is an abstract class and for any $L \in \mathcal{L}$ and any $a \leqslant b \leqslant c$ in L such that $c/a \in \mathcal{C}$, it follows that $b/a \in \mathcal{C}$.*

For any non-empty subclass $\mathcal{C}$ of $\mathcal{M}$ we shall denote by $\mathcal{LC}$ the full subcategory of $\mathcal{LM}$ having $\mathcal{C}$ as the class of its objects.

Proposition 1.5. ([8, Proposition 2.3]). *The following assertions are equivalent for an abstract subclass $\mathcal{C}$ of $\mathcal{M}$.*

(1) *$\mathcal{C}$ is hereditary.*

(2) *For any $L \in \mathcal{C}$, the subobjects of L in the category $\mathcal{LC}$ can be viewed as the initial intervals $a/0$ of $L = 1/0$, $a \in L$.*

In this case, the monomorphisms in the category $\mathcal{LC}$ are precisely the injective linear morphisms. □

Definition 1.6. ([8, Definitions 2.1]). *Let $\mathcal{C}$ be a hereditary subclass of $\mathcal{M}$. A* lattice preradical *on $\mathcal{C}$ is any functor $r : \mathcal{LC} \longrightarrow \mathcal{LC}$ satisfying the following two conditions.*

(1) *$r(L) \leqslant L$, i.e., $r(L)$ is a subobject of L, for any $L \in \mathcal{LC}$.*

(2) *For any morphism $f : L \longrightarrow L'$ in $\mathcal{LC}$, $r(f) : r(L) \longrightarrow r(L')$ is the restriction of f, i.e., $f(r(L)) \subseteq r(L')$.*

In other words, a lattice preradical is nothing else than a subfunctor of the identity functor $1_{\mathcal{LC}}$ of the category $\mathcal{LC}$. □

Let $\mathcal{C}$ be a hereditary subclass of $\mathcal{M}$, and let $r : \mathcal{LC} \longrightarrow \mathcal{LC}$ be a lattice preradical on $\mathcal{C}$. By Proposition 1.5, for every $L \in \mathcal{C}$ and $a \in L$, the subobject $r(a/0)$ of L in $\mathcal{LC}$ is necessarily an initial interval of $a/0$. We denote

$$r(a/0) := a^r/0.$$

If $a \leqslant b$ in L, then $a/0, b/0$ are in $\mathcal{C}$ because $\mathcal{C}$ is hereditary. The inclusion mapping $\iota : a/0 \hookrightarrow b/0$ is clearly a linear morphism, thus it is a morphism in $\mathcal{LC}$. Applying now r we obtain $r(\iota) : a^r/0 \longrightarrow b^r/0$ as a restriction of ι, and so

$$a \leqslant b \Longrightarrow a^r \leqslant b^r.$$

Moreover, since $a^r \leqslant a$, we also have

$$a \leqslant b \Longrightarrow a^r \leqslant a \wedge b^r.$$

For any lattice $L \in \mathcal{C}$ and any a, b in L such that $a \dot{\vee} b = 1$, the mapping

$$q : L \longrightarrow a/0,\ q(x) := (x \vee b) \wedge a,$$

is a surjective linear morphism with kernel b (see [8, Example 0.2(3)]), so q is a morphism in $\mathcal{LC}$. This is the latticial counterpart of the canonical projection $M \oplus M' \longrightarrow M$ for two modules M_R and M'_R.

Because r is a preradical, the linear morphism $q : L \longrightarrow a/0$ entails by restriction the linear morphism

$$r(q) : r(L) = r(1/0) = 1^r/0 \longrightarrow r(a/0) = a^r/0,$$

so $r(q)(1^r) = q(1^r) = (1^r \vee b) \wedge a \leqslant a^r = q(1)^r$, hence

$$a \dot{\vee} b = 1 \Longrightarrow (1^r \vee b) \wedge a \leqslant a^r.$$

We shall use in the sequel the inequalities above without any further reference.

Proposition 1.7. ([8, Proposition 1.3]). *For any lattice $L \in \mathcal{M}$ and any finite independent family $(a_i)_{1 \leqslant i \leqslant n}$ of L, with n a positive integer, one has*

$$\Big(\dot{\bigvee_{1 \leqslant i \leqslant n}} a_i\Big)^r = \dot{\bigvee_{1 \leqslant i \leqslant n}} a_i^r.$$

□

Definition 1.8. *Let $r : \mathcal{LM} \longrightarrow \mathcal{LM}$ be a lattice preradical. We say that r is* hereditary *or* left exact *if for all $L \in \mathcal{LM}$, $a^r = a \wedge 1^r$ for every $a \in L$.* □

For example, by [8, Example 3.6], the assignment $L \mapsto \mathrm{Soc}\,(L)/0$ defines a hereditary preradical on the full subcategory $\mathcal{LM}_u$ of $\mathcal{LM}$ consisting of all upper continuous modular lattices.

2 The main result

In this section we present the latticial counterpart of the main result of [14] concerning the behavior of C_{11} modules under module preradicals.

Lemma 2.1. *Let $\mathcal{C}$ be a hereditary subclass of $\mathcal{M}$, let r be a preradical on $\mathcal{C}$, and let $L \in \mathcal{C}$ be a C_{11} lattice. Then, there exist $a_1, a_2 \in L$ such that*

$$1 = a_1 \dot{\vee} a_2, \ \ 1^r = a_1^r \in E(a_1/0), \ \ and \ \ a_2^r = 0.$$

Moreover, a_1 is a closure of 1^r in L.

Proof. Because L is a C_{11} lattice, by Proposition 0.2, there exist $a_1, a_2 \in L$ such that $a_1 \dot{\vee} a_2 = 1$, $1^r \wedge a_2 = 0$, and $1^r \vee a_2 \in E(L)$. Notice that both $a_1/0$ and $a_2/0$ are members of $\mathcal{C}$ because $\mathcal{C}$ is hereditary. Since $a_2^r \leqslant 1^r \wedge a_2 = 0$ we have $a_2^r = 0$.

Now $1^r \vee a_2 \in E(L)$, so $(1^r \vee a_2) \wedge a_1 \in E(a_1/0)$. But $(1^r \vee a_2) \wedge a_1 \leqslant a_1^r$, thus $a_1^r \in E(a_1/0)$. By Proposition 1.7, we have

$$1^r = a_1^r \vee a_2^r = a_1^r.$$

Because a_1 is a complement in the modular lattice L, it is also a pseudo-complement in L. By [3, Proposition 1.2.16] or [4, Proposition 1.2.16], it follows that $a_1 \in C(L)$, hence a_1 is a closure of 1^r in L. □

Lemma 2.2. *Let L be an essentially closed modular lattice. Suppose that $a \in L$ has a unique closure c in L. Then c is the greatest element of the set $\{ x \in L \mid a \in E(x/0) \}$.*

Proof. By definition, c is a maximal element of the set $C := \{ x \in L \mid a \in E(x/0) \}$.

Let $x \in C$. Since L is essentially closed, x has a closure $\overline{x}$ in L. Then $a \in E(\overline{x}/0)$ because $x \in E(\overline{x}/0)$. Since $\overline{x} \in C(L)$, it follows that $\overline{x}$ is a closure of a in L. But a has a unique closure c in L, so $\overline{x} = c$, and then $x \leqslant c$. Thus, c is the greatest element of C. □

Lemma 2.3. *Let L be an is essentially closed modular lattice. Suppose that $1 = a_1 \dot{\vee} a_2$. Then $a_1/0$ is essentially closed, and for any $x \in a_1/0$, if c is a closure of x in $a_1/0$ then c is a closure of x in L.*

Proof. Because $a_1, a_2 \in D(L)$ we have $a_1, a_2 \in C(L)$. Since L is essentially closed, it follows by [3, Corollary 1.2.23] or [4, Corollary 1.2.23] that $1/a_2$ is also essentially closed. By modularity, the assignment $t \mapsto t \vee a_2$ establishes a lattice isomorphism ψ from $a_1/0 = a_1/(a_1 \wedge a_2)$ to $(a_1 \vee a_2)/a_2 = 1/a_2$, so $a_1/0$ is essentially closed.

Let $x \in a_1/0$, and let c be a closure of x in $a_1/0$. Then $x \in E(c/0)$ and $c \in C(a_1/0)$, so, using the isomorphism ψ above we deduce that $c \vee a_2 \in C(1/a_2)$. Let $e \in L$ with $c \in E(e/0)$. We have $c \leqslant a_1$ and $a_1 \wedge a_2 = 0$, so $c \wedge a_2 = 0$ and consequently, $c \wedge (e \wedge a_2) = 0$. But $c \in E(e/0)$, so $e \wedge a_2 = 0$. Hence, the assignment $s \mapsto s \vee a_2$ produces a lattice isomorphism φ from $e/0 = e/(e \wedge a_2)$ to $(e \vee a_2)/a_2$, and so $c \vee a_2 \in E((e \vee a_2)/a_2)$. Since $c \vee a_2 \in C(1/a_2)$, it follows that $c \vee a_2 = e \vee a_2$ and, by the isomorphism φ above, it follows that $c = e$. Hence $c \in C(L)$, as desired. □

Theorem 2.4. *Let $\mathcal{C}$ be a hereditary subclass of $\mathcal{M}$, and let r be a hereditary preradical on $\mathcal{C}$. Then, the following assertions are equivalent for an essentially closed lattice $L \in \mathcal{C}$ such that 1^r has a unique closure in L.*

(1) *L is a C_{11} lattice.*

(2) *There exist $a_1, a_2 \in L$ such that $1 = a_1 \dot{\vee} a_2$, $a_1^r \in E(a_1/0)$, $a_2^r = 0$, and the lattices $a_1/0$ and $a_2/0$ are both C_{11}.*

Proof. (1)$\Longrightarrow$(2): Using Lemma 2.1, there exist $a_1, a_2 \in L$ such that

$$1 = a_1 \dot{\vee} a_2,\ \ 1^r = a_1^r \in E(a_1/0),\ \ a_2^r = 0,\ \text{ and } a_1 \text{ is a closure of } 1^r \text{ in } L.$$

By hypothesis, a_1 is the unique closure of 1^r in L.

First, we prove that $a_1/0$ is a C_{11} lattice. To do that, let $x \in a_1/0$. Since L is a C_{11} lattice, by Proposition 0.2, there exist $d_1, d_2 \in L$ such that $1 = d_1 \dot{\vee} d_2$, $(x \vee a_2) \wedge d_1 = 0$, and $x \vee a_2 \vee d_1 \in E(L)$. Consider the linear morphism

$$q_1 : L \longrightarrow a_1/0,\ q_1(t) = (t \vee a_2) \wedge a_1,\ \ \forall\, t \in L,$$

which has the kernel a_2. Then, by Proposition 1.2(6), its restriction

$$q_1' : d_1/0 \longrightarrow q_1(d_1)/0,\ \ q_1'(t) = q_1(t),\ \ \forall\, t \in d_1/0,$$

has the kernel $a_2 \wedge d_1$. But $a_2 \wedge d_1 \leqslant (x \vee a_2) \wedge d_1 = 0$, so $a_2 \wedge d_1 = 0$, and hence q_1' is a lattice isomorphism. It follows that $r(q_1') : d_1^r/0 \longrightarrow q_1(d_1)^r/0$ is a lattice isomorphism. Therefore

$$q_1(d_1^r)\ =\ q_1'(d_1^r)\ =\ r(q_1')(d_1^r)\ =\ q_1(d_1)^r.$$

Because r is a hereditary preradical, we have

$$q_1(d_1)^r\ =\ q_1(d_1) \wedge a_1^r\ =\ q_1(d_1) \wedge 1^r,$$

and so,

$$q_1(d_1^r)\ =\ q_1'(d_1^r)\ =\ q_1(d_1) \wedge 1^r.$$

Since $1^r \in E(a_1/0)$ and $q_1(d_1) \leqslant a_1$, we deduce that

$$q_1'(d_1^r) = q_1(d_1) \wedge 1^r \in E(q_1(d_1)/0) = E(q_1'(d_1)/0),$$

and then $d_1^r \in E(d_1/0)$ because q_1' is a lattice isomorphism.

We have $d_2^r \leqslant d_2$ and $d_1 \wedge d_2 = 0$, so $d_1 \wedge d_2^r = 0$. It follows that $d_1^r \vee d_2^r \in E((d_1 \vee d_2^r)/0)$ by [3, Lemma 1.2.7] or [4, Lemma 1.2.7]. By Proposition 1.7, we have $1^r = d_1^r \vee d_2^r$, hence $1^r \in E((d_1 \vee d_2^r)/0)$. Since L is essentially closed and a_1 is the unique closure of 1^r, by Lemma 2.2 it follows that a_1 is the greatest element of the set $\{\, x \in L \,|\, 1^r \in E(x/0) \,\}$. Therefore $d_1 \vee d_2^r \leqslant a_1$, and so $d_1 \leqslant a_1$.

By modularity we have

$$a_1\ =\ d_1 \dot{\vee} (a_1 \wedge d_2),$$

so $d_1 \in D(a_1/0)$, and since $x \vee d_1 \leqslant a_1$, we also have

$$x \vee d_1 \;=\; x \vee d_1 \vee 0 \;=\; (x \vee d_1) \vee (a_2 \wedge a_1) \;=\; (x \vee d_1 \vee a_2) \wedge a_1.$$

Since $x \vee d_1 \vee a_2 \in E(L)$, we deduce that $x \vee d_1 \in E(a_1/0)$. Now observe that $x \wedge d_1 \leqslant (x \vee a_2) \wedge d_1 = 0$, and then, by Proposition 0.2 it follows that $a_1/0$ is a C_{11} lattice.

Now, we are going to prove that $a_2/0$ is a C_{11} lattice. To this end, let $y \in a_2/0$. Since L is a C_{11} lattice, by Proposition 0.2, there exist $e_1, e_2 \in L$ such that $1 = e_1 \dot\vee e_2$, $(y \vee a_1) \wedge e_1 = 0$, and $y \vee a_1 \vee e_1 \in E(L)$.

We have $a_1 \wedge e_1 \leqslant (y \vee a_1) \wedge e_1 = 0$, thus $a_1 \wedge e_1 = 0$. We also have $e_1^r \leqslant e_1 \wedge 1^r$, and since $1^r \leqslant a_1$, we deduce that $e_1 \wedge 1^r \leqslant e_1 \wedge a_1 = 0$, and so $e_1^r = 0$. By Proposition 1.7, we obtain

$$1^r \;=\; e_1^r \vee e_2^r \;=\; e_2^r \;\leqslant\; e_2.$$

By Lemma 2.3, $e_2/0$ is essentially closed. Let f be a closure of 1^r in $e_2/0$. Again by Lemma 2.3, f is also a closure of 1^r in L and, since 1^r has a unique closure a_1 in L, we deduce that $f = a_1$, so $a_1 \leqslant e_2$. By modularity, we have

$$e_2 \;=\; 1 \vee e_2 \;=\; (a_1 \vee a_2) \wedge e_2 \;=\; a_1 \vee (a_2 \wedge e_2),$$

hence

$$1 \;=\; e_1 \vee a_1 \vee (a_2 \wedge e_2).$$

Set $z := (e_1 \vee a_1) \wedge a_2 \in a_2/0$. By modularity, we have

$$a_2 \;=\; 1 \wedge a_2 \;=\; ((e_1 \vee a_1) \vee (a_2 \wedge e_2)) \wedge a_2 \;=\; ((e_1 \vee a_1) \wedge a_2) \vee (a_2 \wedge e_2) \;=\; z \vee (a_2 \wedge e_2),$$

and

$$z \wedge (a_2 \wedge e_2) \;=\; (e_1 \vee a_1) \wedge e_2 \wedge a_2 \;=\; (a_1 \vee (e_1 \wedge e_2)) \wedge a_2 \;=\; (a_1 \vee 0) \wedge a_2 \;=\; a_1 \wedge a_2 \;=\; 0.$$

Therefore

$$a_2 \;=\; z \dot\vee (a_2 \wedge e_2),$$

i.e., $z \in D(a_2/0)$.

In order to prove that $a_2/0$ is a C_{11} lattice, by Proposition 0.2, it is sufficient to show that $y \wedge z = 0$ and $y \vee z \in E(a_2/0)$.

By modularity, we have

$$\begin{aligned} y \wedge z = y \wedge ((e_1 \vee a_1) \wedge a_2) = y \wedge (e_1 \vee a_1) = y \wedge (y \vee a_1) \wedge (e_1 \vee a_1) = \\ = y \wedge (((y \vee a_1) \wedge e_1) \vee a_1) = y \wedge (0 \vee a_1) = y \wedge a_1 \leqslant a_2 \wedge a_1 = 0. \end{aligned}$$

so $y \wedge z = 0$.

In order to prove that $y \vee z \in E(a_2/0)$, observe that

$$y \vee z \;=\; y \vee z \vee (a_1 \wedge a_2) \;=\; (y \vee z \vee a_1) \wedge a_2.$$

By modularity, we have,

$$z \vee a_1 \;=\; ((e_1 \vee a_1) \wedge a_2) \vee a_1 \;=\; (e_1 \vee a_1) \wedge (a_2 \vee a_1) \;=\; (e_1 \vee a_1) \wedge 1 \;=\; e_1 \vee a_1,$$

and so

$$y \vee z \;=\; (y \vee z \vee a_1) \wedge a_2 \;=\; (y \vee e_1 \vee a_1) \wedge a_2.$$

But, by the choice of e_1 and e_2, we have $y \vee e_1 \vee a_1 \in E(L)$ and, thus $y \vee z \in E(a_2/0)$, that ends the proof.

(2)$\Longrightarrow$(1): Apply Proposition 0.3. □

Corollary 2.5. *Let $\mathcal{C}$ be a hereditary subclass of $\mathcal{M}$, and let r be a hereditary preradical on $\mathcal{C}$. Then, following assertions are equivalent for an essentially closed lattice $L \in \mathcal{C}$ such that 1^r is closed in L.*

(1) *L is a C_{11} lattice.*

(2) *There exists $a \in L$ with $1 = 1^r \dot{\vee} a$ such that the lattices $1^r/0$ and $a/0$ are both C_{11}.*

Proof. (1)$\Longrightarrow$(2): Notice that 1^r being closed, it has a unique closure, so we can apply Theorem 2.4 to deduce that there exist $a_1, a_2 \in L$ such that

$$1 = a_1 \dot{\vee} a_2,\;\; a_1^r \in E(a_1/0),\;\; a_2^r = 0, \text{ and both lattices } a_1/0 \text{ and } a_2/0 \text{ are } C_{11}.$$

Moreover, $1^r = (a_1 \dot{\vee} a_2)^r = a_1^r \dot{\vee} a_2^r = a_1^r$, thus $1^r \in E(a_1/0)$. By hypothesis, $1^r \in C(L)$, hence $a_1 = 1^r$. If we set $a = a_2$ we obtain (2).

(2)$\Longrightarrow$(1): Apply 0.3. □

3 Weakly lattice preradicals

According to our strategy explained in the Introduction, we are going to apply the latticial results from the previous section to Grothendieck categories and to module categories equipped with a hereditary torsion theory. However, the classes of lattices that are involved in these categorical and relative cases are not abstract (i.e., they are not necessarily closed under lattice isomorphisms), so, we have to adjust the latticial concepts and results of Section 2 to them, This was our first attempt to weaken our construction and to describe it is exactly the purpose of this section.

Thus, we present in this section weaker versions of the concepts of hereditary class of lattices and lattice preradical and show that any weakly lattice preradical r on a weakly hereditary class $\mathcal{C}$ of modular lattices can be uniquely extended to a lattice preradical $\widetilde{r}$ on the smallest hereditary class $\widetilde{\mathcal{C}}$ of lattices which includes $\mathcal{C}$. As a consequence of this fact, it follows that any result on a lattice preradical also holds for a weakly lattice preradical.

We hoped that, in particular, this will allow us to apply in the next sections the weak versions of Theorem 2.4 and its Corollary 2.5 to Grothendieck categories and module categories equipped with hereditary torsion theories. Example 4.1 at the beginning of the next section will show that our hope can not be fulfilled. We developed in our paper [7] another weaker version of lattice preradicals, that allows us to perform the desired specializations.

Definition 3.1. *We say that a subclass $\varnothing \neq \mathcal{C} \subseteq \mathcal{L}$, not necessarily abstract, is* weakly hereditary *if $a/0 \in \mathcal{C}$ for any $L \in \mathcal{C}$ and $a \in L$.* □

Thus, a hereditary class of lattice is nothing else than a weakly hereditary class which additionally is an abstract class. For example, the class

$$\mathcal{S} = \{\, \mathcal{L}(\mathbb{Z}/p^2\mathbb{Z}) \mid p > 0 \text{ prime number} \,\} \cup \{\, \mathcal{L}(p\,\mathbb{Z}/p^2\mathbb{Z}) \mid p > 0 \text{ prime number} \,\} \cup \{\, 0 \,\}$$

is weakly hereditary but not hereditary, where $\mathcal{L}(G)$ denotes the lattice of all subgroups of the Abelian group G.

Definition 3.2. *For any weakly hereditary class $\mathcal{C} \subseteq \mathcal{M}$ we define a* weakly lattice preradical *on $\mathcal{C}$ as a functor $r : \mathcal{LC} \longrightarrow \mathcal{LC}$ satisfying the following two conditions.*

(1) *$r(L)$ is an initial interval of L for any $L \in \mathcal{LC}$.*

(2) *For any morphism $f : L \longrightarrow L'$ in $\mathcal{LC}$, $r(f) : r(L) \longrightarrow r(L')$ is the restriction of f, i.e., $f(r(L)) \subseteq r(L')$.*

As in the case of "true" lattice preradicals, for a weakly lattice preradical r on the weakly hereditary class $\mathcal{C} \subseteq \mathcal{M}$, we set $r(a/0) = a^r/0$.

Clearly, every lattice preradical is also a weakly lattice preradical. The converse is not true. Indeed, for the class $\mathcal{S}$ defined above, the assignment

$$s(\mathcal{L}(\mathbb{Z}/p^2\mathbb{Z})) = s(\mathcal{L}(p\,\mathbb{Z}/p^2\mathbb{Z})) = \mathcal{L}(p\,\mathbb{Z}/p^2\mathbb{Z}) \text{ for any prime number } p > 0, \text{ and } s(0) = 0$$

defines a weakly preradical, as one can easily see by considering the possible linear morphisms between the members of $\mathcal{S}$.

First, we need the following simple fact.

Lemma 3.3. *Let r be a weakly lattice preradical on the weakly hereditary class $\mathcal{C} \subseteq \mathcal{M}$, and let $\varphi : L \overset{\sim}{\longrightarrow} L'$ be a lattice isomorphism with $L, L' \in \mathcal{C}$. Then $\varphi(1^r) = 1'^r$, where 1 (respectively, $1'$) is the greatest element of L (respectively, L').*

Proof. Notice that any lattice isomorphism is a linear morphism. We have $r(\varphi^{-1})(1'^r) \leqslant 1^r$, so $1'^r = r(\varphi \circ \varphi^{-1})(1'^r) = r(\varphi)(r(\varphi^{-1})(1'^r)) \leqslant r(\varphi)(1^r) \leqslant 1'^r$. Thus $1'^r = r(\varphi)(1^r) = \varphi(1^r)$. □

Proposition 3.4. *Let r be a weakly lattice preradical on the weakly hereditary class $\mathcal{C} \subseteq \mathcal{M}$, and set $\widetilde{\mathcal{C}} := \{\, \widetilde{L} \in \mathcal{M} \mid \exists\, L \in \mathcal{C}\,,\ L \simeq \widetilde{L} \,\}$. Then, the following assertions hold.*

(1) *$\widetilde{\mathcal{C}}$ is the smallest hereditary subclass of $\mathcal{M}$ that includes $\mathcal{C}$.*

(2) *r can be uniquely extended to a lattice preradical $\widetilde{r}$ on $\widetilde{\mathcal{C}}$, i.e., $\widetilde{r}(L) = r(L)$, $\forall\, L \in \mathcal{C}$.*

(3) *If r is a weakly hereditary preradical, then so $\widetilde{r}$ is a hereditary preradical.*

Proof. (1) Let $\widetilde{L} \in \widetilde{\mathcal{C}}$ and $M \in \mathcal{M}$ with $\widetilde{L} \simeq M$. Then, there exists $L \in \mathcal{C}$ such that $L \simeq \widetilde{L}$. Thus $L \simeq M$, and so $M \in \widetilde{\mathcal{C}}$. Hence $\widetilde{\mathcal{C}}$ is an abstract class.

Now, let $\widetilde{L} \in \widetilde{\mathcal{C}}$ and $b \in \widetilde{L}$. There exists $L \in \mathcal{C}$ and an isomorphism $\varphi : L \overset{\sim}{\longrightarrow} \widetilde{L}$. Then $b = \varphi(a)$ for some $a \in L$. Since $\mathcal{C}$ is weakly hereditary, we have $a/0 \in \mathcal{C}$. Clearly, φ induces a lattice isomorphism $a/0 \overset{\sim}{\longrightarrow} b/0$, and so $b/0 \in \widetilde{\mathcal{C}}$, which shows that $\widetilde{\mathcal{C}}$ is a hereditary class. For any hereditary subclass $\mathcal{H}$ of $\mathcal{M}$ such that $\mathcal{C} \subseteq \mathcal{H}$ one has $\widetilde{\mathcal{C}} \subseteq \mathcal{H}$, i.e., $\widetilde{\mathcal{C}}$ is the smallest hereditary subclass of $\mathcal{M}$ which includes $\mathcal{C}$.

(2) Let $\widetilde{L} \in \widetilde{\mathcal{C}}$, and let $\widetilde{1}$ (respectively, $\widetilde{0}$) denote the greatest (respectively least) element of $\widetilde{L}$. As above, there exists $L \in \mathcal{C}$ and a lattice isomorphism $\varphi : L \overset{\sim}{\longrightarrow} \widetilde{L}$. If p is a lattice preradical on $\widetilde{\mathcal{C}}$ that extends r, one has $p(L) = r(L)$, i.e., $1^p/0 = 1^r/0$, so $1^p = 1^r$. By Lemma 3.3 applied to the isomorphism $\varphi : L \overset{\sim}{\longrightarrow} \widetilde{L}$, we have $\varphi(1^r) = \varphi(1^p) = \widetilde{1}^{\,r} = \widetilde{1}^{\,p}$. This shows that for the given weakly lattice preradical r on $\mathcal{C}$ there exists at most one lattice preradical on $\widetilde{\mathcal{C}}$, say $\widetilde{r}$, defined by $\widetilde{r}(\widetilde{L}) := \varphi(1^r)/\,\widetilde{0}$, $\forall\, \widetilde{L} \in \widetilde{\mathcal{C}}$, i.e., $\widetilde{1}^{\,\widetilde{r}} := \varphi(1^r)$.

We are now going to show that $\widetilde{r}$ is actually a lattice preradical on $\widetilde{\mathcal{C}}$. Firstly, $\varphi(1^r)$ depends neither on L nor on φ. Indeed, if we pick another lattice $L' \in \mathcal{C}$ and a lattice isomorphism $\psi : L' \overset{\sim}{\longrightarrow} \widetilde{L}$, then we have an isomorphism $\psi^{-1} \circ \varphi : L \overset{\sim}{\longrightarrow} L'$. By Lemma 3.3, we have $(\psi^{-1} \circ \varphi)(1^r) = 1'^{\,r}$, and so $\varphi(1^r) = \psi(1'^{\,r})$, where $1'$ is the greatest element of L'.

Now, for $\widetilde{L}, \widetilde{L}' \in \widetilde{\mathcal{C}}$, let $g : \widetilde{L} \longrightarrow \widetilde{L}'$ be a linear morphism. In order to define $\widetilde{r}(g)$, it suffices to show that the restriction of g to $\widetilde{1}^{\,\widetilde{r}}/\widetilde{0}$ can be corestricted to $\widetilde{1'}^{\,\widetilde{r}}/\widetilde{0'}$, where $\widetilde{1'}$ (respectively, $\widetilde{0'}$) is the greatest (respectively, least) element of $\widetilde{L}'$, in other words, we have to show that $g(\widetilde{1}^{\,\widetilde{r}}) \leqslant \widetilde{1'}^{\,\widetilde{r}}$. To see this, consider two lattices $L, L' \in \mathcal{C}$ and two isomorphisms $\varphi : L \overset{\sim}{\longrightarrow} \widetilde{L}$ and $\varphi' : L' \overset{\sim}{\longrightarrow} \widetilde{L}'$. Set $f = \varphi'^{-1} \circ g \circ \varphi : L \longrightarrow L'$.

Since r is a weakly preradical and f is a linear morphism, we have $f(1^r) = r(f)(1^r) \leqslant 1'^{\,r}$. Thus

$$g(\widetilde{1}^{\,\widetilde{r}}) = (g \circ \varphi)(1^r) = (\varphi' \circ f)(1^r) \leqslant \varphi'(1'^{\,r}) = \widetilde{1'}^{\,r} = \widetilde{1'}^{\,\widetilde{r}},$$

and we are done.

(3) Let $\widetilde{L} \in \widetilde{\mathcal{C}}$ and $b \in \widetilde{L}$. Then, there exists $L \in \mathcal{C}$, a lattice isomorphism $\varphi : L \overset{\sim}{\longrightarrow} \widetilde{L}$, and $a \in L$ such that $\varphi(a) = b$. Since r is a weakly hereditary preradical, we have $a^r = a \wedge 1^r$. On the other hand, φ induces a lattice isomorphism $a/0 \overset{\sim}{\longrightarrow} b/\widetilde{0}$, thus $b^{\widetilde{r}} = \varphi(a^r)$ by the definition of $\widetilde{r}$ in (2). So, $b^{\widetilde{r}} = \varphi(a^r) = \varphi(a \wedge 1^r) = \varphi(a) \wedge \varphi(1^r) = b \wedge \widetilde{1}^{\,\widetilde{r}}$, as desired. $\square$

If we consider the class $\mathcal{S}$ and the weakly lattice preradical s defined above, then the corresponding hereditary class $\widetilde{\mathcal{S}}$ is the class of all chains with at most three elements, and the corresponding extension $\widetilde{s}$ of s to $\widetilde{\mathcal{S}}$ has an obvious definition. In fact, for any $S \in \widetilde{\mathcal{S}}$, $\widetilde{s}(S)$ is precisely the socle of S.

Remark 3.5. Proposition 3.4 shows that any result $\mathbb{T}$ on a lattice preradical also holds for a weakly lattice preradical. When we apply $\mathbb{T}$ to a weakly lattice preradical, we say that we apply the *weak form* of $\mathbb{T}$. □

4 Applications to Grothendieck categories

In this section we specialize the main result of Section 2 to Grothendieck categories. Recall that a *Grothendieck category* is an Abelian category with exact direct limits and with a generator.

Throughout this section $\mathcal{G}$ will denote a Grothendieck category, and for any object X of $\mathcal{G}$, $\mathcal{L}(X)$ will denote the upper continuous modular lattice of all subobjects of X. The reader is referred to [10] and [15] for more about Grothendieck categories.

We say that $X \in \mathcal{G}$ is a C_{11} object if the lattice $\mathcal{L}(X)$ is C_{11}. More generally, if $\mathbb{P}$ is any property on lattices, we say that an object $X \in \mathcal{G}$ is/has $\mathbb{P}$ if the lattice $\mathcal{L}(X)$ is/has $\mathbb{P}$. Similarly, a subobject Y of an object $X \in \mathcal{G}$ is/has $\mathbb{P}$ if the element Y of the lattice $\mathcal{L}(X)$ is/has $\mathbb{P}$. Thus, we obtain the concepts of an *essential* subobject of an object, *closed* subobject of an object, *complement* subobject of an object, *socle* of an object, etc. For a complement subobject of an object $X \in \mathcal{G}$ one uses the well-established term of a *direct summand* of X.

For any $X' \subseteq X$ in $\mathcal{G}$, we denote by $[X', X]$ the interval in the lattice $\mathcal{L}(X)$, and by

$$\varphi_{X/X'} : [X', X] \xrightarrow{\sim} \mathcal{L}(X/X')$$

the canonical lattice isomorphism $Z \mapsto Z/X'$, which is clearly a linear morphism of lattices.

Let $\mathcal{X}$ be a non-empty class of objects of $\mathcal{G}$. We say that $\mathcal{X}$ is *hereditary,* if it is closed under subobjects, i.e., for any $X \in \mathcal{X}$ and subobject Y of X in $\mathcal{G}$, we have $Y \in \mathcal{X}$. A *preradical* on $\mathcal{X}$ is just a subfunctor of the identity functor $1_{\mathcal{X}}$ of the full subcategory $\mathcal{X}$ of $\mathcal{G}$.

In general, a left exact preradical r on such a class $\mathcal{X}$ does not define a hereditary weakly preradical ϱ on the weakly hereditary class of lattices $\mathcal{L}_{\mathcal{X}} := \{\, \mathcal{L}(X) \,|\, X \in \mathcal{X}\}$ as one may guess by putting $\varrho(\mathcal{L}(X)) := \mathcal{L}(r(X))$, $\forall\, X \in \mathcal{X}$. To see this, for the reader's convenience, we reformulate the idea of [7, Example 2.2].

Example 4.1. Let $\mathcal{G}$ be the category of all Abelian groups, and let $\mathcal{X}$ be the class of all Abelian groups class having at most three subgroups. Then $\mathcal{X}$ is a hereditary class whose objects have totally ordered lattices of subgroups. For $X \in \mathcal{X}$ put $r(X) := \{\, x \in X \,|\, 2x = 0 \,\}$. The assignment $X \mapsto r(X)$ is a left exact preradical on $\mathcal{X}$ as well as on the whole $\mathcal{G}$. Notice that the weakly hereditary class of lattices $\mathcal{L}_{\mathcal{X}} := \{\, \mathcal{L}(X) \,|\, X \in \mathcal{X}\}$ contains only chains with at most three elements and includes the class $\mathcal{S}$ considered in Section 3.

We claim that there is no weakly preradical ϱ on $\mathcal{L}_{\mathcal{X}}$ such that

$$\varrho(\mathcal{L}(X)) = \mathcal{L}(r(X)),\ \forall\, X \in \mathcal{X}.$$

Indeed, suppose that such a ϱ exists, and set $X_1 := \mathbb{Z}/2\mathbb{Z}$, $X_2 := \mathbb{Z}/3\mathbb{Z}$. Then $r(X_1) = X_1 \neq 0$ and $r(X_2) = 0$. On the other hand, $\mathcal{L}(X_1)$ and $\mathcal{L}(X_2)$ are two-element isomorphic lattices in

$\mathcal{L}_{\mathcal{X}}$, so by Lemma 3.3, we deduce that $\varrho(\mathcal{L}(X_1)$ and $\varrho(\mathcal{L}(X_2)$ are isomorphic lattices. But, by definition, we have $\varrho(\mathcal{L}(X_1)) = \mathcal{L}(r(X_1)) = \mathcal{L}(X_1) \neq 0$, and $\varrho(\mathcal{L}(X_2)) = \mathcal{L}(r(X_2)) = \mathcal{L}(0) = 0$, which is a contradiction. $\square$

However, to specialize the main result of Section 2 to Grothendieck categories, by [7, Example 2.7], we may associate to any hereditary subclass $\mathcal{X}$ of $\mathcal{G}$ a so called *linearly closed* subcategory $\mathcal{SC}_{\mathcal{X}}$. Roughly speaking, such a subcategory is a subcategory, not necessarily full, of the category $\mathcal{LM}$ of all linear modular lattices that satisfies four natural conditions (see [7, Definitions 2.3]) naturally occurring when one consider subcategories of locally small Abelian categories or subcategories associated with τ-saturated submodules with respect to a hereditary torsion theory τ on the category Mod-R of all right R-modules over a unital ring R. More precisely, the linearly closed subcategory $\mathcal{SC}_{\mathcal{X}}$ associated to a given hereditary subclass $\mathcal{X}$ of $\mathcal{G}$ has

$$\mathcal{C}_{\mathcal{X}} := \{\, [X', X] \mid X \in \mathcal{X},\, X' \subseteq X\}$$

as class of objects, and as morphisms those mappings that are induced by morphisms $f : X/X' \longrightarrow Y/Y'$ in $\mathcal{G}$, i.e., arise as compositions

$$[X', X] \overset{\varphi_{X/X'}}{\longrightarrow} \mathcal{L}(X/X') \overset{f_*}{\longrightarrow} \mathcal{L}(Y/Y') \overset{\varphi^{-1}_{Y/Y'}}{\longrightarrow} [Y', Y].$$

Recall that for any morphism $f : A \longrightarrow B$ in $\mathcal{G}$ we denoted by f_* the so called *direct image* mapping

$$f_* : \mathcal{L}(A) \longrightarrow \mathcal{L}(B),\; f_*(A') = f(A'),\; \forall\, A' \in \mathcal{L}(A).$$

By [8, Lemma 5.1], any such mapping f_* is a linear morphism of lattices, so, the morphisms in $\mathcal{SC}_{\mathcal{X}}$, as compositions of linear morphisms of lattices, are also linear morphisms of lattices.

Theorem 4.2. *Let $\mathcal{X}$ be a hereditary subclass of a Grothendieck category $\mathcal{G}$, and let r be a left exact preradical on $\mathcal{G}$. Then, the following assertions are equivalent for an object $X \in \mathcal{X}$ such that $r(X)$ has a unique closure in X.*

(1) *X is a C_{11} object.*

(2) *There exist subobjects X_1 and X_2 of X such that $X = X_1 \oplus X_2$, $r(X_1)$ is an essential subobject of X_1, $r(X_2) = 0$, and X_1, X_2 are both C_{11} objects.*

Proof. In view of [7, Proposition 3.3], r canonically yields a preradical ϱ on the linearly closed subcategory $\mathcal{SC}_{\mathcal{X}} := \{\, [X', X] \mid X \in \mathcal{X},\, X' \subseteq X\}$ defined as

$$\varrho : \mathcal{SC}_{\mathcal{X}} \longrightarrow \mathcal{SC}_{\mathcal{X}},\; \varrho([X', X]) := [X', X^r],\; \forall\, [X', X] \in \mathcal{SC}_{\mathcal{X}},$$

where X^r is the subobject of X such that $X' \subseteq X^r \subseteq X$ and $r(X/X') = X^r/X'$. Notice that $X^r \in \mathcal{X}$ because $X \in \mathcal{X}$ and $\mathcal{X}$ is a hereditary subclass of $\mathcal{G}$.

Observe that, for any $L = 1/0 = \mathcal{L}(X)$, $X \in \mathcal{X}$, we have $\varrho(L) = 1^{\varrho}/0$, and also

$$\varrho(L) = \varrho(1/0) = \varrho([0,1]) = \varrho([0,X]) = [0,X^r],$$

On the other hand,

$$r(X) = r(X/0) = X^r/0 = X^r,$$

so

$$[0,X^r] = [0,r(X)] = \mathcal{L}(r(X)).$$

We deduce that $1^\varrho = X^r = r(X)$. By [7, Remarks 3.5(2)], Theorem 2.4 is valid also for the linearly closed subcategory $\mathcal{SC}_{\mathcal{X}}$ and its preradical ϱ, which ends the proof. □

Corollary 4.3. *The following statements are equivalent for an object X of a Grothendieck category $\mathcal{G}$ such that* Soc(X) *has a unique closure in X.*

(1) *X is a C_{11} object.*

(2) *There exist subobjects X_1 and X_2 of X such that $X = X_1 \oplus X_2$,* Soc(X_1) *is an essential subobject of X_1,* Soc$(X_2) = 0$*, and X_1, X_2 are both C_{11} objects.*

Proof. Specialize Theorem 4.2 for $r(X) = \mathrm{Soc}\,(X)$. □

Corollary 4.4. *Let $\mathcal{X}$ be a hereditary subclass of a Grothendieck category $\mathcal{G}$, and let r be a left exact preradical on $\mathcal{G}$. Then, the following assertions are equivalent for an object $X \in \mathcal{X}$ such that $r(X)$ is closed in X.*

(1) *X is a C_{11} object.*

(2) *There exists a subobject Y of X such that $X = r(X) \oplus Y$, and $r(X)$ and Y are both C_{11} objects.*

Proof. By [7, Remarks 3.5(2)], Corollary 2.5 is valid also for the linearly closed subcategory $\mathcal{SC}_{\mathcal{X}}$ and its preradical ϱ, so we may specialize it for the upper continuous modular lattice $L = \mathcal{L}(X) = [0,X]$. □

5 Applications to module categories equipped with torsion theories

In this section we specialize the main result of Section 2 to module categories equipped with hereditary torsion theories.

From now on, R will denote an associative ring with non-zero identity element, Mod-R the category of all unital right R-modules, $\tau = (\mathcal{T}, \mathcal{F})$ a fixed hereditary torsion theory on Mod-R, and $t_\tau(M)$ the τ-torsion submodule of a right R-module M. It is well known that the assignment $M \mapsto t_\tau(M)$, $M \in$ Mod-R, defines a left exact (pre)radical on Mod-R. We shall use the notation M_R to emphasize that M is a right R-module. For any M_R we shall denote

$$\mathrm{Sat}_\tau(M) \ := \ \{ \ N \mid N \leqslant M \text{ and } M/N \in \mathcal{F} \ \},$$

and for any $N \leqslant M$ we shall denote by $\overline{N}$ the τ-*saturation* of N (in M) defined by $\overline{N}/N = t_\tau(M/N)$. The submodule N is called τ-*saturated* if $N = \overline{N}$. Note that

$$\mathrm{Sat}_\tau(M) = \{ N \mid N \leqslant M,\ N = \overline{N} \},$$

so $\mathrm{Sat}_\tau(M)$ is the set of all τ-*saturated* submodules of M. It is well-known that for any M_R, $\mathrm{Sat}_\tau(M)$ is an upper continuous modular lattice. The reader is referred to [15] for more about hereditary torsion theories.

We say that M_R is a τ-C_{11} module if the lattice $\mathrm{Sat}_\tau(M)$ is C_{11}. More generally, if $\mathbb{P}$ is any property on lattices, we say that a module M_R is/has τ-$\mathbb{P}$ if the lattice $\mathrm{Sat}_\tau(M)$ is/has $\mathbb{P}$. We say that a submodule N of M_R is/has τ-$\mathbb{P}$ if its τ-saturation $\overline{N}$, which is an element of $\mathrm{Sat}_\tau(M)$, is/has $\mathbb{P}$. Thus, we obtain the concepts of a τ-*essential* submodule of a module, τ-*closed* submodule of a module, τ-*closure* of a submodule, etc.

According to our definition above of a τ-$\mathbb{P}$ submodule of a module, we say that a non-empty class $\mathcal{H}$ of right R-modules is τ-*hereditary* if for any $M \in \mathcal{H}$ and $N \in \mathrm{Sat}_\tau(M)$ one has $N \in \mathcal{H}$, or equivalently $\overline{N} \in \mathcal{H}$ for any $N \leqslant M$ and any $M \in \mathcal{H}$.

By a *preradical* on a τ-hereditary subclass $\mathcal{H}$ of Mod-R we mean a subfunctor of the identity functor $1_{\mathcal{H}}$ of the full subcategory $\mathcal{H}$ of Mod-R in Mod-R.

Lemma 5.1. *Let $f : M \longrightarrow M'$ be a morphism of right R-modules, $N \leqslant M$, and $N' \leqslant M'$. If $f(N) \subseteq N'$, then $f(\overline{N}) \subseteq \overline{N'}$.*

Proof. Let $g : M/N \longrightarrow M'/N'$, $g(x+N) = f(x) + N'$, $\forall\, x \in M$, be the morphism induced by f. Since $X \mapsto t_\tau(X)$, $X \in$ Mod-R, is a preradical on Mod-R, it follows that

$$g(\overline{N}/N) = g(t_\tau(M/N)) \subseteq t_\tau(M'/N') = \overline{N'}/N'.$$

We deduce that, for $x \in \overline{N}$, one has $f(x) + N' = g(x+N) \in \overline{N'}/N'$, and so, $f(x) \in \overline{N'}$. $\square$

Proposition 5.2. *Let $\mathcal{H}$ be a τ-hereditary class of right R-modules, and let r be a left exact preradical on $\mathcal{H}$. Then, the assignment $M \mapsto \overline{r(M)}$, $\forall\, M \in \mathcal{H}$, defines a preradical $\overline{r}$ on $\mathcal{H}$, and $\overline{r}(N) = N \cap \overline{r}(M)$, $\forall M \in \mathcal{H}$, $N \in \mathrm{Sat}_\tau(M)$.*

Proof. Since $\mathcal{H}$ is τ-hereditary, $\overline{r}(M) := \overline{r(M)} \in \mathcal{H}$, $\forall\, M \in \mathcal{H}$. If $f : M \longrightarrow M'$ is a morphism of modules $M, M' \in \mathcal{H}$, then $f(r(M)) \subseteq r(M')$, and hence $f(\overline{r(M)}) \subseteq \overline{r(M')}$ by Lemma 5.1, so $f(\overline{r}(M)) \subseteq \overline{r}(M')$. This shows that $\bar{r}$ is a preradical on $\mathcal{H}$.

For $N \in \mathrm{Sat}_\tau(M)$, we have $N \cap \overline{r}(M) = \overline{N} \cap \overline{r(M)} = \overline{N \cap r(M)} = \overline{r(N)} = \overline{r}(N)$. $\square$

As in the case of Grothendieck categories, to any τ-hereditary class $\mathcal{H}$ of right R-modules we can associate a linearly closed subcategory $\mathcal{SC}_{\mathcal{H}}$ of $\mathcal{LM}$ as follows.

First, for any $M \in \mathcal{H}$ and $M' \in \mathrm{Sat}_\tau(M)$, we denote by $[M', M]$ the interval in the lattice $\mathrm{Sat}_\tau(M)$, and by

$$\psi_{M/M'} : [M', M] \xrightarrow{\sim} \mathrm{Sat}_{\tau}(M/M'),\ \psi(N) := N/M',\ \forall\, N \in [M', M],$$

the canonical lattice isomorphism in [3, Lemma 3.4.4] or [4, Lemma 3.4.4], which is clearly a linear morphism of lattices.

The linearly closed subcategory $\mathcal{SC}_{\mathcal{H}}$ ol $\mathcal{LM}$ associated to $\mathcal{H}$ has

$$\mathcal{C}_{\mathcal{H}} := \{\, [M', M] \,|\, M \in \mathcal{H},\ M' \in \mathrm{Sat}_{\tau}(M)\}$$

as class of objects and as morphisms the mappings induced by morphisms $f : M/M' \longrightarrow P/P'$ in Mod-R, i.e., arising as compositions

$$[M', M] \overset{\psi_{M/M'}}{\longrightarrow} \mathrm{Sat}_{\tau}(M/M') \xrightarrow{f_{\tau}} \mathrm{Sat}_{\tau}(P/P') \overset{\psi^{-1}_{P/P'}}{\longrightarrow} [P', P].$$

where, for any morphism $f : A \longrightarrow B$ in Mod-R, f_{τ} denotes the mapping

$$f_{\tau} : \mathrm{Sat}_{\tau}(A) \longrightarrow \mathrm{Sat}_{\tau}(B),\ f_{\tau}(X) = \overline{f(X)},\ \forall\, X \in \mathrm{Sat}_{\tau}(A).$$

Notice that f_{τ} is a linear morphism of lattices by [7, Lemma 6.6]. We deduce that the morphisms in $\mathcal{SC}_{\mathcal{H}}$, as compositions of linear morphisms of lattices, are also so.

Theorem 5.3. *Let $\mathcal{H}$ be a τ-hereditary class of right R-modules and let r be a left exact preradical on* Mod-R. *Then, the following assertions are equivalent for a module $M_R \in \mathcal{H} \cap \mathcal{F}$ such that $r(M)$ has a unique τ-closure in M.*

(1) *M is a τ-C_{11} module.*

(2) *There exist submodules N and P of M such that $M/(N+P) \in \mathcal{T}$, $N \cap P \in \mathcal{T}$, $r(N)$ is τ-essential in N, $r(P) \in \mathcal{T}$, and both N and P are τ-C_{11} modules.*

Proof. In view of [7, Proposition 3.4], r canonically yields a preradical ϱ_{τ} on the linearly closed subcategory $\mathcal{SC}_{\mathcal{H}} := \{\, [M', M] \,|\, M \in \mathcal{H},\ M' \in \mathrm{Sat}_{\tau}(M)\}$ of $\mathcal{LM}$, defined as follows:

$$\varrho_{\tau} : \mathcal{SC}_{\mathcal{H}} \longrightarrow \mathcal{SC}_{\mathcal{H}},\ \varrho_{\tau}([M', M]) := [M', M^{r}],\ \forall\, [M', M] \in \mathcal{SC}_{\mathcal{H}},$$

where $M^{r} := \overline{Q}$ and $M' \leqslant Q \leqslant M$ with $r(M/M') = Q/M'$.

Observe that, for any $L = \mathrm{Sat}_{\tau}(M)$, $M \in \mathcal{H}$, the least element 0 of L is $\overline{0} = t_{\tau}(M)$ and the greatest element 1 of L is M, so $L = [\overline{0}, M]$. We deduce that

$$\varrho_{\tau}(L) = \varrho_{\tau}(1/0) = 1^{\varrho_{\tau}}/\overline{0} = [\,\overline{0}, 1^{\varrho_{\tau}}] = \varrho_{\tau}([\,\overline{0}, M]) = [\,\overline{0}, M^{r}],$$

and then, $M^{r} = 1^{\varrho_{\tau}}$. On the other hand, using [1, Lemma 1.14], we have

$$M^{r}/\overline{0} = \overline{Q}/\overline{0} = \overline{Q/\overline{0}} = \overline{r(M/\overline{0}\,)}.$$

It follows that $M^{r} = \overline{r(M)}$, i.e., $1^{\varrho_{\tau}} = \overline{r(M)}$ because $M \in \mathcal{F}$, so $\overline{0} = t_{\tau}(M) = 0$.

Now, notice that

$$r(N) \text{ is } \tau\text{-essential in } N \iff \overline{r(N)} \text{ is } \tau\text{-essential in } N$$

and

$$r(P) \in \mathcal{T} \iff \overline{r(P)} \in \mathcal{T}.$$

As in the proof of Theorem 4.2, by [7, Remarks 3.5(2)] we deduce that Theorem 2.4 is valid also for the linearly closed subcategory $\mathcal{SC}_{\mathcal{H}}$ and its preradical $\varrho_{\mathcal{T}}$. To finish the proof, use the description in [2, Proposition 5.3(5)] of the concept of a τ-complement (or τ-direct summand) of a submodule of a module. □

Corollary 5.4. *The following statements are equivalent for a module $M_R \in \mathcal{F}$ such that $\mathrm{Soc}_{\tau}(M)$ has a unique τ-closure in M.*

(1) *M is a τ-C_{11} module.*

(2) *There exist submodules N and P of X such that $M/(N+P) \in \mathcal{T}$, $N \cap P \in \mathcal{T}$, $\mathrm{Soc}_{\tau}(N)$ is τ-essential in N, $\mathrm{Soc}_{\tau}(P) \in \mathcal{T}$, and N and P are both τ-C_{11} modules.*

Proof. Specialize Theorem 5.3 for $r = \mathrm{Soc}_{\tau}$. □

Corollary 5.5. *Let $\mathcal{H}$ be a τ-hereditary class of right R-modules and let r be a left exact preradical on* Mod-R. *Then, the following assertions are equivalent for a module $M_R \in \mathcal{H} \cap \mathcal{F}$ such that $r(M)$ is a τ-closed submodule of M.*

(1) *M is a τ-C_{11} module.*

(2) *There exists a submodule N of M such that such that $M/(r(M)+N) \in \mathcal{T}$, $r(M) \cap N \in \mathcal{T}$, and $r(M)$ and N are both τ-C_{11} modules.*

Proof. As we stated in [7, Remarks 3.5(2)], Corollary 2.5 can be applied to a lattice from the linearly closed subcategory $\mathcal{SC}_{\mathcal{H}}$ of $\mathcal{LM}$ and its left exact preradical $\varrho_{\mathcal{T}}$, canonically associated to r, as in the proof of Theorem 5.3. Apply this form of Corollary 2.5 for the upper continuous modular lattice $L = \mathrm{Sat}_{\tau}(M) = [\overline{0}, M]$. □

Problem 5.6. We guess that Theorem 5.3 and its subsequent corollaries hold true without the additional condition $M \in \mathcal{F}$. □

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