Tensor products and direct limits of almost Cohen-Macaulay modules

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Abstract. We investigate the almost Cohen-Macaulay property and the Serre-type condition (C_n), $n \in \mathbb{N}$, for noetherian algebras and modules. More precisely, we find permanence properties of these conditions with respect to tensor products and direct limits.

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1. Introduction

All rings considered will be commutative, with unit and noetherian. All modules are supposed to be finitely generated.

Almost Cohen-Macaulay rings appeared from a flaw in Matsumura's book [13] and were first studied by Han [9] and afterwards by Kang [11], who introduced the notion of almost Cohen-Macaulay module. The first author of the present paper considered in [10] the condition (C_n), where n is a natural number, inspired by the well-known condition (S_n) of Serre and characterized almost Cohen-Macaulay rings using this notion. The notion of a module satisfying the condition (C_n) was defined and studied by the second author and A. Mafi [17].

We study the behaviour of the condition (C_n), $n \in \mathbb{N}$, and of almost Cohen-Macaulayness with respect to tensor product of A-modules and of A-algebras. More precisely, in Section 2 we first show that if k is a field and the tensor product of two k-algebras is noetherian and satisfies the condition (C_n) (almost Cohen-Macaulay resp.) then each one of these k-algebras satisfies the condition (C_n) (almost Cohen-Macaulay resp.). We also show that the converse is not necessarily true, unless at least one of these k-algebras satisfies the condition (S_n) (Cohen-Macaulay resp.), see Proposition 2.7, Corollary 2.8 and Example 2.9. In Theorems 2.10 and 2.11 we investigate the permanence of the properties of being almost Cohen-Macaulay or satisfying the condition (C_n) with respect to the tensor product of modules. The main results of Section 3 are Theorems 3.8 and 3.10. It is shown that if M and N are Tor-independent A-modules, M has finite projective dimension and the tensor product of M and N is almost Cohen-Macaulay (resp. (C_n)), then N is almost Cohen-Macaulay (resp. (C_n)) too. As a consequence we show that a ring having a module of finite projective dimension satisfying the condition (C_n), must satisfy the condition (C_n) itself, a property that was already proved for the condition (S_n). Finally, in Section 4 we investigate the behaviour of the condition (C_n) and of almost Cohen-Macaulayness with respect to direct limits. The main result is Theorem 4.3 which shows that under some additional assumptions, a direct limit of almost Cohen-Macaulay rings remains almost Cohen-Macaulay.

2. Tensor products of almost Cohen-Macaulay algebras

We begin by recalling the notions and basic facts that will be needed in the paper.

Definition 2.1. [11, Definition 1.2] Let A be a ring and M an A-module. We say that M is an almost Cohen-Macaulay A-module if for any $P \in \mathrm{Supp}(M)$ we have $\mathrm{depth}(P, M) = \mathrm{depth}_{A_P}(M_P)$. A is called an almost Cohen-Macaulay ring if it is an almost Cohen-Macaulay A-module.

Remark 2.2. Let (A, m) be a noetherian local ring and M a finitely generated A-module. Then it follows at once by [11, Lemma 1.5 and Lemma 2.4] that M is almost Cohen-Macaulay iff $\dim(M) \leq \mathrm{depth}(M) + 1$.

Definition 2.3. [17, Definition 2.1] Let A be a ring, $n \in \mathbb{N}$ and M an $A-$module. We say that M satisfies the condition (C_n) if for every $P \in \mathrm{Supp}(M)$ we have $\mathrm{depth}(M_P) \geq \min(n, \mathrm{ht}_M(P)) - 1$. If A satisfies the condition (C_n) as an A-module, we say that the ring A satisfies (C_n).

Remark 2.4. Recall that an $A-$module M is said to satisfy Serre's condition (S_n) if for every $P \in \mathrm{Supp}(M)$ we have $\mathrm{depth}(M_P) \geq \min(n, \mathrm{ht}_M(P))$. Hence, obviously, if M satisfies the condition (S_n), it satisfies the condition (C_n).

Lemma 2.5. *Let A be a ring, M an A-module and $n \in \mathbb{N}$. If M satisfies the condition (S_n), then it satisfies the condition (C_{n+1}).*

Proof. Let $P \in \mathrm{Supp}(M)$. Then by the condition (S_n) we have $\mathrm{depth}(M_P) \geq \min(\mathrm{ht}_M(P), n)$. Suppose first that $n < \mathrm{ht}_M(P)$. Then $\min(\mathrm{ht}_M(P), n) = n$, hence $\mathrm{depth}(M_P) \geq n$. But $n + 1 \leq \mathrm{ht}_M(P)$, hence $\min(n + 1, \mathrm{ht}_M(P)) = n + 1$. Then $\min(n + 1, \mathrm{ht}_M(P)) - 1 = n + 1 - 1 = n$ and it follows that $\mathrm{depth}(M_P) \geq n = \min(n + 1, \mathrm{ht}_M(P)) - 1$.

Suppose now that $n \geq \mathrm{ht}_M(P)$. Then $\min(n, \mathrm{ht}_M(P)) = \mathrm{ht}_M(P)$, hence by (S_n) we have $\mathrm{depth}(M_P) = \mathrm{ht}_M(P)$. Since $n + 1 > \mathrm{ht}_M(P)$, we obtain $\min(n+1, \mathrm{ht}_M(P)) - 1 = \mathrm{ht}_M(P) - 1 < \mathrm{ht}_M(P) = \mathrm{depth}(M_P)$, as desired. □

Lemma 2.6. *Let $u : A \to B$ be a flat morphism of noetherian rings. If B is almost Cohen-Macaulay, then all the fibers of u are almost Cohen-Macaulay.*

Proof. Let $P \in \operatorname{Spec}(A)$ and $QB_P/PB_P \in \operatorname{Spec}(B_P/PB_P)$. Then the morphism $A_P \to B_Q$ is flat and local and by [10, Proposition 2.2,a)] it follows that $B_Q/PB_Q = (B_P/PB_P)_{QB_P/PB_P}$ is almost Cohen-Macaulay. From [11, Lemma 2.6] we obtain that B_P/PB_P is almost Cohen-Macaulay. □

Proposition 2.7. *Let k be a field, A and B two $k-$algebras such that $A \otimes_k B$ is noetherian and $n \in \mathbb{N}$. Then:*

i) If $A \otimes_k B$ satisfies the condition (C_n), then A and B satisfy the condition (C_n);

ii) If A and B satisfy the condition (C_n) and one of them satisfies the condition (S_n), then $A \otimes_k B$ satisfies the condition (C_n);

iii) If A and B satisfy the condition (S_n), then $A \otimes_k B$ satisfies the condition (C_{n+1}).

Proof. i) Follows from [10, Proposition 3.12].
ii) Follows from [10, Proposition 3.13].
iii) Follows from [18, Theorem 6,b)] and Lemma 2.5. □

Corollary 2.8. *Let k be a field, A and B two $k-$algebras such that $A \otimes_k B$ is noetherian. Then:*

i) If $A \otimes_k B$ is almost Cohen-Macaulay, then A and B are almost Cohen-Macaulay;

ii) If A and B are almost Cohen-Macaulay and moreover one of them is Cohen-Macaulay, then $A \otimes_k B$ is almost Cohen-Macaulay.

Proof. Follows from Proposition 2.7 and [10, Theorem 3.3]. □

Example 2.9. Let $A = k[x^4, x^5, xy, y]_{(x^4,x^5,xy,y)}$. Then A is a noetherian local domain of dimension 2, hence by Remark 2.2 it is almost Cohen-Macaulay. Since $x^5y = x^4xy \in x^4A$ and $x^5(xy)^3 = x^8y^3 \in x^4A$, it follows that $\operatorname{depth}(A) = 1$. But $A \otimes_k A$ is not almost Cohen-Macaulay, because by [7, Lemma 2] we get $\dim(A \otimes_k A) = 4$ and $\operatorname{depth}(A \otimes_k A) = 2$. The first example having this property was given by Tabaâ [16, Exemple].

Theorem 2.10. *Let $u : A \to B$ be a morphism of noetherian rings, M a finitely generated A-module, N a finitely generated B-module and $n \in \mathbb{N}$. Suppose that N is a flat A-module. We consider the structure of $M \otimes_A N$ as a B-module. Then:*

i) If $M \otimes_A N$ satisfies the condition (C_n), then M satisfies the condition (C_n);

ii) If M and N_P/PN_P satisfy the condition (S_n), for any $P \in \operatorname{Spec}(A)$, then $M \otimes_A N$ satisfies the condition (S_n);

iii) If M satisfies the condition (S_n) and N_P/PN_P satisfies the condition (C_n) for any $P \in \operatorname{Supp}(M)$, then $M \otimes_A N$ satisfies the condition (C_n);

iv) If M satisfies the condition (C_n) and N_P/PN_P satisfies the condition (S_n) for any $P \in \operatorname{Supp}(M)$, then $M \otimes_A N$ satisfies the condition (C_n);

v) If M and N_P/PN_P satisfy the condition (S_n), for any $P \in \mathrm{Supp}(M)$, then $M \otimes_A N$ satisfies the condition (C_{n+1}).

Proof. i) Let $P \in \mathrm{Supp}(M)$ and $Q \in \mathrm{Min}(PB)$. Using [4, Proposition 1.2.16 and Theorem A.11] and the flatness of u, we obtain

$$\mathrm{depth}_{B_Q}(M_P \otimes_{A_P} N_Q) = \mathrm{depth}_{A_P}(M_P) + \mathrm{depth}_{B_Q}(N_Q/PN_Q)$$

and

$$\dim_{B_Q}(M_P \otimes_{A_P} N_Q) = \dim_{A_P}(M_P) + \dim_{B_Q}(N_Q/PN_Q).$$

But $\dim_{B_Q}(N_Q/PN_Q) = 0$ and $M_P \otimes_{A_P} N_Q$ satisfies the condition (C_n), hence

$$\mathrm{depth}_{A_P}(M_P) \geq \min(n, \dim_{A_P}(M_P)) - 1,$$

that is M satisfies the condition (C_n).
ii) Let $Q \in \mathrm{Supp}_B(M \otimes_A N)$ and $P = Q \cap A$. As above we have

$$\mathrm{depth}_{B_Q}(M_P \otimes_{A_P} N_Q) \geq \min(n, \dim_{A_P}(M_P)) + \min(n, \dim_{B_Q}(N_Q/PN_Q)) \geq$$

$$\geq \min(n, \dim_{B_Q}(M_P \otimes_{A_P} N_Q)).$$

iii) and iv) The proof is similar to the proof of i).
v) Follows from ii) and Lemma 2.5. □

Theorem 2.11. *Let $u : A \to B$ be a morphism of noetherian rings, M a finitely generated A-module, N a finitely generated B-module and $n \in \mathbb{N}$. Suppose that N is a flat A-module. We consider the structure of $M \otimes_A N$ as a B-module. Then:*

i) If $M \otimes_A N$ is almost Cohen-Macaulay, then M and N_P/PN_P are almost Cohen-Macaulay for any $P \in \mathrm{Supp}(M)$;

ii) If M is almost Cohen-Macaulay and N_P/PN_P is Cohen-Macaulay for any $P \in \mathrm{Supp}(M)$, then $M \otimes_A N$ is almost Cohen-Macaulay;

iii) If M is Cohen-Macaulay and N_P/PN_P is almost Cohen-Macaulay for any $P \in \mathrm{Supp}(M)$, then $M \otimes_A N$ is almost Cohen-Macaulay.

Proof. i) From [10, Theorem 3.3] and Proposition 2.10 i), it follows at once that M is almost Cohen-Macaulay. Let $P \in \mathrm{Supp}_A(M)$. We have

$$\mathrm{Supp}_{B_P/PB_P}(N_P/PN_P) =$$

$$= \{QB_P/PB_P \mid Q \in \mathrm{Supp}_B(N) \cap V(PB), Q \cap (A \setminus P) = \emptyset\}.$$

Hence, let $QB_P/PB_P \in \mathrm{Supp}_{B_P/PB_P}(N_P/PN_P)$. Since $M \otimes_A N$ is an almost Cohen-Macaulay $B-$module and $Q \in \mathrm{Supp}(M \otimes_A N)$, by [11, Lemma 2.6] we get that $M_P \otimes_{A_P} N_Q$ is an almost Cohen-Macaulay B_Q-module. Since u is flat, M_P is a finitely generated A_P-module and N_Q is a finitely generated B_Q-module and a flat A_P-module, by [12, Proposition 2.7] it follows that N_Q/PN_Q is an almost Cohen-Macaulay B_Q/PB_Q-module. Now by [11, Lemma 2.6] we obtain that N_P/PN_P is an almost Cohen-Macaulay B_P/PB_P-module.
ii) and iii) Follow from Theorem 2.10 and [10, Theorem 3.3]. □

Lemma 2.12. *Let R be a commutative ring, A and B two R-algebras such that $A \otimes_R B$ is noetherian. Let $P \in \operatorname{Spec}(A \otimes_R B)$ and set $p := P \cap A, q = P \cap B, r := P \cap R$. Assume that A_p is flat over R_r. Then:*

i) If $(A \otimes_R B)_P$ is almost Cohen-Macaulay, then B_q and A_p/rA_p are almost Cohen-Macaulay;

ii) If A_p/rA_p and B_q are almost Cohen-Macaulay and one of them is Cohen-Macaulay, then $(A \otimes_R B)_P$ is almost Cohen-Macaulay.

Proof. Follows at once from [3, Corollary 2.5]. □

Corollary 2.13. *Let R be a commutative ring, A and B two R-algebras such that $A \otimes_R B$ is noetherian. Let $P \in \operatorname{Spec}(A \otimes_R B)$ and set $p := P \cap A, q = P \cap B, r := P \cap R$. Assume that A_p and B_q are flat over R_r. Then:*

i) If $(A \otimes_R B)_P$ is almost Cohen-Macaulay, then A_p and B_q are almost Cohen-Macaulay;

ii) If A_p and B_q are almost Cohen-Macaulay and one of them is Cohen-Macaulay, then $(A \otimes_R B)_P$ is almost Cohen-Macaulay.

Proof. i) Follows from Lemma 2.12 and the flatness of A_p and B_q over R_r.
ii) Since $R_r \to A_p$ is flat, by [10, Proposition 2.2] it follows that A_p/rA_p is almost Cohen-Macaulay. Note that if A_p is Cohen-Macaulay, then A_p/rA_p is Cohen-Macaulay too. Now the assertion follows from Lemma 2.12, ii). □

Example 2.14. In part ii) of the previous corollary, the assumption that A_p or B_q is Cohen-Macaulay is necessary. Indeed, consider again the ring in Example 2.9, that is $A = B = k[x^4, x^5, xy, y]_{(x^4,x^5,xy,y)}$ and let $R = k[x^4]$. By [3, Proposition 2.1, 2)], there is $P \in \operatorname{Spec}(A \otimes_R B)$ such that $P \cap A$ is the maximal ideal of A. Then by [3, Corollary 2.5] it follows that $(A \otimes_R B)_P$ is not almost Cohen-Macaulay. As we saw in Example 2.9 the ring $A_p = B_q$ is almost Cohen-Macaulay.

Corollary 2.15. *Let R be a commutative ring, A and B two flat R-algebras such that $A \otimes_R B$ is noetherian. Then:*

i) If $A \otimes_R B$ is almost Cohen-Macaulay, then A_p and B_q are almost Cohen-Macaulay for any $p \in \operatorname{Spec}(A)$ and $q \in \operatorname{Spec}(B)$ such that $p \cap R = q \cap R$;

ii) If for any $p \in \operatorname{Spec}(A)$ and $q \in \operatorname{Spec}(B)$ such that $p \cap R = q \cap R$, A_p and B_q are almost Cohen-Macaulay and one of them is Cohen-Macaulay, then $A \otimes_R B$ is almost Cohen-Macaulay.

Proof. i) By [3, Proposition 2.1] there exists $P \in \operatorname{Spec}(A \otimes_R B)$ such that $P \cap A = p, P \cap B = q$. Now apply Corollary 2.13, i).
ii) Follows from Corollary 2.13, ii). □

3. Tensor products of almost Cohen-Macaulay modules

Recall the following definition (cf. [19, Definition 1.1]):

Definition 3.1. If M and N are finitely generated non-zero A-modules, the grade of M with respect to N is defined by

$$\operatorname{grade}(M,N) = \inf\{i \mid \operatorname{Ext}^i_A(M,N) \neq 0\}.$$

Remark 3.2. By [6, Proposition 1.2, h)] we see that $\operatorname{grade}(M,N)$ is the length of a maximal N-regular sequence in $\operatorname{Ann}(M)$.

Remark 3.3. By [4, Proposition 1.2.10, a)] we get

$$\operatorname{grade}(M,N) = \inf\{(\operatorname{depth}(N_P) \mid P \in \operatorname{Supp}(M)\} =$$
$$= \inf\{(\operatorname{depth}(N_P) \mid P \in \operatorname{Supp}(M) \cap \operatorname{Supp}(N)\}.$$

Definition 3.4. If M and N are finitely generated A-modules and $\operatorname{pd}(M) < \infty$, we say that M is N-perfect if $\operatorname{grade}(M,N) = \operatorname{pd}(M)$.

We shall use several times the following fact:

Remark 3.5. (Intersection Theorem cf. [15, Theorem 8.4.4]) If $\operatorname{pd}_A(M) < \infty$, then $\dim(N) \leq \operatorname{pd}(M) + \dim(M \otimes_A N)$.

Proposition 3.6. *Let A be a noetherian local ring, M and N finitely generated non-zero A-modules, $n \in \mathbb{N}$. Assume that:*
i) M is $N-$perfect and $\operatorname{pd}(M) := p \leq n$;
ii) N satisfies the condition (C_n);
iii) $\operatorname{Tor}^A_i(M,N) = 0,\ \forall i > 0$.
Then $M \otimes_A N$ satisfies the condition (C_{n-p}).

Proof. Let $P \in \operatorname{Supp}(M \otimes_A N)$. We have

$$p = \operatorname{grade}(M,N) \leq \operatorname{grade}(M_P,N_P) \leq \operatorname{pd}(M_P) \leq \operatorname{pd}(M) = p,$$

hence $\operatorname{pd}(M_P) = p$. By [2, Theorem 1.2] we obtain

$$\operatorname{depth}(M \otimes_A N)_P = \operatorname{depth}(M_P \otimes_{A_P} N_P) = \operatorname{depth}(N_P) - \operatorname{pd}(M_P) =$$
$$= \operatorname{depth}(N_P) - p \geq \min(\operatorname{ht}_N P, n) - 1 - p = \min(\operatorname{ht}_N P - p, n-p) - 1$$

and it is enough to show that $\min(\operatorname{ht}_N P - p, n-p) \geq \min(\operatorname{ht}_{M\otimes_A N} P, n-p)$. But since $\operatorname{grade}(M_P,N_P) = \operatorname{grade}((M \otimes_A N)_P, N_P)$, by [19, Theorem 2.1] we have $p \leq \operatorname{ht}_N(P) - \operatorname{ht}_{M\otimes_A N}(P)$ and this concludes the proof. □

Corollary 3.7. *Let $n \in \mathbb{N}$, A a noetherian local ring satisfying the condition (C_n) and let M be a perfect A-module such that* $\operatorname{pd}(M) = p \leq n$. *Then M satisfies the condition* (C_{n-p}).

Theorem 3.8. *Let A be a noetherian local ring and let M and N be finitely generated non-zero A-modules such that* $\operatorname{Tor}^A_i(M,N) = 0,\ \forall i > 0$. *If* $\operatorname{pd}(M) < \infty$ *and $M \otimes_A N$ is almost Cohen-Macaulay, then N is almost Cohen-Macaulay.*

Proof. Applying Remark 2.2, Remark 3.5 and [2, Theorem 1.2] we get

$$\dim(N) \leq \operatorname{pd}(M) + \dim(M \otimes_A N) \leq$$
$$\leq \operatorname{pd}(M) + \operatorname{depth}(M \otimes_A N) + 1 = \operatorname{depth}(N) + 1.$$

Hence, by applying again Remark 2.2, it follows that N is almost Cohen-Macaulay. □

Corollary 3.9. (see [15, Corollary 8.4.5]) *Let A be a local ring having an almost Cohen-Macaulay module of finite projective dimension. Then A is an almost Cohen-Macaulay ring.*

We can prove a more general form of Theorem 3.8.

Theorem 3.10. *Let A be a noetherian local ring, $n \in \mathbb{N}$ and M, N finitely generated non-zero A-modules such that $\operatorname{Tor}_i^A(M,N) = 0$, $\forall i > 0$. If $\operatorname{pd}(M) < \infty$ and $M \otimes_A N$ satisfies the condition (C_n), then N satisfies the condition (C_n).*

Proof. Let $P \in \operatorname{Supp}_A(N)$. Assume first that $P \in \operatorname{Supp}_A(M)$. Then $P \in \operatorname{Supp}(M \otimes_A N)$ and we have two cases:
a) $\operatorname{depth}(M \otimes_A N)_P < n-1$. Then $(M \otimes_A N)_P$ is almost Cohen-Macaulay and by Proposition 3.8 it follows that N_P is almost Cohen-Macaulay too. Hence $\operatorname{depth}(N_P) \geq \min(n, \dim(N_P)) - 1$.
b) $\operatorname{depth}(M \otimes_A N)_P \geq n-1$. Then from [2, Theorem 1.2] we obtain:

$$\operatorname{depth}(N_P) = \operatorname{depth}(M \otimes_A N)_P + \operatorname{pd}(M_P) \geq$$

$$\geq n-1 \geq \min(n-1, \operatorname{ht}_N(P) - 1) = \min(n, \operatorname{ht}_N(P)) - 1.$$

Assume now that $P \notin \operatorname{Supp}_A(M)$. Let $Q \in \operatorname{Min}(\operatorname{Ann}_A(M) + P)$. From [19, Theorem 2.1] we get

$$\operatorname{depth}(N_P) \geq \operatorname{grade}(A_Q/PA_Q, N_Q) \geq \operatorname{depth}(N_Q) - \dim(A_Q/PA_Q).$$

By Remark 3.5 we have

$$\dim(A_Q/PA_Q) \leq \operatorname{pd}(M_Q) + \dim(M_Q/PM_Q) = \operatorname{pd}(M_Q),$$

hence by [2, Theorem 1.2] we obtain

$$\operatorname{depth}(N_P) \geq \operatorname{depth}(N_Q) - \operatorname{pd}(M_Q) = \operatorname{depth}(M_Q \otimes_{A_Q} N_Q).$$

If $\operatorname{depth}(M_Q \otimes_{A_Q} N_Q) \geq n-1$ we have

$$\operatorname{depth}(N_P) \geq n-1 \geq \min(n-1, \operatorname{ht}_N(P) - 1) = \min(n, \operatorname{ht}_N(P)) - 1,$$

as required.
In the case $\operatorname{depth}(M_Q \otimes_{A_Q} N_Q) < n-1$, by definition $M_Q \otimes_{A_Q} N_Q$ is almost Cohen-Macaulay, hence from Proposition 3.8 it follows that N_Q is almost Cohen-Macaulay. Then N_P is almost Cohen-Macaulay too and consequently

$$\operatorname{depth}(N_P) \geq \min(n, \operatorname{ht}_N(P)) - 1.$$

□

Corollary 3.11. *Let $n \in \mathbb{N}$ and A a local ring having a module of finite projective dimension satisfying the condition (C_n). Then A satisfies the condition (C_n).*

Remark 3.12. A similar result for the condition (S_n) was proved in [20, Proposition 4.1].

4. Direct limits of almost Cohen-Macaulay rings

In this section we study the behaviour of the almost Cohen-Macaulay property with respect to direct limits.

Proposition 4.1. *Let $(A_i, f_{ij})_{i,j\in\Lambda}$ be a direct system of noetherian rings, $n \in \mathbb{N}$ and set $A := \varinjlim_{i\in\Lambda} A_i$. Assume that:*

i) the ring A is noetherian;
ii) for any $i \le j$, the morphism f_{ij} is flat;
iii) for any $i \in \Lambda$ the ring A_i satisfies the condition (C_n).
Then A satisfies the condition (C_n).

Proof. Let $P \in \operatorname{Spec}(A), B := A_P, k := B/PB$ and for any $i \in \Lambda$, put $P_i := P \cap A_i, B_i := (A_i)_{P_i}$ and $k_i := B_i/P_iB_i$. There exists $i_0 \in \Lambda$ such that $P = P_iB$ for any $i \ge i_0$, hence $k = B \otimes_{B_i} k_i$. But the morphism $B_i \to B$ is flat, hence $\dim(B_i) = \dim(B)$ and $\operatorname{depth}(B_i) = \operatorname{depth}(B)$ for any $i \ge i_0$. The assertion follows from Remark 2.2. □

Corollary 4.2. *Let $(A_i, f_{ij})_{i,j\in\Lambda}$ be a direct system of noetherian rings and let $A := \varinjlim_{i\in\Lambda} A_i$. Assume that:*

i) the ring A is noetherian;
ii) for any $i \le j$ the morphism f_{ij} is flat;
iii) for any $i \in \Lambda$ the ring A_i is almost Cohen-Macaulay.
Then A is almost Cohen-Macaulay.

Proof. Let us give a direct proof, slightly different than the one coming out at once from Proposition 4.1. We will apply [5, Corollary 2.3]. Let $P \subseteq Q$ be two prime ideals in A and for any $i \in \Lambda$ let $P_i := P \cap A_i, Q_i := Q \cap A_i, B_i := (A_i)_{P_i}, C_i := (A_i)_{Q_i}$. Also, let $B := A_P$ and $C := A_Q$. Then, as in the proof of Proposition 4.1, there exists $i_0 \in \Lambda$ such that $P = P_iB$ and $Q = Q_iB$ for any $i \ge i_0$. Since A_i is almost Cohen-Macaulay it follows by [5, Corollary 2.3] that $\operatorname{depth}(B_i) \le \operatorname{depth}(C_i)$ and consequently, by flatness, $\operatorname{depth}(B) = \operatorname{depth}(B_i) \le \operatorname{depth}(C_i) = \operatorname{depth}(C)$. Now we apply again [5, Corollary 2.3]. □

Theorem 4.3. *Let $((A_i, m_i, k_i), f_{ij})_{i,j\in\Lambda}$ be a direct system of noetherian local rings and let $A := \varinjlim_{i\in\Lambda} A_i$. Assume that:*

i) for any $i, j \in \Lambda, i \le j$, we have $m_iA_j = m_j$;
ii) for any $i \in \Lambda$ and for any ideal I of A_i, we have $IA \cap A_i = I$;
iii) for any $i \in \Lambda, A_i$ is an almost Cohen-Macaulay ring.
Then A is almost Cohen-Macaulay.

Proof. By [14] and [8, $(0_{III}, 10.3.1.3)$], the ring A is a noetherian local ring, with maximal ideal $m = m_iA, \forall i \in \Lambda$. Let $i_0 \in \Lambda$ be such that $\dim(A_j) = \dim(A)$, $\forall j \ge i_0$ (see [1, Lemma 3.10]) and let $x_1, \ldots, x_s \in A_{i_0}$ be a system of parameters. Set $Q_j = (x_1, \ldots, x_s)A_j, \forall j \ge i_0$ and $Q = (x_1, \ldots, x_s)A$. Then

$$m = m_{i_0}A = \sqrt{Q_{i_0}}A \subseteq \sqrt{Q_{i_0}A} \subseteq m,$$

whence $x_1, \dots, x_s$ is a system of parameters in A. The same argument shows that $x_1, \dots, x_s$ is a system of parameters in A_j, $\forall j \geq i_0$. If A_i is Cohen-Macaulay for all $i \in \Lambda$, it follows by [1, Corollary 3.7] that A is Cohen-Macaulay and we are done. Assume this is not the case. Since A_j is almost Cohen-Macauly, by [11, Theorem 1.7] there exists $l \in \{1, \dots, s\}$ such that $\{x_1, \dots, \widehat{x_l}, \dots, x_s\}$ is a regular sequence. We can assume $l = 1$, that is $\{x_2, \dots, x_s\}$ is a regular sequence in A_{i_0}. As $\{x_1, \dots, x_{s-1}, x_s\}$ is not a regular sequence in A_{i_0}, it cannot be a regular sequence in A_j, $\forall j \geq i_0$. But $x_1, \dots, x_s$ is a system of parameters in A_j, $\forall j \geq i_0$. We will show that we can assume that $x_2, \dots, x_s$ remains a regular sequence in A_j, $\forall j \geq i_0$. If this is the case, it follows that $x_2, \dots, x_s$ is a regular sequence in A and again by [11, Theorem 1.7] we get that A is almost Cohen-Macaulay. So, assume that there exists $j_1 \geq i_0$ such that $\{x_2, \dots, x_s\}$ is not a regular sequence in A_{j_1}. First we observe that then $\{x_2, \dots, x_s\}$ is not regular in A_k, $\forall k \geq j_1$. Since A_{j_1} is almost Cohen-Macaulay, it follows that, for example, $\{x_1, \widehat{x_2}, x_3, \dots, x_s\} = \{x_1, x_3, \dots, x_s\}$ is a regular sequence in A_{j_1}. Assume that there exists $j_2 \geq j_1$ such that $\{x_1, \widehat{x_2}, x_3, \dots, x_s\}$ is not a regular sequence in A_{j_2}. Then there exists $j_2 \geq j_1$ such that, for example, $\{x_1, x_2, x_4, \dots, x_s\}$ is a regular sequence. Continuing we obtain an index $l_0 \in \Lambda$, such that $\{x_1, \dots, x_{i-1}, \widehat{x_i}, x_{i+1}, \dots, x_s\}$ is not a regular sequence in A_l for any $l \geq l_0$ and for any $i = 1, \dots, s$. Contradiction. □

Example 4.4. Let $K \subset L$ be an infinite algebraic field extension. Then $L = \bigcup_{i \in \Lambda} K_i$, where K_i are the finite field subextensions of L. For any $i \in \Lambda$, let $A_i := K_i[[X^4, X^3Y, XY^3, Y^4]]$. It is easy to see that for any $i \in \Lambda$ we have $\operatorname{depth}(A_i) = 1$ and $\dim(A_i) = 2$, that is A_i is an almost Cohen-Macaulay ring which is not Cohen-Macaulay. Moreover $A := \bigcup_{i \in \Lambda} A_i \subsetneq L[[X^4, X^3Y, XY^3, Y^4]]$ and the family $(A_i)_{i \in \Lambda}$ satisfies the conditions in Theorem 4.3. Hence A is an almost Cohen-Macaulay ring.

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