INSTITUTUL DE MATEMATICA
“SIMION STOILOW”
AL ACADEMIEI ROMANE

PREPRINT SERIES OF THE INSTITUTE OF MATHEMATICS
OF THE ROMANIAN ACADEMY

ISSN 0250 3638

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Preprint nr. 1/2017

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March, 2017

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On the High Weissenberg Number Problem

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Abstract

Most modern industrial materials are obtained by processing several substances in a fluid state. This procedure is limited by the flow instabilities. Since 1960, a large class of numerical methods was used to compute the flow of complex fluids . In the case of the Oldroyd-B fluid, characterized by the Weissenberg numbers W_i, the numerical methods have usually failed for $W_i = O(1)$. Here we study the linear stability of the displacement of an Oldroyd-B fluid by air in a Hele-Shaw cell. We consider a particular class of perturbations, by neglecting some terms in the constitutive relations, due to the very small thickness of the considered Hele-Shaw cell. An approximate formula of the growth rate is presented, displaying a blow-up for $(W_1 - W_2) \approx 0.408$. This result is in agreement with the numerical data obtained by Wilson [18]. Therefore, for the flow geometry considered here, we see that the higher Weissenberg number instability is due to the model, and not to the computational methods. When $(W_1 = W_2)$, our growth rate is quite similar to the Saffman-Taylor formula for a Newtonian liquid displaced by air.

1 Introduction

We consider the flow in a Hele-Shaw cell - that means the flow between two parallel plates with the gap b, $b << l$, where l is the Hele-Shaw length. An important parameter of the problem is $\epsilon = b/l << 1$.

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Hele-Shaw [7] neglected the velocity component in the direction orthogonal on the plates and the partial derivatives (of other two velocity components) in the directions contained in the plates plane. Then the averaged velocities (across the plates) of a Stokes flow are verifying an equation quite similar with the Darcy law for the flow in a porous medium. This model is described in [1] and [2].

The flow of two immiscible fluids, separated by a sharp interface, in a Hele-Shaw cell, can be considered as a model for a displacement process. In [14] was studied the linear stability of the interface between two fluids of Darcy type. A formula for the growth constant was obtained and the well-known Saffman-Taylor stability criterion was given: if the displacing fluid is less viscous, then the interface is unstable.

The above displacement model can be used for the study of oil recovery from a horizontal porous medium, the displacing fluid being some polymer-solute (the gravity effect is neglected). The constitutive relations between the stress and the strain-rate tensors of such polymer-solute fluids being non linear, we have non-Newtonian displacing fluids. As well, the oil in the porous reservoir can often be considered as a non-Newtonian fluid.

The non-Newtonian fluids are studied in a large number of papers - see [6], [13], [15], [16]. A particular type of non-Newtonian fluids, called second order fluids, are studied in [5], [2], [8]. Non-Newtonian flows in Hele-Shaw cells are studied in [12] and [19]. Some numerical methods are used in [9], [10], [18] for study the displacement of Oldroyd-B and Maxwell upper-convected fluids by air in a Hele-Shaw cell. In these papers was obtained a strong destabilizing effect due to the non-Newtonian constitutive relations, compared with the case of a Newtonian fluid displaced by air. In the case of the Oldroyd-B fluid characterized by the Weissenberg numbers W_i, the numerical methods have usually failed for $W_i = O(1)$. In [18] was reported a (possible) blow-up of the growth constant for $W_2 = 0, W_1 > 2.5$. This possible singularity may be related with the fractures observed in the flows of some complex fluids in Hele-Shaw cells - see [12, 19, 11].

In this paper we consider an Oldroyd-B fluid displaced by air in a Hele-Shaw cell and we study the modal linear stability of the interface. The constitutive relations are depending on the Weissenberg numbers W_i and we consider here the case $W_i = O(1)$. A basic steady solution is presented, with a constant pressure gradient in the displacement direction. As in Wilson [18], we consider the full system containing the flow equations and the constitutive relations. In the frame of the linear stability, we get the perturbation system. We consider a particular Fourier expansion of velocity perturbations, depending on the arbitrary parameter α. With some conditions imposed on α - see (42) - we can neglect some terms in the constitutive relations and we get approximate expressions of some components of the extra-stress tensor. The average (across the plates) of these components are inserted in the Laplace's law and it yields an approximate formula of the growth constant. The growth rate formula obtained in this paper gives a blow-up for $W_1 - W_2 \approx 0.408$. Then we get a strong destabilizing effect, compared with the case of a Newtonian fluid displaced by air studied in [14]. Our dispersion curves are similar with those obtained (by using numerical methods) in [18]. The condition (42) imposed on α is used also for justify the form of the amplitude f of the velocity perturbations - see (25) and Proposition 6.

The same method and the same constitutive constitutive relations were used in the talks [3] and [4] given by P. Daripa and G. Paşa, where only the case $W_i = O(\epsilon)$ was considered and a different Fourier expansion was used for the velocity perturbations. The same destabilizing effect was obtained, compared with the case studied in [14].

The most important result of the present paper is following: we present a particular perturbation which leads to the blow-up of the growth constant for $W_1 - W_2 \approx 0.408$. Then all other (possible) perturbations can lead only to a worse situation. Hence, it is possible to have blow-up of the growth constant only for $W_1 - W_2 < 0.408$. Therefore, at least for the flow geometry considered here, the higher Weissenberg number instability is due to the model, and not to the computational methods.

The paper is laid out as follows. In section 2 we present the constitutive equations for Oldroyd-B fluids. In section 3 we present the basic flow about which the stability calculations are performed. The governing system for perturbations is derived in section 4. The linear stability analysis is performed in section 5, leading to the explicit *dimensionless* dispersion relation (47). This formula is quite similar to the Saffman-Taylor formula for the Newtonian fluid when $W_1 = W_2$. Thus we provide a natural extension of the Saffman-Taylor formula to the Oldroyd-B fluid which quantifies the effect of elasticity of Oldroyd-B fluid. Finally, we conclude in section 6.

2 The Oldroyd-B fluid

We consider a Hele-Shaw cell with plates parallel with the xOy plane. The Oz axis is orthogonal on the plates. The gravity effect is neglected. The displacement process occurs in the positive direction of Ox. The gap between plates is denoted by b and the length of the Hele-Shaw cell is l. Our problem is characterized by the small parameter $\epsilon = b/l << 1$. We use the following notations:

The extra-stress and strain-rate tensors are $\underline{\tau}, \underline{\mathbf{S}}$; the velocity, pressure and the fluid viscosity are denoted by μ, $\quad \underline{\mathbf{u}} = (\underline{u}, \underline{v}, \underline{w})$, $\quad \underline{p}$. The relaxation and retardation time constants are denoted by λ_1, λ_2 . $\underline{\mathbf{V}}$ is the matrix containing the velocity derivatives. We have

$$\underline{\mathbf{S}} := \frac{1}{2}(\underline{\mathbf{V}} + \underline{\mathbf{V}}^T), \quad (\underline{\mathbf{V}}_{ij})^T := \underline{\mathbf{V}}_{ji}.$$

The flow equations, the free-divergence relation and the constitutive relations are

$$-\nabla \underline{p} + \nabla \cdot \underline{\tau} = 0, \quad \underline{u}_x + \underline{v}_y + \underline{w}_z = 0, \tag{1}$$

$$\underline{\tau} + \lambda_1 \underline{\tau}^\nabla = 2\mu[\underline{\mathbf{S}} + \lambda_2 \underline{\mathbf{S}}^\nabla], \quad \lambda_1 > \lambda_2 \geq 0. \tag{2}$$

We consider a steady flow (as in [3] and [4]) then the upper convected derivatives involved in (2) become

$$\begin{aligned} \underline{\tau}^\nabla &= \underline{\mathbf{u}} \cdot \nabla \underline{\tau} - (\underline{\mathbf{V}}\underline{\tau} + \underline{\tau}\underline{\mathbf{V}}^T), \\ \underline{\mathbf{S}}^\nabla &= \underline{\mathbf{u}} \cdot \nabla \underline{\mathbf{S}} - (\underline{\mathbf{V}}\underline{\mathbf{S}} + \underline{\mathbf{S}}\underline{\mathbf{V}}^T). \end{aligned} \tag{3}$$

The flow equations containing the components of the extra-stress tensor are

$$\begin{aligned} &\underline{p}_x = \underline{\tau}_{11,x} + \underline{\tau}_{12,y} + \underline{\tau}_{13,z} \\ &\underline{p}_y = \underline{\tau}_{21,x} + \underline{\tau}_{22,y} + \underline{\tau}_{23,z}, \quad \underline{p}_z = \underline{\tau}_{31,x} + \underline{\tau}_{32,y} + \underline{\tau}_{33,z}, \end{aligned} \tag{4}$$

where $\underline{\tau}_{11,x}$ is denoting the x partial derivative of $\underline{\tau}_{11}$.

As boundary conditions, we consider:

a) No-slip condition for the velocity components on the plates.

b) Laplace's law on the (basic) air-fluid interface.

3 The basic flow

We study the linear stability of the following basic flow, denoted by the super index ^{0}:

$$\nabla p^0 = (p_x^0(x), 0, 0), \quad \mathbf{v^0} = (u^0(z), 0, 0), \tag{5}$$

then it follows

$$\mathbf{V}_{13}^0 = u_z^0; \quad \mathbf{V}_{ij}^0 = 0 \quad \forall (ij) \neq (13); \quad \mathbf{S}^0 = \frac{1}{2}(\mathbf{V}^0 + \mathbf{V}^{0T}). \tag{6}$$

The basic extra-stress tensor is given by the following equation

$$\tau^0 - \lambda_1(\mathbf{V}^0\tau^0 + \tau^0\mathbf{V}^{0T}) = \mu\{2\mathbf{S}^0 - \lambda_2(2\mathbf{V}^0\mathbf{S}^0 + 2\mathbf{S}^0\mathbf{V}^{0T})\}, \tag{7}$$

where

$$\mathbf{V}^0\tau^0 + \tau^0\mathbf{V}^{0T} = \begin{pmatrix} 2u_z^0\tau_{31}^0 & u_z^0\tau_{32}^0 & u_z^0\tau_{33}^0 \\ u_z^0\tau_{32}^0 & 0 & 0 \\ u_z^0\tau_{33}^0 & 0 & 0 \end{pmatrix};$$

$$2[\mathbf{V}^0\mathbf{S}^0 + \mathbf{S}^0\mathbf{V}^{0T}] = \begin{pmatrix} 2(u_z^0)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore the components of the basic extra-stress are

$$\tau_{33}^0 = 0, \quad \tau_{22}^0 = 0, \quad \tau_{23}^0 = 0, \quad \tau_{12}^0 = 0, \tag{8}$$

$$\tau_{13}^0 = \mu u_z^0, \quad \tau_{11}^0 = 2(\lambda_1 - \lambda_2)\mu(u_z^0)^2 \tag{9}$$

and we get the basic flow equations:

$$p_x^0 = \tau_{11,x}^0 + \tau_{12,y}^0 + \tau_{13,z}^0, \quad p_y^0 = \tau_{21,x}^0 + \tau_{22,y}^0 + \tau_{23,z}^0, \quad p_z^0 = \tau_{31,x}^0 + \tau_{32,y}^0 + \tau_{33,z}^0. \tag{10}$$

The equations (5) - (10) (used also in [18], [3], [4]) give us

$$p_z^0 = 0, \; p_y^0 = 0, \; p_x^0(x) = \tau_{13,z}^0(z). \tag{11}$$

We conclude that a negative constant G exists such that $p_x^0(x) = \mu u_{zz}^0 = G$, therefore the basic velocity u^0 can be obtained in terms of G:

$$u^0 = \frac{1}{\mu}p_x^0(z^2 - bz)/2 = (1/\mu)G(z^2 - bz)/2. \tag{12}$$

The following characteristic velocity U is introduced

$$U =< u^0 >:= \frac{1}{b}\int_0^b u^0 dz = -\frac{b^2}{12\mu}p_x^0, \tag{13}$$

then it follows

$$u^0 = (G/2\mu)z(z-b) = -(6U/b^2)z(z-b). \tag{14}$$

The (basic) steady air-fluid interface is

$$x =< u^0 > t = Ut. \tag{15}$$

The basic pressure can depend on the time t, as in [18], [3], [4]:

$$p^0 = G(x- < u^0 > t) = G(x - Ut), \quad x > \; < u^0 > t = Ut. \tag{16}$$

In the following we consider the coordinate system moving with the velocity U; with no confusion, the basic interface in the new system is $x = 0$.

4 The perturbations system

The small perturbations of the basic solution are denoted by u, v, w, p and τ, $\mathbf{S}$, $\mathbf{V}$. The perturbation of the basic interface is η. We believe that a fluid element that was originally on the interface remains here, then it follows

$$\eta_t = u \tag{17}$$

(in other words, the interface is material).

In the frame of linear stability, the free-divergence relation is also verified by the components of the velocity perturbation, then $u_x + v_y + w_z = 0$. We integrate across the plates, we use the condition $w = 0$ on $z = 0, z = b$, then we get

$$\int_0^b (u_x + v_y) = 0.$$

In this paper we consider the particular perturbations such that

$$u_x + v_y = 0, \quad w = 0.$$

In [3], [4], the above solution is obtained by using an asymptotic analysis involving the small parameter ϵ; $w = 0$ is obtained by a numerical method in [18].

We introduce the small perturbations in the constitutive equations and in the expressions of the upper convected derivatives and get

$$\tau^0 + \tau + \lambda_1(\tau^0 + \tau)^\nabla = 2\mu[\mathbf{S}^0 + \mathbf{S} + \lambda_2(\mathbf{S}^0 + \mathbf{S})^\nabla], \tag{18}$$

$$(\tau^0 + \tau)^\nabla = u^0\tau_x - [\mathbf{V}^0\tau^0 + \tau^0\mathbf{V}^{0T}] - [\mathbf{V}^0\tau + \mathbf{V}\tau^0 + \tau^0\mathbf{V}^T + \tau\mathbf{V}^{0T}], \tag{19}$$

$$(\mathbf{S}^0 + \mathbf{S})^\nabla = u^0\mathbf{S}_x - [\mathbf{V}^0\mathbf{S}^0 + \mathbf{S}^0\mathbf{V}^{0T}] - [\mathbf{V}^0\mathbf{S} + \mathbf{V}\mathbf{S}^0 + \mathbf{S}^0\mathbf{V}^T + \mathbf{S}\mathbf{V}^{0T}]. \tag{20}$$

In the frame of the linear stability (by neglecting the second order terms in perturbations) it follows

$$\tau + \lambda_1(u^0\tau_x - \mathbf{E}) = \mu\{2\mathbf{S} + \lambda_2(u^0 2\mathbf{S}_x - \mathbf{F})\}, \tag{21}$$

with following expressions for $\mathbf{E}, \mathbf{F}$:

$$\mathbf{E} := \mathbf{V}^0\tau + \mathbf{V}\tau^0 + \tau^0\mathbf{V}^T + \tau\mathbf{V}^{0T},$$

$$\mathbf{F} := 2[\mathbf{V}^0\mathbf{S} + (\mathbf{V}^0\mathbf{S})^T + \mathbf{V}\mathbf{S}^0 + (\mathbf{V}\mathbf{S}^0)^T]. \tag{22}$$

We have

$$\tau^0\mathbf{V}^T = (\mathbf{V}\tau^0)^T, \quad \tau\mathbf{V}^{0T} = (\mathbf{V}^0\tau)^T, \quad \mathbf{S}^0\mathbf{V}^T = (\mathbf{V}\mathbf{S}^0)^T, \quad \mathbf{S}\mathbf{V}^{0T} = (\mathbf{V}^0\mathbf{S})^T,$$

where

$$\mathbf{V}^0\tau = \begin{pmatrix} 0 & 0 & u_z^0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} = \begin{pmatrix} u_z^0\tau_{31} & u_z^0\tau_{32} & u_z^0\tau_{33} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$\mathbf{V}\tau^0 = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tau_{11}^0 & 0 & \tau_{13}^0 \\ 0 & 0 & 0 \\ \tau_{31}^0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} u_x\tau_{11}^0 + u_z\tau_{31}^0 & 0 & u_x\tau_{13}^0 \\ v_x\tau_{11}^0 + v_z\tau_{31}^0 & 0 & v_x\tau_{13}^0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$2\mathbf{V}^0\mathbf{S} = \begin{pmatrix} 0 & 0 & u_z^0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2u_x & u_y + v_x & u_z \\ u_y + v_x & 2v_y & v_z \\ u_z & v_z & 0 \end{pmatrix} = \begin{pmatrix} u_z^0 u_z & u_z^0 v_z & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$2\mathbf{V}\mathbf{S}^0 = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & u_z^0 \\ 0 & 0 & 0 \\ u_z^0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} u_z u_z^0 & 0 & u_x u_z^0 \\ v_z u_z^0 & 0 & v_x u_z^0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore we get the following expressions of **E, F, S** in terms of $\mathbf{u}^0$ and $\mathbf{u}$:

$$\mathbf{E} = \begin{pmatrix} 2(u_z^0\tau_{31} + u_x\tau_{11}^0 + \tau_{13}^0 u_z) & (u_z^0\tau_{32} + v_x\tau_{11}^0 + v_z\tau_{13}^0) & (u_z^0\tau_{33} + u_x\tau_{13}^0) \\ (u_z^0\tau_{32} + v_x\tau_{11}^0 + v_z\tau_{13}^0) & 0 & \tau_{13}^0 v_x \\ (u_z^0\tau_{33} + u_x\tau_{13}^0) & \tau_{13}^0 v_x & 0 \end{pmatrix}, \tag{23}$$

$$\mathbf{F} = \begin{pmatrix} 4u_z^0 u_z & 2u_z^0 v_z & u_x u_z^0 \\ 2u_z^0 v_z & 0 & v_x u_z^0 \\ u_x u_z^0 & v_x u_z^0 & 0 \end{pmatrix}, \quad 2\mathbf{S} = \begin{pmatrix} 2u_x & (u_y + v_x) & u_z \\ (u_y + v_x) & 2v_y & v_z \\ u_z & v_z & 0 \end{pmatrix}. \tag{24}$$

5 Modal linear stability analysis

We introduce the following Fourier expansion:

$$u = f(z)EXP\cos(ny), \quad v = f(z)EXP\sin(ny), \quad EXP = \exp(-n[\alpha + x] + \sigma t),$$

$$f(z) = \beta u^0(z), \quad \beta = O(\epsilon^2), \quad \text{dimension of } \alpha = length. \tag{25}$$

This Fourier expansion was not used in [3], [4], [18]. Near the (perturbed) interface we use the following *Laplace's law* as in [18] (see also (13) and (17)):

$$(G < \eta > + < p >) - < \tau_{11} > = \gamma\,(< \eta_{yy} + \eta_{zz} >), \quad \eta = u/\sigma, \tag{26}$$

where $(\eta_{yy} + \eta_{zz})$ is the approximate expression of the total curvature of the perturbed interface and γ is the surface tension From here we get the growth constant expression σ:

$$\sigma = \frac{\gamma < u_{yy} + u_{zz} > -G < u >}{< p - \tau_{11} >}.$$

The problem is to compute $< p - \tau_{11} >$ in terms of the basic and perturbed velocities. For this, in the following we search a particular solution $\tau_{13}, \tau_{23}, \tau_{12}, p_x, p_z, \tau_{11}$ in terms of u, v, u^0, v^0.

The following dimensionless quantities are introduced

$$x' = x/l, \quad y' = y/l, \quad z' = z/b, \quad \epsilon = b/l << 1, \quad u' = u/U, \quad v' = v/U,$$

$$p' = p(l/\mu U), \quad \gamma' = \gamma(1/\mu U), \quad n' = nl, \quad \sigma' = \sigma(l/U), \quad \alpha' = \alpha/l, \quad t' = t(U/l),$$

$$\{\tau'_{11}, \tau'_{12}, \tau'_{22}\} = \{\tau_{11}, \tau_{12}, \tau_{22}\}(l/\mu U), \quad \{\tau'_{13}, \tau'_{23}, \tau'_{33}\} = \{\tau_{13}, \tau_{23}, \tau_{33}\}(b/\mu U),$$

$$W_i = \lambda_i(U/b), \tag{27}$$

where W_i are the Weissenberg numbers. From (12), (13) and (25) we get

$$u' = \beta u^{0'} EXP' \cos(n'y') = \beta 6z'(1 - z')EXP' \cos(n'y'),$$

$$v' = \beta u^{0'} EXP' \cos(n'y') = \beta 6z'(1 - z')EXP' \sin(n'y'),$$

$$EXP' = \exp(-n'[\alpha' + x'] + \sigma' t').$$

In the following we use only dimensionless quantities, then we omit the $'$.

The flow equations and the dimensionless quantities (27) give us

$$p_x \frac{\mu U}{l^2} - \tau_{11,x} \frac{\mu U}{l^2} = \tau_{12,y} \frac{\mu U}{l^2} + \tau_{13,z} \frac{\mu U}{b^2}, \qquad p_x - \tau_{11,x} = \tau_{12,y} + \tau_{13,z} \frac{1}{\epsilon^2}, \tag{28}$$

$$p - \tau_{11} = (-1/n)\{\tau_{12,y} + \tau_{13,z} \frac{1}{\epsilon^2}\}. \tag{29}$$

5.1 A particular solution

Proposition 1. *We prove that*

$$\tau_{33} = 0, \quad \tau_{31} = u_z, \quad \tau_{32} = v_z \quad \textit{are possible solutions.} \tag{30}$$

Proof. From (21) *and* (27) *we get*

$$\tau_{33} + W_1 \epsilon u^0 \tau_{33,x} = 0, \tag{31}$$

$$\tau_{31} + W_1 \epsilon u^0 \tau_{31,x} - W_1 \epsilon (u_z^0 \tau_{33} + u_x \tau_{13}^0) = u_z + W_2 \epsilon (u^0 u_{zx} - u_x u_z^0), \tag{32}$$

$$\tau_{32} + W_1 \epsilon (u^0 \tau_{32,x} - \tau_{13}^0 v_x) = v_z + W_2 \epsilon (u^0 v_{zx} - v_x u_z^0). \tag{33}$$

The equation (31) *gives us the possible solution* $\tau_{33} = 0$*. We use* (25) *and get* $(u^0 u_{zx} - u_z^0 u_x) = 0$, $(u^0 v_{zx} - v_x u_z^0) = 0$*. If* $\tau_{31} = u_z$ *and* $\tau_{32} = v_z$*, then in the l.h.s. of* (32), (33) *we have*

$$u^0 \tau_{31,x} - u_x \tau_{13}^0 = u^0 u_{zx} - u_x u_z^0 = 0,$$

$$u^0 \tau_{32,x} - v_x \tau_{13}^0 = u^0 v_{zx} - v_x u_z^0 = 0.$$

Therefore the above constitutive relations (32) - (33) *for* τ_{31}, τ_{32} *are verified if the formulas* (30) *hold.* □

As a consequence, we get

$$p_z = \tau_{31,x} + \tau_{32,y} = \mu(u_{zx} + v_{zy}) = 0. \tag{34}$$

From $(30)_2$ we get $\tau_{31,z} = u_{zz}$. We use (13), (29) and (26), then we obtain the dimensionless from of the Laplace's law as follows:

$$\frac{-12\mu U}{b^2} \cdot \frac{< Uu >}{\sigma(U/l)} + \frac{\mu U}{l} < p - \tau_{11} >= \frac{\gamma \mu U}{\sigma(U/l)} [< Uu_{yy} > \frac{1}{l^2} + < Uu_{zz} > \frac{1}{b^2}],$$

$$\frac{-12}{b^2} l \frac{< u >}{\sigma} + \frac{1}{l} < p - \tau_{11} >= l \frac{\gamma}{\sigma} [< u_{yy} > \frac{1}{l^2} + < u_{zz} > \frac{1}{b^2}],$$

$$\frac{-12}{b^2} l^2 \frac{< u >}{\sigma} + < p - \tau_{11} >= l^2 \frac{\gamma}{\sigma} < u_{yy} \frac{1}{l^2} + u_{zz} \frac{1}{b^2} >,$$

$$\frac{-12}{\epsilon^2} \frac{< u >}{\sigma} + (-1/n)\{\tau_{12,y} + u_{zz} \frac{1}{\epsilon^2}\} = \frac{\gamma}{\sigma} < u_{yy} + u_{zz} \frac{1}{\epsilon^2} > . \tag{35}$$

The term $< u_{yy} >$ can not be neglected (is related with the curvature of the perturbed interface).

Proposition 2. *If* $\exp(\sigma t) < 1$*, then*

$$MAX_n\{u_x\} = \frac{3\beta}{2\alpha e}, \quad MAX_n\{u_{xx}\} = \frac{6\beta}{\alpha^2 e^2}, \quad MAX_n\{u_{xxx}\} = \frac{81\beta}{2\alpha^3 e^3}. \tag{36}$$

Proof. Recall (14)*. We have* $u^0 = 6z(1-z) \leq 6 \cdot (1/4) = 3/2$*, then it follows*

$u_x \leq (3\beta/2) n \exp(-n\alpha)$, $u_{xx} \leq (3\beta/2) n^2 \exp(-n\alpha)$, $u_{xxx} \leq (3\beta/2) n^3 \exp(-n\alpha)$.

Consider the functions $F1(n) = n\exp(-n\alpha)$, $F2 = n^2 \exp(-n\alpha)$, $F3 = n^3 \exp(-n\alpha)$*, then*

$F1_n = (n - n\alpha)\exp(-n\alpha)$; $F2_n = (2n - n^2\alpha)\exp(-n\alpha)$; $F3_n = (3n^2 - n^3\alpha)\exp(-n\alpha)$,

where $FI_n, I = 1, 2, 3$ *are the derivatives in terms of* n*. We compute the maximal values of* FI *and obtain the relations* (36)*.* $\square$

As a consequence, we get

$$\alpha > 81/(12e) \Rightarrow MAX_n\{u_{xxx}\} < MAX_n\{u_{xx}\}.$$

In Figures 1, 2, 3, 4 we plot $F2 = 2n^2 \exp(-\alpha n), F4 = \epsilon n^3 \exp(-\alpha n)$ when $\alpha = 2, 1.6$ and $\epsilon = 0.02, 0.002$. We see that $F4$ can be neglected in front of $F2$.

Proposition 3. *If* $\exp(\sigma t) < 1$, $\beta = O(\epsilon^2)$, $(W_1 - W_2) = O(1)$, $\alpha > 11(W_2 - W_1)/e$*, then* $\tau_{12,y}$ *can be approximated by the formula*

$$\tau_{12,y} = (u_y + v_x)_y + 2(W_1 - W_2)u_z^0 v_{zy}/\epsilon. \tag{37}$$

Proof. The constitutive relations (21) *and the dimensionless quantities* (27) *give us*

$$\tau_{12,y}\frac{\mu U}{l^2} + \lambda_1 u^0 \tau_{12,xy}\frac{\mu U^2}{l^3} - \lambda_1[u_z^0 \mu v_{zy}\frac{U^2}{b^2 l} + v_{xy}2\mu(\lambda_1 - \lambda_2)(u_z^0)^2\frac{U^3}{l^2 b^2} + \mu u_z^0 v_{zy}\frac{U^2}{b^2 l}] =$$

$$\mu\{(u_y + v_x)_y\frac{U}{l^2} + \lambda_2 u^0 (u_y + v_x)_{xy}\frac{U^2}{l^3} - 2\lambda_2 u_z^0 v_{zy}\frac{U^2}{b^2 l}\}.$$

Recall $W_i = \lambda_i U/b, \quad \epsilon = b/l$*, then we get*

$$\tau_{12,y} + W_1 u^0 \tau_{12,xy}\epsilon - W_1[2u_z^0 v_{zy}/\epsilon + 2(W_1 - W_2)v_{zy}(u_z^0)^2] = \tag{38}$$

$$(u_y + v_x)_y + W_2 u^0 (u_y + v_x)_{xy}\epsilon - 2W_2 u_z^0 v_{zy}/\epsilon.$$

We introduce the expression (37) *in* (38) *and get*

$$\tau_{12,y} = -W_1 u^0 \epsilon\{(u_y + v_x)_{yx} + 2(W_1 - W_2)u_z^0 v_{zyx}/\epsilon\} + \tag{39}$$

$$(u_y + x_x)_y + 2(W_1 - W_2)u_z^0 v_{zy}/\epsilon + 2W_1(W_1 - W_2)v_{xy}(u_z^0)^2 + W_2 u^0 (u_y + v_x)_{xy}\epsilon.$$

From the Fourier expansion (25) *we have* $2W_1(W_1-W_2)[v_{xy}(u^0_z)^2 - u^0u^0_zv_{zyx}] = 0$, *then the last relation* (39) *becomes*

$$\tau_{12,y} = (u_y+v_x)_y + 2(W_1-W_2)u^0_zv_{zy}/\epsilon + (W_2-W_1)u^0(u_y+v_x)_{xy}\epsilon. \tag{40}$$

The equation (14) *and the dimensionless quantities* (27) *give us* $u^0 = 6(z-z^2) \le 3/2$, $\quad u^0_z = 6(1-2z) \le 6$. *From Proposition 2 it follows*

$$MAX_n(u_y+v_x)_y = 2\cdot\frac{3\beta}{2}\cdot\frac{4}{\alpha^2e^2} = \frac{12\beta}{\alpha^2e^2}, \quad MAX_n(W_1-W_2)u^0_zv_{zy}\frac{1}{\epsilon} = (W_1-W_2)\frac{36\beta}{\alpha e\epsilon},$$

$$MAX_n(W_2-W_1)u^0(u_y+v_x)_{xy}\epsilon = |W_2-W_1|\frac{9\beta}{2}\cdot\frac{27}{\alpha^3e^3}\epsilon = |W_2-W_1|\frac{243\beta}{2\alpha^3e^3}\epsilon. \tag{41}$$

The condition

$$\alpha > 11|W_2-W_1|/e \tag{42}$$

gives us

$$\alpha > \frac{243|W_2-W_1|}{24e}, \qquad \frac{243\beta|W_2-W_1|}{2\alpha^3e^3}\epsilon < \frac{12\beta}{\alpha^2e^2}\epsilon. \tag{43}$$

From (42) - (43) *we get*

$$MAX_n(W_2-W_1)u^0(u_y+v_x)_{xy}\epsilon < MAX_n(u_y+v_x)_y\epsilon << MAX_n(u_y+v_x)_y.$$

We conclude that $(W_2-W_1)u^0(u_y+v_x)_{xy}\epsilon$ *can be neglected in front of* $(u_y+v_x)_y$ *and* (40) *gives the relation* (37). $u_{yxy}, v_{xxy}, u_{yy}, v_{xy}$ *are bounded on terms of* n *and* $(W_2-W_1) = O(1)$. *Then, for large enough* α, *the last and first terms in* (40) *are of order* ϵ^3 *and* ϵ^2. □

Remark 1. *The above result is an important improvement compared with [3], [4], [18], where instead of* (25) *was used the expansion*

$$u = f(z)\exp(-nx+\sigma t\cos(ny), \quad v = f(z)\exp(-nx+\sigma t)\sin(ny), \quad f(z) = z(z-b). \tag{44}$$

By using (44), *the second partial derivatives of* (u,v) *with respect to* x, y *contains the factor* $n^2\exp(-nx+\sigma t)$. *The expression* $n^2\exp(-nx)$ *is not bounded in terms of* n *when* $x \to 0$. *Indeed, consider the function* $I(n,x) = n^2\exp(-nx)$, *then his derivative with respect to* n *is* $dI/dn = (2n-n^2x)\exp(-nx)$ *and we have* $I(n=2/x,x) = MAX_n\{I(n,x)\} = 4/(x^2e^2)$.

From (44) *we see also that* u_{xxx} *contains the factor* $-n^3\exp(-nx)$. *Consider the function* $J(n,x) = -n^3\exp(-nx)$, *we have* $MAX_n|J(n,x)| = 27/(x^3e^3)$. *Then near* $x=0$ *we have* $MAX_n|u_{xxx}| >> MAX_n|u_{xx}|$. *It follows that in the equation* (40) *we can not neglect the* *term* $(W_2 - W_1)\epsilon u^0(u_y + v_x)_x$ *in front of* $(u_y + v_x)_y$ *and we not obtain the approximate expression* (37) *for* τ_{12y}.

On the page 414 of [18] is mentioned that "the representation of the flow field as basic parallel flow plus small disturbances must fail very close to the interface, probably within a distance of order ϵ *(in our notation)" and "this places a restriction on the disturbance wavelength to which the theory can be expected to apply". We have*

$$x = p\epsilon \Rightarrow MAX_n|J(n, p\epsilon)| < MAX_n I(n, p\epsilon) \quad iff \quad p\epsilon > 27/(4e).$$

Then, for neglecting $|u_{xxx}|$ *in front of* $|u_{xx}|$, *is not enough to avoid a distance "of order* ϵ*" from* $x = 0$, *but a distance of order* $27/(4e) \approx 2.48 >> \epsilon$.

The expansion (25) *and the condition* (42) *are useful for avoid the singularity near* $x = 0$ *and for neglecting (in a rigorous way) some terms in the constitutive relations.* □

5.2 The growth constant formula

The relations (28), (37) give us

$$p - \tau = (-1/n)\{(u_y + v_x)_y + 2(W_1 - W_2)u_z^0 v_{zy}\frac{1}{\epsilon} + u_{zz}\frac{1}{\epsilon^2}\} =$$

$$(-1/n)[O(\epsilon^2) + O(\epsilon) + O(1)]. \tag{45}$$

Consider $O(\gamma) = 1$. Then (35) and (45) gives the magnitudes of the terms appearing in the dimensionless Laplace's law :

$$\frac{O(1)}{\sigma} - \frac{1}{n}\{O(\epsilon^2) + O(\epsilon) + O(1)\} = \frac{O(1)}{\sigma}\{O(\epsilon^2) + O(1)\}. \tag{46}$$

We insert (45) *in* (35). As $< u^0 >= 1, < (u_z^0)^2 >= 12$, we get

$$\frac{-12}{\epsilon^2\sigma} + (-1/n)\{-2n^2 + 24n(W_1 - W_2)\frac{1}{\epsilon} - 12\frac{1}{\epsilon^2}\} = \frac{\gamma}{\sigma}(-n^2 - 12\frac{1}{\epsilon^2}).$$

The last relation is giving the following

Proposition 4. *The growth rate* σ *corresponding to the approximate solution* (37) *is*

$$\sigma = \frac{12n - \gamma\epsilon^2 n^3 - 12\gamma n}{2n^2\epsilon^2 - 24n(W_1 - W_2)\epsilon + 12},$$

$$\sigma = \frac{n(1-\gamma) - \gamma(\epsilon^2/12)n^3}{n^2(\epsilon^2/6) - 2n(W_1 - W_2)\epsilon + 1} \tag{47}$$

□

Remark 2. *In the case* $\exp(\sigma t) < 1$, *from* (47) *we obtain the following important results :*

a) The denominator of (47) *is* $\neq 0$ *for* $W_1 - W_2 < 1/\sqrt{6} \approx 0.408$. *Indeed , we have*

$$W_1 - W_2 < 1/\sqrt{6} \Rightarrow \Delta = (W_1 - W_2)\epsilon^2 - (\epsilon^2/6) < 0.$$

b) $\sigma \to \infty$ *when* $W_2 = 0, \quad W_1 < 1/\sqrt{6}, \quad W_1 \to 1/\sqrt{6}.$

c) If $0.3 < W1 < 0.408$ *and* $W_2 = 0$ *we have a destabilization effect compared with the Saffman-Taylor growth constant* σ_{S-T} *below:*

$$\sigma_{S-T} = n - \gamma(\epsilon^2/12)n^3 < \sigma < \infty. \tag{48}$$

In Figures 5, 6 we plot σ *given by* (47) *(on the vertical axis) in terms of* n *(on the horizontal axis), for* $\gamma = 0.9$, $\epsilon = 0.006$, $W_2 = 0$ *and* W_1 *increasing from 0.15 until 0.404. The plots are similar with the numerical results given in [18] . In Figure 5 is also plotted the Saffman-Taylor growth rate* (48). □

5.3 Justification for f used in (25)

Proposition 5. *If the hypothesis of Proposition 3 hold, then* τ_{11x} *can be the approximated by the formula*

$$\tau_{11x} = 2u_{xx} + 4(W_1 - W_2)u^0_z u_{zx}/\epsilon. \tag{49}$$

Proof. *The dimensionless form of the constitutive relations* (21) *is*

$$\tau_{11,x}\frac{\mu U}{l^2} + \lambda_1 u^0 \tau_{11,xx}\frac{\mu U^2}{l^3} - 2\lambda_1[u^0_z \mu u_{zx}\frac{U^2}{b^2 l} + u_{xx}2\mu(\lambda_1 - \lambda_2)(u^0_z)^2\frac{U^3}{l^2 b^2} + \mu u^0_z u_{zx}\frac{U^2}{b^2 l}] =$$

$$\mu\{2u_{xx}\frac{U}{l^2} + \lambda_2 u^0 2u_{xxx}\frac{U^2}{l^3} - 4\lambda_2 u^0_z u_{zx}\frac{U^2}{b^2 l}\},$$

then we obtain

$$\tau_{11,x} + W_1 u^0 \tau_{11,xx}\epsilon - 2W_1[u^0_z u_{zx}/\epsilon + u_{xx}2(W_1 - W_2)(u^0_z)^2 + u^0_z u_{zx}/\epsilon] =$$

$$2u_{xx} + 2W_2 u^0 u_{xxx}\epsilon - 4W_2 u^0_z u_{zx}/\epsilon.$$

We insert the expression (49) *in the last relation and get*

$$\tau_{11,x} + W_1\epsilon u^0[2u_{xxx} + 4(W_1 - W_2)u^0_z u_{zxx}/\epsilon] - 2W_1[u^0_z u_{zx}/\epsilon + u_{xx}2(W_1 - W_2)(u^0_z)^2 + u^0_z u_{zx}/\epsilon] =$$

$$2u_{xx} + 2W_2 u^0 u_{xxx}\epsilon - 4W_2 u^0_z u_{zx}/\epsilon.$$

As $f(z) = \beta u^0$, *we have* $u^0 u^0_z u_{zxx} - u_{xx}(u^0_z)^2 = 0$, *then it follows*

$$\tau_{11,x} = 2u_{xx} + 4(W_1 - W_2)u^0_z u_{zx}/\epsilon + 2(W_2 - W_1)\epsilon u^0_z u_{xxx}.$$

As in Proposition 3, the condition (42) *allows us to neglect* $\{2(W_2 - W_1)\epsilon u_z^0 u_{xxx}\}$ *in front of* $\{2u_{xx}\}$. *Then the last above expresion of* $\tau_{11,x}$ *gives us the equation* (49). $\square$

We compute now the expression of p_x, by using (37) and (49) :

$$p_x = \tau_{11,x} + \tau_{12,y} + \tau_{13,z}/\epsilon^2 =$$

$$2u_{xx} + 4(W_1 - W_2)u_z^0 u_{zx}/\epsilon + (u_y + v_x)_y + 2(W_1 - W_2)u_z^0 v_{zy}/\epsilon + u_{zz}/\epsilon^2.$$

As $u_{xx} + u_{yy} = 0, \quad u_{xx} + v_{xy} = 0, \quad u_{zx} + v_{zy} = 0$, it follows

$$p_x = 2(W_1 - W_2)u_z^0 u_{zx}/\epsilon + u_{zz}/\epsilon^2.$$

We obtained $p_z = 0$ - see (34) - then the last above relation gives us

$$p_{xz} = 2(W_1 - W_2)(u_z^0 u_{zx})_z/\epsilon + u_{zzz}/\epsilon^2 = 0. \tag{50}$$

Proposition 6. *As before, we suppose* $\exp(\sigma t) < 1, \quad (W_1 - W_2) = O(1)$. *If the condition* (42) *holds and*

$$\beta = \epsilon^2/27, \tag{51}$$

then $f(z) = \beta u^0(z)$ *verifies* (50) *with the precision order* $O(\epsilon)$.

Proof. We have $MAX_n\{n\exp(-n\alpha)\} = 1/(\alpha e)$, *then* $(u_z^0 u_{zx})_z \leq 144\beta/(\alpha e)$. *It follows*

$$PXZ1 := 2(W_1 - W_2)(u_z^0 u_{zx})_z \frac{1}{\epsilon} \leq 2(W_1 - W_2)144 \frac{\epsilon}{27} \cdot \frac{1}{11(W_1 - W_2)} < \epsilon << 1.$$

Therefore the conditions (42) *and* (51) *allow us to neglect* $PXZ1$ *in the equation* (50). *As* $u_{zzz}^0 = 0$, *we conclude that* $f(z) = \beta u^0(z)$ *verifies* (50) *with the precision order* $O(\epsilon)$. *The condition* (51) *is in agreement with the hypothesis* $\beta = O(\epsilon^2)$ *used in the decomposition* (25). $\square$

6 Conclusions

We study the modal linear stability of the displacement of an Oldroyd-B fluid by air in a Hele-Shaw cell. The full system of flow equations and constitutive relations are used. The basic flow (8)-(16) is considered. In the frame of linear stability, we obtain the perturbation system (21)-(24).

Roughly speaking, the decomposition (25) gives us bounded values for $u_{yxy}, v_{xxy}, u_{yy}, v_{xy}$ in terms of $n, \forall x \geq 0$. For large enough α, the last and first terms in (40) are of order ϵ^3

and ϵ^2 and we get the approximate expression (37) for $\tau_{12,y}$. Therefore, the condition (42) imposed on the parameter α, appearing in the Fourier decomposition (25) for the velocity perturbations, allows us to obtain the following results:

- the approximate expressions of $\tau_{12,y},\ \tau_{11,x}$ are given in Propositions 3 and 5;

- $\tau_{12,y}$ is used in the Laplace's law (35) and we get the growth constant (47);

- we justify the form of the amplitude f appearing in (25) - see Proposition 6.

If $\alpha = 0$, then the Fourier decomposition (25) lead us to very high values of the derivatives of the velocity perturbations near the interface $x = 0$. In this case, the estimates and the expressions given in Propositions 3 and 5 can not be obtained - also Remark 1. The amplitude of the perturbations introduced by (25) is very small.

The main result is the following. The growth constant formula (47) gives us a strong destabilization effect, compared with the case studied in [14], when $W_1 - W_2 \approx 0.408$ - see Remark 2 and Figures 5 - 6. Most of the numerical results concerning the flow of Oldroyd-B fluids failed when the Weissenberg is near 1. We prove that, for the flow geometry considered here, the higher Weissenberg number instability is due to the model, and not to the numerical methods.

When $W_1 = W_2$, the growth rate according to our formula (47) is quite similar with the Saffman-Taylor formula for a Newtonian fluid displaced by air in a Hele-Shaw cell. In this case, due to our 3D approach and Fourier modes decomposition (25), the formula (47) contains the new term $(-\gamma n)$ in the numerator and the new term $n^2(\epsilon^2/6)$ in the denominator.

In Appendix we prove that our basic flow is Newtonian if $W_1 = W_2$.

Acknowledgments

This paper was made possible by an NPRP grant # 08-777-1-141 to one of the authors (Prabir Daripa) from the Qatar National Research Fund (a member of the Qatar Foundation). The second author (Gelu Pasa) acknowledges financial support through this grant for travel to TAMUQ, Qatar for a two day workshop on “International Workshop on Enhanced Oil Recovery and Porous Media Flows” organized by the first author (Prabir Daripa) during July 31st and August 1 of 2013. The statements made herein are solely the responsibility of the authors.

Appendix: When $\lambda_1 = \lambda_2$

Proposition 7. *The basic steady flow is Newtonian if $\lambda_1 = \lambda_2$.*

Proof. The constitutive relations (7) *become*

$$\tau^0 - \lambda_1(\mathbf{V}^0\tau^0 + \tau^0\mathbf{V}^{0T}) = \mu\{\mathbf{2S}^0 - \lambda_1(\mathbf{V}^0\mathbf{2S}^0 + \mathbf{2S}^0\mathbf{V}^{0T})\}. \tag{52}$$

We introduce $\mathbf{A} = \tau^0 - \mu 2\mathbf{S}^0 = a_{ij}$, then $\mathbf{A} = \mathbf{A}^T$ and we get

$$\mathbf{A} - \lambda_1(\mathbf{V}^0\mathbf{A} + \mathbf{A}\mathbf{V}^{0T}) = 0. \tag{53}$$

We claim that from (53) *it follows $\mathbf{A} = 0$. Indeed, we have:*

$$\mathbf{V}^0 = \begin{pmatrix} 0 & 0 & u_z^0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{V}^0\mathbf{A} + \mathbf{A}\mathbf{V}^{0T} = \begin{pmatrix} 2u_z^0 a_{31} & u_z^0 a_{32} & u_z^0 a_{33} \\ u_z^0 a_{32} & 0 & 0 \\ u_z^0 a_{33} & 0 & 0 \end{pmatrix} \tag{54}$$

and (53) *becomes*

$$\begin{pmatrix} a_{11} - 2\lambda_1 u_z^0 a_{31} & a_{12} - \lambda_1 u_z^0 a_{32} & a_{13} - \lambda_1 u_z^0 a_{33} \\ a_{21} - \lambda_1 u_z^0 a_{32} & a_{22} & a_{23} \\ a_{31} - \lambda_1 u_z^0 a_{33} & a_{32} & a_{33} \end{pmatrix} = 0. \tag{55}$$

First we get $a_{22} = a_{23} = a_{33} = 0$. The third row entries of the above relation give $a_{31} = 0$. Using these values in the second row entries gives $a_{21} = 0$. Finally, the first row entries give $a_{11} = 0$. Thus we have proved that $\mathbf{A} = 0$ which means $\tau^0 - \mu 2\mathbf{S}^0 = 0$. Then the basic flow is indeed Newtonian when $\lambda_1 = \lambda_2$.

□

References

[1] J. Bear, Dynamics of Fluids in Porous Media, Courier corporation, 1972.

[2] R. B. Bird, W. E. Stewart, E. N., Transport phenomena, Vol. 1: Fluid Mechanics, John Wiley and Sons, N Y, 1960.

[3] P. Daripa and G. Pasa, *Saffman-Taylor instability for an Oldroyd-B fluid,* "Enhanced Oil Recovery and Porous media Flows", July 31 - August 1, 2013, Doha, Qatar.

[4] P. Daripa and G. Pasa, *Saffman-Taylor Instability for a non-Newtonian fluid,* 66th Annual Meeting of the APS Division of Fluid Dynamics **58**(18), Session M30, Nov. 26, 2013, Pittsburgh, Pennsylvania, USA.

[5] R. L. Fosdik and K. D. Rajakopal, *Anomalous features in the model of "second order fluids"*, Arch. Rat. Mech. Anal., **70**(1979), 145-152.

[6] C. Guillope, J.C. Saut, *Mathematical problems arising in differential models for viscoelastic fluids, J.F. Rodrigues and A. Sequeira eds.*, Mathematical Topics in Fluid Mechanics, Longman, Halow, 1992, 64-92.

[7] H.S. Hele-Shaw, *The flow of water,* Nature, **28**(1898), 34-36.

[8] D. S. Miroshnichenko, *Anisotropic second-order fluid,* J. Mathematical Sciences, **103**(2001), 43-58.

[9] S. Mora and M. Manna, *Saffman-Taylor instability for generalized Newtonian fluids*, Phys. Rev. E, **80** (2009), 016308.

[10] ——, *Saffman-Taylor instability of viscoelastic fluids: From viscous fingering to elastic fractures*, Phys. Rev. E, **81** (2010), 026305.

[11] J. Nase, A. Lindner, and C. Creton, *On the respective roles of low surface tension and non-Newtonian rheological properties in fractal fingering*, Phys. Rev. Lett., **101** (2008), 02863.

[12] J. Nittman, G. Daccord, and H. E. Stanley, *Fractal growth viscous fingers: quantitative characterization of a fluid instability phenomenon*, Nature, **314** (1985), 141–144.

[13] M. Renardy, Mathematical analysis of viscoelastic flow, CBMS-NSF Regional Conference Series in Applied Mathematics, 73 (2000). Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA.

[14] P. G. Saffman and G. Taylor, *The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid*, Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences, **245** (1958), 312-329.

[15] W.R. Schowalter, Mechanics of Non-Newtonian Fluids, Pergamon Press, New York, 1978.

[16] C. Truesdell and W. Noll, The non-linear filed theories of mechanics, *Encyclopedia of Physics (ed. S. Fluggel)* vol III/3, Springer Verlag, 1965.

[17] H. Van Damme, E. Alsac, C. Laroche, and L. Gatineau, *On the respective roles of low surface tension and non-Newtonian rheological properties in fractal fingering*, EPL (Europhysics Letters), **5** (1988), 25-38.

[18] S. D. R. Wilson, *The Taylor-Saffman problem for a non-Newtonian liquid*, Journal of Fluid Mechanics, **220** (1990), 413-425.

[19] H. Zhao and J. V. Maher, *Associating-polymer effects in a Hele-Shaw experiment*, Phys. Rev. E, **47** (1993), 4278–4283.

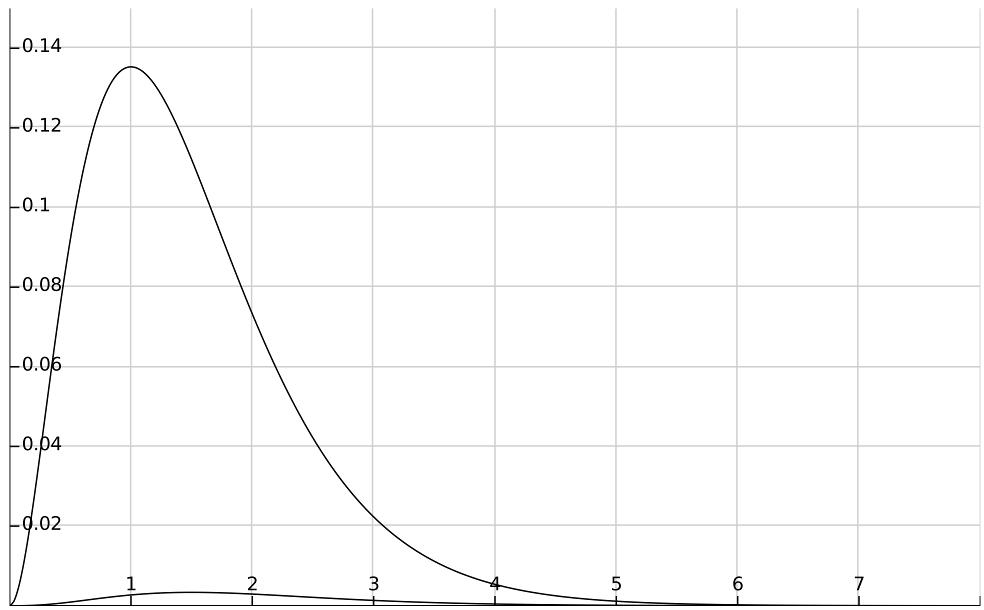

Figure 1: $n^2 \exp(-n\alpha)(upper)$ and $\epsilon n^3 \exp(-n\alpha)(lower)$ for $\alpha = 2, \epsilon = 0.02$.

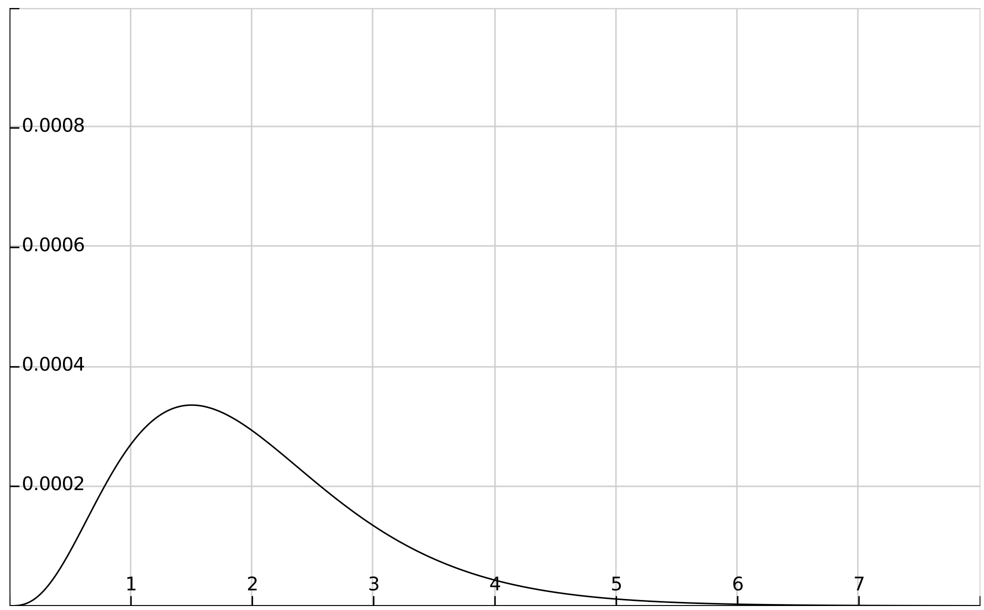

Figure 2: $\epsilon n^3 \exp(-n\alpha)$ for $\alpha = 2, \epsilon = 0.002$.

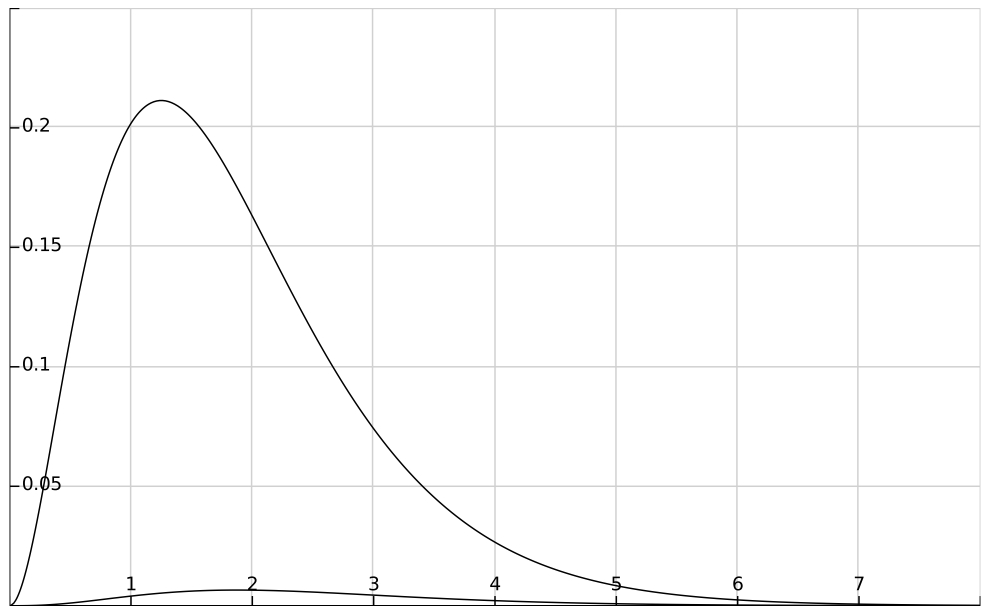

Figure 3: $n^2 \exp(-n\alpha)(upper)$ and $\epsilon n^3 \exp(-n\alpha)(lower)$ for $\alpha = 1.6, \epsilon = 0.02$.

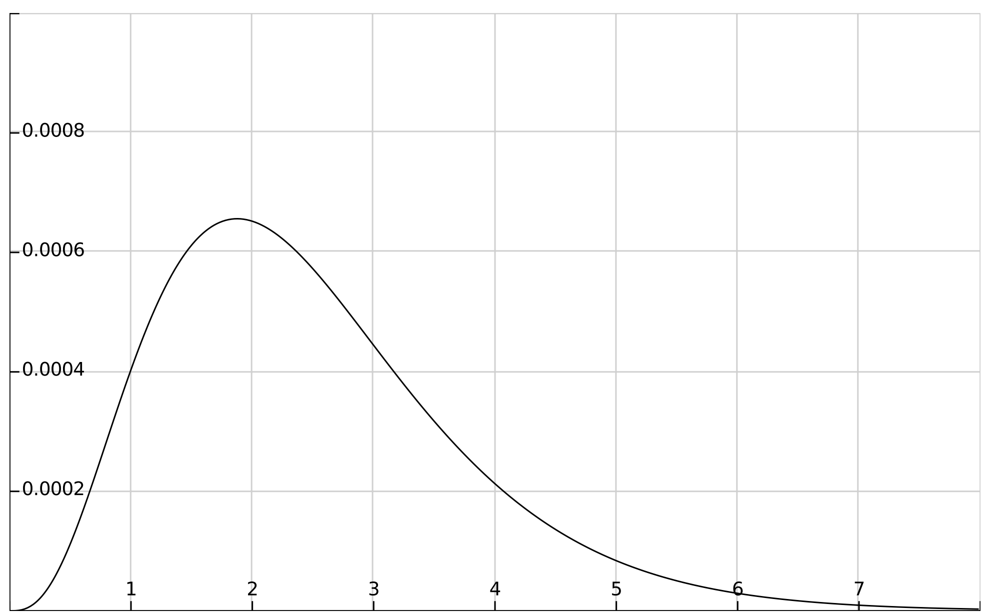

Figure 4: $\epsilon n^3 \exp(-n\alpha)$ for $\alpha = 1.6, \epsilon = 0.002$.

(48)

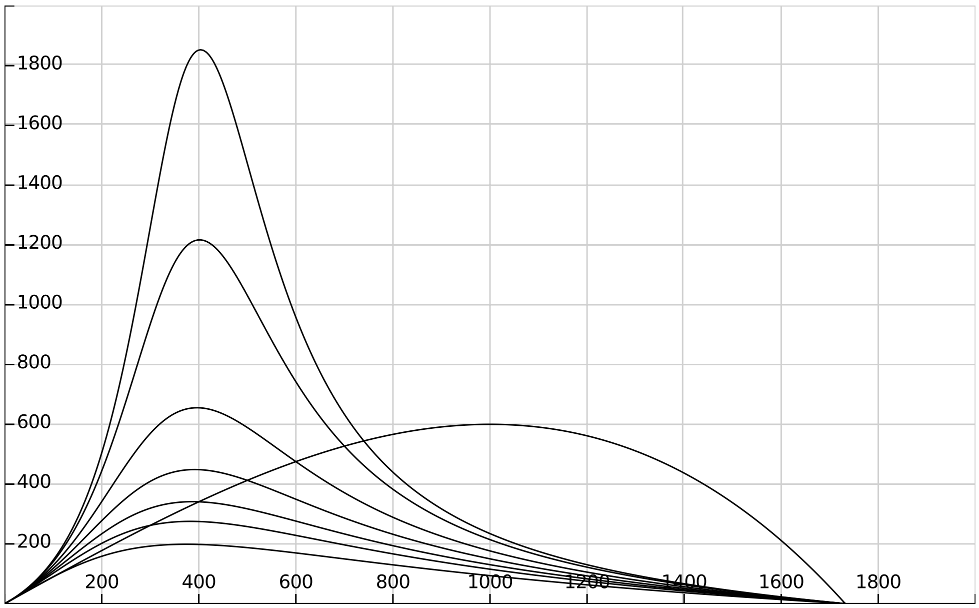

Figure 5: Plot of (47) compared with (48). $\epsilon = 0.006, \gamma = 0.9$,
$W_2 = 0, W_1 = 0.05(lower), 0.15, 0.2, 0.25, 0.3, 0.35, 0.37(upper)$

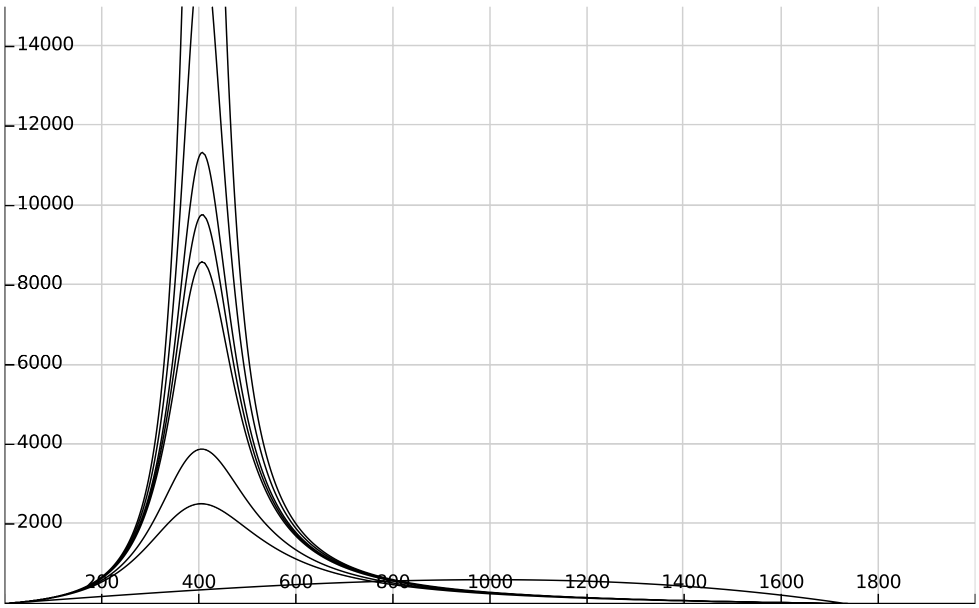

Figure 6: Plot of (47). $\epsilon = 0.006, \gamma = 0.9$,
$W_2 = 0, W_1 = 0.38(lower), 0.39, 0.4, 0.401, 0.402, 0.403, 0.404(upper)$