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by

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Abstract

Starting from quasi-Wajsberg algebras (which are generalizations of Wajsberg algebras), whose regular sets are Wajsberg algebras, we introduce a theory of quasi-algebras versus, in parallel, a theory of regular algebras. We introduce the quasi-RM, quasi-RML, quasi-BCI, quasi-BCK, quasi-Hilbert and quasi-Boolean algebras as generalizations of RM, RML, BCI, BCK, Hilbert and Boolean algebras respectively.

In Part II, the second part of the theory of quasi-algebras - versus the second part of a theory of regular algebras - is presented. We introduce the positive implicative, commutative and quasi-implicative quasi-BCK algebras and the quasi-Hilbert algebras and we prove that the quasi-Hilbert algebras coincide with positive implicative quasi-BCK algebras.

Keywords: quasi-MV algebra, quasi-Wajsberg algebra, MV algebra, Wajsberg algebra, BCK algebra, Hilbert algebra

AMS classification (2010): 06F35, 03G25, 06A06

1 Introduction

The quasi-MV algebras were introduced in 2006 [20], as generalizations of MV algebras introduced in 1958 [4], following an investigation into the foundations of quantum computing. Since then, many papers investigated them [1], [24], [11], [19].

The quasi-Wajsberg algebras were introduced in 2010 [2], as generalizations of Wajsberg algebras introduced in 1984 [6]; they are term-equivalent to quasi-MV algebras, just as Wajsberg algebras are term equivalent to MV algebras. The regular set $R(A)$ of any quasi-Wajsberg algebra A is a Wajsberg algebra. Remark that any Wajsberg algebra A has in the signature an implication $\rightarrow$ and a constant 1 that verify the following two properties, among many others: for all $x \in A$,

(Re) $x \rightarrow x = 1$, (M) $1 \rightarrow x = x$,

while any quasi-Wajsberg algebra A has in the signature an implication $\rightarrow$ and a constant 1 that verify the following two properties, among many others: for all $x, y \in A$,

(Re) $x \rightarrow x = 1$, (qM) $1 \rightarrow (x \rightarrow y) = x \rightarrow y$.

Note that (M) implies (qM) and this is the most important reason why the quasi-Wajsberg algebras are generalizations of Wajsberg algebras.

We have introduced in 2013 [15] many new generalizations of BCI, of BCK and of Hilbert algebras, in a general investigation of algebras $(A, \rightarrow, 1)$ of type $(2, 0)$ that can verify properties in a given list of properties. Among the new generalizations, the most general one is the RM algebra, i.e. an algebra $(A, \rightarrow, 1)$ verifying the properties (Re), (M).

Based mainly on the results in [2] and in [15] and on the above remarks, we have developed a theory of quasi-algebras (including the lists qA, qB, qC of properties, with many connections) versus, in parallel, a theory of regular algebras (including the lists A, B, C of properties, with many connections). We have introduced new quasi-algebras: the quasi-RM, quasi-RML, quasi-BCI, quasi-BCK, quasi-Hilbert algebras and the quasi-Boolean algebras, as generalizations of the corresponding regular algebras: RM, RML, BCI, BCK, Hilbert and Boolean algebras. We have made the connection with the quasi-Wajsberg algebras.

In Part I, the first part of the theory of quasi-algebras is presented [16], including the list qA of basic properties and many connections - versus the first part of a theory of regular algebras, including the list A of basic properties and many connections. We introduce the quasi-order and the quasi-Hasse diagram - versus the regular order and the Hasse diagram - and we study the quasi-ordered algebras (structures). We introduce the quasi-RM and the quasi-RML algebras and we present two equivalent definitions of quasi-BCI and of quasi-BCK algebras.

In Part II, the second part of the theory of quasi-algebras is presented, including the list qB of particular properties and many connections - versus the second part of a theory of regular algebras, including the list B of particular properties and many connections. We introduce the positive implicative, commutative and quasi-implicative quasi-BCK algebras and the quasi-Hilbert algebras and we prove that the quasi-Hilbert algebras coincide with positive implicative quasi-BCK algebras.

This paper, Part II, is organized as follows:

In Section 2, we recall the first part of a theory of regular algebras (including the List A of basic properties) and the corresponding first part of the theory of quasi-algebras [16] (including the List qA of basic quasi-properties).

In Section 3, we present the second part of a theory of regular algebras, including, the list B of particular properties and subsequent connections.

In Section 4, we present the corresponding second part of the theory of quasi-algebras, including the list qB of particular properties and subsequent connections. We present some new quasi-algebras: the positive implicative, commutative and quasi-implicative quasi-BCK algebras and the quasi-Hilbert algebras.

In Section 5, we present some examples of finite quasi-algebras introduced in Section 4.

2 Quasi-algebras versus regular algebras (structures) - Part I (recallings)

Let $\mathcal{A} = (A, \to, 1)$ be an *algebra* of type $(2, 0)$ through this section, where a binary relation $\leq$ can be defined: for all $x, y \in A$,

$$(dfrelR) \qquad x \leq y \overset{df.}{\Leftrightarrow} x \to y = 1.$$

Equivalently, let $\mathcal{A} = (A, \leq, \to, 1)$ be a *structure* where $\leq$ is a binary relation on A, $\to$ is a binary operation (an implication called "residuum" in some conditions) on A and $1 \in A$, all connected by the equivalence: for all $x, y \in A$,

$$(EqrelR) \qquad x \leq y \Leftrightarrow x \to y = 1.$$

Remarks 2.1 (1) We explain the equivalence:
If $A, \to, 1)$ is an algebra of type $(2, 0)$, then we can define a binary relation $\leq$ by (dfrelR); thus, $(A, \leq, \to, 1)$ is a structure, where $\leq$ and $\to, 1$ are connected by the equivalence (EqrelR).
Conversely, if $(A, \leq, \to, 1)$ is a structure where $\leq$ is a binary relation on A, $\to$ is a binary operation on A and $1 \in A$ and all are connected by the equivalence (EqrelR), then the reduct $(A, \to, 1)$ is an algebra of type $(2, 0)$ where we can define a binary relation $\leq$ by (dfrelR).

(2) In other words, roughly speaking, the equivalence (EqrelR) can be used either:
- to define a binary relation $\leq$ in the algebra $(A, \to, 1)$ of type $(2, 0)$, by (dfrelR), or
- to connect the binary relation $\leq$ and $\to$, 1 from the structure $(A, \leq, \to, 1)$.

(3) In these conditions, we simply write: "let $(A, \to, 1)$ be an algebra (or, equivalently, let $(A, \leq, \to, 1)$ be a structure)" or, even shorter, "let $(A, \to, 1)$ $((A, \leq, \to, 1))$ be an algebra (structure)".

In the first subsection, we shall recall from [16] the Part I of the theory of regular algebras (structures).

In the second subsection, we shall recall from [16] the Part I of the theory of quasi-algebras (quasi-structures).

2.1 Introduction to a theory of regular algebras (structures) - Part I (recallings)

Recall first the definitions:

Definitions 2.2 [16]

(1) The algebra $(A, \rightarrow, 1)$ (or, equivalently, the structure $(A, \leq, \rightarrow, 1)$) is called *regular*, if it satisfies the property (M): $1 \rightarrow x = x$, for all $x \in A$.

(1') Any algebra (structure) $\mathcal{A}' = (A, \sigma)$ whose signature σ contains $\rightarrow$, 1 ($\leq$, $\rightarrow$, 1, respectively) is also called *regular*, if it satisfies the property (M).

(1") Any algebra (structure) $\mathcal{A}'' = (A, \tau)$ which is term equivalent to a regular algebra (structure) $\mathcal{A}' = (A, \sigma)$, is also called *regular*.

(2) The implication $\rightarrow$ from a regular algebra (structure) is called *regular implication*.

(3) The binary relation $\leq$ of a regular algebra (structure) is called *binary regular relation*.

Remark 2.3 [16] By (M), we have that: $V_M = V = U = A$, and this is the basic, definable property of regular algebras (structures), where

$$U \stackrel{df.}{=} \{x \rightarrow y \mid x, y \in A\},\ V \stackrel{df.}{=} \{1 \rightarrow x \mid x \in A\},\ V_M \stackrel{df.}{=} \{x \in A \mid x \stackrel{(M)}{=} 1 \rightarrow x\}.$$

2.1.1 The list A of basic properties [16]

Recall the following **List A** of basic properties (those from [15] plus two new properties, (#) and (##)), that can be satisfied by the algebra $\mathcal{A} = (A, \rightarrow, 1)$ (the structure $\mathcal{A} = (A, \leq, \rightarrow, 1)$) (in fact, the properties in the List A are the most important properties satisfied by a BCK algebra (see [15])); some of the properties are presented in two equivalent forms, determined by the above equivalence (EqrelR):

$$(EqrelR) \qquad x \leq y \Leftrightarrow x \rightarrow y = 1.$$

We divided the list into two parts: the properties in Part 1 are those that will be generalized to quasi-properties, when considering the quasi-algebras (quasi-structures).

List A, Part 1

(An) (Antisymmetry) $x \rightarrow y = 1 = y \rightarrow x \implies x = y$,
(An') (Antisymmetry) $x \leq y,\ y \leq x \implies x = y$;
(M) $1 \rightarrow x = x$;
(N) $1 \rightarrow x = 1 \implies x = 1$,
(N') $1 \leq x \implies x = 1$;
(Re) (Reflexivity) $x \rightarrow x = 1$ (we prefer here notation (Re) instead of (I) in the theory of BCI algebras),
(Re') (Reflexivity) $x \leq x$;
(L) (Last element) $x \rightarrow 1 = 1$,
(L') (Last element) $x \leq 1$.

List A, Part 2

(EqrelR) $x \leq y \Leftrightarrow x \rightarrow y = 1$,
(dfrelR) $x \leq y \stackrel{df.}{\Leftrightarrow} x \rightarrow y = 1$;

(11-1) $1 \rightarrow 1 = 1$,
(11-1') $1 \leq 1$;
(B) $(y \rightarrow z) \rightarrow [(x \rightarrow y) \rightarrow (x \rightarrow z)] = 1$,
(B') $y \rightarrow z \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$,
(BB) $(y \rightarrow z) \rightarrow [(z \rightarrow x) \rightarrow (y \rightarrow x)] = 1$,

(BB') $y \to z \leq (z \to x) \to (y \to x)$;
(*) $y \to z = 1 \implies (x \to y) \to (x \to z) = 1$,
(*') $y \leq z \implies x \to y \leq x \to z$;
(**) $y \to z = 1 \implies (z \to x) \to (y \to x) = 1$,
(**') $y \leq z \implies z \to x \leq y \to x$;
(C) $[x \to (y \to z)] \to [y \to (x \to z)] = 1$,
(C') $x \to (y \to z) \leq y \to (x \to z)$;
(D) $y \to [(y \to x) \to x] = 1$,
(D') $y \leq (y \to x) \to x$;
(Ex) (Exchange) $x \to (y \to z) = y \to (x \to z)$;
(K) $x \to (y \to x) = 1$,
(K') $x \leq y \to x$;
(S) $x = y \implies x \to y = 1$,
(S') $x = y \implies x \leq y$;
(Tr) (Transitivity) $x \to y = 1 = y \to z \implies x \to z = 1$,
(Tr') (Transitivity) $x \leq y,\ y \leq z \implies x \leq z$;
(#) $x \to (y \to z) = 1 \Longrightarrow y \to (x \to z) = 1$,
(#') $x \leq y \to z \Longrightarrow y \leq x \to z$;
(Eq#) $x \to (y \to z) = 1 \Longleftrightarrow y \to (x \to z) = 1$,
(Eq#') $x \leq y \to z \Longleftrightarrow y \leq x \to z$;
($) $x \to (y \to z) = 1 \Longrightarrow (x \to y) \to (x \to z) = 1$,
($') $x \leq y \to z \Longrightarrow x \to y \leq x \to z$.

Note that in List A from [16], (Eq#) was (##), while (EqrelR) and (dfrelR) were numbers and hence were not included in List A.

Remarks 2.4 [16]
(0) (M) $\Longrightarrow$ (11-1); (Re) $\Longrightarrow$ (11-1); (L) $\Longrightarrow$ (11-1).
(i) The central role of property (M) in the study of regular algebras (structures) [15] is given by the fact that it determines that $V_M = V = U = A$, i.e. all the elements of A appear compulsory inside the table of $\to$ ($\to$: $A \times A \longrightarrow A$).

Remark 2.5 Note that, for example,
(EqrelR) $\Longrightarrow$ ((An) $\Leftrightarrow$ (An')); (An) + (An') $\Longrightarrow$ (EqrelR).

2.1.2 Connections between the properties in the list A [16]

Proposition 2.6 *[15] [16] Let $(A, \to, 1)$ be an algebra of type $(2, 0)$. Then the following are true:*
(A0) (Re) $\Longrightarrow$ (S);
(A00) (M) $\Longrightarrow$ (N);
(A1) (L) + (An) $\Longrightarrow$ (N);
(A2) (K) + (An) $\Longrightarrow$ (N);
(A3) (C) + (An) $\Longrightarrow$ (Ex); (A3') (Ex) + (Re) $\Longrightarrow$ (C);
(A4) (Re) + (Ex) $\Longrightarrow$ (D); (A4') (D) + (Re) + (An) $\Longrightarrow$ (N);
(A5) (Re) + (Ex) + (An) $\Longrightarrow$ (M);
(A6) (Re) + (K) $\Longrightarrow$ (L);
(A7) (N) + (K) $\Longrightarrow$ (L); (A7') (M) + (K) $\Longrightarrow$ (L);
(A8) (Re) + (L) + (Ex) $\Longrightarrow$ (K);
(A9) (M) + (L) + (B) $\Longrightarrow$ (K); (A9') (M) + (L) + (**) $\Longrightarrow$ (K);
(A10) (Ex) $\Longrightarrow$ (B) $\Leftrightarrow$ (BB);
(A10') (Ex) + (B) $\Longrightarrow$ (BB); (A10") (Ex) + (BB) $\Longrightarrow$ (B);
(A11) (Re) + (Ex) + (*) $\Longrightarrow$ (BB);
(A12) (N) + (B) $\Longrightarrow$ (*); (A12') (M) + (B) $\Longrightarrow$ (*);

(A13) (N) + (*) $\Longrightarrow$ (Tr); *(A13') (M) + (*) $\Longrightarrow$ (Tr);*
(A14) (N) + (B) $\Longrightarrow$ (Tr); *(A14') (M) + (B) $\Longrightarrow$ (Tr);*
(A15) (N) + (BB) $\Longrightarrow$ (**); *(A15') (M) + (BB) $\Longrightarrow$ (**);*
(A16) (N) + (**) $\Longrightarrow$ (Tr); *(A16') (M) + (**) $\Longrightarrow$ (Tr);*
(A17) (N) + (BB) $\Longrightarrow$ (Tr); *(A17') (M) + (BB) $\Longrightarrow$ (Tr);*
(A18) (M) + (BB) $\Longrightarrow$ (Re); *(A18') (M) + (BB) $\Longrightarrow$ (D);*
(A19) (M) + (B) $\Longrightarrow$ (Re);
(A20) (BB) + (D) + (N) $\Longrightarrow$ (C); *(A20') (M) + (BB) $\Longrightarrow$ (C);*
(A21) (BB) + (D) + (N) + (An) $\Longrightarrow$ (Ex);
(A21') (BB) + (D) + (L) + (An) $\Longrightarrow$ (Ex);
(A21") (M) + (BB) + (An) $\Longrightarrow$ (Ex);
(A22) (K) + (Ex) + (M) $\Longrightarrow$ (Re);
(A23) (C) + (K) + (An) $\Longrightarrow$ (Re);
(A24) (Re) + (Ex) + (Tr) $\Longrightarrow$ (**).

Proposition 2.7 *[16] Let $(A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. Then the following are true:*
(A9") (M) + (L) + (BB) $\Longrightarrow$ (K);
(A18") (M) + (D) $\Longrightarrow$ (Re);
(A25) (D) + (K) + (N) + (An) $\Longrightarrow$ (M);
(A26) (#) $\Longleftrightarrow$ (Eq#) ;
(A27) (M) + (C) $\Longrightarrow$ (#);
(A28) (Ex) $\Longrightarrow$ (Eq#);
(A29) (BB) + (#) $\Longrightarrow$ (B), *(A29') (B) + (#) $\Longrightarrow$ (BB),*
(A29") (#) $\Longrightarrow$ ((B) $\Leftrightarrow$ (BB));
(A30) (Re) + (B) + (Tr) + (#) $\Longrightarrow$ (C);
(A31) (Re) + (#) $\Longrightarrow$ (D).

Theorem 2.8 *[15], [16] (Generalization of ([3], Lemma 1.2 and Proposition 1.3))*
If properties (Re), (M), (Ex) hold, then: $(BB) \Leftrightarrow (B) \Leftrightarrow (*)$.

Theorem 2.9 *[15], [16]*
If properties (Re), (M), (Ex) hold, then: $(**) \Leftrightarrow (Tr)$.

Theorem 2.10 *[15], [16]*
If properties (M), (B), (An) hold, then: $(Ex) \Leftrightarrow (BB)$.

Theorem 2.11 *[15], [16] (Michael Kinyon) In any algebra $(A, \rightarrow, 1)$, we have:*
(i) (M) + (BB) $\Longrightarrow$ (B),
(ii) (M) + (B) $\Longrightarrow$ (**).

By Kinyon's Theorem 2.11(i) and (A12'), we obtained immediately that:

Corollary 2.12 [15], [16] (M) + (BB) $\Longrightarrow$ (*).

Concluding, by above Kinyon's Theorem 2.11 and (A12'), (A13'), (A16'), we have obtained:

Corollary 2.13 [15], [16] In any algebra $(A, \rightarrow, 1)$ verifying (M), we have:

$$(BB) \implies (B) \implies (*), (**) \implies (Tr).$$

2.1.3 Some regular algebras: the RM, RML, BCI and BCK algebras

Recall the following definitions:

Definition 2.14 An algebra $(A, \rightarrow, 1)$ is a:
- *RM algebra*, if it verifies the axioms (Re), (M) [15];
- *RML algebra*, if it verifies the axioms (Re), (M), (L) [15];
- *BCI algebra*, if it verifies the axioms (BB), (D), (Re), (An) [17], or, equivalently [15], (B), (C), (Re), (An);
- *BCK algebra*, if it verifies the axioms (BB), (D), (Re), (L), (An) [17], [13], [18], or, equivalently [15], (B), (C), (K), (An).

Recall that the BCK algebras verify all the properties in List A.

2.2 Introduction to a theory of quasi-algebras (quasi-structures) - Part I (recallings)

Recall first the definitions:

Definitions 2.15 [16]

(q1) The algebra $(A, \rightarrow, 1)$ (or, equivalently, the structure $(A, \leq, \rightarrow, 1)$) is called *quasi-algebra* (*quasi-structure*, respectively) if it satisfies the properties (qM) ($1 \rightarrow (x \rightarrow y) = x \rightarrow y$, for all $x, y \in A$) and (11-1) ($1 \rightarrow 1 = 1$).

(q1') Any algebra (structure) $\mathcal{A}' = (A, \sigma)$ whose signature σ contains $\rightarrow$, 1 ($\leq$, $\rightarrow$, 1, respectively) is also called *quasi-algebra* (*quasi-structure*), if it satisfies the properties (qM) and (11-1).

(q1") Any algebra (structure) $\mathcal{A}'' = (A, \tau)$ which is term equivalent to a quasi-algebra (structure) $\mathcal{A}' = (A, \sigma)$, is also called *quasi-algebra* (*quasi-structure*).

(q2) The implication $\rightarrow$ from a quasi-algebra (quasi-structure) is called *quasi-implication.*

(q3) The binary relation $\leq$ of a quasi-algebra (quasi-structure) is called *binary quasi-relation.*

Remark 2.16 [16] (qM) is different of (M) if and only if $V_M = V = U \subset A$, and this is the basic, definable property of quasi-algebras (quasi-structures), where

$$U \overset{df.}{=} \{x \rightarrow y \mid x, y \in A\},\ V \overset{df.}{=} \{1 \rightarrow x \mid x \in A\},\ V_M \overset{df.}{=} \{x \in A \mid x \overset{(M)}{=} 1 \rightarrow x\}.$$

Definitions 2.17 [16]

(1) For every quasi-algebra (quasi-structure) $\mathcal{A}$, the subset $V_M = V = U$ of A will be called the *regular* set of $\mathcal{A}$ and will be denoted by $R(A)$:

$$R(A) \overset{df.}{=} V_M = V = U.$$

The elements of $R(A)$ are called the *regular elements* of A.

(2) The quasi-algebra (quasi-structure) $\mathcal{A}$ is called *proper* if $R(A) \neq A$ (i.e. (M) $\not\Longleftrightarrow$ (qM)); otherwise, $\mathcal{A}$ is a regular algebra (structure).

Theorem 2.18 [16] Let $\mathcal{A} = (A, \rightarrow, 1)$ be a proper quasi-algebra (or, equivalently, let $\mathcal{A} = (A, \leq, \rightarrow, 1)$ be a proper quasi-structure). Then, $\mathcal{R}(\mathcal{A}) = (R(A), \rightarrow, 1)$ is a regular algebra (or, equivalently, $\mathcal{R}(\mathcal{A}) = (R(A), \leq, \rightarrow, 1)$ is a regular structure, respectively).

Definition 2.19 [16] We call *proper quasi-properties* the following nine: (qAn), (qM), (qM($1 \rightarrow y$)), (qN), (qN($1 \rightarrow y$)), (qRe), (qRe($1 \rightarrow y$)), (qL), (qL($1 \rightarrow y$)) which form the Part 1 of List qA (corresponding to the five properties (An), (M), (N), (Re), (L) respectively, which form the Part 1 of List A).

2.2.1 The list qA of basic quasi-properties [16]

The **List qA** of "quasi-properties" that can be satisfied by the algebra $\mathcal{A} = (A, \to, 1)$ (by the structure $\mathcal{A} = (A, \leq, \to, 1)$) has three parts, and follows closely the list A of properties. The proper quasi-properties in Part 1 of List qA are generalizations of the properties in Part 1 of List A; the "quasi-properties" in Part 2 of List qA are the properties in Part 2 of List A; a Part 3, containing the special (specific) quasi-properties, is added.

List qA, Part 1

(qAn) (quasi-Antisymmetry) $x \to y = 1 = y \to x \implies 1 \to x = 1 \to y$,
(qAn') (quasi-Antisymmetry) $x \leq y,\ y \leq x \implies 1 \to x = 1 \to y$;
(qM) $1 \to (x \to y) = x \to y$;
(qM($1 \to x$)) $1 \to (1 \to x) = 1 \to x$;
(qN) $1 \to (x \to y) = 1 \implies x \to y = 1$,
(qN') $1 \leq x \to y \implies x \to y = 1$;
(qN($1 \to x$)) $1 \to (1 \to x) = 1 \implies 1 \to x = 1$,
(qN($1 \to x$)') $1 \leq 1 \to x \implies 1 \to x = 1$;
(qRe) (quasi-Reflexivity) $(x \to y) \to (x \to y) = 1$,
(qRe') (quasi-Reflexivity) $x \to y \leq x \to y$;
(qRe($1 \to x$)) $(1 \to x) \to (1 \to x) = 1$,
(qRe($1 \to x$)') $1 \to x \leq 1 \to x$;
(qL) $(x \to y) \to 1 = 1$,
(qL') $x \to y \leq 1$;
(qL($1 \to x$)) $(1 \to x) \to 1 = 1$,
(qL($1 \to x$)') $1 \to x \leq 1$.

List qA, Part 2

(EqrelR), (dfrelR);
(11-1), (B), (BB), (*), (**), (C), (D), (Ex), (K), (S), (Tr);
(#), (Eq#).

List qA, Part 3

(qR) $(x \to y) \to ((1 \to x) \to (1 \to y)) = 1$,
(qR') $x \to y \leq (1 \to x) \to (1 \to y)$;
(qR1) $(1 \to x) \to x = 1$,
(qR1') $1 \to x \leq x$;
(qR2) $x \to (1 \to x) = 1$,
(qR2') $x \leq 1 \to x$;
(qR3) $(x \to (1 \to y)) \to (1 \to (x \to y)) = 1$,
(qR3') $x \to (1 \to y) \leq 1 \to (x \to y)$;
(qR4) $((1 \to x) \to y) \to (1 \to (x \to y)) = 1$,
(qR4') $(1 \to x) \to y \leq 1 \to (x \to y)$;

(qI) $x \to y = (1 \to x) \to (1 \to y)$,
(qI1) $x \to y = (1 \to x) \to y$,
(qI2) $x \to y = x \to (1 \to y)$,
(qI3) $(1 \to x) \to (1 \to y) = (1 \to x) \to y$,
(qI4) $(1 \to x) \to (1 \to y) = x \to (1 \to y)$;

(qEqI) $x \to y = 1 \Leftrightarrow (1 \to x) \to (1 \to y) = 1$,

(qEqI') $x \leq y \Leftrightarrow 1 \to x \leq 1 \to y$;
(qEqI1) $x \to y = 1 \Leftrightarrow (1 \to x) \to y = 1$,
(qEqI1') $x \leq y \Leftrightarrow 1 \to x \leq y$;
(qEqI2) $x \to y = 1 \Leftrightarrow x \to (1 \to y) = 1$,
(qEqI2') $x \leq y \Leftrightarrow x \leq 1 \to y$;
(qEqI1-2) $(1 \to x) \to y = 1 \Leftrightarrow x \to (1 \to y) = 1$,
(qEqI1-2') $1 \to x \leq y \Leftrightarrow x \leq 1 \to y$.

Note that: (qR4), (qI4) and (qEqI1-2) are new, they were not in List qA of [16]; (qEqI), (qEqI1), (qEqI2) were (qrelI, (qrelI1), (qrelI2) respectively in List qA of [16].

2.2.2 Connections between the properties in List A and the proper quasi-properties in List qA [16]

Theorem 2.20 *[16] Let $(A, \to, 1)$ be an algebra of type $(2, 0)$. Then the following are true:*
(i) (An) $\Longrightarrow$ (qAn);
(ii) (M) $\Longrightarrow$ (qM) $\Longrightarrow$ (qM$(1 \to x)$);
(iii) (N) $\Longrightarrow$ (qN) $\Longrightarrow$ (qN$(1 \to x)$);
(iv) (Re) $\Longrightarrow$ (qRe) $\Longrightarrow$ (qRe$(1 \to x)$);
(v) (L) $\Longrightarrow$ (qL) $\Longrightarrow$ (qL$(1 \to x)$);
(vi) (M) + (qAn) $\Longrightarrow$ (An);
(vii) (M) + (qRe$(1 \to x)$) $\Longrightarrow$ (Re);
(viii) (M) + (qL$(1 \to x)$) $\Longrightarrow$ (L);
(ix) (qM) + (qRe$(1 \to x)$) $\Longrightarrow$ (qRe);
(x) (qM) + (qL$(1 \to x)$) $\Longrightarrow$ (qL);
(xi) (qL$(1 \to x)$) + (qI1) $\Longrightarrow$ (L).

Proof. (xi): $x \to 1 \overset{(qI1)}{=} (1 \to x) \to 1 \overset{(qL(1 \to x))}{=} 1$. □

Proposition 2.21 *[16] (See Proposition 2.6)*
Let $(A, \to, 1)$ be an algebra of type $(2, 0)$. Then the following are true (following the numbering from Proposition 2.6):
(qA00) (qM) $\Longrightarrow$ (qN); (qA00') (qM$(1 \to y)$) $\Longrightarrow$ (qN $(1 \to y)$);
(qA3) (C) + (qM) + (qAn) $\Longrightarrow$ (Ex);
(qA4) (Ex) + (qRe) $\Longrightarrow$ (D);
(qA7) (qN) + (K) $\Longrightarrow$ (L); (qA7') (qM) + (K) $\Longrightarrow$ (L);
(qA8) (Re) + (qL$(1 \to x)$) + (Ex) + (qI1) $\Longrightarrow$ (K);
(qA12) (qN) + (B) $\Longrightarrow$ (*); (qA12') (qM) + (B) $\Longrightarrow$ (*);
(qA13) (qN) + (*) $\Longrightarrow$ (Tr); (qA13') (qM) + (*) $\Longrightarrow$ (Tr);
(qA14) (qN) + (B) $\Longrightarrow$ (Tr); (qA14') (qM) + (B) $\Longrightarrow$ (Tr);
(qA15) (qN) + (BB) $\Longrightarrow$ (**); (qA15') (qM) + (BB) $\Longrightarrow$ (**);
(qA16) (qN) + (**) $\Longrightarrow$ (Tr); (qA16') (qM) + (**) $\Longrightarrow$ (Tr);
(qA17) (qN) + (BB) $\Longrightarrow$ (Tr); (qA17') (qM) + (BB) $\Longrightarrow$ (Tr);
(qA18) (qM) + (BB) $\Longrightarrow$ (qRe$(1 \to y)$);
(qA19) (qM) + (B) $\Longrightarrow$ (qRe$(1 \to y)$);
(qA20) (BB) + (D) + (qN) $\Longrightarrow$ (C);
(qA20') (BB) + (D) + (qM) $\Longrightarrow$ (C);
(qA21) (BB) + (D) + (qM) + (qAn) $\Longrightarrow$ (Ex);
(qA22) (K) + (Ex) + (qM) $\Longrightarrow$ (Re);
(qA23) (C) + (K) + (qM) + (qAn) $\Longrightarrow$ (Re).

Proof.
(qA8): By above (xi), (qL($1 \to x$)) + (qI1) $\Longrightarrow$ (L); then, by (A8), (Re) + (L) + (Ex) $\Longrightarrow$ (K); thus, (K) holds. □

Proposition 2.22 *[16] (See Proposition 2.7)*
Let $(A, \to, 1)$ be an algebra of type $(2, 0)$. We have the additional properties (following the numbering from Proposition 2.7):
(qA18") (qM) + (D) $\Longrightarrow$ (qR1);
(qA25) (D) + (K) + (qN) + (qAn) $\Longrightarrow$ (qM($1 \to y$));
(qA27) (qM) + (C) $\Longrightarrow$ (#).

Now we recall from [16] the corresponding theorems of (above recalled from [15]) Theorems 2.8, 2.9, 2.10, 2.11.

Theorem 2.23 *[16]*
If properties (Re), (qM), (Ex) hold, then: $(BB) \Leftrightarrow (B) \Leftrightarrow (*)$.

Theorem 2.24 *[16]*
If properties (Re), (qM), (Ex) hold, then: $(**) \Leftrightarrow (Tr)$.

Theorem 2.25 *[16]*
If properties (B), (D), (qM), (qAn) hold, then: $(Ex) \Leftrightarrow (BB)$.

Theorem 2.26 *[16] In any algebra $(A, \to, 1)$ we have:*
(i) (qM) + (BB) + (D) imply (B),
(ii) (qM) + (B) imply (**).

Concluding, by above Theorem 2.26 and (qA12'), (qA13'), (qA16'), we have immediately obtained:

Corollary 2.27 [16] In any algebra $(A, \to, 1)$ verifying (qM), we have:

$$(BB) + D \Longrightarrow (B) \Longrightarrow (*), (**) \Longrightarrow (Tr).$$

Proposition 2.28 *[16] Let $(A, \to, 1)$ be an algebra of type $(2, 0)$. Then we have the following additional quasi-properties, with an independent numbering (note that (qAA20) - (qAA26) are new, they were not in [16]):*
(qAA1) (qM) + (BB) $\Longrightarrow$ (11-1); (qAA1') (qM) + (B) $\Longrightarrow$ (11-1);
(qAA1") (qM) + (K) $\Longrightarrow$ (11-1);
(qAA2) (Ex) + (qRe($1 \to y$)) + (qM) $\Longrightarrow$ (qR1);
(qAA3) (B) $\Longrightarrow$ (qR);
(qAA4) (K) $\Longrightarrow$ (qR2);
(qAA5) (qR1) + (BB) $\Longrightarrow$ (qR3);
(qAA6) (qR1) + (K) + (**) + (qM) + (qAn) $\Longrightarrow$ (qI1);
(qAA7) (qRe($1 \to y$)) + (Ex) + (K) + (**) + (qM) + (qAn) $\Longrightarrow$ (qI1);
(qAA7') (D) + (K) + (**) + (qM) + (qAn) $\Longrightarrow$ (qI1);
(qAA8) (Ex) + (qM) $\Longrightarrow$ (qI2) + (qI3);
(qAA9) (qI1) + (qI3) $\Longrightarrow$ (qI);
(qAA10) (qR1) + (qR2) + (BB) + (qM) + (qAn) $\Longrightarrow$ (qI1) + (qI2);
(qAA11) (qR3) + (K) + (*) + (qM) + (qAn) $\Longrightarrow$ (qI2);
(qAA12) (qI1) + (qI2) $\Longrightarrow$ (qI);
(qAA13) (qI) + (BB) + (qM) $\Longrightarrow$ (Re); (see (A18), (qA18))
(qAA14) (qI1) + (BB) + (L) + (qM) $\Longrightarrow$ (K); (see (A9"))
(qAA15) (B) + (Ex) + (K) + (**) + (qM) + (qAn) $\Longrightarrow$ (qI);
(qAA15') (qRe($1 \to y$)) + (Ex) + (K) + (**) + (qM) + (qAn) $\Longrightarrow$ (qI);
(qAA15") (Re) + (Tr) + (Ex) + (L) + (qM) + (qAn) $\Leftrightarrow$
(qRe($1 \to y$)) + (Ex) + (K) + (**) + (qM) + (qAn);

(qAA15"') (Re) + (Tr) + (Ex) + (L) + (qM) + (qAn) $\Longrightarrow$ (qI);
(qAA16) (qI) + (qRe($1 \to y$)) $\Longrightarrow$ (Re);
(qAA17) (qI) $\Longrightarrow$ ((qRe($1 \to y$)) $\Leftrightarrow$ (Re));
(qAA18) (#) + (qM) + (qR1) $\Longrightarrow$ (qRe($1 \to x$));
(qAA18') (#) + (qM) + (qRe($1 \to x$)) $\Longrightarrow$ (qR1);
(qAA18") (#) + (qM) $\Longrightarrow$ ((qR1) $\Leftrightarrow$ (qRe($1 \to x$)));
(qAA19) (qI) $\Longrightarrow$ (qEqI);
(qAA19') (qI1) $\Longrightarrow$ (qEqI1);
(qAA19") (qI2) $\Longrightarrow$ (qEqI2);

(qAA20) (qI) + (qM($1 \to x$)) $\Longrightarrow$ (qI1) + (qI2);
(qAA21) (qI) + (qRe) $\Longrightarrow$ (qR);
(qAA21') (qI1) + (Re) $\Longrightarrow$ (qR1);
(qAA21") (qI2) + (Re) $\Longrightarrow$ (qR2);
(qAA22) (qI2) + (qR2) $\Longrightarrow$ (qR3);
(qAA22') (qI1) + (qR2) $\Longrightarrow$ (qR4);
(qAA23) (qI) + (qI1) $\Longrightarrow$(qI3); (qAA23') (qI) + (qI2) $\Longrightarrow$ (qI4);
(qAA24) (qI1) $\Longrightarrow$ ((qL($1 \to x$)) $\Leftrightarrow$ (L));
(qAA25) (qR1) + (qR2) + (Tr) $\Longrightarrow$ (qEqI1-2);
(qAA26) (qEqI1) + (qEqI2) $\Longrightarrow$ (qEqI1-2).

Proof.

(qAA20): $(1 \to x) \to y \overset{(qI)}{=} (1 \to (1 \to x)) \to (1 \to y) \overset{(qM(1\to x))}{=} (1 \to x) \to (1 \to y) \overset{(qI)}{=} x \to y$; thus, (qI1) holds.

$x \to (1 \to y) \overset{(qI)}{=} (1 \to x) \to (1 \to (1 \to y)) \overset{(qM(1\to x))}{=} (1 \to x) \to (1 \to y) \overset{(qI)}{=} x \to y$; thus, (qI2) holds.

(qAA21): $(x \to y) \to [(1 \to x) \to (1 \to y)] \overset{(qI)}{=} (x \to y) \to (x \to y) \overset{(qRe)}{=}$; thus, (qR) holds.

(qAA21'): $(1 \to x) \to x \overset{(qI1)}{=} x \to x \overset{(Re)}{=} 1$; thus, (qR1) holds.

(qAA21"): $x \to (1 \to x) \overset{(qI2)}{=} x \to x \overset{(Re)}{=} 1$; thus, (qR2) holds.

(qAA22): $(x \to (1 \to y)) \to (1 \to (x \to y)) \overset{(qI2)}{=} (x \to y) \to (1 \to (x \to y)) \overset{(qR2)}{=} 1$; thus, (qR3) holds.

(qAA22'): $(1 \to x) \to y) \to (1 \to (x \to y)) \overset{(qI1)}{=} (x \to y) \to (1 \to (x \to y)) \overset{(qR2)}{=} 1$; thus, (qR4) holds.

(qAA23): Obviously. (qAA23'): Obviously.

(qAA24): $x \to 1 \overset{(qI1)}{=} (1 \to x) \to 1 \overset{(qL(1\to x))}{=}$; thus, (L) holds.

$(1 \to x) \to 1 \overset{(qI1)}{=} x \to 1 \overset{(L)}{=} 1$; thus, (qL($1 \to x$)) holds.

(qAA25): If $x \leq 1 \to y$, then since, by (qR1), $1 \to y \leq y$ and $1 \to x \leq x$, it follows, by (Tr), that $1 \to x \leq y$. Conversely, if $1 \to x \leq y$, then, since by (qR2), $x \leq 1 \to x$ and $y \leq 1 \to y$, it follows that $x \leq 1 \to y$.

(qAA26): Obviously. □

2.2.3 Quasi-ordered algebras

Let $\mathcal{A}$ be a proper quasi-algebra (quasi-structure) (i.e. (qM), which differs from (M), and (11-1) hold) through this subsubsection; then its subalgebra $\mathcal{R}(\mathcal{A})$ is a regular algebra (structure) (i.e. (M) holds), by Theorem 2.18.

Definitions 2.29 Consider the following properties of $\to$ ($\leq$): (Re), (qAn), (Tr) ((Re'), (qAn'), (Tr'), respectively). Then, we shall say that $\mathcal{A}$ is:
- *reflexive*, if property (Re) (or (Re')) is satisfied,
- *quasi-antisymmetric*, if property (qAn) (or (qAn')) is satisfied,
- *transitive*, if property (Tr) (or (Tr')) is satisfied;
- *quasi-pre-ordered*, and $\leq$ is a *quasi-pre-order*, if it is reflexive and transitive;

- *quasi-ordered*, and $\leq$ is a *quasi-order* (or a *q-order* for short), if it is reflexive, quasi-antisymmetric and transitive.

A quasi-ordered quasi-algebra (quasi-structure) will be simply called "a quasi-ordered algebra (structure)".

Remark 2.30 Let $\mathcal{A} = (A, \rightarrow, 1)$ be a quasi-ordered algebra (or $\mathcal{A} = (A, \leq, \rightarrow, 1)$ be a quasi-ordered structure). Since (qM) coincides with (M) on $R(A)$, by Theorem 2.18, it follows that (qAn) coincides with (An) on $R(A)$. Consequently,
- the q-order relation $\leq$ on A becomes an order relation on $R(A)$;
- $\mathcal{R}(\mathcal{A}) = (R(A), \rightarrow, 1)$ is an ordered regular algebra (or, equivalently, $\mathcal{R}(\mathcal{A}) = (R(A), \leq, \rightarrow, 1)$ is an ordered regular structure, respectively).

Definition 2.31 We say that $a, b \in A$ have *the same height*, or are *parallel*, and we denote this by $a \parallel b$, if $a \rightarrow b = 1$ and $b \rightarrow a = 1$ (or, equivalently, $a \leq b$ and $b \leq a$).

Note that if (M) holds, then $a \parallel b \Leftrightarrow a = b$, by (An).
Note also that if $a \parallel b$, then $1 \rightarrow a = 1 \rightarrow b$, by (qAn).

Corollary 2.32 *If (Ex), (L) hold, then*

$$a \parallel b \quad \Longleftrightarrow \ 1 \rightarrow a = 1 \rightarrow b.$$

Proposition 2.33 *The relation $\parallel$ is an equivalence relation of $\mathcal{A}$.*

Proposition 2.34 *If properties (*), (**) also hold, then $\parallel$ is a congruence relation of $\mathcal{A}$.*

Proposition 2.35 *If (*), (**) also hold and if $b \parallel a$, then, in the table of $\rightarrow$, we have:*
(i) the row of b coincides with the row of a;
(ii) the column of b coincides with the column of a.

For each $x \in A$, we denote its equivalence class by

$$\mid x \mid \stackrel{notation}{=} \{y \in A \ \mid \ y \parallel x\}$$

and we denote by $A/\parallel$ the quotient set (i.e. the set of all equivalence classes):

$$A/\parallel \stackrel{notation}{=} \{\mid x \mid \ \mid \ x \in A\}.$$

Note that in the particular case of quasi-MV algebras [20], the equivalence relation $\parallel$ is denoted by χ and the equivalence classes determined by χ are called *clouds*. We have, more generally:

Lemma 2.36 *If properties (Ex), (L) also hold, then every cloud in a quasi-ordered algebra (structure) $\mathcal{A}$ contains exactly one regular element.*

Remark 2.37 Given a finite regular ordered algebra $(X, \rightarrow, 1)$, we can obtain, **in general**, an infinity of finite quasi-ordered algebras: $(A_1, \rightarrow, 1)$, $(A_2, \rightarrow, 1)$, ... such that $R(A_1) = R(A_2) = ... = X$, by adding one or more elements parallel with some (all) elements of X (see the examples in Section 6).
We can quickly draw the table of $\rightarrow$ for such a finite quasi-ordered algebra $(A, \rightarrow, 1)$ with $R(A) = X$ by using either:
- property (qI), if (Ex), (L) hold or
- Proposition 2.35, if (*), (**) hold.

A quasi-order relation $\leq$ on A will be represented graphically by a *quasi-Hasse diagram*, i.e.:
- a regular element is represented by a bullet $\bullet$,
- an element parallel with a regular element is represented by a bigcirc $\bigcirc$,
- the fact that $x < y$ (i.e. $x \leq y$ and $x \neq y$) and there is no z with $x < z < y$ is represented by:
· a line connecting the two points, y being higher than x, if the elements x, y are regular,
· a horizontal line connecting the two points, if the elements x, y have the same height (are parallel).
Consequently, the regular order relation $\leq$ on $R(A)$ will be represented graphically by a Hasse diagram.

2.2.4 New quasi-algebras

In [16], there were introduced the quasi-RM, quasi-RML, quasi-BCI and quasi-BCK algebras. We recall here only the definitions of the quasi-BCK algebras, because they are needed in the sequel.

Definition 2.38 [16] Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2, 0)$ (or, equivalently, let $\mathcal{A} = (A, \leq, \rightarrow, 1)$ be a structure), as before. $\mathcal{A}$ is called a *quasi-BCK algebra* (or a *qBCK algebra*, for short) if one of the two following equivalent groups of properties is satisfied:
(qBCK-1) (BB), (D), (Re), (L), (qM), (qAn) and
(qBCK-2) (B), (C), (K), (qM), (qAn).

Recall that the quasi-BCK algebras verify all the quasi-properties in List qA and that if (M) holds, then any quasi-BCK algebra is a BCK algebra.

3 Introduction to a theory of regular algebras - Part II

3.1 The List B of particular properties

We present the resuming List B of particular properties that will be used in this section and in the sequel. As for the List A of basic properties, we divided the List B into two parts: the properties in Part I are those that will be generalized to quasi-properties (when considering the quasi-algebras, in the next section); the properties in Part II are those that remain unchanged in the quasi-algebras case.

List B, Part 1

(impl) (implicative) $(x \rightarrow y) \rightarrow x = x$;
(pi) $x \rightarrow (x \rightarrow y) = x \rightarrow y$;

(Vid) (idempotency of $\vee$) $x \vee x = x$;
(EqV) $x \rightarrow y = 1 \Leftrightarrow x \vee y = y$,
(EqV') $x \leq y \Leftrightarrow x \vee y = y$,
(dfrelV) $x \leq y \overset{df.}{\Leftrightarrow} x \vee y = y$;
(V=) $x \rightarrow z = 1,\ y \rightarrow z = 1 \implies x \vee y \leq z$,
(V=') $x \leq z,\ y \leq z \implies x \vee y \leq z$;

(G) (Gödel) $x \odot x = x$;
(P1-1) $x \odot 1 = 1 \odot x = x$: (P-1) $x \odot 1 = x$, (P1-) $1 \odot x = x$;
(EqP) $x \rightarrow y = 1 \Leftrightarrow x \odot y = x$,
(EqP') $x \leq y \Leftrightarrow x \odot y = x$,
(dfrelP) $x \leq y \overset{df.}{\Leftrightarrow} x \odot y = x$;
(P=) $z \rightarrow x = 1,\ z \rightarrow y = 1 \implies z \leq x \odot y$,
(P=') $z \leq x,\ z \leq y \implies z \leq x \odot y$;

(Wid) (idempotency of $\wedge$) $x \wedge x = x$;
(W1-1) $x \wedge 1 = 1 \wedge x = x$: (W-1) $x \wedge 1 = x$, (W1-) $1 \wedge x = x$;
(EqW) $x \rightarrow y = 1 \Leftrightarrow x \wedge y = x$,
(EqW') $x \leq y \Leftrightarrow x \wedge y = x$,
(dfrelW) $x \leq y \overset{df.}{\Leftrightarrow} x \wedge y = x$;
(W=) $z \rightarrow x = 1,\ z \rightarrow y = 1 \implies z \leq x \wedge y$,
(W=') $z \leq x,\ z \leq y \implies z \leq x \wedge y$;

(Pabs1) (P-absorption-1) $x \odot (x \vee y) = x$, (Pabs2) (P-absorption-2) $x \vee (x \odot y) = x$;

(Wabs1) (W-absorption-1) $x \wedge (x \vee y) = x$,
(Wabs2) (W-absorption-2) $x \vee (x \wedge y) = x$;

(EqVW) $x \vee y = y \Leftrightarrow x \wedge y = x$.

List B, Part 2

(pimpl) (positive implicative) $x \to (y \to z) = (x \to y) \to (x \to z)$;
(pimpl-1) $[x \to (y \to z)] \to [(x \to y) \to (x \to z)] = 1$,
(pimpl-1') $x \to (y \to z) \leq (x \to y) \to (x \to z)$;
(pimpl-2) $[(x \to y) \to (x \to z)] \to [x \to (y \to z)] = 1$,
(pimpl-2') $(x \to y) \to (x \to z) \leq x \to (y \to z)$;
($) $x \to (y \to z) = 1 \Longrightarrow (x \to y) \to (x \to z) = 1$,
($') $x \leq y \to z \Longrightarrow x \to y \leq x \to z$;

(comm) (commutative) $(x \to y) \to y = (y \to x) \to x$;
(comm-1) $[(x \to y) \to y] \to x = (x \to (y \to x)) \to (y \to x)$;

(Vee) $x \vee y = (x \to y) \to y$,
(dfV) (df. of $\vee$ (vee, join)) $x \vee y \stackrel{df.}{=} (x \to y) \to y$;

(Vcomm) (commutativity of $\vee$ (vee, join)) $x \vee y = y \vee x$;
(Vassoc) (associativity of $\vee$ (vee, join)) $(x \vee y) \vee z = x \vee (y \vee z)$;
(V1-1) $x \vee 1 = 1 \vee x = 1$: (V-1) $x \vee 1 = 1$, (V1-) $1 \vee x = 1$;
(V-) $x \to y = 1 \Longrightarrow (x \vee a) \to (y \vee a) = 1$,
(V-') $x \leq y \Longrightarrow x \vee a \leq y \vee a$;
(V- -) $x \to y = 1,\ a \to b = 1 \ \Longrightarrow (x \vee a) \to (y \vee b) = 1$,
(V- -') $x \leq y,\ a \leq b \ \Longrightarrow x \vee a \leq y \vee b$;
(Vgeq) $x \to (x \vee y) = 1,\ y \to (x \vee y) = 1$,
(Vgeq') $x, y \leq x \vee y$;

(VV) $z \to (x \vee y) = (x \to y) \to (z \to y)$;
(VVV) $(z \vee y) \to (x \vee y) = (x \to y) \to (z \to y)$;

(P) (Product) $\exists\, x \odot y = \min\{z \mid x \to (y \to z) = 1\}$,
(P') (Product) $\exists\, x \odot y = \min\{z \mid x \leq y \to z\}$;
(dfP) (df. of Product) $\exists\, x \odot y \stackrel{df.}{=} \min\{z \mid x \to (y \to z) = 1\}$,
(dfP') (df. of Product) $\exists\, x \odot y \stackrel{df.}{=} \min\{z \mid x \leq y \to z\}$;
(PP) $x \to [y \to (x \odot y)] = 1$,
(PP') $x \leq y \to (x \odot y)$;
(R) (Residuum) $\exists y \to z = \max\{x \mid (x \odot y) \to z = 1\}$,
(R') (Residuum) $\exists y \to z = \max\{x \mid x \odot y \leq z\}$;
(RR) $[(y \to z) \odot y] \to z = 1$,
(RR') $(y \to z) \odot y \leq z$;
(RP)=(PR) $(x \odot y) \to z = 1 \Leftrightarrow x \to (y \to z) = 1$,
(RP')=(PR') $x \odot y \leq z \Leftrightarrow x \leq y \to z$;

(Pcomm) (commutativity of $\odot$ (Product)) $x \odot y = y \odot x$;
(Passoc) (associativity of $\odot$ (Product)) $(x \odot y) \odot z = x \odot (y \odot z)$;
(P-) $x \to y = 1 \Longrightarrow (x \odot a) \to (y \odot a) = 1$,

(P-') $x \leq y \Longrightarrow x \odot a \leq y \odot a$;
(P- -) $x \to y = 1,\ a \to b = 1 \ \Longrightarrow (x \odot a) \to (y \odot b) = 1$,
(P- -') $x \leq y,\ a \leq b \ \Longrightarrow x \odot a \leq y \odot b$;
(Pleq) $(x \odot y) \to x = 1,\ (x \odot y) \to y = 1$,
(Pleq') $x \odot y \leq x, y$;

(PEx) $(x \odot y) \to z = x \to (y \to z)$;
(PB) $[(y \to z) \odot (x \to y)] \to (x \to z) = 1$,
(PB') $(y \to z) \odot (x \to y) \leq x \to z$;
(PBB) $[(x \to y) \odot (y \to z)] \to (x \to z) = 1$,
(PBB') $(x \to y) \odot (y \to z) \leq x \to z$;
(PD) $[y \odot (y \to x)] \to x = 1$,
(PD') $y \odot (y \to x) \leq x$;

(Wcomm) (commutativity of $\wedge$ (wedge, meet)) $x \wedge y = y \wedge x$;
(Wassoc) (associativity of $\wedge$ (wedge, meet)) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$;
(W-) $x \to y = 1 \Longrightarrow (x \wedge a) \to (y \wedge a) = 1$,
(W-') $x \leq y \Longrightarrow x \wedge a \leq y \wedge a$;
(W- -) $x \to y = 1,\ a \to b = 1 \Longrightarrow (x \wedge a) \to (y \wedge b) = 1$,
(W- -') $x \leq y,\ a \leq b \Longrightarrow x \wedge a \leq y \wedge b$;
(Wleq) $(x \wedge y) \to x = 1; (x \wedge y) \to y = 1$,
(Wleq') $x \wedge y \leq x, y$;

(Pdis1) (P-distributivity-1) $(x \vee y) \odot z = (x \odot z) \vee (y \odot z)$;
(Pdis1-p) (Pdis1-partial-1) $[(x \odot z) \vee (y \odot z)] \to [(x \vee y) \odot z] = 1$,
(Pdis1-p') (Pdis1-partial-1') $(x \odot z) \vee (y \odot z) \leq (x \vee y) \odot z$;
(Pdis1-pp) (Pdis1-partial-2) $[(x \vee y) \odot z] \to [(x \odot z) \vee (y \odot z)] = 1$,
(Pdis1-pp') (Pdis1-partial-2') $(x \vee y) \odot z \leq (x \odot z) \vee (y \odot z)$;

(Pdis2) (P-distributivity-2) $(x \odot y) \vee z = (x \vee z) \odot (y \vee z)$;
(Pdis2-p) (Pdis2-partial-1) $[(x \odot y) \vee z] \to [(x \vee z) \odot (y \vee z)] = 1$,
(Pdis2-p') (Pdis2-partial-1') $(x \odot y) \vee z \leq (x \vee z) \odot (y \vee z)$;
(Pdis2-pp) (Pdis2-partial-2) $[(x \vee z) \odot (y \vee z)] \to [(x \odot y) \vee z] = 1$,
(Pdis2-pp') (Pdis2-partial-2') $(x \vee z) \odot (y \vee z) \leq (x \odot y) \vee z$;

(Wdis1) (W-distributivity-1) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$;
(Wdis1-p) (Wdis1-partial-1) $[(x \wedge z) \vee (y \wedge z)] \to [(x \vee y) \wedge z] = 1$,
(Wdis1-p') (Wdis1-partial-1') $(x \wedge z) \vee (y \wedge z) \leq (x \vee y) \wedge z$;
(Wdis1-pp) (Wdis1-partial-2) $[(x \vee y) \wedge z] \to [(x \wedge z) \vee (y \wedge z)] = 1$,
(Wdis1-pp') (Wdis1-partial-2') $(x \vee y) \wedge z \leq (x \wedge z) \vee (y \wedge z)$;

(Wdis2) (W-distributivity-2) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$;
(Wdis2-p) (Wdis2-partial-1) $[(x \wedge y) \vee z] \to [(x \vee z) \wedge (y \vee z)] = 1$,
(Wdis2-p') (Wdis2-partial-1') $(x \wedge y) \vee z \leq (x \vee z) \wedge (y \vee z)$;
(Wdis2-pp) (Wdis2-partial-2) $[(x \vee z) \wedge (y \vee z)] \to [(x \wedge y) \vee z] = 1$,
(Wdis2-pp') (Wdis2-partial-2') $(x \vee z) \wedge (y \vee z) \leq (x \wedge y) \vee z$.

3.2 Ordered regular algebras (structures). Dedekind regular-join-semilattices. Dedekind regular-meet-semilattices. Dedekind regular-lattices.

Recall that a poset (partially ordered set) $\mathcal{A} = (A, \leq)$ is called an *Ore lattice*, if for each two elements $x, y \in A$, there exist $\inf(x, y)$ and $\sup(x, y)$. If there exist $1 \in A$ such that $x \leq 1$ for all $x \in A$, then $\mathcal{A}$ is said to be an Ore lattice with last (top) element 1. An Ore lattice can be represented by the Hasse diagram.

In an Ore lattice, the following are equivalent: for all $x, y \in A$,
(i) $x \leq y$, (ii) $\sup(x, y) = y$, (iii) $\inf(x, y) = x$.

Recall also that an algebra $(A, \wedge, \vee)$ or $(A, \vee, \wedge)$ of type $(2, 2)$ ($(A, \wedge, \vee, 1)$ or $(A, \vee, \wedge, 1)$ of type $(2, 2, 0)$) is a *Dedekind lattice (with last element* 1*)* if the properties (Wid), (Vid), (Wcomm), (Vcomm), (Wassoc), (Vassoc), (Wabs1), (Wabs2) ((W1-1) and (V1-1), respectively) hold, where: for all $x, y, z \in A$,

(Vid) (idempotency of $\vee$) $x \vee x = x$,
(Vcomm) (commutativity of $\vee$) $x \vee y = y \vee x$,
(Vassoc) (associativity of $\vee$) $(x \vee y) \vee z = x \vee (y \vee z)$,
(V1-1) $x \vee 1 = 1 \vee x = 1$: (V-1) $x \vee 1 = 1$, (V1-) $1 \vee x = 1$;

(Wid) (idempotency of $\wedge$) $x \wedge x = x$,
(Wcomm) (commutativity of $\wedge$) $x \wedge y = y \wedge x$,
(Wassoc) (associativity of $\wedge$) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$,
(W1-1) $x \wedge 1 = 1 \wedge x = x$: (W-1) $x \wedge 1 = x$, (W1-) $1 \wedge x = x$;

(Wabs1) (W-absorption-1) $x \wedge (x \vee y) = x$,
(Wabs2) (W-absorption-2) $x \vee (x \wedge y) = x$.

In a Dedekind lattice, we have the equivalence: for all $x, y \in A$,
(EqVW) $x \vee y = y \Leftrightarrow x \wedge y = x$.

Recall finally that the two definitions of lattices are equivalent:

Theorem 3.1

(1) Let $\mathcal{A} = (A, \leq)$ be an Ore lattice. Define $\Phi(\mathcal{A}) \stackrel{df.}{=} (A, \wedge, \vee)$, where for all $x, y \in A$:
$x \wedge y \stackrel{df.}{=} \min(x, y)$ and $x \vee y \stackrel{df.}{=} \max(x, y)$.
Then, $\Phi(\mathcal{A})$ is a Dedekind lattice.

(1') Let $\mathcal{A} = (A, \wedge, \vee)$ be a Dedekind lattice. Define $\Psi(\mathcal{A})) \stackrel{df.}{=} (L, \leq)$, where for all $x, y \in A$:
(dfrelV) $x \leq y \stackrel{df.}{\Leftrightarrow} x \vee y = y$ or, equivalently,
(dfrelW) $x \leq y \stackrel{df.}{\Leftrightarrow} x \wedge y = x$.
Then, $\Psi(\mathcal{A}))$ is an Ore lattice.

(2) The two maps, Φ and Ψ are mutually inverse.

In the sequel, we shall work with the Dedekind lattices, and in particular with the Dedekind join-semilattices and the Dedekind meet-semilattices.

3.2.1 Dedekind regular-join-semilattices

Recall that an algebra $(A, \vee)$ of type (2) ($(A, \vee, 1)$ of type $(2, 0)$) is a *Dedekind $\vee$-semilattice (or Dedekind join-semilattice) (with last (top) element* 1*)* if $\vee$ verifies the properties (Vid), (Vcomm), (Vassoc) ((V1-1), respectively).

Let us introduce the following new definitions.

Definitions 3.2 Let $\mathcal{A} = (A, \vee, \rightarrow, 1)$ be an algebra of type $(2, 2, 0)$ (or $\mathcal{A} = (A, \vee, \leq, \rightarrow, 1)$ be a structure) such that:

- the reduct $(A, \rightarrow, 1)$ $((A, \leq, \rightarrow, 1))$ is a regular algebra (structure) (i.e. property (M) holds) and
- the operations $\rightarrow$, 1 and $\vee$ are connected by: for all $x, y \in A$,
(EqV) $\quad x \rightarrow y = 1 \;\Leftrightarrow\; x \vee y = y$,
- $\leq$ and $\vee$ are connected by: for all $x, y \in A$,
(EqV') $\quad x \leq y \;\Leftrightarrow\; x \vee y = y$.
In these conditions:

(1) we say that $\mathcal{A}$ is a *Dedekind regular-$\vee$-semilattice* (or *Dedekind regular-join-semilattice*), if the reduct $(A, \vee)$ is a Dedekind $\vee$-semilattice.

(1') we say that $\mathcal{A}$ is a *Dedekind regular-$\vee$-semilattice with last (top) element* 1, if the reduct $(A, \vee, 1)$ is a Dedekind $\vee$-semilattice with last (top) element 1.

Remark 3.3 Given a Dedekind regular-join-semilattice $(A, \vee, \rightarrow, 1)$, then the reduct $(A, \vee)$ is a Dedekind join-semilattice. Conversely, given a Dedekind join-semilattice $\mathcal{A} = (A, \vee)$, then there can exist one or more Dedekind regular-join-semilattices whose reduct $(A, \vee)$ coincides with $\mathcal{A}$ (see the next Examples 3.7).

• Let us introduce the new properties:
(dfrelV) $x \leq y \overset{df.}{\Leftrightarrow} x \vee y = y$;

(V-) $x \rightarrow y = 1 \Longrightarrow (x \vee a) \rightarrow (y \vee a) = 1$,
(V-') $x \leq y \Longrightarrow x \vee a \leq y \vee a$;
(V- -) $x \rightarrow y = 1,\ a \rightarrow b = 1 \;\Longrightarrow (x \vee a) \rightarrow (y \vee b) = 1$,
(V- -') $x \leq y,\ a \leq b \;\Longrightarrow x \vee a \leq y \vee b$;
(Vgeq) $x \rightarrow (x \vee y) = 1,\ y \rightarrow (x \vee y) = 1$,
(Vgeq') $x, y \leq x \vee y$;

(V=) $x \rightarrow z = 1,\ y \rightarrow z = 1 \;\Longrightarrow x \vee y \leq z$,
(V=') $x \leq z,\ y \leq z \;\Longrightarrow x \vee y \leq z$.

Remarks 3.4 We have the following connections:
(i) (EqrelR) $\Longrightarrow$ ((EqV) $\Leftrightarrow$ (EqV')); (EqV) + (EqV') $\Longrightarrow$ (EqrelR);
(ii) (EqV) $\Longrightarrow$ ((dfrelR) $\Leftrightarrow$ (dfrelV)); (dfrelR) + (dfrelV) $\Longrightarrow$ (EqV).

Other connections are presented in the next proposition.

Proposition 3.5 *Let $(A, \vee, \rightarrow, 1)$ be an algebra of type $(2, 2, 0)$ (or, equivalently, by (EqrelR) and (dfrelR), let $(A, \vee, \leq, \rightarrow, 1)$ be a structure). Then, we have:*
(BV1) (EqV) $\Longrightarrow$ ((Vid) $\Leftrightarrow$ (Re)),
(BV1') (EqV) $\Longrightarrow$ ((V-1) $\Leftrightarrow$ (L));
(BV2) (EqV) + (Vcomm) $\Longrightarrow$ (An);
(BV3) (EqV) + (Vassoc) $\Longrightarrow$ (Tr);
(BV3') (EqV) + (Vassoc) $\Longrightarrow$ (V=);
(BV4) (EqV) + (Vcomm) + (Vassoc) + (Vid) $\Longrightarrow$ (Vgeq),
(BV5) (EqV) + (Vcomm) + (Vassoc) + (Vid) $\Longrightarrow$ (V-);
(BV6) (V-) + (Vcomm) + (Tr) $\Longrightarrow$ (V- -);
(BV7) (Vid) + (V- -) $\Longrightarrow$ (V=).

Proof.
(BV1): $x \vee x \overset{(Vid)}{=} x \overset{(EqV)}{\Leftrightarrow} x \rightarrow x \overset{(Re)}{=} 1$.

(BV1'): $x \vee 1 \overset{(V-1)}{=} 1 \overset{(EqV)}{\Leftrightarrow} x \rightarrow 1 \overset{(L)}{=} 1$.

(BV2): If $x \leq y$ and $y \leq x$, i.e. $x \vee y = y$ and $y \vee x = x$, by (EqV'), then $x = y \vee x \overset{(Vcomm)}{=} x \vee y = y$; thus (An) holds.

(BV3): If $x \leq y$ and $y \leq z$, i.e. $x \vee y = y$ and $y \vee z = z$, by (EqV'), then $x \vee z = x \vee (y \vee z) \overset{(Vassoc)}{=} (x \vee y) \vee z = y \vee z = z$, hence $x \leq z$, by (EqV'); thus, (Tr') holds.

(BV4): $x \vee (x \vee y) \overset{(Vassoc)}{=} (x \vee x) \vee y \overset{(Vid)}{=} x \vee y$; then, by (EqV), $x \leq x \vee y$.
$y \vee (x \vee y) \overset{(Vcomm)}{=} (x \vee y) \vee y \overset{(Vassoc)}{=} x \vee (y \vee y) \overset{(Vid)}{=} x \vee y$; then, by (EqV), $y \leq x \vee y$. Thus, (Vgeq) holds.

(BV5): If $x \leq y$, i.e. $x \vee y = y$, by (EqV'), then $(x \vee z) \vee (y \vee z) \overset{(Vcomm)}{=} (x \vee z) \vee (z \vee y) \overset{(Vassoc)}{=} x \vee (z \vee z) \vee y \overset{(Vid)}{=} x \vee z \vee y \overset{(Vcomm)}{=} x \vee y \vee z = y \vee z$; hence, by (EqV'), $x \vee z \leq y \vee z$; thus, (V-) holds.

(BV6): Let $x \leq y$ and $a \leq b$; then, by (V-), $x \vee a \leq y \vee a$ and, by (V-) again, $a \vee y \leq b \vee y$; but $y \vee a \overset{(Vcomm)}{=} a \vee y$, hence $x \vee a \leq b \vee y \overset{(Vcomm)}{=} y \vee b$, by (Tr). Thus, (V- -') holds.

(BV7): Let $x \leq z$ and $y \leq z$; then, by (V- -), $x \vee y \leq z \vee z \overset{(Vid)}{=} z$; thus (V=) holds. □

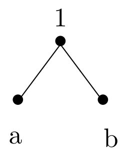
Then we have:

Theorem 3.6 *Let $\mathcal{A} = (A, \vee, \rightarrow, 1)$ ($\mathcal{A} = (A, \vee, \leq, \rightarrow, 1)$) be a Dedekind regular-$\vee$-semilattice with last (top) element 1 (i.e. properties (M), (EqV), (Vid), (Vcomm), (Vassoc), (V1-1) hold).*
Then, the following properties hold: (Re), (L), (Tr), (An), (V-), (V- -), (V=), (Vgeq).

Proof.
(Re): By (BV1), (EqV) $\Longrightarrow$ ((Vid) $\Leftrightarrow$ (Re)); thus, (Re) holds.
(L): By (BV1'), (EqV) $\Longrightarrow$ ((V-1) $\Leftrightarrow$ (L)); thus, (L) holds.
(Tr): By (BV3), (EqV) + (Vassoc) $\Longrightarrow$ (Tr); thus, (Tr) holds.
(An): By (BV2), (EqV) + (Vcomm) $\Longrightarrow$ (An); thus, (An) holds.
(V-): By (BV5), (EqV) + (Vcomm) + (Vassoc) + (Vid) $\Longrightarrow$ (V-); thus, (V-) holds.
(V- -): By (BV6), (V-) + (Vcomm) + (Tr) $\Longrightarrow$ (V- -); thus, (V- -) holds.
(V=): By (BV3'), (EqV) + (Vassoc) $\Longrightarrow$ (V=); thus, (V=) holds.
(Vgeq): By (BV4), (EqV) + (Vcomm) + (Vassoc) + (Vid) $\Longrightarrow$ (Vgeq); thus, (Vgeq) holds. □

Note that by the above Theorem 3.6, since the properties (Re), (M), (L), (An), (Tr) hold, then the reduct $(A, \rightarrow, 1)$ ($(A, \leq, \rightarrow, 1)$) is an oRML algebra (in fact an oRML join-semillatice), as defined in [15].

Examples 3.7 Consider the following poset (partially-ordered set) with top element $\mathcal{A} = (A = \{a, b, 1\}, \leq, 1)$ represented by the Hasse diagram from Figure 1 (hence is an **Ore join-semilattice with last (top) element** 1).



$\mathcal{A}$

Figure 1: The Hasse diagram of the given poset with 1

Hence, the corresponding **Dedekind join-semilattice with last (top) element** 1: $(A = \{a, b, 1\}, \vee, 1)$ is given by the following table:

$\vee$	a	b	1
a	a	1	1
b	1	b	1
1	1	1	1

Since $x \leq y \Leftrightarrow x \rightarrow y = 1$ (EqrelR) and we want that $1 \rightarrow x = x$ (M) be varified, it follows that the table of the *general implication* $\rightarrow_g$ (residuum, in certain cases) corresponding to the given Ore join-semilattice with 1 from Figure 1 is as follows:

$\to_g$	a	b	1
a	1	r_{12}	1
b	r_{21}	1	1
1	a	b	1

Since there are two free spaces (r_{12}, r_{21}) that can be filled with one element of $\{a, b\}$, it follows that $\to_g$ induces $2^2 = 4$ implications, $\to_1$ - $\to_4$, namely (by generating the elements (r_{12}, r_{21}) in increasing lexicographical order):
$\to_1$: (r_{12}, r_{21})=(a,a); $\to_2$: (r_{12}, r_{21})=(a,b); $\to_3$: (r_{12}, r_{21})=(b,a); $\to_4$: (r_{12}, r_{21})=(b,b).

Consequently, there exist four Dedekind regular-join-semilattices with top element 1**:** $\mathcal{A}_1 = (A = \{a, b, 1\}, \vee, \to_1, 1)$ **-** $\mathcal{A}_4 = (A = \{a, b, 1\}, \vee, \to_4, 1)$**, corresponding to the given Dedekind join-semilattice with top element** 1**:** $(A = \{a, b, 1\}, \vee, 1)$**.**

Note that the properties (Re), (M), (L), (An) and (Tr) hold; hence, all the four reduct algebras $(A = \{a, b, 1\}, \to_g, 1)$ are regular **oRML algebras** (in fact regular oRML join-semilattices or oRML regular-join-semilattices), as defined in [15].

By checking (with a PASCAL programm) the other possible considered properties ((Ex), (pimpl), (pi), (BB), (**), (B), (*), (Tr), (D), (comm), (impl)) that can be verified by $\to_g$, we have obtained:

$\to_1$: verifies only (*), (Tr), (D), hence $(A = \{a, b, 1\}, \to_1, 1)$ is a ***aRML algebra with (D)** [15]. It has no product, because $a \odot b = \min\{a, b, 1\}$ does not exist.

$\to_2$: verifies only (*), (Tr), (D), hence $(A = \{a, b, 1\}, \to_2, 1)$ is a ***aRML algebra with (D)**. It has no product, because $b \odot b = \min\{a, b, 1\}$ does not exist.

$\to_3$: verifies (Ex), (pimpl), (pi), (BB), (**), (B), (*), (Tr), (D), (comm), (impl), hence $(A = \{a, b, 1\}, \to_3, 1)$ is an **implicative BCK algebra**. We have $x \vee y = (x \to y) \to y$. It has no product, because $a \odot b = \min\{a, b, 1\}$ does not exist.

$\to_4$: verifies only (*), (Tr), (D), hence $(A = \{a, b, 1\}, \to_4, 1)$ is a ***aRML algebra with (D)**. It has no product, because $b \odot b = \min\{a, b, 1\}$ does not exist.

3.2.2 Dedekind regular-meet-semilattices

The results of this subsubsection are the dual of the results from the precedent subsubsection.

Recall that an algebra $(A, \wedge)$ of type (2) ($(A, \wedge, 1)$ of type $(2, 0)$) is a *Dedekind* $\wedge$*-semilattice (or Dedekind meet-semilattice) (with last (top) element* 1*)*, if $\wedge$ verifies the properties (Wid), (Wcomm), (Wassoc) ((W1-1), respectively).

Let us introduce now the following definitions.

Definitions 3.8 Let $\mathcal{A} = (A, \wedge, \to, 1)$ be an algebra of type $(2, 2, 0)$ (or $\mathcal{A} = (A, \wedge, \leq, \to, 1)$ be a structure) such that:
- the reduct $(A, \to, 1)$ ($(A, \leq, \to, 1)$) is a regular algebra (structure) (i.e. property (M) holds) and
- the operations $\to$, 1 and $\wedge$ are connected by: for all $x, y \in A$,
(EqW) $\quad x \to y = 1 \;\Leftrightarrow\; x \wedge y = x$,
- $\leq$ and $\wedge$ are connected by: for all $x, y \in A$,
(EqW') $\quad x \leq y \;\Leftrightarrow\; x \wedge y = x$.
In these conditions:

(1) We say that $\mathcal{A}$ is a *Dedekind regular-*$\wedge$*-semilattice* (or *Dedekind regular-meet-semilattice*), if the reduct $(A, \wedge)$ is a Dedekind $\wedge$-semilattice.

(1') We say that $\mathcal{A}$ is a *Dedekind regular-*$\wedge$*-semilattice* (or *Dedekind regular-meet-semilattice*) with last (top) element 1, if the reduct $(A, \wedge, 1)$ is a Dedekind $\wedge$-semilattice with top element 1.

Remark 3.9 Given a Dedekind regular-meet-semilattice $(A, \wedge, \to, 1)$, then the reduct $(A, \wedge)$ is a Dedekind meet-semilattice. Conversely, given a Dedekind meet-semilattice $\mathcal{A} = (A, \wedge)$, then there can exist one or more Dedekind regular-meet-semilattices whose reduct $(A, \wedge)$ coincides with $\mathcal{A}$.

• Let us introduce now the following new properties:
(dfrelW) $x \leq y \overset{df.}{\Leftrightarrow} x \wedge y = x$;

(W-) $x \to y = 1 \Longrightarrow (x \wedge a) \to (y \wedge a) = 1$,
(W-') $x \leq y \Longrightarrow x \wedge a \leq y \wedge a$;
(W- -) $x \to y = 1,\ a \to b = 1 \Longrightarrow (x \wedge a) \to (y \wedge b) = 1$,
(W- -') $x \leq y,\ a \leq b \Longrightarrow x \wedge a \leq y \wedge b$;
(Wleq) $(x \wedge y) \to x = 1, \quad (x \wedge y) \to y = 1$,
(Wleq') $x \wedge y \leq x, y$;

(W=) $z \to x = 1,\ z \to y = 1 \ \Longrightarrow z \to (x \wedge y) = 1$,
(W=') $z \leq x,\ z \leq y \ \Longrightarrow z \leq x \wedge y$.

Remarks 3.10 We have the following connections:
(i) (EqrelR) $\Longrightarrow$ ((EqW) $\Leftrightarrow$ (EqW')); (EqW) + (EqW') $\Longrightarrow$ (EqrelR);
(ii) (EqW) $\Longrightarrow$ ((dfrelR) $\Leftrightarrow$ (dfrelW)); (dfrelR) + (dfrelW) $\Longrightarrow$ (EqW).

Other connections are presented in the next proposition.

Proposition 3.11 *(See the dual Proposition 3.5)*
Let $(A, \wedge, \to, 1)$ be an algebra of type $(2, 2, 0)$ (or, equivalently, by (EqrelR) and (dfrelR), let $(A, \wedge, \leq, \to, 1)$ be a structure). Then, we have:
(BW1) (EqW) $\Longrightarrow$ ((Wid) $\Leftrightarrow$ (Re)),
(BW1') (EqW) $\Longrightarrow$ ((W-1) $\Leftrightarrow$ (L));
(BW2) (EqW) + (Wcomm) $\Longrightarrow$ (An);
(BW3) (EqW) + (Wassoc) $\Longrightarrow$ (Tr),
(BW3') (EqW) + (Wassoc) $\Longrightarrow$ (W=);
(BW4) (EqW) + (Wcomm) + (Wassoc) + (Wid) $\Longrightarrow$ (Wleq),
(BW5) (EqW) + (Wcomm) + (Wassoc) + (Wid) $\Longrightarrow$ (W-);
(BW6) (W-) + (Wcomm) + (Tr) $\Longrightarrow$ (W- -);
(BW7) (Wid) + (W- -) $\Longrightarrow$ (W=).

Proof.
(BW1): $x \wedge x \overset{(Wid)}{=} x \overset{(EqW)}{\Leftrightarrow} x \to x \overset{(Re)}{=} 1$.
(BW1'): $x \wedge 1 \overset{(W-1)}{=} x \overset{(EqW)}{\Leftrightarrow} x \to 1 \overset{(L)}{=} 1$.
(BW2): If $x \leq y$ and $y \leq x$, i.e. $x \wedge y = x$ and $y \wedge x = y$, by (EqW'), then $x = x \wedge y \overset{(Wcomm)}{=} y \wedge x = y$; thus (An) holds.
(BW3): If $x \leq y$ and $y \leq z$, i.e. $x \wedge y = x$ and $y \wedge z = y$, by (EqW'), then $x \wedge z = (x \wedge y) \wedge z \overset{(Wassoc)}{=} x \wedge (y \wedge z) = x \wedge y = x$, hence, by (EqW'), $x \leq z$; thus, (Tr') holds.
(BW4): $(x \wedge y) \wedge y \overset{(Wassoc)}{=} x \wedge (y \wedge y) \overset{(Wid)}{=} x \wedge y$; hence, by (EqW), $x \wedge y \leq y$.
$(x \wedge y) \wedge x \overset{(Wcomm)}{=} (y \wedge x) \wedge x \overset{(Wassoc)}{=} y \wedge (x \wedge x) \overset{(Wid)}{=} y \wedge x \overset{(Wcomm)}{=} x \wedge y$; hence, by (EqW), $x \wedge y \leq x$. Thus, (Wleq) holds.
(BW5): If $x \leq y$, i.e. $x \wedge y = x$, by (EqW'), then $(x \wedge z) \wedge (y \wedge z) \overset{(Wcomm)}{=} (x \wedge z) \wedge (z \wedge y) \overset{(Wassoc)}{=} x \wedge (z \wedge z) \wedge y \overset{(Wid)}{=} x \wedge z \wedge y \overset{(Wcomm)}{=} (x \wedge y) \wedge z = x \wedge z$; hence, by (EqW'), $x \wedge z \leq y \wedge z$; thus, (W-) holds.
(BW6): Let $x \leq y$ and $a \leq b$; then, by (W-), $x \wedge a \leq y \wedge a$ and, by (W-) again, $a \wedge y \leq b \wedge y$; but $y \wedge a \overset{(Wcomm)}{=} a \wedge y$, hence $x \wedge a \leq b \wedge y \overset{(Wcomm)}{=} y \wedge b$, by (Tr). Thus, (W- -') holds.
(BW7): Let $x \leq z$ and $y \leq z$; then, by (W- -), $x \wedge y \leq z \wedge z \overset{(Wid)}{=} z$; thus (W=) holds. □
Then we have:

Theorem 3.12 *(See the dual Theorem 3.6)*
Let $\mathcal{A} = (A, \wedge, \to, 1)$ ($\mathcal{A} = (A, \wedge, \leq, \to, 1)$) be a Dedekind regular-$\wedge$-semilattice with last (top) element 1 (i.e. properties (M), (EqW), (Wid, (Wcomm), (Wassoc), (W1-1) hold).
Then, the following properties hold: (Re), (L), (Tr), (An), (W-), (W- -), (W=), (Wleq).

Proof.

(Re): By (BW1), (EqW) $\Longrightarrow$ ((Wid) $\Leftrightarrow$ (Re)); thus, (Re) holds.
(L): By (BW1'), (EqW) $\Longrightarrow$ ((W-1) $\Leftrightarrow$ (L)); thus, (L) holds.
(Tr): By (BW3), (EqW) + (Wassoc) $\Longrightarrow$ (Tr); thus, (Tr) holds.
(An): By (BW2), (EqW) + (Wcomm) $\Longrightarrow$ (An); thus (An) holds.
(W-): By (BW5), (EqW) + (Wcomm) + (Wassoc) + (Wid) $\Longrightarrow$ (W-); thus, (W-) holds.
(W- -): By (BW6), (W-) + (Wcomm) + (Tr) $\Longrightarrow$ (W- -); thus, (W- -) holds.
(W=): By (BW7), (Wid) + (W- -) $\Longrightarrow$ (W=); thus, (W=) holds.
(Wleq): By (BW4), (Wid) + (Wcomm) + (Wassoc) + (EqW) $\Longrightarrow$ (Wleq); thus, (Wleq) holds. $\Box$

Note that by the above Theorem 3.12, since the properties (Re), (M), (L), (An), (Tr) hold, then the reduct $(A, \to, 1)$ ($(A, \leq, \to, 1)$) is an oRML algebra (in fact an oRML meet-semillatice), as defined in [15].

3.2.3 Dedekind regular-lattices

Recall firstly that:
- An algebra $\mathcal{A} = (A, \wedge, \vee)$ or $\mathcal{A} = (A, \vee, \wedge)$ of type $(2, 2)$ is a *Dedekind lattice* (more precisely a *Dedekind* $\wedge\vee$*-lattice* or a *Dedekind* $\vee\wedge$*-lattice*, respectively), if the reduct $(A, \vee)$ is a Dedekind $\vee$-semilattice, the reduct $(A, \wedge)$ is a Dedekind $\wedge$-semilattice and the properties of absorptions, (Wabs1) and (Wabs2), are verified.
- An algebra $\mathcal{A} = (A, \wedge, \vee, 1)$ or $\mathcal{A} = (A, \vee, \wedge, 1)$ of type $(2, 2, 0)$ is called a *Dedekind lattice with top element* 1 (more precisely a *Dedekind* $\wedge\vee$*-lattice with* 1 or a *Dedekind* $\vee\wedge$*-lattice with* 1, respectively), if the reduct $(A, \vee, 1)$ is a Dedekind $\vee$-semilattice with 1, the reduct $(A, \wedge, 1)$ is a Dedekind $\wedge$-semilattice with 1 and the properties (Wabs1) and (Wabs2) are verified.

We shall introduce now the notion of *Dedekind regular-lattice*.

Definitions 3.13 (See Definitions 3.2 and 3.8)

Let $\mathcal{A} = (A, \wedge, \vee, \to, 1)$ or $\mathcal{A} = (A, \vee, \wedge, \to, 1)$ be an algebra of type $(2, 2, 2, 0)$ (or $\mathcal{A} = (A, \wedge, \vee, \leq, \to, 1)$ or $\mathcal{A} = (A, \vee, \wedge, \leq, \to, 1)$, respectively, be a structure) such that:
- the reduct $(A, \wedge, \to, 1)$ is a Dedekind regular-$\wedge$-semilattice (with last element 1) and
- the reduct $(A, \vee, \to, 1)$ is a Dedekind regular-$\vee$-semilattice (with last element 1).
In these conditions, we say that $\mathcal{A}$ is a *Dedekind regular-lattice (with last element* 1) (more precisely, a *Dedekind regular-*$\wedge\vee$*-lattice with* 1 or a *Dedekind regular-*$\vee\wedge$*-lattice with* 1, respectively) if the additional properties of absorption, (Wabs1) and (Wabs2), hold.

Hence, a Dedekind regular-lattice (with 1) verifies (M), (EqW), (EqV) and (Wid), (Vid), (Wcomm), (Vcomm), (Wassoc), (Vassoc), (Wabs1), (Wabs2) ((W1-1), (V1-1) respectively).

Note that a Dedekind regular-lattice can be defined alternatively by:

Definition 3.14 Let $\mathcal{A} = (A, \wedge, \vee, \to, 1)$ or $\mathcal{A} = (A, \vee, \wedge, \to, 1)$ be an algebra of type $(2, 2, 2, 0)$ (or $\mathcal{A} = (A, \wedge, \vee, \leq, \to, 1)$ or $\mathcal{A} = (A, \vee, \wedge, \leq, \to, 1)$, respectively, be a structure) such that:
- the reduct $(A, \to, 1)$ ($(A, \leq, \to, 1)$) is a regular algebra (structure),
- the operations $\to$, 1 and $\vee$ are connected by (EqV),
- the operations $\to$, 1 and $\wedge$ are connected by (EqW).
In these conditions, we say that $\mathcal{A}$ is a *Dedekind regular-algebra* (*with* 1) if the reduct $(A, \wedge, \vee)$ or $(A, \vee, \wedge)$ ($(A, \wedge, \vee, 1)$ or $(A, \vee, \wedge, 1)$, respectively) is a Dedekind lattice (with 1).

Remark 3.15 Given a Dedekind regular-lattice, say $(A, \wedge, \vee, \to, 1)$, then the reduct $(A, \wedge, \vee)$ is a Dedekind lattice. Conversely, given a Dedekind lattice $\mathcal{A} = (A, \wedge, \vee)$, then there can exist one or more Dedekind regular-lattices whose reduct $(A, \wedge, \vee)$ coincides with $\mathcal{A}$.

• Let us introduce the following properties:
(Wdis1) (W-distributivity-1) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$;
(Wdis1-p) (Wdis1-partial-1) $[(x \wedge z) \vee (y \wedge z)] \to [(x \vee y) \wedge z] = 1$,
(Wdis1-p') (Wdis1-partial-1') $(x \wedge z) \vee (y \wedge z) \leq (x \vee y) \wedge z$;

(Wdis1-pp) (Wdis1-partial-2) $[(x \vee y) \wedge z] \to [(x \wedge z) \vee (y \wedge z)] = 1$,
(Wdis1-pp') (Wdis1-partial-2') $(x \vee y) \wedge z \leq (x \wedge z) \vee (y \wedge z)$;

(Wdis2) (W-distributivity-2) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$;
(Wdis2-p) (Wdis2-partial-1) $[(x \wedge y) \vee z] \to [(x \vee z) \wedge (y \vee z)] = 1$,
(Wdis2-p') (Wdis2-partial-1') $(x \wedge y) \vee z \leq (x \vee z) \wedge (y \vee z)$;
(Wdis2-pp) (Wdis2-partial-2) $[(x \vee z) \wedge (y \vee z)] \to [(x \wedge y) \vee z] = 1$,
(Wdis2-pp') (Wdis2-partial-2') $(x \vee z) \wedge (y \vee z) \leq (x \wedge y) \vee z$.

Then we have:

Proposition 3.16 *Let* $\mathcal{A} = (A, \vee, \wedge, \to, 1)$ *be an algebra of type* $(2, 2, 2, 0)$ *(or, equivalently,* $\mathcal{A} = (A, \vee, \wedge, \leq, \to, 1)$ *be a structure) or, dually, let* $\mathcal{A} = (A, \wedge, \vee, \to, 1)$ *be an algebra of type* $(2, 2, 2, 0)$ *(or* $\mathcal{A} = (A, \wedge, \vee, \leq, \to, 1)$ *be a structure) such that the reduct* $(A, \to, 1)$ *(*$(A, \leq, \to, 1)$*) is an algebra (structure) verifying the properties (Re) and (Tr) (i.e.* $\leq$ *is a pre-order). Then, the following hold:*
(BVW1) (Wdis1-p) + (Wdis1-pp) + (An) $\Longrightarrow$ *(Wdis1);*
(BVW1') (Vgeq) + (V=) + (W-) $\Longrightarrow$ *(Wdis1-p); (Wleq) + (W=) + (V- -) + (V=)* $\Longrightarrow$ *(Wdis1-p);*

(BVW2) (Wdis2-p) + (Wdis2-pp) + (An) $\Longrightarrow$ *(Wdis2);*
(BVW2') (Wleq) + (W=) + (V-) $\Longrightarrow$ *(Wdis2-p); (Vgeq) + (V=) + (W- -) + (W=)* $\Longrightarrow$ *(Wdis2-p);*
(BVW2") (Wdis1) + (Wleq) + (Wcomm) + (Vcomm) + (Vassoc) + (V-) + (V=) $\Longrightarrow$ *(Wdis2-pp);*

(BVW3) (Vgeq) + (EqW) $\Longrightarrow$ *(Wabs1);*
(BVW4) (Wleq) + (EqV) + (Vcomm) $\Longrightarrow$ *(Wabs2).*

Proof.
(BVW1): Obviously.
(BVW1'): First proof: $x, y \leq x \vee y$, by (Vgeq'); then, $x \wedge z,\ y \wedge z \leq (x \vee y) \wedge z$, by (W-), hence $(x \wedge z) \vee (y \wedge z) \leq (x \vee y) \wedge z$, by (V=); thus, (Wdis1-p) holds.
Second proof: On the one hand, we have $x \wedge z \leq x$ and $y \wedge z \leq y$, by (Wleq); then, $(x \wedge z) \vee (y \wedge z) \leq x \vee y$, by (V- -). On the other hand, we have $x \wedge z \leq z$ and $y \wedge z \leq z$, by (Wleq); then, $(x \wedge z) \vee (y \wedge z) \leq z$, by (V=). Consequently, $(x \wedge z) \vee (y \wedge z) \leq (x \vee y) \wedge z$, by (W=); thus, (Wdis1-p) holds.
(BVW2): Obviously.
(BVW2'): First proof: $x \wedge y \leq x,\ y$, by (Wleq); then, $(x \wedge y) \vee z \leq x \vee z,\ y \vee z$, by (V-), hence $(x \wedge y) \vee z \leq (x \vee z) \wedge (y \vee z)$, by (W=); thus, (Wdis2-p) holds.
Second proof: On the one hand, we have: $x \leq x \vee z$ and $y \leq y \vee z$, by (Vgeq); then, $x \wedge y \leq (x \vee z) \wedge (y \vee z)$, by (W- -). On the other hand, we have $z \leq x \vee z$ and $z \leq y \vee z$, by (Vgeq); then, $z \leq (x \vee z) \wedge (y \vee z)$, by (W=). Consequently, $x \wedge y) \vee z \leq (x \vee z) \wedge (y \vee z)$, by (V=); thus, (Wdis2-p) holds.
(BVW2"): Denote $Z \overset{notation}{=} (x \vee z) \wedge (y \vee z)$; then
$Z \overset{(Wdis1)}{=} (x \wedge (y \vee z)) \vee (z \wedge (y \vee z))$
$\overset{(Wcomm)}{=} ((y \vee z) \wedge x) \vee ((y \vee z) \wedge z)$
$\overset{(Wdis1)}{=} ((y \wedge x) \vee (z \wedge x)) \vee ((y \wedge z) \vee (z \wedge z))$
$\overset{(Vassoc)}{=} (y \wedge x) \vee [(z \wedge x) \vee (y \wedge z) \vee (z \wedge z)]$
$\overset{(Vcomm)}{=} [(z \wedge x) \vee (y \wedge z) \vee (z \wedge z)] \vee (y \wedge x)$.
But $z \wedge x \leq z$, $y \wedge z \leq z$, $z \wedge z \leq z$, by (Wleq);
hence, $(z \wedge x) \vee (y \wedge z) \vee (z \wedge z) \leq z$, by (V=);
hence, $Z \leq z \vee (y \wedge x) \overset{(Wcomm),(Vcomm)}{=} (x \wedge y) \vee z$, by (V-); thus (Wdis2-pp) holds.
(BVW3): $x \overset{(Vgeq)}{\leq} x \vee y \overset{(EqW')}{\Leftrightarrow} x \wedge (x \vee y) = x$; thus (Wabs1) holds.
(BVW4): $x \wedge y \overset{(Wleq)}{\leq} x \overset{(EqV)}{\Leftrightarrow} (x \wedge y) \vee x = x$, hence $x \vee (x \wedge y) = x$, by (Vcomm); thus, (Wabs2) holds. □

Recall that a Dedekind lattice is called to be *distributive* if the distributivity properties, (Wdis1) and (Wdis2), hold.

A Dedekind regular-lattice (with last element 1) will be said to be *distributive* if the distributivity properties, (Wdis1) and (Wdis2), hold.

We can prove now the following result.

Theorem 3.17 *Let $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 1)$ be a Dedekind regular-$\wedge\vee$-lattice with 1 or $\mathcal{A} = (A, \vee, \wedge, \rightarrow, 1)$ be a Dedekind regular-$\vee\wedge$-lattice with 1. Then, the following properties hold: (Re), (L), (Tr), (An), (V-), (V- -), (V=), (Vgeq), (W-), (W- -), (W=), (Wleq), (Wdis1-p), (Wdis2-p).*

Proof. By Theorem 3.12, (Re), (L), (Tr), (An), (W-), (W- -), (W=), (Wleq) hold.
By Theorem 3.6, (V-), (V- -), (V=), (Vgeq) hold.
By (BVW1'), (Vgeq) + (V=) + (W-) $\Longrightarrow$ (Wdis1-p), hence (Wdis1-p) holds.
By (BVW2'), (Wleq) + (W=) + (V-) $\Longrightarrow$ (Wdis2-p), hence (Wdis2-p) holds. □

Note that by the above Theorem 3.17, since the properties (Re), (M), (L), (An), (Tr) hold, then the reduct $(A, \rightarrow, 1)$ ($(A, \leq, \rightarrow, 1)$) is an oRML algebra (oRML lattice) [15].

It is a further research to define the Ore regular-lattices and to prove that Dedekind regular-lattices and ore regular-lattices are equivalent.

3.3 Positive implicative, commutative and implicative regular algebras (structures)

Let us introduce the following definitions.

Definitions 3.18 Let $\mathcal{A}$ be a regular algebra (structure). We say that $\mathcal{A}$ is:
- *positive implicative*, if the following property (pimpl) is satisfied: for all $x, y, z \in A$,

$$(pimpl) \quad x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z);$$

- *generalized positive implicative*, if the following property (pi) is satisfied: for all $x, y \in A$,

$$(pi) \quad x \rightarrow (x \rightarrow y) = x \rightarrow y;$$

- *commutative*, if the following property (comm) is satisfied: for all $x, y \in A$,

$$(comm) \quad (x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x;$$

- *implicative*, if the following property (impl) is satisfied: for all $x, y \in A$,

$$(impl) \quad (x \rightarrow y) \rightarrow x = x.$$

• Let us introduce also the following new properties: for all $x, y \in A$,
(pimpl-1) $[x \rightarrow (y \rightarrow z)] \rightarrow [(x \rightarrow y) \rightarrow (x \rightarrow z)] = 1$,
(pimpl-1') $x \rightarrow (y \rightarrow z) \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$;

(pimpl-2) $[(x \rightarrow y) \rightarrow (x \rightarrow z)] \rightarrow [x \rightarrow (y \rightarrow z)] = 1$,
(pimpl-2') $(x \rightarrow y) \rightarrow (x \rightarrow z) \leq x \rightarrow (y \rightarrow z)$;

(\$) $x \rightarrow (y \rightarrow z) = 1 \Longrightarrow (x \rightarrow y) \rightarrow (x \rightarrow z) = 1$,
(\$') $x \leq y \rightarrow z \Longrightarrow x \rightarrow y \leq x \rightarrow z$;

(comm-1) $[(x \rightarrow y) \rightarrow y] \rightarrow x = (x \rightarrow (y \rightarrow x)) \rightarrow (y \rightarrow x)$.

Recall the following general connections between properties in List A and List B from [15]:

Proposition 3.19 *(See [15] Remark 6.2, Propositions 6.4, 6.9, 6.8, Theorems 6.13, 6.16, Remarks 6.20 (ii), Proposition 6.21, Theorems 6.23, 6.25)*
Let $(A, \rightarrow, 1)$ ($(A, \leq, \rightarrow, 1)$) be an algebra (a structure). Then, we have:

(B0) (pimpl-1) + (pimpl-2) + (An) $\Longrightarrow$ (pimpl);
(B1) (pimpl) + (Re) $\Longrightarrow$ (L);
(B2) (pi) + (Re) $\Longrightarrow$ (L);
(B3) (pimpl) + (Re) + (M) $\Longrightarrow$ (pi);
(B4) (pimpl) + (Re) + (L) $\Longrightarrow$ (K);
(B5) (pimpl) + (K) $\Longrightarrow$ (B);
(B6) (pimpl) + (Re) $\Longrightarrow$ (B);
(B7) (pimpl) + (Re) + (M) $\Longrightarrow$ (*), (**);
(B8) (pimpl) + (L) $\Longrightarrow$ (*) (Michael Kinyon);
(B9) (pimpl) + (Re) + (M) $\Longrightarrow$ (BB) (Michael Kinyon);
(B10) (pimpl) + (Re) + (M) $\Longrightarrow$ (C) (Michael Kinyon);
(B11) (pimpl) + (Re) + (M) + (An) $\Longrightarrow$ (Ex) (Michael Kinyon);

(B12) (comm) + (M) $\Longrightarrow$ (An);

(B13) (Re) + (L) + (Ex) + (**) $\Longrightarrow$ (pimpl-2);
(B14) (pi) + (Ex) + (B) + (*) $\Longrightarrow$ (pimpl-1);
(B15) (pi) + (Re) + (Ex) + (B) + (An) $\Longrightarrow$ (pimpl);
(B16) (pi) + (Re) + (M) + (B) + (An) $\Longrightarrow$ ((Ex) $\Leftrightarrow$ (BB) $\Leftrightarrow$ (pimpl));
(B17) (pi) + (Re) + (M) + (Ex) + (An) $\Longrightarrow$ ((BB) $\Leftrightarrow$ (B) $\Leftrightarrow$ (*) $\Leftrightarrow$ (pimpl)).

Remark that the above property (B15) is simplified compared with that from [15], because the properties (**), (*) and (L) follow. Indeed, by (B2), (pi) + (Re) $\Longrightarrow$ (L); by (A5), (Re) + (Ex) + (An) $\Longrightarrow$ (M), by (A12'), (M) + (B) $\Longrightarrow$ (*); by (A10'), (Ex) + (B) $\Longrightarrow$ (BB) and by (A15'), (M) + (BB) $\Longrightarrow$ (**).

We now add the following new connections between the properties in Lists A and B:

Proposition 3.20 *Let $(A, \rightarrow, 1)$ ($(A, \leq, \rightarrow, 1)$) be an algebra (a structure). Then, the following hold:*

(B18) (K) + (**) + (C) + (Tr) $\Longrightarrow$ (pimpl-2);

(B19) (pimpl-1) + (K) + (N) $\Longrightarrow$ (Re); *(B19') (pimpl-1) + (K) + (M) $\Longrightarrow$ (Re);*
(B20) (pimpl-1) + (L) + (N) $\Longrightarrow$ (*); *(B20') (pimpl-1) + (L) + (M) $\Longrightarrow$ (*);*
(B21) (pimpl-1) + (L) + (N) $\Longrightarrow$ (Tr); *(B21') (pimpl-1) + (L) + (M) $\Longrightarrow$ (Tr);*
(B22) (pimpl-1) + (K) + (Tr) $\Longrightarrow$ (B);
(B23) (pimpl-1) + (Re) + (*) + (K) + (M) $\Longrightarrow$ (D);
(B24) (pimpl-1) + (*) + (K) + (M) $\Longrightarrow$ (**);
(B25) (pimpl-1) + (K) + (**) + (Tr) $\Longrightarrow$ (C);
(B26) (pimpl-1) + (K) + (An) + (C) + (**) + (Tr) $\Longrightarrow$ (pimpl);
(B27) (pimpl-1) + (N) $\Longrightarrow$ (\$); *(B27') (pimpl-1) + (M) $\Longrightarrow$ (\$);*
(B28) (\$) + (K) + (Tr) $\Longrightarrow$ (#);

(B29) (comm) + (L) + (M) $\Longrightarrow$ (Re);
(B30) (comm) + (Re) + (M) + (Ex) $\Longrightarrow$ (L);
(B31) (comm) + (Re) + (M) + (Ex) $\Longrightarrow$ (*);
(B32) (comm) + (M) + (BB) $\Longrightarrow$ (L);
(B33) (comm) + (K) + (BB) + (Tr) + (M) $\Longrightarrow$ (#);
(B34) (comm) $\Longrightarrow$ (comm-1);

(B35) (Re) + (Ex) + (B) + (M) + (An) $\Longrightarrow$ ((pimpl) $\Leftrightarrow$ (pi)).

Proof:

(B18): $y \overset{(K')}{\leq} x \to y$, then $(x \to y) \to (x \to z) \overset{(**')}{\leq} y \to (x \to z) \overset{(C)}{\leq} x \to (y \to z)$; hence, by (Tr), we obtain that $(x \to y) \to (x \to z) \leq x \to (y \to z)$, i.e. (pimpl-2) holds.

(B19): From (pimpl-1) ($[x \to (y \to z)] \to [(x \to y) \to (x \to z)] = 1$), for $y = x \to x$ and $z = x$ we obtain: $[x \to ((x \to x) \to x)] \to [(x \to (x \to x)) \to (x \to x)] = 1$; by (K), $x \to ((x \to x) \to x) = 1$ and $x \to (x \to x) = 1$, hence, $1 \to [1 \to (x \to x)] = 1$; then, applying (N) twice, we obtain that $x \to x = 1$, i.e. (Re) holds.

(B19'): By (A00), (M) $\Longrightarrow$ (N), then apply (B19).

(B20): Suppose that $y \to z = 1$; we must prove that $(x \to y) \to (x \to z) = 1$. Indeed, from (pimpl-1) ($[x \to (y \to z)] \to [(x \to y) \to (x \to z)] = 1$) we obtain: $(x \to 1) \to [(x \to y) \to (x \to z)] = 1$, hence, by (L) and (N), we obtain: $(x \to y) \to (x \to z) = 1$; thus (*) holds.

(B20'): By (A00), (M) $\Longrightarrow$ (N), then apply (B20).

(B21): By (B20), (pimpl-1) + (L) + (N) $\Longrightarrow$ (*) and by (A13), (N) + (*) $\Longrightarrow$ (Tr). Thus, (Tr) holds.

(B21'): By (A00), (M) $\Longrightarrow$ (N), then apply (B21).

(B22): $y \to z \overset{(K')}{\leq} x \to (y \to z) \overset{(pimpl-1)}{\leq} (x \to y) \to (x \to z)$; hence, by (Tr), we obtain $y \to z \leq (x \to y) \to (x \to z)$, i.e. (B') holds.

(B23): From (pimpl-1) ($[x \to (y \to z)] \to [(x \to y) \to (x \to z)] = 1$), for $x = y \to z$ we obtain:
$[(y \to z) \to (y \to z)] \to [((y \to z) \to y) \to ((y \to z) \to z)] = 1$; then, by (Re) and (M), we obtain:
$((y \to z) \to y) \to ((y \to z) \to z) = 1$ i.e. $(y \to z) \to y \leq (y \to z) \to z$; now, apply (*') and obtain:
$y \to [(y \to z) \to y] \leq y \to [(y \to z) \to z]$; then, by (K) and (M), we obtain:
$y \to [(y \to z) \to z] = 1$, i.e. (D) holds.

(B24): Suppose that $x \to y = 1$; we shall prove that $(y \to z) \to (x \to z) = 1$. Indeed,
(pimpl-1) ($[x \to (y \to z)] \to [(x \to y) \to (x \to z)] = 1$) gives:
$[x \to (y \to z)] \to [1 \to (x \to z)] = 1$; then, by (M), we obtain:
$[x \to (y \to z)] \to (x \to z) = 1$, i.e. $x \to (y \to z) \leq x \to z = 1$; now, by (*'), we obtain:
$(y \to z) \to [x \to (y \to z)] \leq (y \to z) \to (x \to z)$; now, by (K) and (M), we obtain: $(y \to z) \to (x \to z) = 1$.
Thus, (**) holds.

(B25): We have $x \to (y \to z) \overset{(pimpl-1')}{\leq} (x \to y) \to (x \to z)$. But, $y \overset{(K')}{\leq} x \to y$ implies, by (**'), $(x \to y) \to (x \to z) \leq y \to (x \to z)$.
Then, by (Tr), we obtain: $x \to (y \to z) \leq y \to (x \to z)$, i.e. (C') holds.

(B26): $y \overset{(K)}{\leq} x \to y$ implies, by (**'), $(x \to y) \to (x \to z) \leq y \to (x \to z)$. But $y \to (x \to z) \overset{(C')}{\leq} x \to (y \to z)$. Then, by (Tr), we obtain:
$(x \to y) \to (x \to z) \leq x \to (y \to z)$, on the one hand.
On the other hand, $x \to (y \to z) \overset{(pimpl-1')}{\leq} (x \to y) \to (x \to z)$.
Consequently, by (An), $x \to (y \to z) = (x \to y) \to (x \to z)$, i.e. (pimpl) holds.

(B27): By (pimpl-1), $[x \to (y \to z)] \to [(x \to y) \to (x \to z)] = 1$; if $x \to (y \to z) = 1$, then by (N), $(x \to y) \to (x \to z) = 1$, i.e. ($) holds.

(B27'): By (A00), (M) $\Longrightarrow$ (N), then apply (B27).

(B28): If $x \leq y \to z$, then, by ($\$'$), $x \to y \leq x \to z$; but, by (K'), $y \leq x \to y$; hence, by (Tr), $y \leq x \to z$. Thus, (#) holds.

(B29): $x \to x \overset{(M)}{=} (1 \to x) \to x \overset{(comm)}{=} (x \to 1) \to 1 \overset{(L)}{=} 1$, i.e. (Re) holds.

(B30): $(x \to 1) \to 1 \overset{(comm)}{=} (1 \to x) \to x \overset{(M)}{=} x \to x \overset{(Re)}{=} 1$; then, $x \to 1 = x \to [(x \to 1) \to 1] \overset{(Ex)}{=} (x \to 1) \to (x \to 1) \overset{(Re)}{=} 1$, i.e. (L) holds.

(B31): By (B30), (comm) + (Re) + (M) + (Ex) $\Longrightarrow$ (L). Suppose now that $y \to z = 1$; then, $z \overset{(M)}{=} 1 \to z = (y \to z) \to z \overset{(comm)}{=} (z \to y) \to y$, hence $x \to z = x \to [(z \to y) \to y] \overset{(Ex)}{=} (z \to y) \to (x \to y)$. Then, $(x \to y) \to (x \to z) = (x \to y) \to [(z \to y) \to (x \to y)] \overset{(Ex)}{=} (z \to y) \to [(x \to y) \to (x \to y)] \overset{(Re)}{=} (z \to y) \to 1 \overset{(L)}{=} 1$. Thus, (*) holds.

(B32): By (A18), (M) + (BB) $\Longrightarrow$ (Re); then $x \to 1 \overset{(Re)}{=} x \to (x \to x) \overset{(M)}{=} x \to ((1 \to x) \to x)$
$\overset{(comm)}{=} x \to ((x \to 1) \to 1)$
$\overset{(M)}{=} (1 \to x) \to [(x \to 1) \to (1 \to 1)] \overset{(BB)}{=} 1$, i.e. (L) holds.

(B33): Firstly, note that $y \overset{(K')}{\leq} (z \to y) \to y$.
Second, suppose that $x \to (y \to z) = 1$; then,
$((z \to y) \to y) \to (x \to z) \overset{(comm)}{=} ((y \to z) \to z) \to (x \to z)$
$\overset{(M)}{=} 1 \to [((y \to z) \to z) \to (x \to z)] = [x \to (y \to z)] \to [((y \to z) \to z) \to (x \to z)] \overset{(BB)}{=} 1$, i.e. $(z \to y) \to y \leq x \to z$.
Now, by (Tr), we obtain that $y \leq x \to z$, i.e. $(\#')$ holds.

(B34): (See the proof of (qW15) from [2]) $[(x \to y) \to y] \to x \overset{(comm)}{=} [(y \to x) \to x] \to x \overset{(comm)}{=} (x \to (y \to x)) \to (y \to x)$, i.e. (comm-1) holds.

(B35): By (B3), (pimpl) + (Re) + (M) $\Longrightarrow$ (pi), and by (B15), (pi) + (Re) + (Ex) + (B) + (An) $\Longrightarrow$ (pimpl). □

Let $\mathcal{A} = (A, \to, 1)$. Recall from [15] that:
- a RME algebra (= CI algebra) is an algebra $\mathcal{A}$ verifying the properties (Re), (M), (Ex) (i.e. is a RM algebra verifying (Ex)); hence, by Theorem 2.8, $(BB) \Leftrightarrow (B) \Leftrightarrow (*)$.
- a BE algebra is a RME algebra verifying (L) (i.e. is a RML algebra verifying (Ex)),
- a BCI algebra is just a RME algebra verifying (An), (*) and
- a BCK algebra is just a BE algebra verifying (An), (*) or, equivalently, a BCI algebra verifying (L).

Then we obtain the following result:

Theorem 3.21 *Commutative RME, BE, BCI, BCK algebras coincide.*

Proof. Let $\mathcal{A} = (a, \to, 1)$ be a commutative RME (CI) algebra, i.e. the properties (Re), (M), (Ex) and (comm) hold. Then, by (B12), (comm) + (M) imply (An); by (B30), (comm) + (Re) + (M) + (Ex) imply (L) and, by (B31), (comm) + (Re) + (M) + (Ex) imply (*); hence, the properties (L), (An) and (*) hold too, i.e. $\mathcal{A}$ is a (commutative) BE, BCI, BCK algebra. □

Proposition 3.22 *Let $(A, \to, 1)$ $((A, \leq, \to, 1))$ be an algebra (a structure). Then, the following hold (with an independent numbering):*
(BIM1) (impl) $\Longrightarrow$ (pi);
(BIM2) (comm) + (pi) + (Re) + (K) + (M) $\Longrightarrow$ (impl);
(BIM2') (comm) + (L) + (K) + (M) $\Longrightarrow$ ((pi) $\Leftrightarrow$ (impl));
(BIM3) (impl) + (Ex) + (B) + (An) $\Longrightarrow$ (comm);
(BIM4) (impl) + (Re) $\Longrightarrow$ (M);
(BIM5) (impl) + (L) $\Longrightarrow$ (M);
(BIM5') (impl) + (M) $\Longrightarrow$ (L);
(BIM5") (impl) $\Longrightarrow$ ((L) $\Leftrightarrow$ (M));
(BIM6) (impl) + (K) $\Longrightarrow$ (Re);
(BIM7) (L) + (K) + (M) + (Ex) + (B) + (An) $\Longrightarrow$ ((impl) $\Leftrightarrow$ ((comm) + (pi))).

Proof.
(BIM1): $x \to (x \to y) \overset{(impl)}{=} [(x \to y) \to x] \to (x \to y) \overset{(impl)}{=} x \to y$, i.e. (pi) holds.

(BIM2): Firstly, by (B12), (comm) + (M) $\Longrightarrow$ (An). Then, $((x \to y) \to x) \to x \overset{(comm)}{=} [x \to (x \to y)] \to (x \to y) \overset{(pi)}{=} (x \to y) \to (x \to y) \overset{(Re)}{=} 1$. Hence, $(x \to y) \to x \leq x$. But, we also have that $x \overset{(K)}{\leq} (x \to y) \to x$. Then, by (An), it follows that $x = (x \to y) \to x$, i.e. (impl) holds.

(BIM2'): By (BIM2) and (BIM1);

(BIM3): $[(x \to y) \to y] \to [(y \to x) \to x]$
$\overset{(impl)}{=} [(x \to y) \to y] \to [(y \to x) \to [(x \to y) \to x]]$

$\stackrel{(Ex)}{=} (y \to x) \to [[(x \to y) \to y] \to [(x \to y) \to x]] \stackrel{(B)}{=} 1.$
Thus, we obtained that

$$[(x \to y) \to y] \to [(y \to x) \to x] = 1. \quad (1)$$

Similarly,

$$[(y \to x) \to x] \to [(x \to y) \to y] = 1. \quad (2)$$

From (1) and (2), by (An), we obtain $(x \to y) \to y = (y \to x) \to x$, i.e. (comm) holds.

(BIM4): $1 \to x \stackrel{(Re)}{=} (x \to x) \to x \stackrel{(impl)}{=} x$, i.e. (M) holds.

(BIM5): $1 \to x \stackrel{(L)}{=} (x \to 1) \to x \stackrel{(impl)}{=} x$, i.e. (M) holds.

(BIM5'): $y \to 1 \stackrel{(M)}{=} (1 \to y) \to 1 \stackrel{(impl)}{=} 1$, i.e. (L) holds.

(BIM5"): By (BIM5) and (BIM5').

(BIM6): $x \to x \stackrel{(impl)}{=} x \to [(x \to y) \to x] \stackrel{(K)}{=} 1$, i.e. (Re) holds.

(BIM7): By (BIM3), (impl) + (Ex) + (B) + (An) $\Longrightarrow$ (comm), and by (BIM1), (impl) $\Longrightarrow$ (pi). Conversely, by (BIM2), (comm) + (pi) + (L) + (K) + (M) $\Longrightarrow$ (impl). □

3.3.1 Regular-join-semilattices from commutative regular algebras (structures)

• Let us introduce the new properties:
(Vee) $x \vee y = (x \to y) \to y$,
(dfV) $x \vee y \stackrel{df.}{=} (x \to y) \to y$;

(VV) $z \to (x \vee y) = (x \to y) \to (z \to y)$;
(VVV) $(x \to y) \to (z \to y) = (z \vee y) \to (x \vee y)$.

Remark 3.23 Just as the equivalence (EqrelR) ($x \leq y \Leftrightarrow x \to y = 1$) was used either:
- to define the binary relation $\leq$ in the algebra $A, \to, 1)$, by (dfrelR) ($x \leq y \stackrel{df.}{\Leftrightarrow} x \to y = 1$), or
- to connect $\leq$ and $\to$, 1 of the structure $(A, \leq, \to, 1)$,
the same happens with the equality (Vee) ($x \vee y = (x \to y) \to y$), that can be used either:
- to define a new binary operation $\vee$ in an algebra $(A, \to, 1)$ (in a structure $(A, \leq, \to, 1)$), by (dfV) ($x \vee y \stackrel{df.}{=} (x \to y) \to y$), or
- to connect $\vee$ and $\to$, 1 of an algebra $(A, \vee, \to, 1)$ or $(A, \to, \vee, 1)$ of type $(2, 2, 0)$ (of a structure $(A, \leq, \vee, \to, 1)$ or $(A, \leq, \to, \vee, 1)$, respectively).

Some connections are presented in the following Proposition 3.24.

Proposition 3.24 *Let $(A, \to, 1)$ ($(A, \leq, \to, 1)$ be an algebra (a structure). Define a new binary operation $\vee$ by (dfV). Then, we have:*
(BdfV1) (dfV) + (comm) $\Longrightarrow$ (Vcomm);
(BdfV2) (dfV) + (Re) + (M) $\Longrightarrow$ (Vid);
(BdfV3) (dfV) + (D) + (K) $\Longrightarrow$ (Vgeq);
(BdfV4) (dfV) + (L) + (M) + (Re) $\Longrightarrow$ (V1-1);
(BdfV5) (dfV) + (M) + (Ex) + (Re) $\Longrightarrow$ (EqV);
(BdfV5') (dfV) + (M) + (D) $\Longrightarrow$ (EqV);
(BdfV6) (dfV) + (Ex) $\Longrightarrow$ (VV);
(BdfV7) (dfV) + (Vcomm) + (EqV) + (VV) + (K) $\Longrightarrow$ (VVV);
(BdfV8) (dfV) + (Vcomm) + (Ex) + (VVV) $\Longrightarrow$ (Vassoc);
(BdfV9) (dfV) + (**) $\Longrightarrow$ (V-);
(BdfV10) (dfV) + (impl) $\Longrightarrow$ (Vid).

Proof.
(BdfV1): $x \vee y \stackrel{(dfV)}{=} (x \to y) \to y \stackrel{(comm)}{=} (y \to x) \to x \stackrel{(dfV)}{=} y \vee x$; thus (Vcomm) holds.

(BdfV2): $x \vee x \overset{(dfV)}{=} (x \to x) \to x \overset{(Re)}{=} 1 \to x \overset{(M)}{=} x$; thus (Vid) holds.

(BdfV3): $x \overset{(D)}{\leq} (x \to y) \to y \overset{(dfV)}{=} x \vee y$ and $y \overset{(K)}{\leq} (x \to y) \to y \overset{(dfV)}{=} x \vee y$; thus (Vgeq) holds.

(BdfV4): $x \vee 1 \overset{(dfV)}{=} (x \to 1) \to 1 \overset{(L)}{=} 1$ and
$1 \vee x \overset{(dfV)}{=} (1 \to x) \to x \overset{(M)}{=} x \to x \overset{(Re)}{=} 1$; thus (V1-1) holds.

(BdfV5): If $x \leq y$, i.e. $x \to y = 1$, then: $x \vee y \overset{(dfV)}{=} (x \to y) \to y = 1 \to y \overset{(M)}{=} y$.
If $y = x \vee y \overset{(dfV)}{=} (x \to y) \to y$, then: $x \to y = x \to [(x \to y) \to y] \overset{(Ex)}{=} (x \to y) \to (x \to y) \overset{(Re)}{=} 1$, i.e. $x \leq y$. Thus, (EqV) holds.

(BdfV5'): If $x \leq y$, i.e. $x \to y = 1$, then: $x \vee y \overset{(dfV)}{=} (x \to y) \to y = 1 \to y \overset{(M)}{=} y$.
If $y = x \vee y \overset{(dfV)}{=} (x \to y) \to y$, then, since $x \overset{(D)}{\leq} (x \to y) \to y$, it follows that $x \leq y$.

(BdfV6): $z \to (x \vee y) \overset{(dfV)}{=} z \to [(x \to y) \to y] \overset{(Ex)}{=} (x \to y) \to (z \to y)$; thus (VV) holds.

(BdfV7): We must prove that $(x \to y) \to (z \to y) = (z \vee y) \to (x \vee y)$. Indeed,
$y \overset{(K)}{\leq} z \to y \overset{(EqV)}{\Leftrightarrow} y \vee (z \to y) = z \to y$, hence
$(x \to y) \to (z \to y) = (x \to y) \to [y \vee (z \to y)] \overset{(Vcomm)}{=} (x \to y) \to [(z \to y) \vee y]$
$\overset{(VV)}{=} [(z \to y) \to y] \to [(x \to y) \to y] \overset{(dfV)}{=} (z \vee y) \to (x \vee y)$; thus (VVV) holds.

(BdfV8): $A \overset{notation}{=} (x \vee y) \vee z \overset{(dfV)}{=} [(x \to y) \to y] \vee z \overset{(Vcomm)}{=} z \vee [(x \to y) \to y]$
$\overset{(dfV)}{=} (z \to [(x \to y) \to y]) \to [(x \to y) \to y] \overset{(Ex),(dfV)}{=} ((x \to y) \to (z \to y)) \to (x \vee y)$
$\overset{(VVV)}{=} [(z \vee y) \to (x \vee y)] \to (x \vee y) \overset{(dfV)}{=} (z \vee y) \vee (x \vee y)$.
$B \overset{(notation}{=} x \vee (y \vee z) \overset{(Vcomm)}{=} x \vee (z \vee y) \overset{(dfV)}{=} x \vee [(z \to y) \to y]$
$\overset{(dfV)}{=} (x \to [(z \to y) \to y]) \to [(z \to y) \to y] \overset{(Ex),(dfV)}{=} [(z \to y) \to (x \to y)] \to (z \vee y)$
$\overset{(VVV)}{=} [(x \vee y) \to (z \vee y)] \to (z \vee y) \overset{(dfV)}{=} (x \vee y) \vee (z \vee y) \overset{(Vcomm)}{=} (z \vee y) \vee (x \vee y)$.
Hence, $A = B$, i.e. (Vassoc) holds.

(BdfV9): If $x \leq y$, then $y \to a \leq x \to a$, by (**); then $(x \to a) \to a \leq (y \to a) \to a$, by (**) again, i.e. $x \vee a \leq y \vee a$, by (dfV); thus, (V-) holds.

(BdfV10): $x \vee x \overset{(dfV)}{=} (x \to x) \to x \overset{(impl)}{=} x$, i.e. (Vid) holds. □

Then, we have the following theorem:

Theorem 3.25 *Let $\mathcal{A} = (A, \to, 1)$ ($\mathcal{A} = (A, \leq, \to, 1)$) be an algebra (structure) verifying the properties (comm), (Re), (K), (Ex), (M).*
Define a new binary operation $\vee$ by (dfV) ($x \vee y \overset{df.}{=} (x \to y) \to y$).
Then $(A, \vee, \to, 1)$ ($(A, \vee, \leq, \to, 1)$) is a regular-$\vee$-semilattice with last element 1.

Proof. Firstly, note that $\mathcal{A}$ is a regular algebra (structure), since (M) holds. Then,
- by (B12), (comm) + (M) $\Longrightarrow$ (An);
- by (A7'), (K) + (M) $\Longrightarrow$ (L);
- by (BdfV2), (dfV) + (Re) + (M) $\Longrightarrow$ (Vid) •
- by (BdfV1), (dfV) + (comm) $\Longrightarrow$ (Vcomm) •
- by (BdfV4), (dfV) + (L) + (M) + (Re) $\Longrightarrow$ (V1-1) •
- by (BdfV5), (dfV) + (M) + (Ex) + (Re) $\Longrightarrow$ (EqV) •
- by (BdfV6), (dfV) + (Ex) $\Longrightarrow$ (VV);
- by (BdfV7), (dfV) + (Vcomm) + (EqV) + (VV) + (K) $\Longrightarrow$ (VVV);
- by (BdfV8), (dfV) + (Vcomm) + (Ex) + (VVV) $\Longrightarrow$ (Vassoc) • □

3.4 Regular algebras (structures) with product. Commutative residoids and (regular-) monoids

3.4.1 (Regular) algebras (structures) with product. The Galois duality: algebras with (P), (RP) versus X-algebras with (R), (PR)

We shall study here the particular algebras (structures) (with implication $\to$ and 1) which have also a product $\odot$: there are two forms of these algebras (structures):
- $(A, \to, \odot, 1)$ (or, equivalently, $(A, \leq, \to, \odot, 1)$), where $\to$ is the principal operation and there exists $\min\{x, y\}$ in principal; we shall simply call this first form *algebra (structure)*;
- $(A, \odot, \to, 1)$ (or, equivalently, $(A, \leq, \odot, \to, 1)$), where $\odot$ is the principal operation and there exists $\max\{x, y\}$ in principal; we shall call this second form *X-algebra* (*X-structure*, respectively).

In these forms, between the implication $\to$ (called *residuum*, in certain cases), 1 and the product $\odot$, there can be a *Galois connection*, called *residuation*: for all $x, y, z \in A$,
(RP)=(PR) $\quad (x \odot y) \to z = 1 \Leftrightarrow x \to (y \to z) = 1$,
(RP')=(PR') $\quad x \odot y \leq z \Leftrightarrow x \leq y \to z$.
Therefore, we shall say that the two forms: algebras and X-algebras - and the results involving them - are *Galois dual*.

Definition 3.26 Let $\mathcal{A} = (A, \to, 1)$ ($\mathcal{A} = (A, \leq, \to, 1)$) be an algebra (a structure) verifying the properties (Re) and (Tr) ((Re') and (Tr') respectively) (i.e. $\leq$ is a pre-order). For $x, y \in A$, $x < y$ means $x \leq y$ and $x \neq y$.

(1) We define, for all $x, y \in A$: $\min\{x, y\} = x,\ if\ x \leq y,\ \min\{x, y\} = y,\ if\ y < x$.
Hence, $\min\{x, y\} \in \{x, y\}$, when it exists (if x and y are *comparable*).

(2) We define, for all subsets $B \subseteq A$: $\min B = b \in B,\ such\ that\ b \leq x,\ \forall x \in B$.
Hence, $\min B \in B$, when it exists, and is called *the smallest element* of B.

Dually,

(1') We define, for all $x, y \in A$: $\max\{x, y\} = y,\ if\ x \leq y,\ \max\{x, y\} = x,\ if\ y < x$.
Hence, $\max\{x, y\} \in \{x, y\}$, when it exists (if x and y are *comparable*).

(2') We define, for all subsets $B \subseteq A$: $\max B = b \in B,\ such\ that\ x \leq b,\ \forall x \in B$.
Hence, $\max B \in B$, when it exists, and is called *the greatest element* of B.

We introduce now the following definitions:

Definitions 3.27 Let $\mathcal{A} = (A, \to, \odot, 1)$ ($\mathcal{A} = (A, \leq, \to, \odot, 1)$) be an algebra (structure) such that the reduct $(A, \to, 1)$ ($(A, \leq, \to, 1)$) is an algebra (structure) verifying the properties (Re) and (Tr) ((Re') and (Tr') respectively) (i.e. $\leq$ is a pre-order).

(i) We say that $\mathcal{A}$ is an *algebra with (P)* (a *structure with (P')*) (i.e. *with product*) if, for any $x, y \in A$, there exists the smallest element of the set $\{z \mid x \to (y \to z) = 1\}$ ($\{z \mid x \leq y \to z\}$) and it equals $x \odot y$, i.e. $\mathcal{A}$ satisfies the following property:
(P) there exists, for all $x, y \in A$, $x \odot y = \min\{z \mid x \to (y \to z) = 1\}$,
(P') there exists, for all $x, y \in A$, $x \odot y = \min\{z \mid x \leq y \to z\}$, respectively.

Alternatively, we say that $(A, \to, 1)$ ($(A, \leq, \to, 1)$) is an *algebra with (P)* (a *structure with (P')*) (i.e. *with product*) if, for any $x, y \in A$, there exists the smallest element of the set $\{z \mid x \to (y \to z) = 1\}$ ($\{z \mid x \leq y \to z\}$) and we define a new operation, called *product*, by: (dfP) there exists, for all $x, y \in A$, $x \odot y \overset{df.}{=} \min\{z \mid x \to (y \to z) = 1\}$,
(dfP') there exists, for all $x, y \in A$, $x \odot y \overset{df.}{=} \min\{z \mid x \leq y \to z\}$, respectively.

(ii) We say that $\mathcal{A}$ is an *algebra with (RP)* (a *structure with (RP')*) (i.e. *with residuum and product*), if the operations $\to$, 1 and $\odot$ are connected by the property:
(RP) for all $x, y, z \in A$, $(x \odot y) \to z = 1 \Leftrightarrow x \to (y \to z) = 1$,
(RP') for all $x, y, z \in A$, $x \odot y \leq z \Leftrightarrow x \leq y \to z$, respectively.

Remark 3.28 Just as the equivalence (EqrelR) ($x \leq y \Leftrightarrow x \to y = 1$) was used either:
- to define the binary relation $\leq$ in the algebra $A, \to, 1)$, by (dfrelR) ($x \leq y \overset{df.}{\Leftrightarrow} x \to y = 1$), or

- to connect $\leq$ and $\rightarrow$, 1 of the structure $(A, \leq, \rightarrow, 1)$,
the same happens with the equality (P), that can be used either:
- to define a new binary operation $\odot$ in an algebra $(A, \rightarrow, 1)$ (in a structure $(A, \leq, \rightarrow, 1)$), by (dfP), or
- to connect $\odot$ and $\rightarrow$, 1 of an algebra $(A, \rightarrow, \odot, 1)$ of type $(2, 2, 0)$ (of a structure $(A, \leq, \rightarrow, \odot, 1)$, respectively).

Remark 3.29 By Definition 3.27 (i), we have $x \odot y \in \{z \mid x \rightarrow (y \rightarrow z) = 1\}$ $(x \odot y \in \{z \mid x \leq y \rightarrow z\})$, hence we have the property:
(PP) $x \rightarrow [y \rightarrow (x \odot y)] = 1$,
(PP') $x \leq y \rightarrow (x \odot y)$, respectively.
Consequently, (P) $\Longrightarrow$ (PP).

Conventions. In order to simplify the writting:
- if the algebra $(A, \rightarrow, 1)$ verifies the property (Re), for example, then we shall no more mention that the associated structure $(A, \leq, \rightarrow, 1)$ verifies the associated property (Re'). Also, we shall freely use (Re) or (Re') in a proof;
- we shall write "algebra (structure) with (P)" instead of "algebra with (P) (structure with (P'))" and
- we shall write "algebra (structure) with (RP)" instead of "algebra with (RP) (structure with (RP'))".

Then, we introduce the following definitions, Galois dually:

Definitions 3.30 (See the Galois dual Definition 3.27)
Let $\mathcal{A} = (A, \odot, \rightarrow, 1)$ $(\mathcal{A} = (A, \leq, \odot, \rightarrow, 1))$ be an X-algebra (X-structure) such that the reduct $(A, \rightarrow, 1)$ $((A, \leq, \rightarrow, 1))$ is an algebra (structure) verifying the properties (Re) and (Tr) (i.e. $\leq$ is a pre-order).
(i') We say that $\mathcal{A}$ is an *X-algebra with (R)* (*X-structure with (R')*) (i.e. *with residuum*) if, for any $x, y \in A$, there exists the greatest element of the set $\{x \mid (x \odot y) \rightarrow z = 1\}$ $(\{x \mid x \odot y \leq z\})$ and it equals $y \rightarrow z$, i.e. $\mathcal{A}$ satisfies the following property:
(R) there exists, for all $y, z \in A$, $y \rightarrow z = \max\{x \mid (x \odot y) \rightarrow z = 1\}$,
(R') there exists, for all $y, z \in A$, $y \rightarrow z = \max\{x \mid x \odot y \leq z\}$, respectively.
(ii') We say that $\mathcal{A}$ is an *X-algebra with (PR)* (*X-structure with (PR')*) (i.e. *with product and residuum*) if the operations $\rightarrow$, 1 and $\odot$ are connected by the property:
(PR) for all $x, y, z \in A$, $x \rightarrow (y \rightarrow z) = 1 \Leftrightarrow (x \odot y) \rightarrow z = 1$,
(PR') for all $x, y, z \in A$, $x \leq y \rightarrow z \Leftrightarrow x \odot y \leq z$ respectively.

Note that the properties (RP) and (PR) coincide (we write (RP)=(PR)) and that this property (often called the *residuation property*) reflects a *Galois connection.*

Remark 3.31 By Definition 3.30 (i'), we have $y \rightarrow z \in \{x \mid (x \odot y) \rightarrow z = 1\}$ $(y \rightarrow z \in \{x \mid x \odot y \leq z\})$, hence we have the property:
(RR) $[(y \rightarrow z) \odot y] \rightarrow z = 1$,
(RR') $(y \rightarrow z) \odot y \leq z$ respectively.
Consequently, (R) $\Longrightarrow$ (RR).

Conventions. In order to simplify the writting:
- we shall write "X-algebra (X-structure) with (R)" instead of "X-algebra with (RR) (X-structure with (RR'))" and
- we shall write "X-algebra (X-structure) with (PR)" instead of "X-algebra with (PR) (X-structure with (PR'))".

• Let us introduce the following new properties:
(G) (Gödel) (the idempotency of $\odot$) for all $x \in A$, $x \odot x = x$;

(EqP) $x \rightarrow y = 1 \Leftrightarrow x \odot y = x$,
(EqP') $x \leq y \Leftrightarrow x \odot y = x$,
(dfrelP) $x \leq y \overset{df.}{\Leftrightarrow} x \odot y = x$;

(P=) $z \to x = 1,\ z \to y = 1 \implies z \le x \odot y$,
(P=') $z \le x,\ z \le y \implies z \le x \odot y$;

(P-) $x \to y = 1 \Longrightarrow (x \odot a) \to (y \odot a) = 1$,
(P-') $x \le y \Longrightarrow x \odot a \le y \odot a$;
(P- -) $x \to y = 1,\ a \to b = 1 \implies (x \odot a) \to (y \odot b) = 1$,
(P- -') $x \le y,\ a \le b \implies x \odot a \le y \odot b$;
(Pleq) $(x \odot y) \to x = 1,\ (x \odot y) \to y = 1$,
(Pleq') $x \odot y \le x, y$.

Remarks 3.32 We have the following connections:
(i) (EqrelR) $\Longrightarrow$ ((EqP) $\Leftrightarrow$ (EqP')); (EqP) + (EqP') $\Longrightarrow$ (EqrelR);
(ii) (EqP) $\Longrightarrow$ ((dfrelR) $\Leftrightarrow$ (dfrelP)); (dfrelR) + (dfrelP) $\Longrightarrow$ (EqP).

We prove now the following two Galois dual Lemmas 3.33 and 3.34.

Lemma 3.33 *Let $\mathcal{A} = (A, \to, \odot, 1)$ ($\mathcal{A} = (A, \le, \to, \odot, 1)$) be an algebra (a structure) with (P) (i.e. (Re), (Tr) and (P) hold).*
If the property (*) also holds, then $\mathcal{A}$ is an algebra (a structure) with (RP).

Proof. If $x \odot y \le z$, then, it follows by (*), that $y \to (x \odot y) \le y \to z$ and since we also have, by (PP), that $x \le y \to (x \odot y)$, we get $x \le y \to z$, by (Tr). Conversely, if $x \le y \to z$ then, by (P), $x \odot y \le z$. Thus, (RP) holds. □

Lemma 3.34 *(See the Galois dual Lemma 3.33)*
Let $\mathcal{A} = (A, \odot, \to, 1)$ ($\mathcal{A} = (A, \le, \odot, \to, 1)$) be an X-algebra (X-structure) with (R) (i.e. properties (Re), (Tr) and (R) hold).
If the property (P-) also holds, then $\mathcal{A}$ is an X-algebra (X-structure) with (PR).

Proof. If $x \odot y \le z$, then, by (R), $x \le y \to z$. Conversely, if $x \le y \to z$, it follows, by (P-), that $x \odot y \le (y \to z) \odot y$ and since we also have, by (RR), that $(y \to z) \odot y \le z$, we get, by (Tr), that $x \odot y \le z$. Thus, (PR) holds. □

A basic result is the following lemma.

Lemma 3.35 *Let $\mathcal{A} = (A, \to, \odot, 1)$ be an algebra ($\mathcal{A} = (A, \le, \to, \odot, 1)$ be a structure) or, Galois dually, let $\mathcal{A} = (A, \odot, \to, 1)$ be an X-algebra ($\mathcal{A} = (A, \le, \odot, \to, 1)$ be an X-structure) such that the reduct $(A, \to, 1)$ ($(A, \le, \to, 1)$) is an algebra (structure) verifying the properties (Re) and (Tr) (i.e. $\le$ is a pre-order).*
If the property (RP)=(PR) holds, then the properties (P), (R), (PP), (RR), (*), (P-) also hold.

Proof.
(P): Since $x \odot y \overset{(Re)}{\le} x \odot y$, then, by (RP), we obtain that $x \le y \to (x \odot y)$, i.e. $x \odot y \in \{z \mid x \le y \to z\}$. If $z' \in \{z \mid x \le y \to z\}$, i.e. $x \le y \to z'$, then, by (RP), $x \odot y \le z'$. Thus, $\min\{z \mid x \le y \to z\}$ exists and equals $x \odot y$, i.e. (P) holds.
(R): Since $y \to z \overset{(Re)}{\le} y \to z$, then, by (RP), we obtain that $(y \to z) \odot y \le z$, i.e. $y \to z \in \{x \mid x \odot y \le z\}$. If $x' \in \{x \mid x \odot y \le z\}$, i.e. $x' \odot y \le z$, then, by (RP), $x' \le y \to z$. Thus, $\max\{x \mid x \odot y \le z\}$ exists and equals $y \to z$, i.e. (R) holds.
(PP): by Remark 3.29.
(RR): by Remark 3.31.
(*): By (RR), $(z \to x) \odot z \le x$ and since $x \le y$, it follows that $(z \to x) \odot z \le y$, by (Tr); hence, $z \to x \le z \to y$, by (RP), i.e. property (*) holds.
(P-): By (PP), $y \le z \to (y \odot z)$ and since $x \le y$, it follows that $x \le z \to (y \odot z)$, by (Tr); hence, $x \odot z \le y \odot z$, by (RP), i.e. property (P-) holds. □

• **The basic general equivalences**
Now the following three theorems follow immediately:

Theorem 3.36 *Let $\mathcal{A} = (A, \rightarrow, \odot, 1)$ ($\mathcal{A} = (A, \leq, \rightarrow, \odot, 1)$) be an algebra (structure) verifying (Re), (Tr). Then*

$$(P) \;+\; (*) \;\Leftrightarrow\; (RP).$$

Proof. $\Longrightarrow$: by Lemma 3.33.
$\Longleftarrow$: by Lemma 3.35, (P) and (*) hold. □

Galois dually we have:

Theorem 3.37 *Let $\mathcal{A} = (A, \odot, \rightarrow, 1)$ ($\mathcal{A} = (A, \leq, \odot, \rightarrow, 1)$) be an X-algebra (X-structure) verifying (Re), (Tr). Then*

$$(R) \;+\; (P-) \;\Leftrightarrow\; (PR).$$

Proof. $\Longrightarrow$: by Lemma 3.34.
$\Longleftarrow$: by Lemma 3.35, (R) and (P-) hold. □

The connection between the Galois dual algebras "algebra with (RP)" and "X-algebra with (PR)" is the following:

Theorem 3.38
1) Let $\mathcal{A} = (A, \rightarrow, \odot, 1)$ be an algebra (or, equivalently, let $\mathcal{A} = (A, \leq, \rightarrow, \odot, 1)$ be a structure) with (RP). Define

$$\alpha(\mathcal{A}) \overset{df.}{=} (A, \odot, \rightarrow, 1) \quad (\alpha(\mathcal{A}) \overset{df.}{=} (A, \leq, \odot, \rightarrow, 1)).$$

Then, $\alpha(\mathcal{A})$ is an X-algebra (X-structure, respectively) with (PR).
1') Conversely, let $\mathcal{A} = (A, \odot, \rightarrow, 1)$ be an X-algebra (or, equivalently, let $\mathcal{A} = (A, \leq, \odot, \rightarrow, 1)$ be an X-structure) with (PR). Define

$$\beta(\mathcal{A}) \overset{df.}{=} (A, \rightarrow, \odot, 1) \quad (\beta(\mathcal{A}) \overset{df.}{=} (A, \leq, \rightarrow, \odot, 1)).$$

Then, $\beta(\mathcal{A})$ is an algebra (structure, respectively) with (RP).
2) The above defined mappings are mutually inverse.

Proof. Obviously. □

By Theorems 3.36, 3.37 and 3.38, we obtain the announced basic equivalences, presented in next Figure 2. Note that these equivalences remain valid also for the particular cases of regular algebras (structures) and of quasi-algebras (quasi-structures).

$$\begin{array}{c}\textbf{Algebra with (P)} \\ + (*)\end{array} \Leftrightarrow \textbf{Algebra with (RP)} \; \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \; \textbf{X-algebra with (PR)} \Leftrightarrow \begin{array}{c}\textbf{X-algebra with (R)} \\ + (P-)\end{array}$$

Figure 2: The basic general equivalences

Note that any algebra with (P) (or with (RP)) and any X-algebra with (R) (or with (PR)) that verify the property (M) is a regular algebra (X-algebra).

3.4.2 Commutative residoids and (regular-) monoids

Recall the following definitions.

Definitions 3.39
(1) An algebra $(A, \rightarrow, 1)$ (a structure $(A, \leq, \rightarrow, 1)$) is a *commutative (or abelian) residoid* if the following properties hold ([14], Definition 1.4.7): for all $x, y, z \in A$,
(M) $1 \rightarrow x = x$,
(BB) $(y \rightarrow z) \rightarrow [(z \rightarrow x) \rightarrow (y \rightarrow x)] = 1$.
(2) An algebra $(A, \odot, 1)$ of type $(2, 0)$ is a *commutative (or abelian) monoid* if the following properties hold: for all $x, y, z \in A$,
(Pcomm) $x \odot y = y \odot x$,
(Passoc) $(x \odot y) \odot z = x \odot (y \odot z)$,
(P1-1) $x \odot 1 = 1 \odot x = x$: (P-1) $x \odot 1 = x$ and (P1-) $1 \odot x = x$.

Note that any commutative residoid is a regular algebra (structure) (i.e. property (M) holds), while the commutative monoid is not.
We then introduce the following new definition.

Definition 3.40 An algebra $\mathcal{A} = (A, \rightarrow, \odot, 1)$ or an X-algebra $\mathcal{A} = (A, \odot, \rightarrow, 1)$ (a structure $\mathcal{A} = (A, \leq, \rightarrow, \odot, 1)$ or an X-structure $\mathcal{A} = (A, \leq, \odot, \rightarrow, 1)$ respectively) is a *commutative (or abelian) regular-monoid* if the reduct $(A, \rightarrow, 1)$ ($(A, \leq, \rightarrow, 1)$) is a regular algebra (structure) and the reduct $(A, \odot, 1)$ is a commutative monoid.

Remark 3.41 Given a commutative regular-monoid, say $(A, \rightarrow, \odot, 1)$, then the reduct $(A, \odot, 1)$ is a commutative monoid. Conversely, given a commutative monoid $\mathcal{A} = (A, \odot, 1)$, there can exist none, one or more commutative regular-monoids whose reduct $(A, \odot, 1)$ coincides with $\mathcal{A}$.

• Let us introduce the following new properties:
(PEx) for all $x, y, z \in A$, $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$;

(PBB) for all $x, y, z \in A$, $[(x \rightarrow y) \odot (y \rightarrow z)] \rightarrow (x \rightarrow z) = 1$,
(PBB') for all $x, y, z \in A$, $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$;

(PB) for all $x, y, z \in A$, $[(y \rightarrow z) \odot (x \rightarrow y)] \rightarrow (x \rightarrow z) = 1$,
(PB') for all $x, y, z \in A$, $(y \rightarrow z) \odot (x \rightarrow y) \leq x \rightarrow z$;

(PD) for all $x, y \in A$, $[y \odot (y \rightarrow x)] \rightarrow x = 1$,
(PD') for all $x, y \in A$, $y \odot (y \rightarrow x) \leq x$.

The following three Propositions 3.42, 3.43, 3.44 present some connections.

Proposition 3.42 *Let $\mathcal{A} = (A, \rightarrow, \odot, 1)$ be an algebra (equivalently, $\mathcal{A} = (A, \leq, \rightarrow, \odot, 1)$ be a structure) or, Galois dually, let $\mathcal{A} = (A, \odot, \rightarrow, 1)$ be an X-algebra ($\mathcal{A} = (A, \leq, \odot, \rightarrow, 1)$ be an X-structure) such that the reduct $(A, \rightarrow, 1)$ ($(A, \leq, \rightarrow, 1)$) is an algebra (structure) verifying the properties (Re) and (Tr) (i.e. $\leq$ is a pre-order). The following hold:*
(BP1) (PEx) $\Longrightarrow$ ((PB) $\Leftrightarrow$ (B));
(BP2) (PEx) $\Longrightarrow$ ((PBB) $\Leftrightarrow$ (BB));
(BP3) (PEx) $\Longrightarrow$ ((PD) $\Leftrightarrow$ (D));
(BP4) (Pcomm) $\Longrightarrow$ ((PB) $\Leftrightarrow$ (PBB));
(BP5) (Pcomm) $\Longrightarrow$ ((PD) $\Leftrightarrow$ (RR));
(BP6) (PEx) + (Re) $\Longrightarrow$ (PP);
(BP7) (PEx) + (Pcomm) $\Longrightarrow$ (Ex);
(BP8) (P-) + (Pcomm) + (Tr) $\Longrightarrow$ (P- -);
(BP9) (EqP) $\Longrightarrow$ ((L) $\Leftrightarrow$ (P-1)).

Proof.
(BP1): $[(y \rightarrow z) \odot (x \rightarrow y)] \rightarrow (x \rightarrow z) = 1 \overset{(PEx)}{\Leftrightarrow} (y \rightarrow z) \rightarrow [(x \rightarrow y) \rightarrow (x \rightarrow z)] = 1$, i.e. (PB) $\Leftrightarrow$ (B), by (PEx).

(BP2): $[(y \to z) \odot (z \to x)] \to (y \to x) = 1 \stackrel{(PEx)}{\Leftrightarrow} (y \to z) \to [(z \to x) \to (y \to x)] = 1$, i.e. (PBB) $\Leftrightarrow$ (BB), by (PEx).

(BP3): $[y \odot (y \to x) \to x = 1 \stackrel{(PEx)}{\Leftrightarrow} y \to [(y \to x) \to x] = 1$, i.e. (PD) $\Leftrightarrow$ (D), by (PEx).

(BP4): $[(y \to z) \odot (x \to y)] \to (x \to z) = 1 \stackrel{(Pcomm)}{\Leftrightarrow} [(x \to y) \odot (y \to z)] \to (x \to z) = 1$, i.e. (PB) $\Leftrightarrow$ (PBB), by (Pcomm).

(BP5): $[y \odot (y \to z) \to z = 1 \stackrel{(Pcomm)}{\Leftrightarrow} [(y \to z) \odot y] \to z = 1$, i.e. (PD) $\Leftrightarrow$ (RR), by (Pcomm).

(BP6): $x \to [y \to (x \odot y)] = 1 \stackrel{(PEx)}{\Leftrightarrow} (x \odot y) \to (x \odot y) = 1$, which is true by (Re).

(BP7): $x \to (y \to z) \stackrel{(PEx)}{\Leftrightarrow} (x \odot y) \to z \stackrel{(Pcomm)}{\Leftrightarrow} (y \odot x) \to z \stackrel{(PEx)}{\Leftrightarrow} y \to (x \to z)$, i.e. (Ex) holds.

(BP8): $x \leq y \stackrel{(P-)}{\Longrightarrow} x \odot a \leq y \odot a$ and $a \leq b \stackrel{(P-)}{\Longrightarrow} a \odot y \leq b \odot y$; by (Pcomm), $x \odot a \leq y \odot a \leq y \odot b$, hence, by (Tr), $x \odot a \leq y \odot b$; thus, (P- -) holds.

(BP9): $x \odot 1 \stackrel{(P-1)}{=} x \stackrel{(EqP)}{\Leftrightarrow} x \to 1 \stackrel{(L)}{=} 1$. □

Proposition 3.43 *Under the hypothesis from Proposition 3.42, the following hold:*
(BPR1) (PR) + (K) + (Re) + (L) $\Longrightarrow$ (Pleq);
(BPR1') (PR) + (Pcomm) + (K) $\Longrightarrow$ (Pleq);
(BPR2) (PR) + (Re) + (Ex) + (An) $\Longrightarrow$ (PEx);
(BPR3) (PEx) $\Longrightarrow$ (RP);
(BPR4) (PR) $\Longrightarrow$ (B) $\Leftrightarrow$ (PB);
(BPR5) (PR) $\Longrightarrow$ (BB) $\Leftrightarrow$ (PBB);
(BPR6) (PR) $\Longrightarrow$ (D) $\Leftrightarrow$ (PD);
(BPR7) (PR) + (M) + (Re) $\Longrightarrow$ (P1-1);
(BPR8) (PR) + (pi) + (Re) + (An) $\Longrightarrow$ (G);
(BPR9) (PR) + (Re) + (An) + (Eq#) $\Longrightarrow$ (Pcomm);
(BPR9') (PR) + (Pcomm) $\Longrightarrow$ (Eq#);
(BPR9") (PR) + (Re) + (An) $\Longrightarrow$ ((Eq#) $\Leftrightarrow$ (Pcomm)).

Proof.

(BPR1): $x \odot y \leq x \stackrel{(PR)}{\Leftrightarrow} x \leq y \to x$, which is true by (K).
$x \odot y \leq y \stackrel{(PR)}{\Leftrightarrow} x \leq y \to y \stackrel{(Re)}{=} 1$, which is true by (L).

(BPR1'): $x \odot y \leq x \stackrel{(PR)}{\Leftrightarrow} x \leq y \to x$, which is true by (K).
$x \odot y \leq y \stackrel{(Pcomm)}{\Leftrightarrow} y \odot x \leq y \stackrel{(PR)}{\Leftrightarrow} y \leq x \to y$, which is true by (K).

(BPR2): Firstly, by (A28), (Ex) $\Longrightarrow$ (##); then $a \leq (x \odot y) \to z \stackrel{(\#\#)}{\Leftrightarrow} x \odot y \leq a \to z$ and $a \leq x \to (y \to z) \stackrel{(\#\#)}{\Leftrightarrow} x \leq a \to (y \to z) \stackrel{(Ex)}{\Leftrightarrow} x \leq y \to (a \to z) \stackrel{(PR)}{\Leftrightarrow} x \odot y \leq a \to z$; hence $(x \odot y) \to z = x \to (y \to z)$, by (Re) and (An), i.e. (PEx) holds.

(BPR3): If $x \odot y \leq z$, i.e. $(x \odot y) \to z = 1$, then, by (PEx), $x \to (y \to z) = 1$, i.e. $x \leq y \to z$. Conversely, if $x \leq y \to z$, i.e. $x \to (y \to z) = 1$, then, by (PEx), $(x \odot y) \to z = 1$, i.e. $x \odot y \leq z$. Thus, (PR) holds.

(BPR4): $(y \to z) \odot (x \to y) \leq x \to z \stackrel{(PR)}{\Leftrightarrow} y \to z \leq (x \to y) \to (x \to z)$.

(BPR5): $(x \to y) \odot (y \to z) \leq x \to z \stackrel{(PR)}{\Leftrightarrow} x \to y \leq (y \to z) \to (x \to z)$.

(BPR6): $y \odot (y \to x) \leq x \stackrel{(PR)}{\Leftrightarrow} y \leq (y \to x) \to x$.

(BPR7): By Lemma 3.35, (PR) $\Longrightarrow$ (P); by (A00), (M) $\Longrightarrow$ (N); by (Re), $x \leq x$. Hence, we obtain: $x \odot 1 \stackrel{(P)}{=} \min\{z \mid x \leq 1 \to z\} \stackrel{(M)}{=} \min\{z \mid x \leq z\} = x$ and $1 \odot x = \min\{z \mid 1 \leq x \to z\} \stackrel{(N)}{=} \min\{z \mid 1 = x \to z\} = \min\{z \mid x \leq z\} = x$.

(BPR8): By (B2), (pi) + (Re) $\Longrightarrow$ (L).
$x \odot x \leq x \stackrel{(PR)}{\Leftrightarrow} x \leq x \to x \stackrel{(Re)}{=} 1$, which is true by (L).
$x \leq x \odot x$; indeed, by Lemma 3.35, (PR) $\Longrightarrow$ (P) and (P) $\Longrightarrow$ (PP); by (PP), $x \to [y \to (x \odot y)] = 1$;

hence $x \to (x \odot x) \overset{(pi)}{=} x \to [x \to (x \odot x)] = 1$.
Since $x \odot x \leq x$ and $x \leq x \odot x$, then $x \odot x = x$, by (An), i.e. (G) holds.

(BPR9): $x \odot y \leq a \overset{(PR)}{\Leftrightarrow} x \leq y \to a \overset{(Eq\#)}{\Leftrightarrow} y \leq x \to a \overset{(PR)}{\Leftrightarrow} y \odot x \leq a$; then, by (Re) and (An), we obtain (Pcomm).

(BPR9'): $x \leq y \to z \overset{(PR)}{\Leftrightarrow} x \odot y \leq z \overset{(Pcomm)}{\Leftrightarrow} y \odot x \leq z \overset{(PR)}{\Leftrightarrow} y \leq x \to z$; thus, (Eq#) holds.

(BPR9"): By (BPR9) and (BPR9'). □

Proposition 3.44 *Under the hypothesis from Proposition 3.42, the following hold:*
(BG1) (G) + (P–) $\Longrightarrow$ (P=);
(BG2) (G) + (P-) + (Pcomm) + (Pleq) + (An) $\Longrightarrow$ (EqP);
(BG3) (EqP) $\Longrightarrow$ ((Re) $\Leftrightarrow$ (G));
(BG4) (pi) + (Pleq) + (PEx) + (Re) + (An) $\Longrightarrow$ (G);
(BG5) (pimpl-1) + (Pleq) + (PEx) + (Re) + (M) + (An) $\Longrightarrow$ (G);
(BG6) (impl) + (Pleq) + (PEx) + (Re) + (M) + (**) + (An) $\Longrightarrow$ (G).

Proof.
(BG1): $z \leq x$ and $z \leq y$ imply $z \odot z \leq x \odot y$, by (P- -); hence $z \leq x \odot y$, by (G). Thus (P=) holds.

(BG2): If $x \leq y$, then by (P-), $x \odot x \leq y \odot x$, hence $x \leq x \odot y$, by (G) and (Pcomm); we also have that $x \odot y \leq x$, by (Pleq); then, by (An), $x \odot y = x$. Conversely, if $x \odot y = x$, then $x \leq y$, since $x \odot y \leq y$, by (Pleq).

(BG3): Obviously.

(BG4): (a) $x \odot x \leq x$. Indeed, this holds by (Pleq).
(b) $x \leq x \odot x$. Indeed, $H \overset{notation}{=} (x \odot x) \to y \overset{(PEx)}{=} x \to (x \to y) \overset{(pi)}{=} x \to y$; take now $y = x \odot x$ in H; we obtain: $1 \overset{(Re)}{=} (x \odot x) \to (x \odot x) = x \to (x \odot x)$, hence (b) holds.
Finally, (a) + (b) + (An) imply $x \odot x = x$, i.e. (G) holds.

(BG5): (a) $x \odot x \leq x$. Indeed, this holds by (Pleq).
(b) $x \leq x \odot x$. Indeed, $(x \odot x) \to z \overset{(PEx)}{=} x \to (x \to z) \overset{(pimpl-1)}{\leq} (x \to x) \to (x \to z) \overset{(Re)}{=} 1 \to (x \to z) \overset{(M)}{=} x \to z$, hence $(x \odot x) \to z \leq x \to z$, which for $z = x \odot x$, gives:
$1 \overset{(Re)}{=} (x \odot x) \to (x \odot x) \leq x \to (x \odot x)$, i.e. $x \leq x \odot x$, by (M); thus (b) holds.
Finally, (a) + (b) + (An) imply $x \odot x = x$, i.e. (G) holds.

(BG6): (a) $x \odot x \leq x$. Indeed, this holds by (Pleq).
(b) $x \leq x \odot x$. Indeed, by (A0), (Re) $\Longrightarrow$ (S) and since $[x \to (x \odot x)] \to x \overset{(impl)}{=} x$, then, by (S'), we obtain: $[x \to (x \odot x)] \to x \leq x$; then, by (**'), we obtain:
$H \overset{notation}{=} x \to [x \to (x \odot x)] \leq ([x \to (x \odot x)] \to x) \to [x \to (x \odot x)] \overset{(impl)}{=} x \to (x \odot x)$
and since $H \overset{(PEx)}{=} (x \odot x) \to (x \odot x) \overset{(Re)}{=} 1$,
it follows that $1 \leq x \to (x \odot x)$, i.e. $1 \to [x \to (x \odot x)] = 1$, hence $x \to (x \odot x) = 1$, by (M). Thus, (b) holds.
Finally, (a) + (b) + (An) imply $x \odot x = x$, i.e. (G) holds. □

We can prove now the following two Galois dual Propositions 3.45 and 3.47 and their corollaries.

Proposition 3.45 *(See [**?**], [**?**])*
Let $\mathcal{A} = (A, \to, \odot, 1)$ ($\mathcal{A} = (A, \leq, \to, \odot, 1)$) be an algebra (structure) with (RP) (i.e. (Re), (Tr) and (RP) hold), verifying the properties (An) and (Ex).
Then, the reduct $(A, \odot, 1)$ is an abelian monoid verifying (M) and (PEx) (i.e. $\mathcal{A}$ is in fact an abelian regular-monoid verifying (PEx)).

Proof. Firstly, by (A5), (Re) + (Ex) + (An) $\Longrightarrow$ (M), hence $(A, \to, 1)$ ($(A, \leq, \to, 1)$) is a regular algebra (structure).

By (A28), (Ex) $\Longrightarrow$ (Eq#).

(Pcomm): By (BPR9), (PR)=(RP) + (Re) + (An) + (Eq#) $\Longrightarrow$ (Pcomm).

(Passoc): $(x \odot y) \odot z \leq a \overset{(RP)}{\Leftrightarrow} x \odot y \leq z \to a \overset{(RP)}{\Leftrightarrow} x \leq y \to (z \to a)$
$\overset{(Eq\#)}{\Leftrightarrow} y \leq x \to (z \to a) \overset{(Ex)}{\Leftrightarrow} y \leq z \to (x \to a)$ and
$x \odot (y \odot z) \leq a \overset{(RP)}{\Leftrightarrow} x \leq (y \odot z) \to a \overset{(\#\#)}{\Leftrightarrow} y \odot z \leq x \to a \overset{(RP)}{\Leftrightarrow} y \leq z \to (x \to a)$.
Thus, $(x \odot y) \odot z = x \odot (y \odot z)$, by (Re) and (An).

(P1-1): Firstly, by (A5), (Re) + (Ex) + (An) imply (M). Then,
$x \odot 1 \leq a \overset{(RP)}{\Leftrightarrow} x \leq 1 \to a \overset{(M)}{\Leftrightarrow} x \leq a$. Thus, $x \odot 1 = x$, by (Re) and (An).
$1 \odot x \leq a \overset{(RP)}{\Leftrightarrow} 1 \leq x \to a \overset{(M)}{\Leftrightarrow} x \leq a$. Thus, $1 \odot x = x$, by (Re) and (An).

Consequently, $(A, \odot, 1)$ is an abelian monoid. Hence, $(A, \to, \odot, 1)$ $((A, \leq, \to, \odot, 1)$ is an abelian regular-monoid.

Finally, by (BPR2), (PR) + (Re) + (Ex) + (An) $\Longrightarrow$ (PEx). □

Corollary 3.46 *Let $\mathcal{A} = (A, \to, \odot, 1)$ $(\mathcal{A} = (A, \leq, \to, \odot, 1))$ be an algebra (structure) with (RP) (i.e. (Re), (Tr) and (RP) hold), such that the reduct $(A, \to, 1)$ $((A, \leq, \to, 1))$ is an abelian residoid (i.e. (M), (BB) hold), verifying also (An) (i.e. $\leq$ is an order).*
Then, the reduct $(A, \odot, 1)$ is an abelian monoid (i.e. $\mathcal{A}$ is in fact an abelian regular-monoid).

Proof. By (A21”), (M) + (BB) + (An) imply (Ex). Then apply the above Proposition 3.45. □

Galois dually, we have:

Proposition 3.47
Let $\mathcal{A} = (A, \odot, \to, 1)$ $(\mathcal{A} = (A, \leq, \odot, \to, 1))$ be an X-algebra (X-structure) with (PR) (i.e. (Re), (Tr), (RP)=(PR) hold), such that the reduct $(A, \odot, 1)$ is an abelian monoid.
Then, the properties (M), (PEx), (PBB) hold (hence $\mathcal{A}$ is in fact an abelian regular-monoid).

Proof. Note firstly that (R), (RR) and (P-) also hold, by Lemma 3.35.

(M): $1 \to x = x \overset{(R)}{\Longleftrightarrow} \max\{y \mid y \odot 1 \leq x\} = x \overset{(P1-1)}{\Longleftrightarrow} \max\{y \mid y \leq x\} = x$, which is true, by (Re). Thus, (M) holds and hence $(A, \to, 1)$ $((A, \leq, \to, 1))$ is a regular algebra (structure). Consequently, $\mathcal{A}$ is in fact an abelian regular-monoid.

(PEx): $(x \odot y) \to z \overset{(R)}{=} \max\{u \mid u \odot (x \odot y) \leq z\} \overset{(Passoc)}{=} \max\{u \mid (u \odot x) \odot y \leq z\}$
$\overset{(RP)}{=} \max\{u \mid u \odot x \leq y \to z\} \overset{(R)}{=} x \to (y \to z)$.

(PBB’): $x \to y \overset{(R)}{=} \max\{u \mid u \odot x \leq y\}$, $y \to z \overset{(R)}{=} \max\{v \mid v \odot y \leq z\}$. Hence, we get that $(x \to y) \odot x \leq y$ and $(y \to z) \odot y \leq z$, by (RR).
Then, by (P-), $[(x \to y) \odot x] \odot (y \to z) \leq y \odot (y \to z) \leq z$.
Hence, by (Tr) and (Passoc), $[(y \to z) \odot (x \to y)] \odot x \leq z$.
Then, by (RP), $(y \to z) \odot (x \to y) \leq x \to z$, i.e. $(x \to y) \odot (y \to z) \leq x \to z$, by (Pcomm). □

Corollary 3.48 *Let $\mathcal{A} = (A, \odot, \to, 1)$ $(\mathcal{A} = (A, \leq, \odot, \to, 1))$ be an X-algebra (X-structure) with (PR) (i.e. (Re), (Tr) and (PR) hold), such that the reduct $(A, \odot, 1)$ is an abelian monoid.*
Then, the reduct $(A, \to, 1)$ $((A, \leq, \to, 1)$ is an abelian residoid (i.e. (M) and (BB) hold), verifying (Ex), (B), (D) (hence, in fact, $\mathcal{A}$ is an abelian regular-monoid).

Proof. By Proposition 3.47, the properties (M), (PEx) and (PBB) hold. Since (M) holds, then $\mathcal{A}$ is an abelian regular-monoid.

By (BP2), (PEx) $\Longrightarrow$ ((PBB) $\Leftrightarrow$ (BB)), hence (BB) holds. Thus, the reduct $(A, \to, 1)$ $((A, \leq, \to, 1))$ is an abelian residoid.

By (BP7), (PEx) + (Pcomm) $\Longrightarrow$ (Ex), hence (Ex) holds.

By (BP4), (Pcomm) $\Longrightarrow$ ((PB) $\Leftrightarrow$ (PBB)), hence (PB) holds; by (BP1), (PEx) $\Longrightarrow$ ((PB) $\Leftrightarrow$ (B)), hence (B) holds.

By Lemma 3.35, (RR) holds; by (BP5), (Pcomm) $\Longrightarrow$ ((PD) $\Leftrightarrow$ (RR)), hence (PD) holds too; finally, by (BP3), (PEx) $\Longrightarrow$ ((PD) $\Leftrightarrow$ (D)), hence (D) holds. □

• **The equivalence between some regular algebras with (RP) and some regular X-algebras with (PR)**

The announced equivalence is given in the next Theorem 3.49.

Theorem 3.49 *(See Theorem 3.38)*

(1) Let $\mathcal{A} = (A, \to, \odot, 1)$ be an algebra (or $\mathcal{A} = (A, \leq, \to, \odot, 1)$ be a structure) with (RP) (i.e. (Re), (Tr) and (RP) hold), verifying also the property (An) and such that the reduct $(A, \to, 1)$ ($(A, \leq, \to, 1)$) is an abelian residoid (i.e. (M) and (BB) hold). Define

$$\alpha(\mathcal{A}) \stackrel{df.}{=} (A, \odot, \to, 1) \quad (\alpha(\mathcal{A}) \stackrel{df.}{=} (A, \leq, \odot, \to, 1)).$$

Then, $\alpha(\mathcal{A})$ is an X-algebra (X-structure) with (PR) (i.e. (Re), (Tr), (PR)=(RP) hold) that is an abelian regular-monoid.

(1') Conversely, let $\mathcal{A} = (A, \odot, \to, 1)$ be an X-algebra (or $\mathcal{A} = (A, \leq, \odot, \to, 1)$) be an X-structure) with (PR) (i.e. (Re), (Tr) and (PR) hold) that is an abelian regular-monoid verifying the property (An). Define

$$\beta(\mathcal{A}) \stackrel{df.}{=} (A, \to, \odot, 1) \quad (\beta(\mathcal{A}) \stackrel{df.}{=} (A, \leq, \to, \odot, 1)).$$

Then, $\beta(\mathcal{A})$ is an algebra (structure) with (RP) (i.e. (Re), (Tr) and (RP) hold), such that the reduct $(A, \to, 1)$ ($(A, \leq, \to, 1)$) is an abelian residoid verifying (Ex), (B), (D).

(2) The above defined mappings are mutually inverse.

Proof.

(1): By Corollary 3.46.

(1'): By Corollary 3.48.

(2): Obviously. □

Note that $\leq$ in Theorem 3.49 is an order relation and that the algebras and the X-algebras involved are regular ones.

3.4.3 Regular-meet-semilattices from regular algebras with product

We then obtain the following two Galois dual Theorems 3.50 and 3.52:

Theorem 3.50 *Let $\mathcal{A} = (A, \to, \odot, 1)$ ($\mathcal{A} = (A, \leq, \to, \odot, 1)$) be an algebra (structure) with (PR) (i.e. (Re), (Tr) and (RP) hold), verifying also the properties (An), (Ex), (K) and (G).*

Define $\wedge \stackrel{df.}{=} \odot$.

Then, $(A, \wedge, \to, 1)$ ($(A, \wedge, \leq, \to, 1)$ is a regular-$\wedge$-semilattice with top element 1.

Proof. Firstly, by (A5), (Re) + (Ex) + (An) $\Longrightarrow$ (M), hence the reduct $(A, \to, 1)$ ($(A, \leq, \to, 1)$) is a regular algebra (structure).

Since (An), (Ex) hold, then, by Proposition 3.45, $(A, \odot, 1)$ is an abelian monoid, i.e. properties (Pcomm), (Passoc), (P1-1) hold. Then,

- by (BPR1'), (RP) + (Pcomm) + (K) $\Longrightarrow$ (Pleq);
- by Lemma 3.35, the property (P-) holds;
- by (BG2), (P-) + (G) + (Pcomm) + (Pleq) + (An) $\Longrightarrow$ (EqP); thus, by the above definition of $\wedge$, it follows that (EqW) holds.

Note that: (Wid)≡(G), (Wcomm)≡(Pcomm), (Wassoc)≡(Passoc), (W1-1)≡ (P1-1), hence the properties (Wid), (Wcomm), (Wassoc), (W1-1) hold too. □

Corollary 3.51 *Let $\mathcal{A} = (A, \to, \odot, 1)$ ($\mathcal{A} = (A, \leq, \to, \odot, 1)$) be an algebra (structure) with (RP) (i.e. (Re), (Tr) and (RP) hold), such that the reduct $(A, \to, 1)$ is an abelian residoid (i.e. (M), (BB) hold) and the properties (An), (L) and (G) hold.*

Define $\wedge \stackrel{df.}{=} \odot$.

Then, $(A, \to, \wedge, 1)$ ($(A, \leq, \to, \wedge, 1)$ is a regular-$\wedge$-semilattice with top element 1.

Proof. By (A21"), (M) + (BB) + (An) $\Longrightarrow$ (Ex) and by (A9"), (M) + (L) + (BB) $\Longrightarrow$ (K). Since (An), (Ex), (K), (G) hold, now apply Theorem 3.50. $\Box$

Theorem 3.52 *Let $\mathcal{A} = (A, \odot, \rightarrow, 1)$ ($\mathcal{A} = (A, \leq, \odot, \rightarrow, 1)$) be an X-algebra (X-structure) with (PR) (i.e. (Re), (Tr) and (PR) hold), such that the reduct $(A, \odot, 1)$ is an abelian monoid (i.e. (Pcomm), (Passoc), (P1-1) hold) and the properties (An), (K) and (G) hold.*
Define $\wedge \overset{df.}{=} \odot$.
Then, $(A, \wedge, \rightarrow, 1)$ ($(A, \wedge, \leq, \rightarrow, 1)$) is a regular-$\wedge$-semilattice with top element 1.

Proof. Firstly, by Theorem 3.47, the property (M) holds, hence the reduct $(A, \rightarrow, 1)$ ($(A, \leq, \rightarrow, 1)$) is a regular algebra (structure). Consequently, $\mathcal{A}$ is an abelian regular-monoid. Then,
- by Lemma 3.35, the property (P-) holds;
- by (BPR1'), (RP) + (Pcomm) + (K) $\Longrightarrow$ (Pleq);
- by (BG2), (P-) + (G) + (Pcomm) + (Pleq) + (An) $\Longrightarrow$ (EqP), hence (EqP) holds; by the above definition of $\wedge$, (EqW)$\equiv$ (EqP), hence (EqW) holds.

Since $\wedge \overset{df.}{=} \odot$, then note that: (Wid)$\equiv$ (G), (Wcomm)$\equiv$ (Pcomm), (Wassoc)$\equiv$ (Passoc), (W1-1)$\equiv$ (P1-1); hence, the properties (Wid), (Wcomm), (Wassoc), (W1-1) hold. $\Box$

3.5 Commutative regular algebras (structures) with product

In the commutative regular algebras (structures) (i.e. (M) holds) having product, we can define a new operation, the join $\vee$, by (dfV); then, by (BdfV1), (dfV) + (comm) $\Longrightarrow$ (Vcomm), hence (Vcomm) holds.

In certain conditions, we can obtain a regular-$\vee$-semilattice with top element (see Theorem 3.25).

Open problem 3.53 *It is an open problem to find conditions (if they exist) in which, by defining $\wedge \overset{df.}{=} \odot$, or otherwise, we can obtain a regular-$\wedge$-semilattice with top element from a commutative regular algebra with product.*

• Let us introduce now the following new properties:
(Pdis1) (P-distributivity-1) $(x \vee y) \odot z = (x \odot z) \vee (y \odot z)$;
(Pdis1-p) (Pdis1-partial-1) $[(x \odot z) \vee (y \odot z)] \rightarrow [(x \vee y) \odot z] = 1$,
(Pdis1-p') (Pdis1-partial-1') $(x \odot z) \vee (y \odot z) \leq (x \vee y) \odot z$;
(Pdis1-pp) (Pdis1-partial-2) $[(x \vee y) \odot z] \rightarrow [(x \odot z) \vee (y \odot z)] = 1$,
(Pdis1-pp') (Pdis1-partial-2') $(x \vee y) \odot z \leq (x \odot z) \vee (y \odot z)$;

(Pdis2) (P-distributivity-2) $(x \odot y) \vee z = (x \vee z) \odot (y \vee z)$;
(Pdis2-p) (Pdis2-partial-1) $[(x \odot y) \vee z] \rightarrow [(x \vee z) \odot (y \vee z)] = 1$,
(Pdis2-p') (Pdis2-partial-1') $(x \odot y) \vee z \leq (x \vee z) \odot (y \vee z)$;
(Pdis2-pp) (Pdis2-partial-2) $[(x \vee z) \odot (y \vee z)] \rightarrow [(x \odot y) \vee z] = 1$,
(Pdis2-pp') (Pdis2-partial-2') $(x \vee z) \odot (y \vee z) \leq (x \odot y) \vee z$;

(Pabs1) (P-absorption-1) $x \odot (x \vee y) = x$,
(Pabs2) (P-absorption-2) $x \vee (x \odot y) = x$.

Then we have:

Proposition 3.54 *(See the above similar Proposition 3.16)*

Let $\mathcal{A} = (A, \vee, \rightarrow, \odot, 1)$ be an algebra of type $(2, 2, 2, 0)$ (or, equivalently, $\mathcal{A} = (A, \leq, \vee, \rightarrow, \odot, 1)$ be a structure) or, Galois dually, let $\mathcal{A} = (A, \odot, \rightarrow, \vee, 1)$ be an algebra of type $(2, 2, 2, 0)$ (or $\mathcal{A} = (A, \leq, \odot, \rightarrow, \vee, 1)$ be a structure) such that the reduct $(A, \rightarrow, 1)$ ($(A, \leq, \rightarrow, 1)$) is an algebra (structure) verifying the properties (Re) and (Tr) (i.e. $\leq$ is a pre-order).
Then, the following hold:

(BVP1) (Pdis1-p) + (Pdis1-pp) + (An) $\Longrightarrow$ (Pdis1);

(BVP1') (Vgeq) + (V=) + (P-) $\Longrightarrow$ (Pdis1-p); (Pleq) + (P=) + (V- -) + (V=) $\Longrightarrow$ (Pdis1-p);
(BVP1") (RP) + (Vgeq) + (V=) $\Longrightarrow$ (Pdis1-pp);

(BVP2) (Pdis2-p) + (Pdis2-pp) + (An) $\Longrightarrow$ (Pdis2);
(BVP2') (Pleq) + (P=) + (V-) $\Longrightarrow$ (Pdis2-p); (Vgeq) + (V=) + (P- -) + (P=) $\Longrightarrow$ (Pdis2-p);
(BVP2") (Pdis1) + (Pleq) + (Pcomm) + (Vcomm) + (Vassoc) + (V-) + (V=) $\Longrightarrow$ (Pdis2-pp);

(BVP3) (Vgeq) + (EqP) $\Longrightarrow$ (Pabs1);
(BVP4) (Pleq) + (EqV) + (Vcomm) $\Longrightarrow$ (Pabs2).

Proof.
(BVP1): Obviously.
(BVP1'): First proof: $x, y \leq x \vee y$, by (Vgeq'); then, $x \odot z,\ y \odot z \leq (x \vee y) \odot z$, by (P-), hence $(x \odot z) \vee (y \odot z) \leq (x \vee y) \odot z$, by (V=); thus, (Pdis1-p) holds.
Second proof: On the one hand, we have $x \odot z \leq x$ and $y \odot z \leq y$, by (Pleq); then, $(x \odot z) \vee (y \odot z) \leq x \vee y$, by (V- -). On the other hand, we have $x \odot z \leq z$ and $y \odot z \leq z$, by (Pleq); then, $(x \odot z) \vee (y \odot z) \leq z$, by (V=). Consequently, $(x \odot z) \vee (y \odot z) \leq (x \vee y) \odot z$, by (P=); thus, (Pdis1-p) holds.
(BVP1"): Denote $Z \overset{notation}{=} (x \odot z) \vee (y \odot z)$; then $x \odot z, y \odot z \leq Z$, by (Vgeq'); then, $x \leq z \to Z$ and $y \leq z \to Z$, by (RP); then, $x \vee y \leq z \to Z$, by (V=), hence $(x \vee y) \odot z \leq Z$, by (RP), i.e. (Pdis1-pp) holds.
(BVP2): Obviously.
(BVP2'): First proof: $x \odot y \leq x,\ y$, by (Pleq); then, $(x \odot y) \vee z \leq x \vee z,\ y \vee z$, by (V-), hence $(x \odot y) \vee z \leq (x \vee z) \odot (y \vee z)$, by (P=); thus, (Pdis2-p) holds.
Second proof: On the one hand, we have: $x \leq x \vee z$ and $y \leq y \vee z$, by (Vgeq); then, $x \odot y \leq (x \vee z) \odot (y \vee z)$, by (P- -). On the other hand, we have $z \leq x \vee z$ and $z \leq y \vee z$, by (Vgeq); then, $z \leq (x \vee z) \odot (y \vee z)$, by (P=). Consequently, $x \odot y) \vee z \leq (x \vee z) \odot (y \vee z)$, by (V=); thus, (Pdis2-p) holds.
(BVP2"): Denote $Z \overset{notation}{=} (x \vee z) \odot (y \vee z)$; then
$Z \overset{(Pdis1)}{=} (x \odot (y \vee z)) \vee (z \odot (y \vee z))$
$\overset{(Pcomm)}{=} ((y \vee z) \odot x) \vee ((y \vee z) \odot z)$
$\overset{(Pdis1)}{=} ((y \odot x) \vee (z \odot x)) \vee ((y \odot z) \vee (z \odot z))$
$\overset{(Vassoc)}{=} (y \odot x) \vee [(z \odot x) \vee (y \odot z) \vee (z \odot z)]$
$\overset{(Vcomm)}{=} [(z \odot x) \vee (y \odot z) \vee (z \odot z)] \vee (y \odot x)$.
But $z \odot x \leq z$, $y \odot z \leq z$, $z \odot z \leq z$, by (Pleq);
hence, $(z \odot x) \vee (y \odot z) \vee (z \odot z) \leq z$, by (V=);
hence, $Z \leq z \vee (y \odot x) \overset{(Pcomm),(Vcomm)}{=} (x \odot y) \vee z$, by (V-); thus (Pdis2-pp) holds.
(BVP3): $x \overset{(Vgeq)}{\leq} x \vee y \overset{(EqP')}{\Leftrightarrow} x \odot (x \vee y) = x$; thus (Pabs1) holds.
(BVP4): $x \odot y \overset{(Pleq)}{\leq} x \overset{(EqV)}{\Leftrightarrow} (x \odot y) \vee x = x$, hence $x \vee (x \odot y) = x$, by (Vcomm); thus, (Pabs2) holds. □

Remark that the two Propositions 3.16 and 3.54 are quite identique, if $\odot = \wedge$, and are only similar, if $\odot \neq \wedge$ (see Proposition 3.79 for such a case).

Open problem 3.55 *It is an open problem to find conditions (if they exist) in which, by defining $\wedge \overset{df.}{=} \odot$, or otherwise, we can obtain a Dedekind regular-lattice (with last element 1) from a commutative regular algebra with product.*

Note that, for commutative (regular) BCK(P) algebras, there exists in the literature a positive answer to this open problem (see Theorem 3.79 in the sequel).

3.6 Positive implicative regular algebras (structures) with product

In this subsection, we study the regular algebras (structures) having product and being positive implicative.

We have obtained the following two Galois dual Theorems 3.56 and 3.57.

Theorem 3.56 *Let $\mathcal{A} = (A, \rightarrow, \odot, 1)$ ($\mathcal{A} = (A, \leq, \rightarrow, \odot, 1)$) be an algebra (structure) with (RP) (i.e. (Re), (Tr) and (RP)=(PR) hold), verifying also the properties (An), (M) and (pimpl).*
Define $\wedge \stackrel{df.}{=} \odot$.
Then, $(A, \wedge, \rightarrow, 1)$ ($(A, \wedge, \leq, \rightarrow, 1)$ is a regular-$\wedge$-semilattice with top element 1.

Proof. Note that since (M) holds, $(A, \rightarrow, 1)$ is a regular algebra.
By (B1), (pimpl) + (Re) $\Longrightarrow$ (L).
By (B4), (pimpl) + (Re) + (L) $\Longrightarrow$ (K).
By (B11), (pimpl) + (Re) + (M) + (An) $\Longrightarrow$ (Ex).
By (A0), (Re) $\Longrightarrow$ (S) and by (S), (pimpl) $\Longrightarrow$ (pimpl-1).
Now,
- by (BPR1), (RP) + (K) + (Re) + (L) $\Longrightarrow$ (Pleq);
- by (BPR2), (RP) + (Re) + (Ex) + (An) $\Longrightarrow$ (PEx);
- by (BG5), (pimpl-1) + (Pleq) + (PEx) + (Re) + (M) + (An) $\Longrightarrow$ (G).
Since (An), (Ex), (K), (G) hold, now apply Theorem 3.50. □

Theorem 3.57 *Let $\mathcal{A} = (A, \odot, \rightarrow, 1)$ ($\mathcal{A} = (A, \leq, \odot, \rightarrow, 1)$) be an X-algebra (X-structure) with (RP) (i.e. (Re), (TR) and (PR)=(RP) hold), such that the reduct $(A, \odot, 1)$ is an abelian monoid (i.e. (Pcomm), (Passoc), (P1-1) hold) and the properties (An) and (pimpl) (or (pi)) also hold.*
Define $\wedge \stackrel{df.}{=} \odot$.
Then, $(A, \wedge, \rightarrow, 1)$ is a regular-$\wedge$-semilattice with 1.

Proof. By Proposition 3.47, the properties (M), (PEx), (PBB) hold. Since (M) holds, it follows that $\mathcal{A}$ is a regular X-algebra.
By Corollary 3.48, the reduct $(A, \rightarrow, 1)$ is an abelian residoid (i.e. (M), (BB) hold) verifying (Ex), (B), (D). By (B35), (Re) + (M) + (Ex) + (B) + (An) $\Longrightarrow$ ((pimpl) $\Leftrightarrow$ (pi)).
Now, by (B1), (pimpl) + (Re) $\Longrightarrow$ (L); by (A8), (Re) + (L) + (Ex) $\Longrightarrow$ (K).
By (BPR1'), (RP) + (Pcomm) + (K) $\Longrightarrow$ (Pleq).
By (A0), (Re) $\Longrightarrow$ (S); (pimpl) + (S) $\Longrightarrow$ (pimpl-1).
By (BG5), (pimpl-1) + (Pleq) + (PEx) + (Re) + (M) + (An) $\Longrightarrow$ (G).
Since (An), (K), (G) hold, now apply Theorem 3.52. □

3.7 Implicative regular algebras (structures) with product

In this subsection, we shall study the regular algebras (structures) having product and being implicative.

We have obtained the following two Galois dual Theorems 3.58 and 3.59.

Theorem 3.58 *Let $\mathcal{A} = (A, \rightarrow, \odot, 1)$ ($\mathcal{A} = (A, \leq, \rightarrow, \odot, 1)$) be an algebra (structure) with (RP) (i.e. (Re), (Tr) and (RP) hold), such that the reduct $(A, \rightarrow, 1)$ is an abelian residoid (i.e. (M), (BB) hold) verifying (An) and (impl).*
Define $x \vee y \stackrel{(dfV)}{=} (x \rightarrow y) \rightarrow y$ and $\wedge \stackrel{df.}{=} \odot$.
Then, $(A, \vee, \rightarrow, \wedge, 1)$ ($(A, \leq, \vee, \rightarrow, \wedge, 1)$) is a distributive Dedekind regular-$\vee\wedge$-lattice with last element 1.

Proof. We must prove that (M); (EqW), (Wid), (Wcomm), (Wassoc), (W1-1); (EqV), (Vid), (Vcomm), (Vassoc), (V1-1); (Wabs1), (Wabs2); (Wdis1), (Wdis2) hold.

(M) holds; hence, $\mathcal{A}$ is a regular algebra. We must prove the rest. Indeed, we have:
- by (A21"), (M) + (BB) + (An) $\Longrightarrow$ (Ex);
- by (A10"), (Ex) + (BB) $\Longrightarrow$ (B);
- by (BIM3), (impl) + (Ex) + (B) + (An) $\Longrightarrow$ (comm);

- by (BIM5'), (impl) + (M) $\Longrightarrow$ (L);
- by (A8), (Re) + (L) + (Ex) $\Longrightarrow$ (K).

Then, by Theorem 3.25, since (comm), (Re), (M), (K), (Ex) hold, then $(A, \vee, \rightarrow, 1)$ $((A, \leq, \vee, \rightarrow, 1))$ is a regular-$\vee$-semilattice with 1, i.e. (M), (EqV), (Vid), (Vcomm), (Vassoc), (V1-1) hold. Then, by Theorem 3.6, (V-), (V- -), (Vgeq), (V=) also hold.

By (BPR1), (PR) + (K) + (Re) + (L) $\Longrightarrow$ (Pleq);
- by (BPR2), (PR) + (Re) + (Ex) + (An) $\Longrightarrow$ (PEx);
- by (A15'), (M) + (BB) $\Longrightarrow$ (**);
- by (BG6), (impl) + (Pleq) + (PEx) + (Re) + (M) + (**) + (An) $\Longrightarrow$ (G).

Since (M), (BB), (An), (L) and (G) hold, then, by Corollary 3.51, $(A, \rightarrow, \wedge, 1)$ $((A, \leq, \rightarrow, \wedge, 1))$ is a regular-$\wedge$-semilattice with top element 1, i.e. (M), (EqW) $\equiv$ (EqP), (Wid)$\equiv$(G), (Wcomm)$\equiv$(Pcomm), (Wassoc)$\equiv$(Passoc), (W1-1)$\equiv$(P1-1) hold. Then, by Theorem 3.12, (W-)$\equiv$(P-), (W- -)$\equiv$(P- -), (Wleq)$\equiv$(Pleq), (W=)$\equiv$ (P=) also hold.

Resuming, $(A, \vee, \rightarrow, \wedge, 1)$ $((A, \leq, \vee, \rightarrow, \wedge, 1))$ is both a regular-$\vee$-semilattice and a regular-$\wedge$-semilattice with last element 1.

Now, by (BVP3), (Vgeq) + (EqP) $\Longrightarrow$ (Pabs1) and by (BVP4), (Pleq) + (EqV) + (Vcomm) $\Longrightarrow$ (Pabs2); hence, (Wabs1) and (Wabs2) hold. Thus, $\mathcal{A}$ is a Dedekind regular-$\vee\wedge$-lattice with 1.

Finally, we prove the distributivity:
By (BVP1'), (Vgeq) + (V=) + (P-) $\Longrightarrow$ (Pdis1-p);
by (BVP1"), (RP) + (Vgeq) + (V=) $\Longrightarrow$ (Pdis1-pp);
then, by (BVP1), (Pdis1-p) + (Pdis1-pp) + (An) $\Longrightarrow$ (Pdis1); thus (Wdis1) holds.
By (BVP2'), (Pleq) + (P=) + (V-) $\Longrightarrow$ (Pdis2-p);
by (BVP2"), (Pdis1) + (Pleq) + (Pcomm) + (Vcomm) + (Vassoc) + (V-) + (V=) $\Longrightarrow$ (Pdis2-pp);
then, by (BVP2), (Pdis2-p) + (Pdis2-pp) + (An) $\Longrightarrow$ (Pdis2), i.e. (Wdis2) also holds. □

Theorem 3.59 *Let $\mathcal{A} = (A, \odot, \rightarrow, 1)$ ($\mathcal{A} = (A, \leq, \odot, \rightarrow, 1)$ be an X-algebra (X-structure) with (RP) (i.e. (Re), (Tr), (RP)=(PR) hold), such that the reduct $(A, \odot, 1)$ is an abelian monoid (i.e. (Pcomm), (Passoc), (P1-1) hold) and also the properties (K), (An) and (impl) hold.*
Define $\wedge \overset{df.}{=} \odot$ and $x \vee y \overset{(dfV)}{=} (x \rightarrow y) \rightarrow y$.
Then $\mathcal{A} = (A, \wedge, \rightarrow, \vee, 1)$ ($\mathcal{A} = (A, \leq, \wedge, \rightarrow, \vee, 1)$) is a distributive Dedekind regular-$\wedge\vee$-lattice with 1.

Proof. We must prove that (M); (EqW), (Wid), (Wcomm), (Wassoc), (W1-1); (EqV), (Vid), (Vcomm), (Vassoc), (V1-1); (Wabs1), (Wabs2); (Wdis1), (Wdis2) hold.

By Proposition 3.47, the properties (M), (PEx) hold. Since (M) holds, then $(A, \rightarrow, 1)$ is a regular algebra, hence $\mathcal{A}$ is an abelian regular-monoid.

By Corollary 3.48, the properties (BB), (Ex), (B) hold.
By (BPR1'), (PR) + (Pcomm) + (K) $\Longrightarrow$ (Pleq); by (A15'), (M) + (BB) $\Longrightarrow$ (**); by (BG6), (impl) + (Pleq) + (PEx) + (Re) + (M) + (**) + (An) $\Longrightarrow$ (G).
Since $(A, \odot, 1)$ is an abelian monoid and the properties (K), (An) and (G) hold, then, by Theorem 3.52, $(A, \wedge, \rightarrow, 1)$ is a regular-$\wedge$-semilattice with 1, i.e. (M), (EqW) $\equiv$ (EqP), (Wid)$\equiv$(G), (Wcomm)$\equiv$(Pcomm), (Wassoc)$\equiv$(Passoc), (W1-1)$\equiv$(P1-1) hold. Then, by Theorem 3.12, (W-)$\equiv$(P-), (Wleq)$\equiv$(Pleq), (W=)$\equiv$ (P=) also hold.

By (BIM3), (impl) + (Ex) + (B) + (An) $\Longrightarrow$ (comm).
Since (comm), (Re), (M), (K), (Ex) hold, then, by Theorem 3.25, $(A, \rightarrow, \vee, 1)$ is a regular-$\vee$-semilattice with 1, i.e. (M), (EqV), (Vid), (Vcomm), (Vassoc), (V1-1) hold. Then, by Theorem 3.6, (V-), (Vgeq), (V=) also hold.

The rest of the proof is the same as the proof of Theorem 3.58. □

3.8 Hierarchies of regular algebras

The regular algebras (structures) and their particular cases: the positive implicative ones, the commutative ones, the implicative ones, on the one hand, and the regular algebras (structures) with product, the commutative regular algebras (structures) with product, the positive implicative regular algebras (structures) with product and the implicative regular algebras (structures) with product, on the other hand, are connected by the hierarchies presented in Figure 3 (where the word "regular" is ommited by lack of space).

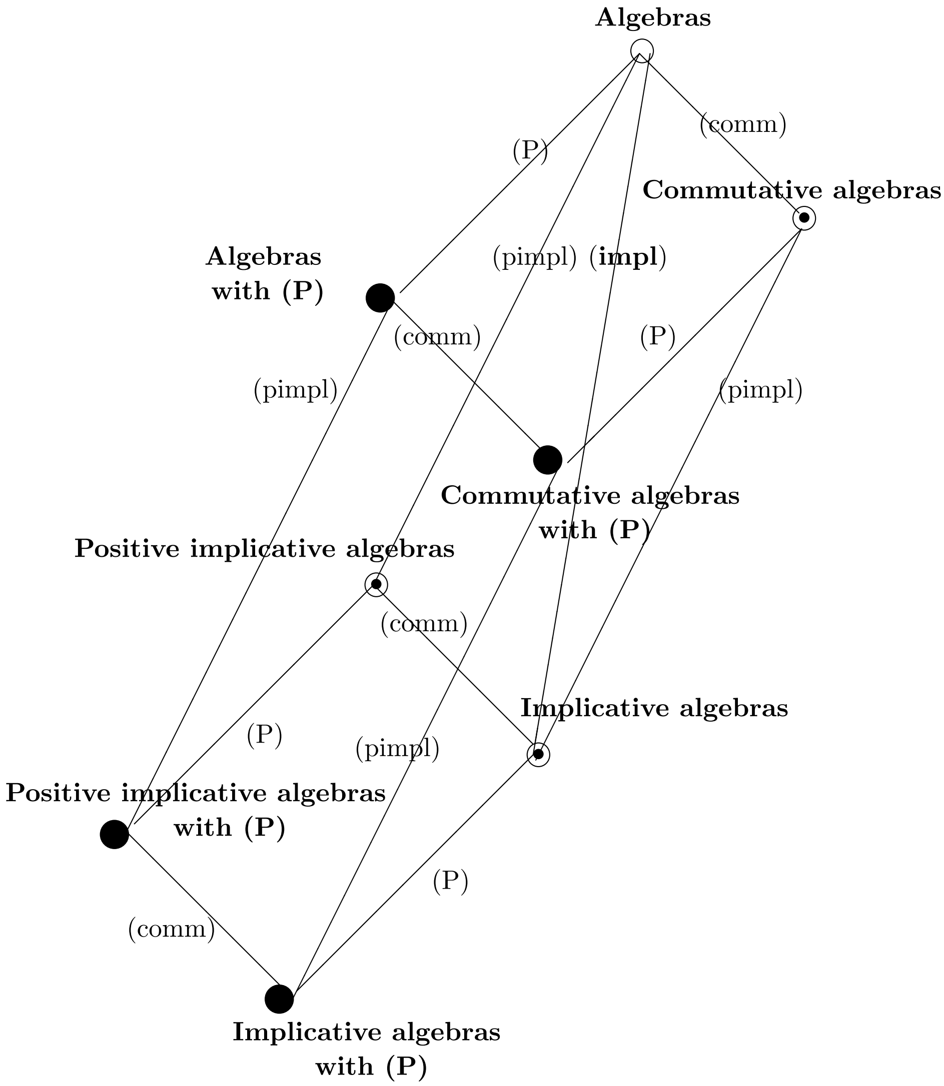


Figure 3: Hierarchies of regular algebras

3.9 Other examples of regular algebras

3.9.1 Positive implicative, commutative, implicative BCK algebras. Hilbert algebras

We recall here definitions and results concerning the positive implicative, commutative, implicative BCK algebras and the Hilbert algebras, results obtained by applying the above theory of regular algebras.

- **Positive implicative, commutative, implicative BCK algebras**

Definition 3.60 An algebra $(A, \rightarrow, 1)$ (a structure $(A, \leq, \rightarrow, 1)$) is a *BCK algebra*, if it verifies the axioms [17] (BB), (D), (Re), (L), (An) or, equivalently [15], (B), (C), (K), (An).

Recall that BCK algebras are regular algebras and verify all the properties in List A.
See more about BCK algebras in the books [23], [14].

Definitions 3.61 [18] Let $\mathcal{A} = (A, \rightarrow, 1)$ be a BCK algebra. We say that $\mathcal{A}$ is
- *positive implicative*, if property (pimpl) is satisfied;
- *commutative*, if property (comm) is satisfied;
- *implicative*, if property (impl) is satisfied.

Theorem 3.62 *([18], Theorem 8)*
A BCK algebra is positive implicative if and only if the property (pi) holds (or, in a BCK algebra the properties (pimpl) and (pi) are equivalent).

Proof. By (B35), (Re) + (M) + (Ex) + (B) + (An) $\Longrightarrow$ ((pimpl) $\Leftrightarrow$ (pi)). □

Following ([18], Remark 1), commutative BCK algebras were introduced by S. Tanaka [27]; moreover, H. Yutani proved that [28]: *the class of commutative BCK algebras is a variety* and found the following short equivalent system of axioms [29]: (comm), (M), (Re), (Ex).

Remark 3.63 By (B12), (comm) + (M) $\Longrightarrow$ (An); consequently, a commutative BCK algebra can be defined equivalently as an algebra $(A, \rightarrow, 1)$ verifying the axioms: (comm), (M), (B), (C), (K).

Theorem 3.64 *([18], Theorem 9)*
In a commutative BCK algebra, the properties (pi) and (impl) are equivalent.

Proof. Indeed, by (BIM2'), (comm) + (L) + (K) + (M) $\Longrightarrow$ ((pi) $\Leftrightarrow$ (impl)). □

Theorem 3.65 *([18], Theorem 10)*
Any implicative BCK algebra is commutative and positive implicative.

Proof. Indeed, by (BIM3), (impl) + (Ex) + (B) + (An) $\Longrightarrow$ (comm). By (BIM1), (impl) $\Longrightarrow$ (pi), and by Theorem 3.62, (pi) and (pimpl) are equivalent. □

We add now the following results.

Theorem 3.66 *Any commutative and positive implicative BCK algebra is implicative.*

Proof. By Theorem 3.62, a BCK algebra is positive implicative if and only if the property (pi) holds. Then, by (BIM2), (comm) + (pi) + (L) + (K) + (M) $\Longrightarrow$ (impl). □

Corollary 3.67 *In a BCK algebra we have:*

$$(impl) \;\Leftrightarrow\; ((comm) \;+\; (pi)\; (\Leftrightarrow\; (pimpl)))$$

Proof. By Theorems 3.65 and 3.66. □

- **Hilbert algebras**

Definition 3.68 [5] A *Hilbert algebra* is an algebra $(A, \rightarrow, 1)$ of type $(2, 0)$ satisfying: for all $x, y, z \in A$,
(K) $x \rightarrow (y \rightarrow x) = 1$,
(pimpl-1) $[x \rightarrow (y \rightarrow z)] \rightarrow [(x \rightarrow y) \rightarrow (x \rightarrow z)] = 1$,
(An) $x \rightarrow y = 1 = y \rightarrow x \;\Longrightarrow\; x = y$.

Theorem 3.69 *([14], Remarks 2.1.32 (1))*
Hilbert algebras are categorically equivalent to positive implicative BCK algebras.

Proof.

$\Longrightarrow$: Let $\mathcal{A} = (A, \to, 1)$ be a Hilbert algebra, i.e. properties (K), (pimpl-1), (An) hold. We must prove that $\mathcal{A}$ is a positive implicative BCK algebra, i.e. (B), (C), (K), (An), (pimpl) hold. It remains to prove that (B), (C), (pimpl) hold.

Indeed, by (A2), (K) + (An) $\Longrightarrow$ (N); by (A7), (N) + (K) $\Longrightarrow$ (L); by (B19), (pimpl-1) + (K) + (N) $\Longrightarrow$ (Re); by (B27), (pimpl-1) + (N) $\Longrightarrow$ (\$); by (B21), (pimpl-1) + (L) + (N) $\Longrightarrow$ (Tr); by (B22), (pimpl-1) + (K) + (Tr) $\Longrightarrow$ (B); thus (B) holds. Then, by (B28), (\$) + (K) + (Tr) $\Longrightarrow$ (#); by (A31), (Re) + (#) $\Longrightarrow$ (D); by (A25), (D) + (K) + (N) + (An) $\Longrightarrow$ (M), hence $\mathcal{A}$ is a regular algebra; by (A29'), (B) + (#) $\Longrightarrow$ (BB); by (A21), (BB) + (D) + (N) + (An) $\Longrightarrow$ (Ex); by (A3'), (Re) + (Ex) $\Longrightarrow$ (C); thus (C) holds.

Finally, by (A15), (N) + (BB) $\Longrightarrow$ (**); by (B13), (Re) + (L) + (Ex) + (**) $\Longrightarrow$ (pimpl-2); by (B0), (pimpl-1) + (pimpl-2) + (An) $\Longrightarrow$ (pimpl); thus (pimpl) holds too.

$\Longleftarrow$: Let $\mathcal{A} = (A, \to, 1)$ be a positive implicative BCK algebra, i.e. (B), (C), (K), (An), (pimpl) hold. We must prove that $\mathcal{A}$ is a Hilbert algebra, i.e. (K), (An), (pimpl-1) hold. It remains to prove (pimpl-1).

Indeed, by (A23), (C) + (K) + (An) $\Longrightarrow$ (Re) and by (A0), (Re) implies (S); then (pimpl) + (S) $\Longrightarrow$ (pimpl-1). $\Box$

Note that by (the proof of) above theorem, Hilbert algebras are regular algebras.

Corollary 3.70 *Any commutative Hilbert algebra is implicative.*

Proof. By above Theorem 3.69 and by Theorem 3.66. $\Box$

Denote by **RM**, **RML**, **BCI**, **BCK**, **Hilbert** the classes of RM algebras, of RML algebras, of BCI algebras, of BCK algebras and of Hilbert algebras, respectively. We have then [15] the hierarchy from Figure 4.

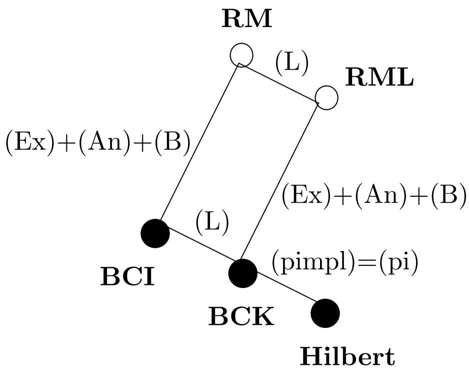


Figure 4: Hierarchy of RM algebras

- **Regular-join-semilattices from commutative BCK algebras**

Theorem 3.71 *Let* $\mathcal{A} = (A, \to, 1)$ *(*$\mathcal{A} = (A, \leq, \to, 1)$*) be a commutative BCK algebra.*
Define $x \vee y \stackrel{dfV}{=} (x \to y) \to y$.
Then, $(A, \vee, \to, 1)$ *is a regular-*$\vee$*-semilattice with last element* 1.

Proof. By Theorem 3.25. $\Box$

3.9.2 BCK algebras (structures) with product

We rediscover here old results by applying the above theory of regular algebras.

Definitions 3.72 (See Definitions 1.5.8, 1.5.10 from [14])

(i) A *BCK algebra (structure) with (P)* - or a *BCK(P) algebra (structure)* for short - is an algebra $\mathcal{A} = (A, \rightarrow, \odot, 1)$ (a structure $\mathcal{A} = (A, \leq, \rightarrow, \odot, 1)$) with (P) (by Definition 3.27 (i)) such that the reduct $(A, \rightarrow, 1)$ ($(A, \leq, \rightarrow, 1)$) is a BCK algebra (structure).

(ii) A *BCK algebra (structure) with (RP)* - or a *BCK(RP) algebra (structure)* for short - is an algebra $\mathcal{A} = (A, \rightarrow, \odot, 1)$ (a structure $\mathcal{A} = (A, \leq, \rightarrow, \odot, 1)$) with (RP) (by Definition 3.27 (ii)) such that the reduct $(A, \rightarrow, 1)$ ($(A, \leq, \rightarrow, 1)$) is a BCK algebra (structure).

Galois dually, we have the following definitions.

Definitions 3.73 (See Definitions 1.5.19, 1.5.20 from [14])

(i) An *X-BCK algebra (structure) with (R)* - or an *X-BCK(R) algebra (structure)* for short - is an X-algebra $\mathcal{A} = (A, \odot, \rightarrow, 1)$ (an X-structure $\mathcal{A} = (A, \leq, \odot, \rightarrow, 1)$) with (R) (by Definition 3.30 (i)), verifying (An), (L), (P-) and such that the reduct $(A, \odot, 1)$ is an abelian monoid.

(ii) An *X-BCK algebra (structure) with (RP)* - or an *X-BCK(PR) algebra (structure)* for short - is an X-algebra $\mathcal{A} = (A, \odot, \rightarrow, 1)$ (an X-structure $\mathcal{A} = (A, \leq, \odot, \rightarrow, 1)$) with (PR) (by Definition 3.30 (ii)), verifying (An), (L) and such that the reduct $(A, \odot, 1)$ is an abelian monoid.

Note that an X-BCK(R) algebra is in fact what usually is called a *pocrim* (*partially-ordered commutative residuated integral monoid*).

We reobtain the following three Theorems 3.74, 3.75, 3.76 from [14]:

Theorem 3.74 *(See Theorem 1.5.13 from [14])*

The BCK(P) algebras (structures) coincide with the BCK(RP) algebras (structures).

Proof. Any BCK algebra (structure) verifies (*). Then apply Theorem 3.36. □

Galois dually we have:

Theorem 3.75 *(See Theorem 1.5.23 from [14])*

The X-BCK(R) algebras (structures) (i.e. the pocrims) coincide with the X-BCK(PR) algebras (structures).

Proof. Any X-BCK(R) algebra (structure) verifies (P-). Then apply Theorem 3.37. □

The next theorem expresses the Galois duality between BCK(RP) and X-BCK(PR) algebras (structures).

Theorem 3.76 *(See Theorem 1.5.29 from [14])*

(1) Let $\mathcal{A} = (A, \rightarrow, \odot, 1)$ be a BCK(RP) algebra (or $\mathcal{A} = (A, \leq, \rightarrow, \odot, 1)$ be a BCK(RP) structure). Define

$$\alpha(\mathcal{A}) \stackrel{df.}{=} (A, \odot, \rightarrow, 1) \quad (\alpha(\mathcal{A}) \stackrel{df.}{=} (A, \leq, \odot, \rightarrow, 1)).$$

Then, $\alpha(\mathcal{A})$ is an X-BCK(PR) algebra (structure).

(1') Conversely, let $\mathcal{A} = (A, \odot, \rightarrow, 1)$ be an X-BCK(PR) algebra (or $\mathcal{A} = (A, \leq, \odot, \rightarrow, 1)$) be an X-BCK(PR) structure). Define

$$\beta(\mathcal{A}) \stackrel{df.}{=} (A, \rightarrow, \odot, 1) \quad (\beta(\mathcal{A}) \stackrel{df.}{=} (A, \leq, \rightarrow, \odot, 1)).$$

Then, $\beta(\mathcal{A})$ is a BCK(RP) algebra (structure).

(2) The above defined mappings are mutually inverse.

Proof. By Theorem 3.49. □

We denote by **BCK(P)**, **BCK(RP)**, **X-BCK(R)**, **X-BCK(PR)** the classes of BCK(P) algebras (structures), BCK(RP) algebras (structures), X-BCK(R) algebras (structures) and X-BCK(PR) algebras (structures, respectively).

By above Theorems 3.74, 3.75 and 3.76, we obtain in next Figure 5 the following known basic equivalences for BCK algebras with product (see the Figure 1.2 from [14]) as a particular case of the equivalences from the above Figure 2:

$$\mathbf{BCK(P)} \Longleftrightarrow \mathbf{BCK(RP)} \overset{\alpha}{\underset{\beta}{\rightleftarrows}} \mathbf{X\text{-}BCK(PR)} \Longleftrightarrow \mathbf{X\text{-}BCK(R)} \text{ (pocrims)}$$

Figure 5: The basic equivalences for BCK algebras with product

Finally, we have the following two Galois dual Propositions 3.77 and 3.78:

Proposition 3.77
Let $\mathcal{A} = (A, \to, \odot, 1)$ ($\mathcal{A} = (A, \leq, \to, \odot, 1)$) be a BCK(RP) algebra (structure). Then, $\mathcal{A}$ is an abelian regular-monoid verifying (PEx).

Proof. By Proposition 3.45. □

Proposition 3.78
Let $\mathcal{A} = (A, \odot, \to, 1)$ ($\mathcal{A} = (A, \leq, \odot, \to, 1)$) be an X-BCK(PR) algebra (structure). Then, the properties (M), (PEx), (PBB) hold (hence $\mathcal{A}$ is in fact an abelian regular-monoid).

Proof. By Proposition 3.47. □

3.9.3 Commutative BCK algebras with product

Recall the following old result which answers to the open problem 3.55 in the case of commutative BCK(P) algebras.

Theorem 3.79 *(See Corollary 2.1.23 from [14]) Let $\mathcal{A} = (A, \to, \odot, 1)$ be a commutative BCK(P) algebra (or $\mathcal{A} = (A, \leq, \to, \odot, 1)$ be a commutative BCK(P) structure).
Then, we have:*
*(1) $(A, \leq)$ is a lattice (i.e. $\mathcal{A}$ is a Dedekind regular-$\vee\wedge$ lattice), where for any $x, y \in A$,
$x \vee y = (x \to y) \to y$, i.e. $\vee$ is given by (dfV);
$x \wedge y = ([x \to (x \odot y)] \vee [y \to (x \odot y)]) \to (x \odot y)$.*
(2) $(x \to y) \vee (y \to x) = 1$, for any $x, y \in A$.

3.9.4 Positive implicative BCK algebras with product

We obtained the following two Galois dual Theorems 3.80 and 3.81.

Theorem 3.80 *Let $\mathcal{A} = (A, \to, \odot, 1)$ ($\mathcal{A} = (A, \leq, \to, \odot, 1)$) be a BCK(P) algebra (structure) verifying (pimpl) (hence a positive implicative BCK(P) algebra = Hilbert (P) algebra = Hertz algebra).
Define $\wedge \overset{df.}{=} \odot$.
Then, $(A, \to, \wedge, 1)$ ($(A, \leq, \to, \wedge, 1)$) is a regular-$\wedge$-semilattice with top element 1.*

Proof. By Theorems 3.74 and 3.56. □

Theorem 3.81 *Let $\mathcal{A} = (A, \odot, \to, 1)$ ($\mathcal{A} = (A, \leq, \odot, \to, 1)$) be an X-BCK(PR) algebra (structure) verifying (pimpl) (or (pi)).
Define $\wedge \overset{df.}{=} \odot$.
Then, $(A, \wedge, \to, 1)$ ($(A, \leq, \wedge, \to, 1)$) is a regular-$\wedge$-semilattice with 1.*

Proof. By Theorem 3.57 or by Theorems 3.76, 3.74, 3.80. □

3.9.5 Implicative BCK algebras with product

We have the following two Galois dual Theorems 3.82 and 3.83:

Theorem 3.82 *Let $\mathcal{A} = (A, \rightarrow, \odot, 1)$ ($\mathcal{A} = (A, \leq, \rightarrow, \odot, 1)$) be a BCK(P) algebra (structure) verifying (impl).*
Define $x \vee y \stackrel{(dfV)}{=} (x \rightarrow y) \rightarrow y$ and $\wedge \stackrel{df.}{=} \odot$.
Then, $(A, \vee, \rightarrow, \wedge, 1)$ ($(A, \leq, \vee, \rightarrow, \wedge, 1)$) is a distributive Dedekind regular-$\vee\wedge$-lattice with last element 1.

Proof. By Theorems 3.74 and 3.58. □

Theorem 3.83 *Let $\mathcal{A} = (A, \odot, \rightarrow, 1)$ ($\mathcal{A} = (A, \leq, \odot, \rightarrow, 1)$ be an X-BCK(PR) algebra (structure) verifying (impl).*
Define $\wedge \stackrel{df.}{=} \odot$ and $x \vee y \stackrel{(dfV)}{=} (x \rightarrow y) \rightarrow y$.
Then $\mathcal{A} = (A, \wedge, \rightarrow, \vee, 1)$ ($\mathcal{A} = (A, \leq, \wedge, \rightarrow, \vee, 1)$) is a distributive Dedekind regular-$\wedge\vee$-lattice with 1.

Proof. By Theorem 3.59 or by Theorems 3.76, 3.74, 3.82. □

Remark that the Dedekind regular-$\vee\wedge$-lattices and the Dedekind regular-$\wedge\vee$-lattices are Galois dual.

3.9.6 Hierarchies between different classes of BCK algebras

Following the hierarchies from Figure 3, the BCK algebras (structures) and their particular cases: the positive implicative ones (i.e. the Hilbert algebras (structures)), the commutative ones, the implicative ones, on the one hand, and the BCK(P) algebras (structures), the commutative BCK(P) algebras (structures), the positive implicative BCK(P) algebras (structures) (i.e. the Hilbert(P) algebras (structures) or the Hertz algebras) and the implicative BCK(P) algebras (structures), on the other hand, are connected by the hierarchies presented in next Figure 6.

3.10 The duality: left-algebras versus right-algebras

The (regular) algebras studied or recalled until now in Part I and Part II have essentially an associated order, or pre-order, relation (denoted by $\leq$). The presence of the relation $\leq$ implies the presence of the duality. Thus, each (regular) algebra has a dual algebra, where the dual order, or pre-order, relation (denoted by $\geq$) acts: for each x, y,

$$x \geq y \quad if\ and\ only\ if \quad y \leq x.$$

Recall that we have given names to the algebras of the pair of dual algebras: *left-algebras* and *right-algebras* (see [**?**], [**?**], [**?**], ([14], Definition 1.4.14)).

Hence, the "left" and the "right" notions are dual; they are connected with the left-continuity of a t-norm and with the right-continuity of a t-conorm, respectively. We can also say that they are connected with the "negative (left)" cone and with the "positive (right) cone", respectively, of a commutative partially-ordered, or lattice-ordered, group.

Hence, the logic of *truth*, where the truth is denoted by 1, the implication is denoted by $\rightarrow$ (or $\rightarrow^L$) and the tautologies are those formulas that are always true, can be called a *left-logic*, while, dually, the logic of *false*, where the false is denoted by 0, the implication is denoted by $\rightarrow^R$ and the tautologies are those formulas that are always false, can be called a *right-logic*.

Thus, in the "left-world", starting with the implication $\rightarrow$ (that can be denoted by ra^L) (residuum) and $1 \in A^L$, we can have an associated product $\odot$, verifying the proprieties (P), (RP); we can also have the associated operations $\vee^L$ and $\wedge^L$ and we can obtain a Dedekind regular-$\vee^L\wedge^L$-lattice, where $\wedge^L = \odot$. Galois dually, starting with the product $\odot$, we can have an associated implication $\rightarrow^L$ (residuum), verifying (R), (PR)=(RP); we can also have the associated operations $\wedge^L$ and $\vee^L$ and we can obtain a Dedekind regular-$\wedge^L\vee^L$-lattice, where $\wedge^L = \odot$.

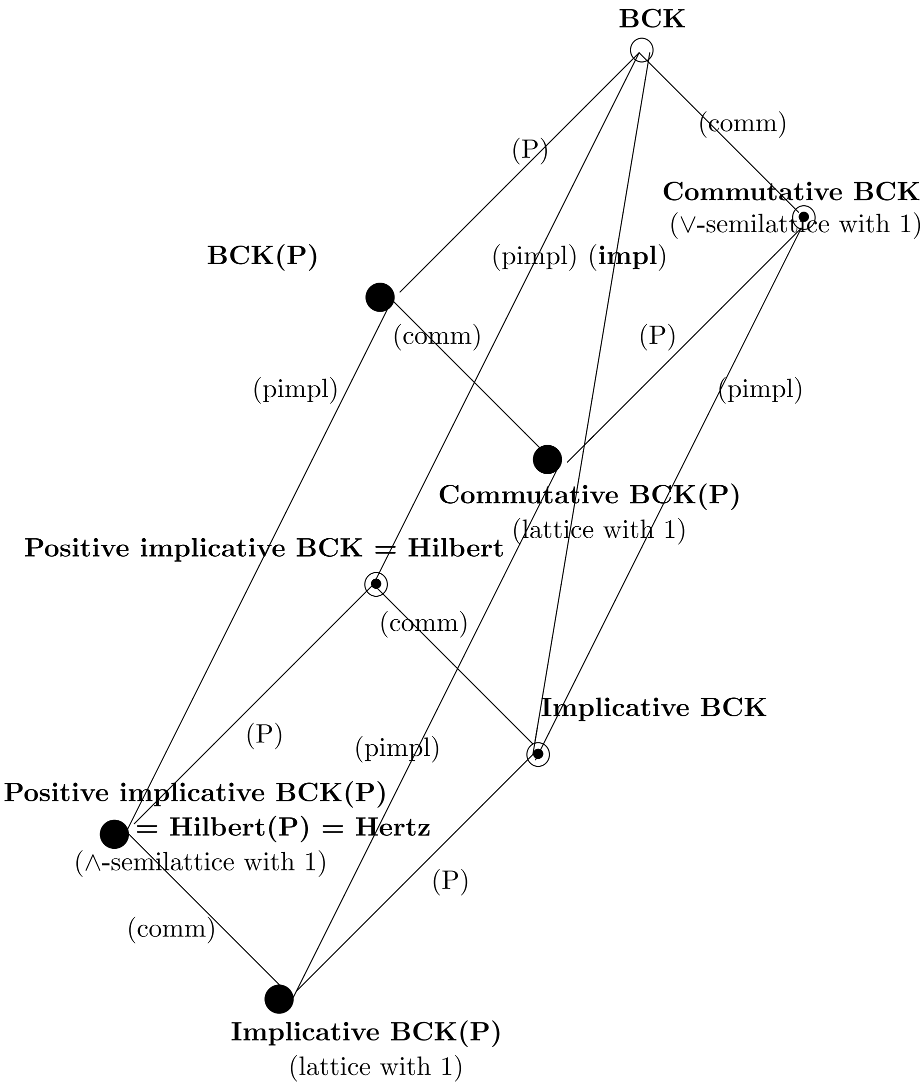


Figure 6: Hierarchies of classes of BCK algebras

We can have a BCK algebra $(A^L, \to^L, 1)$ (recalled "left-BCK algebra"), a left-BCK(P) algebra, a left-BCK(RP) algebra and all the rest. In a (regular) left-algebra, we can define the $\to$-deductive systems and the filters.

Dually, in the "right-world", starting with the implication $\to^R$ (coresiduum) and $0 \in A^R$, we can have an associated sum $\oplus$, verifying the proprieties (S), (corRS), where: for all x, y, z,
(S) there exists $\max\{z \mid x \geq y \to^R z\} = x \oplus y$;
(coRS) $x \oplus y \geq z \Longleftrightarrow x \geq y \to^R z$;
we can also have the associated operations $\wedge^R$ and $\vee^R$ and we can obtain a Dedekind regular-$\wedge^R\vee^R$-lattice, where $\vee^L = \oplus$.
Galois dually, starting with the sum $\oplus$, we can have an associated implication $\to^R$ (coresiduum), verifying (coR), (ScoR)=(corRS), where: for all x, y, z,
(coR) $\min\{x \mid x \oplus y \geq z\} = y \to^R z$;
(ScoR) $x \geq y \to^R z \Longleftrightarrow x \oplus y \geq z$;
we can also have the associated operations $\vee^R$ and $\wedge^R$ and we can obtain a Dedekind regular-$\vee^R\wedge^R$-lattice, where $\vee^L = \oplus$.
We can have a "right-BCK algebra" $(A^R, \to^R, 0)$, a right-BCK(S) algebra, a right-BCK(coRS) algebra and all the rest. In a (regular) right-algebra, we can define the $\to^R$-deductive systems and the ideals.

A resuming table of "left" - "right-" notions mentioned in this subsection is the following:

"Left-" notion	"Right-" notion
$\leq$	$\geq$
the implication (residuum) $\to$ (or $\to^L$)	the implication (coresiduum) $\to^R$
(last element) 1	(first element) 0
the product $\odot$	the sum $\oplus$
condition (P) for all $x, y \in A^L$, there exists $\min\{z \mid x \leq y \to z\} = x \odot y$	condition (S) for all $x, y \in A^R$, there exists $\max\{z \mid x \geq y \to^R z\} = x \oplus y$
condition (RP) for all $x, y, z \in A^L$, $x \odot y \leq z \Longleftrightarrow x \leq y \to z$	condition (coRS) for all $x, y, z \in A^R$, $x \oplus y \geq z \Longleftrightarrow x \geq y \to^R z$
condition (R) for all $y, z \in A^L$, there exists $\max\{x \mid x \odot y \leq z\} = y \to z$	condition (coR) for all $y, z \in A^R$, there exists $\min\{x \mid x \oplus y \geq z\} = y \to^R z$
condition (PR) for all $x, y, z \in A^L$, $x \leq y \to z \;\Leftrightarrow\; x \odot y \leq z$	condition (ScoR) for all $x, y, z \in A^R$, $x \geq y \to^R z \Longleftrightarrow x \oplus y \geq z$
$\wedge^L$ $(= \odot)$, $\vee^L$	$\vee^R$ $(= \oplus)$, $\wedge^R$
Dedekind (regular-) $\vee^L\wedge^L$-lattice	Dedekind (regular-) $\wedge^R\vee^R$-lattice
or Dedekind (regular-) $\wedge^L\vee^L$-lattice	or Dedekind (regular-) $\vee^R\wedge^R$-lattice
left-BCK algebra $(A^L, \to, 1)$	right-BCK algebra $(A^R, \to^R, 0)$
left-BCK(P), left-BCK(RP)	right-BCK(S), right-BCK(coRS)
$(\to)$-deductive system filter	$(\to^R)$-deductive system ideal

Remarks 3.84 Since, in the "left-world", we can have either a Dedekind (regular-) $\vee^L\wedge^L$-lattice or, Galois dually, a Dedekind (regular-) $\wedge^L\vee^L$-lattice, and since the operation $\wedge^L$ is connected to the product $\odot$ (we can have $\wedge^L = \odot$), hence to the implication (residuum) $\to$ and 1, then, by convention, we shall write the two operations $\wedge^L$, $\vee^L$ of a "left" Dedekind (regular-) lattice in this order: $\wedge^L$ $\vee^L$ (i.e. $\wedge^L$ on the first place); hence, we shall say, for example: "let $(A^L, \wedge^L, \vee^L, 1)$ be a (left-) Dedekind lattice with 1".

Since, dually, in the "right-world", we can have either a Dedekind (regular-) $\wedge^R\vee^R$-lattice or, Galois dually, a Dedekind (regular-) $\vee^R\wedge^R$-lattice, and since the operation $\vee^R$ is connected to the sum $\oplus$ (we can have $\vee^R = \oplus$), hence to the implication (coresiduum) $\rightarrow^R$ and 0, then, by convention, we shall write the two operations $\wedge^R$, $\vee^R$ of a "right" Dedekind (regular-) lattice in this order: $\vee^R$ $\wedge^R$ (i.e. $\vee^R$ on the first place); hence, we shall say, for example: "let $(A^R, \vee^R, \wedge^R, 0)$ be a (right-) Dedekind lattice with 0".

Note that, usually, both $\wedge^L$, $\vee^L$ and $\wedge^R$, $\vee^R$ are denoted the same, by $\wedge$, $\vee$, and that A^L can coincide with A^R. In this case, we shall say, for example: "let $(A, \wedge, \vee, 1)$ be a (left-) Dedekind lattice with 1" and, dually, "let $(A, \vee, \wedge, 0)$ be a (right-) Dedekind lattice with 0".

These remarks remain valid for quasi-algebras.

The theory of regular algebras (structures) will be continued with the bounded ones and with the study of the negation, in Part III.

4 Introduction to a theory of quasi-algebras - Part II

4.1 The List qB of particular quasi-properties

We present here the resuming List qB of particular quasi-properties then will appear in the sequel. As for the List qA of basic quasi-properties, the List qB has three parts: Part I contains the proper quasi-properties corresponding to the properties from the Part I of List B; Part II contains the properties from List B that are not changed in quasi-algebras; Part III contains special quasi-properties (quasi-properties that are identiqually satisfied if (M) holds).

List qB, Part 1

(q-impl) (quasi-implicative) $(x \rightarrow y) \rightarrow x = 1 \rightarrow x$;
(q-pi) $(1 \rightarrow x) \rightarrow (x \rightarrow y) = 1 \rightarrow (x \rightarrow y)$;

(qM($\vee$)) $1 \rightarrow (x \vee y) = x \vee y$;
(qM($\odot$)) $1 \rightarrow (x \odot y) = x \odot y$;
(qM($\wedge$)) $1 \rightarrow (x \wedge y) = x \wedge y$;

(qVid) (q-idempotency of $\vee$) $x \vee x = 1 \rightarrow x$;
(qEqV) $x \rightarrow y = 1 \Leftrightarrow x \vee y = 1 \rightarrow y$,
(qEqV') $x \leq y \Leftrightarrow x \vee y = 1 \rightarrow y$,
(qdfrelV) $x \leq y \overset{df.}{\Leftrightarrow} x \vee y = 1 \rightarrow y$;
(qV=) $x \rightarrow z = 1,\ y \rightarrow z = 1 \implies (x \vee y) \rightarrow (1 \rightarrow z) = 1$,
(qV=') $x \leq z,\ y \leq z \implies x \vee y \leq 1 \rightarrow z$;

(qG) (quasi-Gödel) $x \odot x = 1 \rightarrow x$;
(qP1-1) $x \odot 1 = 1 \odot x = 1 \rightarrow x$: (qP-1) $x \odot 1 = 1 \rightarrow x$, (qP1-) $1 \odot x = 1 \rightarrow x$;
(qEqP) $x \rightarrow y = 1 \Leftrightarrow x \odot y = 1 \rightarrow x$,
(qEqP') $x \leq y \Leftrightarrow x \odot y = 1 \rightarrow x$,
(qdfrelP) $x \leq y \overset{df.}{\Leftrightarrow} x \odot y = 1 \rightarrow x$;
(qP=) $z \rightarrow x = 1,\ z \rightarrow y = 1 \implies (1 \rightarrow z) \rightarrow (x \odot y) = 1$,
(qP=') $z \leq x,\ z \leq y \implies 1 \rightarrow z \leq x \odot y$;

(qWid) (q-idempotency of $\wedge$) $x \wedge x = 1 \rightarrow x$;
(qW1-1) $x \wedge 1 = 1 \wedge x = 1 \rightarrow x$: (qW-1) $x \wedge 1 = 1 \rightarrow x$, (qW1-) $1 \wedge x = 1 \rightarrow x$;
(qEqW) $x \leq y \Leftrightarrow x \wedge y = 1 \rightarrow x$,
(qEqW') $x \rightarrow y = 1 \Leftrightarrow x \wedge y = 1 \rightarrow x$,

(qdfrelW) $x \leq y \overset{df.}{\Leftrightarrow} x \wedge y = 1 \to x$;
(qW=) $z \to x = 1,\ z \to y = 1 \implies (1 \to z) \to (x \wedge y) = 1$,
(qW=') $z \leq x,\ z \leq y \implies 1 \to z \leq x \wedge y$;

(qPabs1) (qP-absorption-1) $x \odot (x \vee y) = 1 \to x$,
(qPabs2) (qP-absorption-2) $x \vee (x \odot y) = 1 \to x$;

(qWabs1) (qW-absorption-1) $x \wedge (x \vee y) = 1 \to x$,
(qWabs2) (qW-absorption-2) $x \vee (x \wedge y) = 1 \to x$;

(qEqVW) $x \vee y = 1 \to y \Leftrightarrow x \wedge y = 1 \to x$.

List qB, Part 2

(pimpl), (pimpl-1), (pimpl-2),($) ;
(comm), (comm-1);
(Vee), (dfV), (Vcomm), (Vassoc), (V1-1): (V-1) and (V1-); (V-), (V- -), (Vgeq), (VV), (VVV);
(P), (dfP), (PP), (R), (RR), (RP)=(PR), (Pcomm), (Passoc), (P-), (P- -), (Pleq);
(PEx), (PB), (PBB), (PD);
(Wcomm), (Wassoc), (W-), (W- -), (Wleq);
(Pdis1), (Pdis1-p), (Pdis1-pp), (Pdis2), (Pdis2-p), (Pdis2-pp);
(Wdis1), (Wdis1-p), (Wdis1-pp), (Wdis2), (Wdis2-p), (Wdis2-pp).

List qB, Part 3

(qVI) $x \vee y = (1 \to x) \vee (1 \to y)$,
(qVI1) $x \vee y = (1 \to x) \vee y$,
(qVI2) $x \vee y = x \vee (1 \to y)$;

(qPI) $x \odot y = (1 \to x) \odot (1 \to y)$;
(qPI1) $x \odot y = (1 \to x) \odot y$;
(qPI2) $x \odot y = x \odot (1 \to y)$;

(qWI) $x \wedge y = (1 \to x) \wedge (1 \to y)$,
(qWI1) $x \wedge y = (1 \to x) \wedge y$,
(qWI2) $x \wedge y = x \wedge (1 \to y)$.

4.2 Quasi-ordered algebras. Dedekind quasi-join-semilattices. Dedekind quasi-meet-semilattices. Dedekind quasi-lattices

4.2.1 Quasi-ordered algebras

Recall from [16] that quasi-ordered algebras are in fact proper quasi-algebras that are quasi-ordered, i.e. the quasi-properties (qM), (11-1), (Re), (qAn), (Tr) hold.

4.2.2 Dedekind quasi-join-semilattices

Let us introduce the following new definitions.

Definitions 4.1 (See Definitions 3.2 in the regular case)

Let $\mathcal{A} = (A, \vee, \rightarrow, 1)$ be an algebra of type $(2, 2, 0)$ (or $\mathcal{A} = (A, \vee, \leq, \rightarrow, 1)$ be a structure) such that:
- the reduct $(A, \rightarrow, 1)$ ($(A, \leq, \rightarrow, 1)$) is a quasi-algebra (quasi-structure) (i.e. properties (qM) and (11-1) hold) and
- the operations $\rightarrow$, 1 (the relation $\leq$) and $\vee$ are connected by (qM($\vee$)) and (qEqV) ((qEqV'), respectively), where for all $x, y \in A$:
(qM($\vee$)) $\quad 1 \rightarrow (x \vee y) = x \vee y$;
(qEqV) $\quad x \rightarrow y = 1 \Leftrightarrow x \vee y = 1 \rightarrow y$,
(qEqV') $\quad x \leq y \Leftrightarrow x \vee y = 1 \rightarrow y$.
In these conditions:

(1) We say that $\mathcal{A}$ is a *Dedekind quasi-$\vee$-semilattice* (*Dedekind quasi-join-semilattice*), if $\vee$ verifies the properties (qVid), (Vcomm), (Vassoc), where: for all $x \in A$,
(qVid) $x \vee x = 1 \rightarrow x$.

(1') We say that $\mathcal{A}$ is a *Dedekind quasi-$\vee$-semilattice with last (top) element* 1, if $\vee$ verifies the properties (qVid), (Vcomm), (Vassoc), (V1-1).

Note that if (M) holds, then any Dedekind quasi-join-semilattice is a Dedekind regular-join-semilattice.

• Let us introduce the new quasi-properties:
(qdfrelV) $x \leq y \overset{df.}{\Leftrightarrow} x \vee y = 1 \rightarrow y$;

(qV=) $x \rightarrow z = 1,\ y \rightarrow z = 1 \Longrightarrow (x \vee y) \rightarrow (1 \rightarrow z) = 1$,
(qV=') $x \leq z,\ y \leq z \Longrightarrow x \vee y \leq 1 \rightarrow z$,
and the special quasi-properties:
(qVI) $x \vee y = (1 \rightarrow x) \vee (1 \rightarrow y)$,
(qVI1) $x \vee y = (1 \rightarrow x) \vee y$,
(qVI2) $x \vee y = x \vee (1 \rightarrow y)$.

Remarks 4.2 (See Remarks 3.4 in the regular case)
We have the following connections:
(i) (EqrelR) $\Longrightarrow$ ((qEqV) $\Leftrightarrow$ (qEqV')); (qEqV) + (qEqV') $\Longrightarrow$ (EqrelR);
(ii) (qEqV) $\Longrightarrow$ ((dfrelR) $\Leftrightarrow$ (qdfrelV)); (dfrelR) + (qdfrelV) $\Longrightarrow$ (qEqV).

We present some other connections in the next two Propositions 4.3 and 4.4.

Proposition 4.3 *Let $(A, \vee, \rightarrow, 1)$ or $(A, \rightarrow, \vee, 1)$ be an algebra of type $(2, 2, 0)$ (or, equivalently, by (EqrelR) and (dfrelR), let $(A, \leq, \vee, \rightarrow, 1)$ or $(A, \leq, \rightarrow, \vee, 1)$ be a structure). Then, we have (with an independent numbering):*
(qBBV1) (Vcomm) + (qVI1) $\Longrightarrow$ (qVI2),
(qBBV1') (Vcomm) + (qVI2) $\Longrightarrow$ (qV11),
(qBBV1") (Vcomm) $\Longrightarrow$ ((qVI1) $\Leftrightarrow$ (qVI2));
(qBBV2) (Vcomm) + (qVI1) $\Longrightarrow$ (qVI),
(qBBV2') (Vcomm) + (qVI2) $\Longrightarrow$ (qVI);
(qBBV3) (qVid) + (Vcomm) + (Vassoc) + (qM($\vee$)) $\Longrightarrow$ (qVI);
(qBBV4) (qVI) + (qM) $\Longrightarrow$ (qVI1), (qVI2).

Proof.
(qBBV1): $x \vee (1 \rightarrow y) \overset{(Vcomm)}{=} (1 \rightarrow y) \vee x \overset{(qVI1)}{=} y \vee x \overset{(Vcomm)}{=} x \vee y$; thus, (qVI2) holds.
(qBBV1'): $(1 \rightarrow x) \vee y \overset{(Vcomm)}{=} y \vee (1 \rightarrow x) \overset{(qVI2)}{=} y \vee x \overset{(Vcomm)}{=} x \vee y$; thus, (qVI1) holds.
(qBBV1"): By (qBBV1) and (qBBV1').
(qBBV2): $(1 \rightarrow x) \vee (1 \rightarrow y) \overset{(qVI1)}{=} x \vee (1 \rightarrow y) \overset{(Vcomm)}{=} (1 \rightarrow y) \vee x \overset{(qVI1)}{=} y \vee x \overset{(Vcomm)}{=} x \vee y$; thus, (qVI) holds.
(qBBV2'): Obviously.

(qBBV3): $(1 \to x) \vee (1 \to y) \overset{(qVid)}{=} (x \vee x) \vee (y \vee y) \overset{(Vassoc)}{=} x \vee (x \vee y) \vee y \overset{(Vcomm)}{=} x \vee (y \vee (x \vee y)) \overset{(Vassoc)}{=} (x \vee y) \vee (x \vee y) \overset{(qVid)}{=} 1 \to (x \vee y) \overset{(qM(\vee))}{=} x \vee y$; thus, (qVI) holds.

(qBBV4): $(1 \to x) \vee y \overset{(qVI)}{=} (1 \to (1 \to x)) \vee (1 \to y) \overset{(qM)}{=} (1 \to x) \vee (1 \to y) \overset{(qVI)}{=} x \vee y$; thus, (qVI1) holds.

$x \vee (1 \to y) \overset{(qVI)}{=} (1 \to x) \vee (1 \to (1 \to y)) \overset{(qM)}{=} (1 \to x) \vee (1 \to y) \overset{(qVI)}{=} x \vee y$; thus, (qVI2) holds. □

Proposition 4.4 *(See Proposition 3.5 in the regular case)*

Let $(A, \vee, \to, 1)$ be an algebra of type $(2, 2, 0)$ (or, equivalently, by (EqrelR) and (dfrelR), let $(A, \vee, \leq, \to, 1)$ be a structure).
Then, we have (following the numbering from Proposition 3.5):

(qBV1) (qEqV) $\Longrightarrow$ ((qVid) $\Leftrightarrow$ (Re)),
(qBV1') (qEqV) $\Longrightarrow$ ((V-1) + (11-1) $\Leftrightarrow$ (L));
(qBV2) (qEqV) + (Vcomm) $\Longrightarrow$ (qAn);
(qBV3) (qEqV) + (Vassoc) + (qVI) $\Longrightarrow$ (Tr);
(qBV4) (qEqV) + (Vcomm) + (Vassoc) + (qVid) + (qM($\vee$)) + (qVI1) $\Longrightarrow$ (Vgeq),
(qBV4') (qEqV) + (Vassoc) + (qVid) + (qM($\vee$)) + (Vgeq) $\Longrightarrow$ (qVI1),
(qBV4") (qEqV) + (Vcomm) + (Vassoc) + (qVid) + (qM($\vee$)) $\Longrightarrow$ ((qVI1) $\Leftrightarrow$ (Vgeq));
(qBV5) (qEqV) + (Vcomm) + (Vassoc) + (qVid) + (qM($\vee$)) + (qVI) $\Longrightarrow$ (V-);
(qBV7) (qVid) + (V- -) $\Longrightarrow$ (qV=).

Proof.

(qBV1): $x \vee x \overset{(qVid)}{=} 1 \to x \overset{(qEqV)}{\Leftrightarrow} x \to x \overset{(Re)}{=} 1$.

(qBV1'): $x \vee 1 \overset{(V-1)}{=} 1 \overset{(11-1)}{=} 1 \to 1 \overset{(qEqV)}{\Leftrightarrow} x \to 1 \overset{(L)}{=} 1$.

(qBV2): If $x \leq y$ and $y \leq x$, i.e. $x \vee y = 1 \to y$ and $y \vee x = 1 \to x$, by (qEqV'), then $1 \to x = y \vee x \overset{(Vcomm)}{=} x \vee y = 1 \to y$; thus (qAn) holds.

(qBV3): If $x \leq y$ and $y \leq z$, i.e. $x \vee y = 1 \to y$ and $y \vee z = 1 \to z$, by (qEqV), then:

$$\begin{aligned}
x \vee z &\overset{(qVI)}{=} (1 \to x) \vee (1 \to z) = (1 \to x) \vee (y \vee z) \\
&\overset{(qVI)}{=} (1 \to x) \vee [(1 \to y) \vee (1 \to z)] \\
&\overset{(Vassoc)}{=} [(1 \to x) \vee (1 \to y)] \vee (1 \to z) \\
&\overset{(qVI)}{=} (x \vee y) \vee (1 \to z) = (1 \to y) \vee (1 \to z) \\
&\overset{(qVI)}{=} y \vee z = 1 \to z;
\end{aligned}$$

hence, by (qEqV), $x \leq z$; thus, (Tr) holds.

(qBV4): $x \vee (x \vee y) \overset{(Vassoc)}{=} (x \vee x) \vee y \overset{(qVid)}{=} (1 \to x) \vee y \overset{(qVI1)}{=} x \vee y \overset{(qM(\vee))}{=} 1 \to (x \vee y)$, i.e. $x \leq x \vee y$, by (qEqV).

$y \vee (x \vee y) \overset{(Vcomm)}{=} y \vee (y \vee x) \overset{(Vassoc)}{=} (y \vee y) \vee x \overset{(qVid)}{=} (1 \to y) \vee x \overset{(qVI1)}{=} y \vee x \overset{(Vcomm)}{=} x \vee y$, i.e. $y \leq x \vee y$, by (qEqV). Thus, (Vgeq) holds.

(qBV4'): $(1 \to x) \vee y \overset{(qVid)}{=} (x \vee x) \vee y \overset{(Vassoc)}{=} x \vee (x \vee y) = 1 \to (x \vee y) \overset{(qM(\vee))}{=} x \vee y$, since $x \overset{(Vgeq)}{\leq} x \vee y \overset{(qEqV)}{\Leftrightarrow} x \vee (x \vee y) = 1 \to (x \vee y$.

(qBV4"): By (qBV4) and (qBV4').

(qBV5): If $x \leq y$, i.e. $x \vee y = 1 \to y$, by (qEqV), then $(x \vee z) \vee (y \vee z) \overset{(Vcomm)}{=} (x \vee z) \vee (z \vee y) \overset{(Vassoc)}{=} x \vee (z \vee z) \vee y \overset{(qVid)}{=} x \vee (1 \to z) \vee y \overset{(Vcomm)}{=} (x \vee y) \vee (1 \to z) = (1 \to y) \vee (1 \to z) = y \vee z \overset{(qM(\vee))}{=} 1 \to (y \vee z)$; hence, by (qEqV), $x \vee z \leq y \vee z$; thus, (V-) holds.

(qBV7): Let $x \leq z$ and $y \leq z$; then, by (V- -), $x \vee y \leq z \vee z \overset{(qVid)}{=} 1 \to z$; thus (qV=) holds. □

Then we have:

Theorem 4.5 *(See Theorem 3.6 in the regular case)*

Let $\mathcal{A} = (A, \vee, \to, 1)$ ($\mathcal{A} = (A, \vee, \leq, \to, 1)$) be a Dedekind quasi-$\vee$-semilattice with last (top) element 1 (i.e. properties (qM), (11-1), (qM($\vee$)), (qEqV), (qVid), (Vcomm), (Vassoc), (V1-1) hold).
Then, the following properties hold: (Re), (L), (Tr), (qAn), (V-), (V- -), (qV=), (Vgeq).

Proof. Firstly, by (qBBV3), (qVid) + (Vcomm) + (Vassoc) + (qM($\vee$)) $\Longrightarrow$ (qVI); then, by (qBBV4), (qVI) + (qM) $\Longrightarrow$ (qVI1).

(Re): By (qBV1), (qEqV) $\Longrightarrow$ ((qVid) $\Leftrightarrow$ (Re)); thus, (Re) holds.

(L): By (qBV1'), (qEqV) $\Longrightarrow$ ((V-1) + (11-1) $\Leftrightarrow$ (L)); thus, (L) holds.

(Tr): By (qBV3), (qEqV) + (Vassoc) + (qVI) $\Longrightarrow$ (Tr); thus, (Tr) holds.

(qAn): By (qBV2), (qEqV) + (Vcomm) $\Longrightarrow$ (qAn); thus, (qAn) holds.

(V-): By (qBV5), (qEqV) + (Vcomm) + (Vassoc) + (qVid) + (qM($\vee$)) + (qVI) $\Longrightarrow$ (V-); thus, (V-) holds.

(V- -): By (BV6), (V-) + (Vcomm) + (Tr) $\Longrightarrow$ (V- -); thus (V- -) holds.

(qV=): By (qBV7), (V- -) + (qVid) $\Longrightarrow$ (qV=); thus, (qV=) holds.

(Vgeq): By (qBV4), (qEqV) + (Vcomm) + (Vassoc) + (qVid) + (qM($\vee$)) + (qVI1) $\Longrightarrow$ (Vgeq); thus, (Vgeq) holds. □

Examples 4.6 Starting from the poset with 1 $\mathcal{A} = (A = \{a, b, 1\}, \leq, 1)$ from Examples 3.7, consider the following quasi-ordered sets with 1 : $\mathcal{A}_1 = (A_1 = \{a, b, c, 1\}, \leq, 1)$, $\mathcal{A}_2 = (A_2 = \{a, b, c, d, 1\}, \leq, 1)$, where $c \parallel a$ and $d \parallel 1$, represented by the quasi-Hasse diagrams from Figure 7.

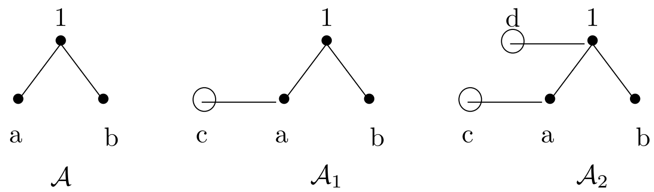


Figure 7: The Hasse diagram of $\mathcal{A}$ and the quasi-Hasse diagrams of $\mathcal{A}_1$, $\mathcal{A}_2$

Since $c \parallel a$ and $d \parallel 1$, it follows that $1 \to_{g_1} c = 1 \to a = a$, $1 \to_{g_2} c = 1 \to a = a$ and $1 \to_{g_2} d = 1 \to 1 = 1$, where the tables of the corresponding general implications $\to_g$, $\to_{g_1}$, $\to_{g_2}$ are the following (with $r_{12}, r_{21} \in \{a, b\}$):

$\mathcal{A}$

$\to_g$	a	b	1
a	1	r_{12}	1
b	r_{21}	1	1
1	a	b	1

$\mathcal{A}_1$

$\to_{g_1}$	**a**	**b**	c	**1**
a	**1**	r_{12}	1	**1**
b	r_{21}	**1**	r_{21}	**1**
c	1	r_{12}	1	1
1	**a**	**b**	a	**1**

$\mathcal{A}_2$

$\to_{g_2}$	**a**	**b**	c	d	**1**
a	**1**	r_{12}	1	1	**1**
b	r_{21}	**1**	r_{21}	1	**1**
c	1	r_{12}	1	1	1
d	a	b	a	1	1
1	**a**	**b**	a	1	**1**

It follows that we have the following corresponding tables of $\vee$, $\vee_1$ and $\vee_2$:

$\mathcal{A}$

$\vee$	a	b	1
a	a	1	1
b	1	b	1
1	1	1	1

$\mathcal{A}_1$

$\vee_1$	**a**	**b**	c	**1**
a	**a**	**1**	a	**1**
b	**1**	**b**	1	**1**
c	a	1	a	1
1	**1**	**1**	1	**1**

$\mathcal{A}_2$

$\vee_2$	**a**	**b**	c	d	**1**
a	**a**	**1**	a	1	**1**
b	**1**	**b**	1	1	**1**
c	a	1	a	1	1
d	1	1	1	1	1
1	**1**	**1**	1	1	**1**

For exemples, $b \vee_2 d = (1 \to_{g_2} b) \vee (1 \to_{g_2} d) = b \vee 1 = 1$ and $d \vee_2 d = 1 \to_{g_2} d = 1 \to 1 = 1$.

We have seen in Exemples 3.7 that there are four Dedekind regular-join-semilattices with top element 1 that can be deduced from the general (or generic) Dedekind regular-join-semilattice with 1 $(A, \vee, \to_g, 1)$.

It follows that there are four corresponding Dedekind quasi-join-semilattices with top element 1 that can be deduced from the general (or generic) Dedekind quasi-join-semilattice $(A_1, \vee_1, \to_{g_1}, 1)$ and that there are four corresponding Dedekind quasi-join-semilattices with 1 that can be deduced from the general Dedekind quasi-join-semilattice $(A_2, \vee_2, \rightarrow_{g_2}, 1)$.

For examples, for $(r_{12}, r_{21}) = (b, a)$, we obtain the particular implication $\rightarrow_3$ and we have seen in Examples 3.7 that $(A, \rightarrow_3, 1)$ is an implicative BCK algebra and $(A, \vee, \rightarrow_3, 1)$ is a Dedekind regular-join-semilattice with 1. It follows that both $(A_1, \rightarrow_3, 1)$ and $(A_2, \rightarrow_3, 1)$ are quasi-BCK algebras and that both $(A_1, \vee_1, \rightarrow_3, 1)$ and $(A_2, \vee_2, \rightarrow_3, 1)$ are Dedekind quasi-join-semilattices with 1.

Note that $R(A_1) = R(A_2) = A$ and $\mathcal{R}((A_1, \vee_1, \rightarrow_{g_1}, 1)) = (A, \vee, \rightarrow_g, 1)$ and $\mathcal{R}((A_2, \vee_2, \rightarrow_{g_2}, 1)) = (A, \vee, \rightarrow_g, 1)$.

4.2.3 Dedekind quasi-meet-semilattices

The results from this subsubsection are dual to those from the precedent subsubsection.

Definitions 4.7 (See Definitions 3.8 in the regular case)

Let $\mathcal{A} = (A, \wedge, \rightarrow, 1)$ be an algebra (or $\mathcal{A} = (A, \wedge, \leq, \rightarrow, 1)$ be a structure) such that:
- the reduct $(A, \rightarrow, 1)$ ($(A, \leq, \rightarrow, 1)$) is a quasi-algebra (structure) (i.e. properties (qM) and (11-1) hold) and
- the operations $\rightarrow$, 1 (the quasi-relation $\leq$) and $\wedge$ are connected by (qM($\wedge$)) and (qEqW) ((qEqW'), respectively), where, for all $x, y \in A$:

(qM($\wedge$)) $1 \rightarrow (x \wedge y) = x \wedge y$;
(qEqW) $\quad x \rightarrow y = 1 \;\Leftrightarrow\; x \wedge y = 1 \rightarrow x$,
(qEqW') $\quad x \leq y \;\Leftrightarrow\; x \wedge y = 1 \rightarrow x$.

In these conditions:

(1) We say that $\mathcal{A}$ is a *Dedekind quasi-$\wedge$-semilattice* (or *Dedekind quasi-meet-semilattice*) if $\wedge$ verifies the properties (qWid), (Wcomm), (Wassoc), where: for all $x \in A$,
(qWid) $x \wedge x = 1 \rightarrow x$.

(1') We say that $\mathcal{A}$ is a *Dedekind quasi-$\wedge$-semilattice with last (top) element* 1 if $\wedge$ verifies the properties (qWid), (Wcomm), (Wassoc), (qW1-1), where: for all $x \in A$,
(qW1-1) $x \wedge 1 = 1 \wedge x = 1 \rightarrow x$: (qW-1) $x \wedge 1 = 1 \rightarrow x$, (qW1-) $1 \wedge x = 1 \rightarrow x$.

Note that if (M) holds, then any Dedekind quasi-meet-semilattice is a Dedekind regular-meet-semilattice.

• Let us introduce the new quasi-properties:
(qdfrelW) $x \leq y \overset{df.}{\Leftrightarrow} x \wedge y = 1 \rightarrow x$;

(qW=) $z \rightarrow x = 1,\; z \rightarrow y = 1 \implies (1 \rightarrow z) \rightarrow (x \wedge y) = 1$,
(qW=') $z \leq x,\; z \leq y \implies 1 \rightarrow z \leq x \wedge y$,

and the following special quasi-properties:
(qWI) $x \wedge y = (1 \rightarrow x) \wedge (1 \rightarrow y)$,
(qWI1) $x \wedge y = (1 \rightarrow x) \wedge y$,
(qWI2) $x \wedge y = x \wedge (1 \rightarrow y)$.

Remarks 4.8 (See Remarks 3.10 in the regular case)
We have the following connections:

(i) (EqrelR) $\Longrightarrow$ ((qEqW) $\Leftrightarrow$ (qEqW')); (qEqW) + (qEqW') $\Longrightarrow$ (EqrelR);
(ii) (qEqW) $\Longrightarrow$ ((dfrelR) $\Leftrightarrow$ (qdfrelW)); (dfrelR) + (qdfrelW) $\Longrightarrow$ (qEqW).

Other connections are presented in the next three Propositions 4.9, 4.10 and 4.11.

Proposition 4.9 *(See the dual Proposition 4.3)*

Let $(A, \wedge, \rightarrow, 1)$ be an algebra of type $(2, 2, 0)$ (or, equivalently, by (EqrelR) and (dfrelR), let $(A, \wedge, \leq, \rightarrow, 1)$ be a structure). Then, we have (with an independent numbering):
(qBBW1) (Wcomm) + (qWI1) $\Longrightarrow$ (qWI2),
(qBBW1') (Wcomm) + (qWI2) $\Longrightarrow$ (qWI1),
(qBBW1") (Wcomm) $\Longrightarrow$ ((qWI1) $\Leftrightarrow$ (qWI2));

(qBBW2) (Wcomm) + (qWI1) $\Longrightarrow$ (qWI),
(qBBW2') (Wcomm) + (qWI2) $\Longrightarrow$ (qWI);
(qBBW3) (qWid) + (Wcomm) + (Wassoc) + (qM($\wedge$)) $\Longrightarrow$ (qWI);
(qBBW4) (qWI) + (qM) $\Longrightarrow$ (qWI1), (qWI2).

Proof.

(qBBW1): $x \wedge (1 \to y) \stackrel{(Wcomm)}{=} (1 \to y) \wedge x \stackrel{(qWI1)}{=} y \wedge x \stackrel{(Wcomm)}{=} x \wedge y$; thus, (qWI2) holds.

(qBBW1'): $(1 \to x) \wedge y \stackrel{(Wcomm)}{=} y \wedge (1 \to x) \stackrel{(qWI2)}{=} y \wedge x \stackrel{(Wcomm)}{=} x \wedge y$; thus, (qWI1) holds.

(qBBW1"): By (qBBW1) and (qBBW1').

(qBBW2): $(1 \to x) \wedge (1 \to y) \stackrel{(qWI1)}{=} x \wedge (1 \to y) \stackrel{(Wcomm)}{=} (1 \to y) \wedge x \stackrel{(qWI1)}{=} y \wedge x \stackrel{(Wcomm)}{=} x \wedge y$; thus, (qWI) holds.

(qBBW2'): Obviously.

(qBBW3): $(1 \to x) \wedge (1 \to y) \stackrel{(qWid)}{=} (x \wedge x) \wedge (y \wedge y) \stackrel{(Wassoc)}{=} x \wedge (x \wedge y) \wedge y \stackrel{(Wcomm)}{=} x \wedge (y \wedge (x \wedge y)) \stackrel{(Wassoc)}{=} (x \wedge y) \wedge (x \wedge y) \stackrel{(qWid)}{=} 1 \to (x \wedge y) \stackrel{(qM(\wedge))}{=} x \wedge y$; thus, (qWI) holds.

(qBBW4): $(1 \to x) \wedge y \stackrel{(qWI)}{=} (1 \to (1 \to x)) \wedge (1 \to y) \stackrel{(qM)}{=} (1 \to x) \wedge (1 \to y) \stackrel{(qWI)}{=} x \wedge y$; thus, (qWI1) holds.

$x \wedge (1 \to y) \stackrel{(qWI)}{=} (1 \to x) \wedge (1 \to (1 \to y)) \stackrel{(qM)}{=} (1 \to x) \wedge (1 \to y) \stackrel{(qWI)}{=} x \wedge y$; thus, (qWI2) holds. □

Proposition 4.10 *Let $(A, \wedge, \to, 1)$ be an algebra of type $(2, 2, 0)$ (or, equivalently, by (EqrelR) and (dfrelR), let $(A, \wedge, \leq, \to, 1)$ be a structure). Then, we have (with an independent numbering):*
(qBBWW1) (K) + (qR1) + (qAn) + (qM($\wedge$)) + (W- -) $\Longrightarrow$ (qWI);
(qBBWW1') (K) + (qR1) + (qAn) + (qM($\wedge$)) + (W-) $\Longrightarrow$ (qWI1);
(qBBWW1") (K) + (qR1) + (qAn) + (qM($\wedge$)) + (W- -) + (Re) $\Longrightarrow$ (qWI2).

Proof.

(qBBWW1): $x \stackrel{(K')}{\leq} 1 \to x$ and $y \stackrel{(K')}{\leq} 1 \to y$, hence, by (W- -), $x \wedge y \leq (1 \to x) \wedge (1 \to y)$. On the other hand, $1 \to x \stackrel{(qR1)}{\leq} x$ and $1 \to y \stackrel{(qR1)}{\leq} y$, hence, by (W- -), $(1 \to x) \wedge (1 \to y) \leq x \wedge y$. Now, by (qAn) and (qM($\wedge$)), $x \wedge y = (1 \to x) \wedge (1 \to y)$, i.e. (qWI) holds.

(qBBWW1'): $x \stackrel{(K')}{\leq} 1 \to x$, hence, by (W-), $x \wedge y \leq (1 \to x) \wedge y$. On the other hand, $1 \to x \stackrel{(qR1)}{\leq} x$, hence, by (W-), $(1 \to x) \wedge y \leq x \wedge y$. Now, by (qAn) and (qM($\wedge$)), $x \wedge y = (1 \to x) \wedge y$, i.e. (qWI1) holds.

(qBBWW1"): $x \stackrel{(Re')}{\leq} x$ and $y \stackrel{(K')}{\leq} 1 \to y$, hence, by (W- -), $x \wedge y \leq x \wedge (1 \to y)$. On the other hand, $x \stackrel{(Re')}{\leq} x$ and $1 \to y \stackrel{(qR1)}{\leq} y$, hence, by (W- -), $x \wedge (1 \to y) \leq x \wedge y$. Now, by (qAn) and (qM($\wedge$)), $x \wedge y = x \wedge (1 \to y)$, i.e. (qWI2) holds. □

Proposition 4.11 *(See Proposition 3.11 in the regular case and the dual Proposition 4.4)*
Let $(A, \wedge, \to, 1)$ be an algebra of type $(2, 2, 0)$ (or, equivalently, by (EqrelR) and (dfrelR), let $(A, \wedge, \leq, \to, 1)$ be a structure). Then, we have (following the numbering from Proposition 3.11):
(qBW1) (qEqW) $\Longrightarrow$ ((qWid) $\Leftrightarrow$ (Re)),
(qBW1') (qEqW) $\Longrightarrow$ ((qW-1) $\Leftrightarrow$ (L));
(qBW2) (qEqW) + (Wcomm) $\Longrightarrow$ (qAn);
(qBW3) (qEqW) + (Wassoc) + (qWI) $\Longrightarrow$ (Tr);
(qBW4) (qEqW) + (Wcomm) + (Wassoc) + (qWid) +(qWI1) + (qM($\wedge$)) $\Longrightarrow$ (Wleq),
(qBW5) (qEqW) + (Wcomm) + (Wassoc) + (qWid) + (qWI) + (qM($\wedge$)) $\Longrightarrow$ (W-);
(qBW7) (qWid) + (W- -) $\Longrightarrow$ (qW=).

Proof.

(qBW1): $x \wedge x \stackrel{(qWid)}{=} 1 \to x \stackrel{(qEqW)}{\Leftrightarrow} x \to x \stackrel{(Re)}{=} 1$.

(qBW1'): $x \wedge 1 \stackrel{(qW-1)}{=} 1 \to x \stackrel{(qEqW)}{\Leftrightarrow} x \to 1 \stackrel{(L)}{=} 1$.

(qBW2): If $x \leq y$ and $y \leq x$, i.e. $x \wedge y = 1 \to x$ and $y \wedge x = 1 \to y$, by (qEqW'), then $1 \to x = x \wedge y \overset{(Wcomm)}{=} y \wedge x = 1 \to y$; thus (qAn) holds.

(qBW3): If $x \leq y$ and $y \leq z$, i.e. $x \wedge y = 1 \to x$ and $y \wedge z = 1 \to y$, by (qEqW), then:

$x \wedge z \overset{(qWI)}{=} (1 \to x) \wedge (1 \to z) = (x \wedge y) \wedge (1 \to z)$
$\overset{(qWI)}{=} [(1 \to x) \wedge (1 \to y)] \wedge (1 \to z)$
$\overset{(Wassoc)}{=} (1 \to x) \wedge [(1 \to y)] \wedge (1 \to z)]$
$\overset{(qWI)}{=} (1 \to x) \wedge (y \wedge z) = (1 \to x) \wedge (1 \to y)$
$\overset{(qWI)}{=} x \wedge y = 1 \to x;$
hence, by (qEqW), $x \leq z$; thus, (Tr) holds.

(qBW4): $(x \wedge y) \wedge x \overset{(Wcomm)}{=} x \wedge (x \wedge y) \overset{(Wassoc)}{=} (x \wedge x) \wedge y \overset{(qWid)}{=} (1 \to x) \wedge y \overset{(qWI1)}{=} x \wedge y \overset{(qM(\wedge))}{=} 1 \to (x \wedge y)$, i.e. $x \wedge y \leq x$, by (qEqW).
$(x \wedge y) \wedge y \overset{(Wassoc)}{=} x \wedge (y \wedge y) \overset{(qWid)}{=} x \wedge (1 \to y) \overset{(Wcomm)}{=} (1 \to y) \wedge x \overset{(qWI1)}{=} y \wedge x \overset{(Wcomm)}{=} x \wedge y \overset{(qM(\wedge))}{=} 1 \to (x \wedge y)$, i.e. $x \wedge y \leq y$, by (qEqW). Thus, (Wleq) holds.

(qBW5): If $x \leq y$, i.e. $x \wedge y = 1 \to x$, by (qEqW), then $(x \wedge z) \wedge (y \wedge z) \overset{(Wcomm)}{=} (x \wedge z) \wedge (z \wedge y) \overset{(Wassoc)}{=} x \wedge (z \wedge z) \wedge y \overset{(qWid)}{=} x \wedge (1 \to z) \wedge y \overset{(Wcomm)}{=} (x \wedge y) \wedge (1 \to z) = (1 \to x) \wedge (1 \to z) \overset{(qWI)}{=} x \wedge z \overset{(qM(\wedge))}{=} 1 \to (x \wedge z)$; hence, by (qEqW), $x \wedge z \leq y \wedge z$; thus, (W-) holds.

(qBW7): Let $x \leq z$ and $y \leq z$; then, by (W- -), $x \wedge y \leq z \wedge z \overset{(qWid)}{=} 1 \to z$; thus (qW=) holds. □

Then we have:

Theorem 4.12 *(See Theorem 3.12 in the regular case and the dual Theorem 4.5)*
Let $\mathcal{A} = (A, \wedge, \to, 1)$ ($\mathcal{A} = (A, \wedge, \leq, \to, 1)$) be a Dedekind quasi-$\wedge$-semilattice with last (top) element 1 (i.e. properties (qM), (11-1), (qM($\wedge$)), (qEqW), (qWid, (Wcomm), (Wassoc), (qW1-1) hold).
Then, the following properties hold: (Re), (L), (Tr), (qAn), (W-), (W- -), (qW=), (Wleq).

Proof. Firstly, note that (qWI) and (qWI1) hold. Indeed, by (qBBW3), (qWid) + (Wcomm) + (Wassoc) + (qM($\wedge$)) $\Longrightarrow$ (qWI) and by (qBBW4), (qWI) + (qM) $\Longrightarrow$ (qWI1).

(Re): By (qBW1), (qEqW) $\Longrightarrow$ ((qWid) $\Leftrightarrow$ (Re)); thus, (Re) holds.

(L): By (qBW1'), (qEqW) $\Longrightarrow$ ((qW-1) $\Leftrightarrow$ (L)); thus, (L) holds.

(Tr): By (qBW3), (qEqW) + (Wassoc) + (qWI) $\Longrightarrow$ (Tr); thus, (Tr) holds.

(qAn): By (qBW2), (qEqW) + (Wcomm) $\Longrightarrow$ (qAn); thus, (qAn) holds.

(W-): By (qBW5), (qEqW) + (Wcomm) + (Wassoc) + (qWid) + (qWI) + (qM($\wedge$)) $\Longrightarrow$ (W-); thus, (W-) holds.

(W- -): By (BW6), (W-) + (Wcomm) + (Tr) $\Longrightarrow$ (W- -); thus (W- -) holds.

(qW=): By (qBW7), (W- -) + (qWid) $\Longrightarrow$ (qW=); thus, (qW=) holds.

(Wleq): By (qBW4), (qEqW) + (Wcomm) + (Wassoc) + (qWid) +(qWI1) + (qM($\wedge$)) $\Longrightarrow$ (Wleq); thus, (Wleq) holds. □

4.2.4 Dedekind quasi-lattices

By the dual Definitions 4.1 and 4.7, we introduce the following new definitions:

Definitions 4.13 (see Definition 3.13 in the regular case)

(1) Let $\mathcal{A} = (A, \wedge, \vee, \to, 1)$ or $\mathcal{A} = (A, \vee, \wedge, \to, 1)$ be an algebra of type $(2, 2, 2, 0)$ (or $\mathcal{A} = (A, \wedge, \vee, \leq, \to, 1)$ or $\mathcal{A} = (A, \vee, \wedge, \leq, \to, 1)$ be a structure) such that:
- the reduct $(A, \wedge, \to, 1)$ is a Dedekind quasi-$\wedge$-semilattice (with last element 1) and
- the reduct $(A, \vee, \to, 1)$ is a Dedekind quasi-$\vee$-semilattice (with last element 1).
In these conditions, we say that $\mathcal{A}$ is a *Dedekind quasi-lattice (with last element* 1*)* (more precisely, a *Dedekind quasi-$\wedge\vee$-lattice (with last element* 1*)* or a *Dedekind quasi-$\vee\wedge$-lattice (with last element* 1*)*, respectively, if the additional properties of absorption, (qWabs1) and (qWabs2), hold.

(2) A Dedekind quasi-lattice (with last element 1) is said to be *distributive*, if the distributivity properties, (Wdis1) and (Wdis2), hold.

Hence, a Dedekind quasi-lattice (with 1) verifies the properties: (qM), (11-1), (qM($\wedge$)), (qM($\vee$)), (qEqW), (qEqV), (qWid), (qVid), (Wcomm), (Vcomm), (Wassoc), (Vassoc), (qWabs1), (qWabs2) ((qW1-1) and (V1-1), respectively).

Note that if (M) holds, then any Dedekind quasi-lattice is a Dedekind regular-lattice.

• Let us introduce the new quasi-properties:
(qWabs1) (qW-absorption-1) $x \wedge (x \vee y) = 1 \to x$,
(qWabs2) (qW-absorption-2) $x \vee (x \wedge y) = 1 \to x$.

Proposition 4.14 *(See Proposition 3.16 in the regular case)*
Let $\mathcal{A} = (A, \vee, \wedge, \to, 1)$ be an algebra of type $(2, 2, 2, 0)$ (or, equivalently, $\mathcal{A} = (A, \vee, \wedge, \leq, \to, 1)$ be a structure) or, Galois dually, let $\mathcal{A} = (A, \wedge, \vee, \to, 1)$ be an algebra of type $(2, 2, 2, 0)$ (or $\mathcal{A} = (A, \wedge, \vee, \leq, \to, 1)$ be a structure) such that the reduct $(A, \to, 1)$ ($(A, \leq, \to, 1)$) is an algebra (structure) verifying the properties (Re) and (Tr) (i.e. $\leq$ is a pre-order).
Then, we have (following the numbering from Proposition 3.16):
(qBVW1) (Wdis1-p) + (Wdis1-pp) + (qAn) + (qM($\vee$)) + (qM($\wedge$)) $\Longrightarrow$ (Wdis1);
(qBVW1') (Vgeq) + (qV=) + (W-) + (qM($\wedge$)) $\Longrightarrow$ (Wdis1-p);
(Wleq) + (qW=) + (qWI2) + (V- -) + (qV=) + (qM($\vee$)) $\Longrightarrow$ (Wdis1-p);
(qBVW2) (Wdis2-p) + (Wdis2-pp) + (qAn) + (qM($\vee$)) + (qM($\wedge$)) $\Longrightarrow$ (Wdis2);
(qBVW2') (Wleq) + (qW=) + (V-) + (qM($\vee$)) $\Longrightarrow$ (Wdis2-p);
(Vgeq) + (qV=) + (qVI2) + (W- -) + (qW=) + (qM($\wedge$)) $\Longrightarrow$ (Wdis2-p);
(qBVW2") (Wdis1) + (Wleq) + (Wcomm) + (Vcomm) + (Vassoc) + (V-) + (qV=) + (qVI2) $\Longrightarrow$ (Wdis2-pp);

(qBVW3) (Vgeq) + (qEqW) $\Longrightarrow$ (qWabs1);
(qBVW4) (Wleq) + (qEqV) + (Vcomm) $\Longrightarrow$ (qWabs2).

Proof.
(qBVW1): Obviously.
(qBVW1'): First proof: $x, y \leq x \vee y$, by (Vgeq'); then, $x \wedge z,\ y \wedge z \leq (x \vee y) \wedge z$, by (W-), hence $(x \wedge z) \vee (y \wedge z) \overset{(qV=)}{\leq} 1 \to [(x \vee y) \wedge z] \overset{(qM(\wedge))}{=} (x \vee y) \wedge z$; thus, (Wdis1-p) holds.
Second proof: On the one hand, we have $x \wedge z \leq x$ and $y \wedge z \leq y$, by (Wleq); then, $(x \wedge z) \vee (y \wedge z) \leq x \vee y$, by (V- -). On the other hand, we have $x \wedge z \leq z$ and $y \wedge z \leq z$, by (Wleq); then, $(x \wedge z) \vee (y \wedge z) \leq 1 \to z$, by (qV=). Consequently, $x \wedge z) \vee (y \wedge z) \overset{(qM(\vee))}{=} 1 \to [x \wedge z) \vee (y \wedge z)] \overset{(qW=)}{\leq} (x \vee y) \wedge (1 \to z) \overset{(qWI2)}{=} (x \vee y) \wedge z$; thus, (Wdis1-p) holds.
(qBVW2): Obviously.
(qBVW2'): First proof: $x \wedge y \leq x,\ y$, by (Wleq); then, $(x \wedge y) \vee z \leq x \vee z,\ y \vee z$, by (V-), hence $(x \wedge y) \vee z \overset{(qM(\vee))}{=} 1 \to [(x \wedge y) \vee z] \overset{(qW=)}{\leq} (x \vee z) \wedge (y \vee z)$; thus, (Wdis2-p) holds.
Second proof: On the one hand, we have: $x \leq x \vee z$ and $y \leq y \vee z$, by (Vgeq); then, $x \wedge y \leq (x \vee z) \wedge (y \vee z)$, by (W- -). On the other hand, we have $z \leq x \vee z$ and $z \leq y \vee z$, by (Vgeq); then, $1 \to z \leq (x \vee z) \wedge (y \vee z)$, by (qW=). Consequently, $x \wedge y) \vee z \overset{(qVI2)}{=} (x \wedge y) \vee (1 \to z) \overset{(qV=)}{\leq} 1 \to [(x \vee z) \wedge (y \vee z)] \overset{(qM(\wedge))}{=} (x \vee z) \wedge (y \vee z)$; thus, (Wdis2-p) holds.
(qBVW2"): Denote $Z \overset{notation}{=} (x \vee z) \wedge (y \vee z)$; then
$Z \overset{(Wdis1)}{=} (x \wedge (y \vee z)) \vee (z \wedge (y \vee z))$
$\overset{(Wcomm)}{=} ((y \vee z) \wedge x) \vee ((y \vee z) \wedge z)$
$\overset{(Wdis1)}{=} ((y \wedge x) \vee (z \wedge x)) \vee ((y \wedge z) \vee (z \wedge z))$
$\overset{(Vassoc)}{=} (y \wedge x) \vee [(z \wedge x) \vee (y \wedge z) \vee (z \wedge z)]$
$\overset{(Vcomm)}{=} [(z \wedge x) \vee (y \wedge z) \vee (z \wedge z)] \vee (y \wedge x)$.

But $z \wedge x \leq z$, $y \wedge z \leq z$, $z \wedge z \leq z$, by (Wleq);
hence, $(z \wedge x) \vee (y \wedge z) \vee (z \wedge z) \leq 1 \to z$, by (qV=);
hence, $Z \overset{(V-)}{\leq} (1 \to z) \vee (y \wedge x) \overset{(Wcomm),(Vcomm)}{=} (x \wedge y) \vee (1 \to z) \overset{(qVI2)}{=} (x \wedge y) \vee z$; thus (Wdis2-pp) holds.

(qBVW3): $x \overset{(Vgeq)}{\leq} x \vee y \overset{(qEqW')}{\Leftrightarrow} x \wedge (x \vee y) = 1 \to x$; thus (qWabs1) holds.

(qBVW4): $x \wedge y \overset{(Wleq)}{\leq} x \overset{(qEqV)}{\Leftrightarrow} (x \wedge y) \vee x = 1 \to x$, hence $x \vee (x \wedge y) = 1 \to x$, by (Vcomm); thus, (qWabs2) holds. □

Then, we can prove:

Theorem 4.15 *(See Theorem 3.17 in the regular case)*

Let $\mathcal{A} = (A, \wedge, \vee, \to, 1)$ be a Dedekind quasi-$\wedge\vee$-lattice with 1 or $\mathcal{A} = (A, \vee, \wedge, \to, 1)$ be a Dedekind quasi-$\vee\wedge$-lattice with 1.
Then, the following properties hold: (Re), (L), (Tr), (qAn), (V-), (V- -), (qV=), (Vgeq), (W-), (W- -), (qW=), (Wleq), (Wdis1-p), (Wdis2-p).

Proof. By Theorem 4.12, (Re), (L), (Tr), (qAn), (W-), (W- -), (qW=), (Wleq) hold.
By Theorem 4.5, (V-), (V- -), (qV=), (Vgeq) hold too.
By (qBVW1'), (Vgeq) + (qV=) + (W-) + (qM($\wedge$)) $\Longrightarrow$ (Wdis1-p), hence (Wdis1-p) holds.
By (qBVW2'), (Wleq) + (qW=) + (V-) + (qM($\vee$)) $\Longrightarrow$ (Wdis2-p), hence (Wdis2-p) holds. □

It is a further research to define the *Ore quasi-lattice* and to prove that Dedekind quasi-lattices and Ore quasi-lattices are equivalent.

4.3 Positive implicative, commutative and quasi-implicative quasi-algebras (quasi-structures)

Definitions 4.16 Let $\mathcal{A}$ be a quasi-algebra (quasi-structure). We say that $\mathcal{A}$ is:
- *positive implicative*, if the following property (pimpl) is satisfied: for all $x, y, z \in A$,

$$(pimpl) \quad x \to (y \to z) = (x \to y) \to (x \to z);$$

- *commutative*, if the following property (comm) is satisfied: for all $x, y \in A$,

$$(comm) \quad (x \to y) \to y = (y \to x) \to x;$$

- *quasi-implicative* or *q-implicative* for short, if the following property (q-impl) is satisfied: for all $x, y \in A$,

$$(q-impl) \quad (x \to y) \to x = 1 \to x.$$

• Let us introduce also the following quasi-property:
(q-pi) $(1 \to x) \to (x \to y) = 1 \to (x \to y)$.

Then, we have the following connections.

Proposition 4.17 *Let $(A, \to, 1)$ ($(A, \leq, \to, 1)$) be an algebra (a structure). Then we have:*
(i) (q-impl) +(M) $\Longrightarrow$ (impl);
(ii) (q-pi) + (M) $\Longrightarrow$ (pi);
(iii) (q-pi) + (qM) + (qI1) $\Longrightarrow$ (pi);
(iii') (pi) + (qM) + (qI1) $\Longrightarrow$ (q-pi);
(iii") (qM) + (qI1) $\Longrightarrow$ ((pi) $\Leftrightarrow$ (q-pi)).

Proof.
(i): Obviously. (ii): Obviously.

(iii): $x \to (x \to y) \overset{(qI1)}{=} (1 \to x) \to (x \to y) \overset{(q-pi)}{=} 1 \to (x \to y) \overset{(qM)}{=} x \to y$; thus (pi) holds.
(iii'): $(1 \to x) \to (x \to y) \overset{(qI1)}{=} x \to (x \to y) \overset{(pi)}{=} x \to y \overset{(qM)}{=} 1 \to (x \to y)$; thus (q-pi) holds.
(iii"): By (iii) and (iii'). □

We present the corresponding "quasi-results" from Propositions 3.19 and 3.20 and new "quasi-results".

Proposition 4.18 *(See Proposition 3.19 in the regular case)*
Let $(A, \to, 1)$ ($(A, \leq, \to, 1)$) be an algebra (a structure). Then, we have (following the numbering from Proposition 3.19):

(qB0) (pimpl-1) + (pimpl-2) + (qM) + (qAn) $\Longrightarrow$ (pimpl);
(qB2) (q-pi) + (Re) + (qI1) $\Longrightarrow$ (L);
(qB3) (pimpl) + (Re) +(qM) $\Longrightarrow$ (pi);
(qB3') (pimpl) + (qR1) + (qI1) $\Longrightarrow$ (q-pi);
(qB7) (pimpl) + (Re) + (qM) $\Longrightarrow$ (*), (**);
(qB9) (pimpl) + (Re) + (qM) $\Longrightarrow$ (BB);
(qB10) (pimpl) + (Re) + (qM) $\Longrightarrow$ (C);
(qB11) (pimpl) + (Re) + (qM) + (qAn) $\Longrightarrow$ (Ex);

(qB12) (comm) $\Longrightarrow$ (qAn);

(qB14) (q-pi) + (Ex) + (B) + (qM) + (qR1) $\Longrightarrow$ (pimpl-1);
(qB15) (pi) + (Re) + (Ex) + (B) + (qAn) + (qM) $\Longrightarrow$ (pimpl);
(qB15') (q-pi) + (Re) + (Ex) + (B) + (qI1) + (qM) + (qAn) $\Longrightarrow$ (pimpl);
(qB16) (pi) + (Re) + (qM) + (B) + (D) +(qAn) $\Longrightarrow$ ((Ex) $\Leftrightarrow$ (BB) $\Leftrightarrow$ (pimpl));
(qB17) (pi) + (Re) + (qM) + (Ex) + (qAn) $\Longrightarrow$ ((BB) $\Leftrightarrow$ (B) $\Leftrightarrow$ (*) $\Leftrightarrow$ (pimpl)).

Proof:
(qB0): Obviously.

(qB2): $x \to (x \to y) \overset{(qI1)}{=} (1 \to x) \to (x \to y) \overset{(q-pi)}{=} 1 \to (x \to y)$; take $y = x$ to obtain: $x \to (x \to x) = 1 \to (x \to x)$, hence $x \to 1 = 1 \to 1 = 1$, by (Re); hence (L) holds.

(qB3): Take $x = y$ in (pimpl); we obtain: $y \to (y \to z) = (y \to y) \to (y \to z)$; then, by (Re), we obtain: $y \to (y \to z) = 1 \to (y \to z)$; then, by (qM), we obtain: $y \to (y \to z) = y \to z$, i.e. (pi) holds.

(qB3'): $(1 \to x) \to (x \to y) \overset{(pimpl)}{=} [(1 \to x) \to x] \to [(1 \to x) \to y] \overset{(qR1)}{=} 1 \to [(1 \to x) \to y] \overset{(qI1)}{=} 1 \to (x \to y)$, i.e. (q-pi) holds.

(qB7): By (B1), (Re) + (pimpl) $\Longrightarrow$ (L); by (B8), (L) + (pimpl) $\Longrightarrow$ (*); hence, (Re) + (pimpl) $\Longrightarrow$ (*). Also, by (B6), (Re) + (pimpl) $\Longrightarrow$ (B); then, by Theorem 2.26 (ii), (qM) + (B) $\Longrightarrow$ (**); hence, (Re) + (qM) + (pimpl) $\Longrightarrow$ (**).

(qB9): (Michael Kinyon's ideea, from [15]) By (B1) and (B3), (Re) + (pimpl) $\Longrightarrow$ (L) and (Re) + (M) + (pimpl) $\Longrightarrow$ (pi), and by (B4), (Re) + (L) + (pimpl) $\Longrightarrow$ (K).
Now, first prove that

$$((x \to y) \to z) \to (x \to (y \to u)) = (x \to y) \to (z \to (x \to u)). \tag{3}$$

Indeed, $(x \to y) \to (z \to (x \to u)) \overset{(pimpl)}{=} ((x \to y) \to z) \to ((x \to y) \to (x \to u)) \overset{(pimpl)}{=} ((x \to y) \to z) \to (x \to (y \to u))$; thus (3) holds.
We prove now that

$$x \to ((y \to x) \to z) = x \to z. \tag{4}$$

Indeed, $x \to ((y \to x) \to z) \overset{(pimpl)}{=} (x \to (y \to x)) \to (x \to z) \overset{(K)}{=} 1 \to (x \to z) \overset{(qM)}{=} x \to z$; thus (4) holds.
We prove now that

$$x \to (y \to (z \to x)) = 1. \tag{5}$$

Indeed, $x \to (y \to (z \to x)) \overset{(pimpl)}{=} (x \to y) \to (x \to (z \to x)) \overset{(K)}{=} (x \to y) \to 1 \overset{(L)}{=} 1$; thus (5) holds.
We prove now that

$$((x \to y) \to z) \to (u \to (y \to z)) = 1. \tag{6}$$

Indeed, $((x \to y) \to z) \to (u \to (y \to z)) \overset{(4)}{=} ((x \to y) \to z) \to (u \to [y \to ((x \to y) \to z)]) \overset{(5)}{=} 1$, for $X = (x \to y) \to z$; thus, (6) holds.
We prove now that

$$((x \to y) \to (y \to z)) \to (u \to (y \to z)) = 1. \tag{7}$$

Indeed, $((x \to y) \to (y \to z)) \to (u \to (y \to z)) \overset{(pi)}{=} ((x \to y) \to (y \to z)) \to (u \to (y \to (y \to z))) \overset{(6)}{=} 1$, with $Z = y \to z$; thus (7) holds.
We are ready to prove now that (BB) holds, i.e. $(x \to y) \to ((y \to z) \to (x \to z)) = 1$. Indeed, for $Z = y \to z$ and $u = z$ in (3), we get $(x \to y) \to ((y \to z) \to (x \to z)) \overset{(3)}{=} ((x \to y) \to (y \to z)) \to (x \to (y \to z)) \overset{(7)}{=} 1$, for $u = x$; thus (BB) holds.

(qB10): (Michael Kinyon's ideea, from [15]) By (B1), (B3), properties (L) and (pi) hold, and by (B4), property (K) holds.

Now, firstly we prove that:

$$x \to (y \to (x \to z)) = x \to (y \to z). \tag{8}$$

Indeed, $x \to (y \to z) \overset{(pimpl)}{=} (x \to y) \to (x \to z) \overset{(pi)}{=} (x \to y) \to (x \to (x \to z)) \overset{(pimpl)}{=} x \to (y \to (x \to z))$.

Then, we prove that:

$$y \to ((x \to y) \to z) = y \to z. \tag{9}$$

Indeed, $y \to ((x \to y) \to z) \overset{(pimpl)}{=} (y \to (x \to y)) \to (y \to z) \overset{(K)}{=} 1 \to (y \to z) \overset{(qM)}{=} y \to z$.

Then, we prove that:

$$((x \to y) \to z) \to (y \to z) = 1. \tag{10}$$

Indeed, $((x \to y) \to z) \to (y \to z) \overset{(9)}{=} ((x \to y) \to z) \to (y \to ((x \to y) \to z)) \overset{(K)}{=} 1$.

Finally, we prove that property (C) holds, i.e. $(x \to (y \to z)) \to (y \to (x \to z)) = 1$.
Indeed, $(x \to (y \to z)) \to (y \to (x \to z)) \overset{(pimpl)}{=} ((x \to y) \to (x \to z)) \to (y \to (x \to z)) \overset{(10)}{=} 1$; thus (C) holds.

(qB11): (Michael Kinyon's ideea, from [15]) By (qB10), (C) holds, and then by (A3), (C) + (An) $\Longrightarrow$ (Ex).

(qB12): If $x \to y = 1 = y \to x$ in (comm) ($(x \to y) \to y = (y \to x) \to x$)), then $1 \to y = 1 \to x$, i.e. (qAn) holds.

(qB14): Firstly, by (qA12'), (qM) + (B) $\Longrightarrow$ (*), by (qA13'), (qM) + (*) $\Longrightarrow$ (Tr).
Then, by (A10'), (Ex) + (B) $\Longrightarrow$ (BB), by (qA15'), (qM) + (BB) $\Longrightarrow$ (**).

$X \overset{notation}{=} (x \to y) \to (x \to z) \overset{(qM)}{=} (x \to y) \to [1 \to (x \to z)]$
$\overset{(q-pi)}{=} (x \to y) \to [(1 \to x) \to (x \to z)] \overset{(Ex)}{=} (1 \to x) \to [(x \to y) \to (x \to z)]$.
By (B'), $y \to z \leq (x \to y) \to (x \to z)$, hence by (*), we obtain:

(a) $x \to (y \to z) \leq x \to [(x \to y) \to (x \to z)]$.

But, by (qR1), $1 \to x \leq x$, hence by (**), we obtain:

(b) $x \to [(x \to y) \to (x \to z)] \leq (1 \to x) \to [(x \to y) \to (x \to z)] = X$.

Finally, (a) + (b) + (Tr) $\Longrightarrow x \to (y \to z) \leq X$, i.e. (pimpl-1) holds.

(qB15): By (qA12'), (qM) + (B) $\Longrightarrow$ (*);
by (A10'), (Ex) + (B) $\Longrightarrow$ (BB) and by (qA15'), (qM) + (BB) $\Longrightarrow$ (**);
by (B2), (pi) + (Re) $\Longrightarrow$ (L).

Now, by (B14), (pi) + (Ex) + (B) + (*) $\Longrightarrow$ (pimpl-1) and,
by (B13), (Re) + (L) + (Ex) + (**) $\Longrightarrow$ (pimpl-2).

Finally, by (qB0), (pimpl) holds.

(qB15'): Since (Re) $\Longrightarrow$ (qRe($1 \to x$)), then by (qAA2), (Ex) + (qRe($1 \to x$)) + (qM) $\Longrightarrow$ (qR1); then by (qB14), (q-pi) + (Ex) + (B) + (qM) + (qR1) $\Longrightarrow$ (pimpl-1).

Now, by (qB2), (q-pi) + (Re) + (qI1) $\Longrightarrow$ (L);
by (A10'), (Ex) + (B) $\Longrightarrow$ (BB) and by (qA15'), (qM) + (BB) $\Longrightarrow$ (**);
hence, by (B13), (Re) + (L) + (Ex) + (**) $\Longrightarrow$ (pimpl-2).

Finally, by (qB0), (pimpl-1) + (pimpl-2) + (qM) + (qAn) $\Longrightarrow$ (pimpl).

(qB16): Firstly, by Theorem 2.25, if the algebra $(A, \to, 1)$ verifies (B), (D), (qM), (qAn), then:

$$(Ex) \ \Leftrightarrow \ (BB).$$

Then, by above (qB9), ((Re) + (qM)) + (pimpl) $\Longrightarrow$ (BB) or, by above (qB11), ((Re) + (qM) + (qAn)) + (pimpl) $\Longrightarrow$ (Ex). Note that (Re) + (pimpl) $\Longrightarrow$ (B), by (B6). Hence, if (Re), (qM), (qAn) hold, then (pimpl) $\Rightarrow$ (Ex) ($\Leftrightarrow$ (BB)).

Conversely, by (B2), (Re) + (pi) $\Longrightarrow$ (L). By (qA12'), (qM) + (B) $\Longrightarrow$ (*). By Theorem 2.26 (ii), (qM) + (B) $\Longrightarrow$(**). By (B13), ((Re) + (L) + (**)) + (Ex) $\Longrightarrow$ (pimpl-2). By (B14), ((B) + (*) + (pi)) + (Ex) $\Longrightarrow$ (pimpl-1). Then, ((Re) + (qM) + (B) + (pi) + (qAn)) + (Ex) $\Longrightarrow$ (pimpl-1) + (pimpl-2) + (qAn) + (qAn) $\Longrightarrow$ (pimpl), by (qB0). Consequently, if (Re), (qM), (B), (D), (qAn), (pi) hold, then ((BB) $\Leftrightarrow$) (Ex) $\Rightarrow$ (pimpl).

Thus, if (Re), (qM), (B), (D), (qAn) and (pi) hold, we have $(BB) \ \Leftrightarrow \ (Ex) \ \Leftrightarrow \ (pimpl)$.

(qB17): Firstly, by Theorem 2.23, if the algebra $(A, \to, 1)$ verifies (Re), (qM), (Ex), then we have:

$$(BB) \ \Leftrightarrow \ (B) \ \Leftrightarrow \ (*). \tag{11}$$

Then, by (Kinyon's) (qB9), (Re) + (qM) + (pimpl) $\Longrightarrow$ (BB). Hence, if (Re), (qM) hold, then

$$(pimpl) \ \Longrightarrow \ (BB)(\Leftrightarrow \ (B) \ \Leftrightarrow \ (*)).$$

Conversely, first, by (B2), (Re) + (pi) $\Longrightarrow$ (L). By (qA15'), (qM) + (BB) $\Longrightarrow$ (**). Hence, by (B13), (Re) + (L) + (Ex) + (**) $\Longrightarrow$ (pimpl-2). Hence, if (Re), (pi), (qM), (Ex) hold, then

$$(BB) \ \Longrightarrow \ (pimpl-2).$$

On the other hand, (Ex) + (pi) + (BB) $\stackrel{(11)}{\Longrightarrow}$ (Ex) + (pi) + (B) ($\Leftrightarrow$ $(*)$) $\Longrightarrow$ (pimpl-1), by (B14). Hence, if (Re), (qM), (Ex), (pi) hold, then

$$(BB) \ \Longrightarrow \ (pimpl-1).$$

Consequently, if (Re), (qM), (Ex), (pi) and (qAn) hold, then, by (qB0), we obtain:

$(BB) \ \Longrightarrow \ ((pimpl-2) \ + \ (pimpl-1) \ + \ (qAn) \ + \ (qM)) \ \Longrightarrow \ (pimpl)$, i.e. (BB) $\Longrightarrow$ (pimpl).

Thus, if (Re), (qM), (Ex), (qAn) and (pi) hold, then we have: $(BB) \ \Leftrightarrow \ (B) \ \Leftrightarrow \ (*) \ \Leftrightarrow \ (pimpl)$. $\square$

Proposition 4.19 *(See Proposition 3.20 in the regular case)*
Let $(A, \to, 1)$ ($(A, \leq, \to, 1)$) be an algebra (a structure). Then, we have (following the numbering from Proposition 3.20):

(qB19) (pimpl-1) + (K) + (qN) $\Longrightarrow$ (Re);
(qB19') (pimpl-1) + (K) + (qM) $\Longrightarrow$ (Re);
(qB20) (pimpl-1) + (L) + (qN) $\Longrightarrow$ (*);
(qB20') (pimpl-1) + (L) + (qM) $\Longrightarrow$ (*);
(qB21) (pimpl-1) + (L) + (qN) $\Longrightarrow$ (Tr);
(qB21') (pimpl-1) + (L) + (qM) $\Longrightarrow$ (Tr);
(qB23) (pimpl-1) + (Re) + (*) + (K) + (qM) $\Longrightarrow$ (D);
(qB24) (pimpl-1) + (*) + (K) + (qM) $\Longrightarrow$ (**);
(qB26) (pimpl-1) + (K) + (qM) + (qAn) + (C) + (**) + (Tr) $\Longrightarrow$ (pimpl);
(qB27) (pimpl-1) + (qN) $\Longrightarrow$ (\$);
(qB27') (pimpl-1) + (qM) $\Longrightarrow$ (\$);

(qB30) (comm) + (Re) + (qM) + (Ex) $\Longrightarrow$ (qL);
(qB31) (comm) + (Re) + (qM) + (Ex) $\Longrightarrow$ (*);
(qB32) (comm) + (qR1) + (BB) + (qM) $\Longrightarrow$ (qL($1 \to x$));
(qB33) (comm) + (K) +(BB) + (Tr) + (qM) $\Longrightarrow$ (#);

(qB35) (Re) + (Ex) + (B) + (qM) + (qAn) $\Longrightarrow$ ((pimpl) $\Leftrightarrow$ (pi));
(qB35') (Re) + (Ex) + (B) + (qI1) + (qM) + (qAn) $\Longrightarrow$ ((pimpl) $\Leftrightarrow$ (pi) $\Leftrightarrow$ (q-pi)).

Proof:
(qB19): From (pimp-1) ($(x \to (y \to z)) \to [(x \to y) \to (x \to z)] = 1$), for $y = x \to x$ and $z = x$ we obtain:
$(x \to ((x \to x) \to x)) \to [(x \to (x \to x)) \to (x \to x)] = 1$; by (K), $x \to ((x \to x) \to x) = 1$ and $x \to (x \to x) = 1$; hence we obtain that $1 \to [1 \to (x \to x)] = 1$; then, applying (qN) twice, we obtain that $x \to x = 1$, i.e. (Re) holds.
(qB19'): By (qA00), (qM) $\Longrightarrow$ (qN), then apply (qB19).
(qB20): Suppose that $y \to z = 1$; we must prove that $(x \to y) \to (x \to z) = 1$. Indeed, from (pimpl-1) ($(x \to (y \to z)) \to [(x \to y) \to (x \to z)] = 1$) we obtain:
$(x \to 1) \to [(x \to y) \to (x \to z)] = 1$, hence, by (L) and (qN), we obtain: $(x \to y) \to (x \to z) = 1$; thus (*) holds.
(qB20'): By (qA00), (qM) $\Longrightarrow$ (qN), then apply (qB20).
(qB21): By (qB20), (pimpl-1) + (L) + (qN) $\Longrightarrow$ (*) and by (qA13), (qN) + (*) $\Longrightarrow$ (Tr). Thus, (Tr) holds.
qB21'): By (qB20'), (pimpl-1) + (L) + (qM) $\Longrightarrow$ (*) and by (qA13'), (qM) + (*) $\Longrightarrow$ (Tr). Thus, (Tr) holds.
(qB23): From (pimp-1) ($(x \to (y \to z)) \to [(x \to y) \to (x \to z)] = 1$), for $x = y \to z$ we obtain:
$((y \to z) \to (y \to z)) \to [((y \to z) \to y) \to ((y \to z) \to z)] = 1$; then, by (Re) and (qM), we obtain:
$((y \to z) \to y) \to ((y \to z) \to z) = 1$ i.e. $(y \to z) \to y \leq (y \to z) \to z$; now, apply (*') and obtain:
$y \to [(y \to z) \to y] \leq y \to [(y \to z) \to z]$; then, by (K) and (qM), we obtain:
$y \to [(y \to z) \to z] = 1$, i.e. (D) holds.
(qB24): Suppose that $x \to y = 1$; we shall prove that $(y \to z) \to (x \to z) = 1$. Indeed,
(pimp-1) ($(x \to (y \to z)) \to [(x \to y) \to (x \to z)] = 1$) gives:
$(x \to (y \to z)) \to [1 \to (x \to z)] = 1$; then, by (qM), we obtain:
$(x \to (y \to z)) \to (x \to z) = 1$, i.e. $x \to (y \to z) \leq x \to z = 1$; now, by (*'), we obtain:
$(y \to z) \to [x \to (y \to z)] \leq (y \to z) \to (x \to z)$; now, by (K) and (qM), we obtain: $(y \to z) \to (x \to z) = 1$. Thus, (**) holds.
(qB26): $y \overset{(K)}{\leq} x \to y$ implies, by (**'), $(x \to y) \to (x \to z) \leq y \to (x \to z)$. But $y \to (x \to z) \overset{(C')}{\leq} x \to (y \to z)$. Then, by (Tr), we obtain:
$(x \to y) \to (x \to z) \leq x \to (y \to z)$, on the one hand.
On the other hand, $x \to (y \to z) \overset{(pimpl-1)}{\leq} (x \to y) \to (x \to z)$.
Consequently, by (qAn) and (qM), $x \to (y \to z) = (x \to y) \to (x \to z)$, i.e. (pimpl) holds.
(qB27): By (pimpl-1), $(x \to (y \to z)) \to [(x \to y) \to (x \to z)] = 1$; if $x \to (y \to z) = 1$, then by (qN), $(x \to y) \to (x \to z) = 1$, i.e. (\$) holds.
(qB27'): By (qA00), (qM) $\Longrightarrow$ (qN), then apply (qB26).
(qB30): $[(x \to y) \to 1] \to 1 \overset{(comm)}{=} [1 \to (x \to y)] \to (x \to y) \overset{(qM)}{=} (x \to y) \to (x \to y) \overset{(Re)}{=} 1$. Then, $(x \to y) \to 1 = (x \to y) \to [((x \to y) \to 1) \to 1] \overset{(Ex)}{=} [(x \to y) \to 1] \to [(x \to y) \to 1] \overset{(Re)}{=} 1$; thus (qL) holds.
(qB31): By (qB30), (comm) + (Re) + (qM) + (Ex) $\Longrightarrow$ (qL). Suppose $y \to z = 1$; then $1 \to z = (y \to z) \to z \overset{(comm)}{=} (z \to y) \to y$, hence $x \to z \overset{(qM)}{=} 1 \to (x \to z) \overset{(Ex)}{=} x \to (1 \to z) = x \to [(z \to y) \to y] \overset{(Ex)}{=} (z \to y) \to (x \to y)$. Then, $(x \to y) \to (x \to z) = (x \to y) \to [(z \to y) \to (x \to y)] \overset{(Ex)}{=} (z \to y) \to [(x \to y) \to (x \to y)] \overset{(Re)}{=} (z \to y) \to 1 \overset{(qL)}{=} 1$. Thus, (*) holds.

(qB32): (See the proof of (qW13) from [2])
Firstly, by (qAA1), (BB) + (qM) $\Longrightarrow$ (11-1). Then, $(1 \to x) \to 1$
$\overset{(qR1)}{=} (1 \to x) \to [(1 \to (1 \to x)) \to (1 \to x)]$
$\overset{(comm)}{=} (1 \to x) \to [((1 \to x) \to 1) \to 1]$
$\overset{(qM)}{=} (1 \to (1 \to x)) \to [((1 \to x) \to 1) \to 1]$
$\overset{(11-1)}{=} (1 \to (1 \to x)) \to [((1 \to x) \to 1) \to (1 \to 1)] \overset{(BB)}{=} 1$, i.e. (qL$(1 \to x)$) holds.

(qB33): (See the proof of (qW31) from [2]) Firstly, note that $y \to ((z \to y) \to y) \overset{(K)}{=} 1$.
Second, suppose that $x \to (y \to z) = 1$; then,
$((z \to y) \to y) \to (x \to z) \overset{(comm)}{=} ((y \to z) \to z) \to (x \to z)$
$\overset{(qM)}{=} 1 \to [((y \to z) \to z) \to (x \to z)] = [x \to (y \to z)] \to [((y \to z) \to z) \to (x \to z)] \overset{(BB)}{=} 1$.
Now, by (Tr), we obtain that $y \to (x \to z) = 1$, i.e. (#) holds.

(qB35): By (qB3), (pimpl) + (Re) + (qM) $\Longrightarrow$ (pi). Conversely, by (qB15), (pi) + (Re) + (Ex) + (B) + (qAn) + (qM) $\Longrightarrow$ (pimpl).

(qB35'): By (qB35), (Re) + (Ex) + (B) + (qM) + (qAn) $\Longrightarrow$ ((pimpl) $\Leftrightarrow$ (pi));
by (iii") of Proposition 4.17, (qM) + (qI1) $\Longrightarrow$ ((pi) $\Leftrightarrow$ (q-pi)). □

Proposition 4.20 *Let $(A, \to, 1)$ ($(A, \leq, \to, 1)$) be an algebra (a structure). Then the additional quasi-property holds (with an independent numbering, because we have no a correspondent regular property):*

(qBB1) (comm) + (qRe) + (BB) + (qM) $\Longrightarrow$ (qR1).

Proof.

(qBB1): (See the proof of (qW12) from [2])
Firstly, by (qAA1), (BB) + (qM) $\Longrightarrow$ (11-1). Then, $(1 \to x) \to x$
$\overset{(comm)}{=} (x \to 1) \to 1$
$\overset{(qRe)}{=} (x \to 1) \to [(x \to 1) \to (x \to 1)]$
$\overset{(qM)}{=} (x \to 1) \to [(1 \to (x \to 1)) \to (x \to 1)]$
$\overset{(comm)}{=} (x \to 1) \to [((x \to 1) \to 1) \to 1]$
$\overset{(11-1)}{=} (x \to 1) \to [((x \to 1) \to 1) \to (1 \to 1)]$
$\overset{(qM)}{=} (1 \to (x \to 1)) \to [((x \to 1) \to 1) \to (1 \to 1)] \overset{(BB)}{=} 1$, i.e. (qR1) holds. □

Proposition 4.21 *(See Proposition 3.22 in the regular case)*
Let $(A, \to, 1)$ ($(A, \leq, \to, 1)$) be an algebra (a structure). Then, we have (following the numbering from Proposition 3.22):

(qBIM1) (q-impl) $\Longrightarrow$ (q-pi);
(qBIM1') (q-impl) + (BB) + (K) + (qM) + (qAn) $\Longrightarrow$ (pi);
(qBIM1") (comm) + (q-impl) + (L) + (K) + (qM) $\Longrightarrow$ (pi);
(qBIM2) (comm) + (pi) + (Re) + (K) + (qM) $\Longrightarrow$ (q-impl);
(qBIM2') (comm) + (q-pi) + (qM) + (BB) + (qR1) + (K) $\Longrightarrow$ (q-impl);
(qBIM2") (comm) + (L) + (Re) + (K) + (qM) $\Longrightarrow$ ((pi) $\Leftrightarrow$ (q-impl));
(qBIM3) (q-impl) + (Ex) + (B) + (qM) + (qAn) $\Longrightarrow$ (comm);
(qBIM5) (q-impl) + (qM) + (11-1) $\Longrightarrow$ (qL$(1 \to y)$);
(qBIM6) (q-impl) + (K) $\Longrightarrow$ (qR2);
(qBIM7) (K) + (L) + (Ex) + (B) ($\Leftrightarrow$ (BB)) + (qM) + (qAn) $\Longrightarrow$ ((q-impl) $\Leftrightarrow$ ((comm) + (pi))).

Proof.

(qBIM1): $(1 \to x) \to (x \to y) \overset{(q-impl)}{=} [(x \to y) \to x] \to (x \to y) \overset{(q-impl)}{=} 1 \to (x \to y)$, i.e. (q-pi) holds.

(qBIM1'): $[y \to (y \to x)] \to (y \to x)$
$\overset{(qM)}{=} [y \to (y \to x)] \to [1 \to (y \to x)]$

$\overset{(q-impl)}{=} [y \to (y \to x)] \to [((y \to x) \to x) \to (y \to x)] \overset{(BB)}{=} 1.$

Thus, $[y \to (y \to x)] \to (y \to x) = 1$. But, we also have $(y \to x) \to [y \to (y \to x)] \overset{(K)}{=} 1$.
Then, by (qAn), we obtain that $1 \to [y \to (y \to x)] = 1 \to (y \to x)$, hence, by (qM), $y \to (y \to x) = y \to x$, i.e. (pi) holds.

(qBIM1"): Firstly, by (qB12), (comm) + (qM) $\Longrightarrow$ (qAn). Then,
$[y \to (y \to x)] \to (y \to x) \overset{(comm)}{=} [(y \to x) \to y] \to y \overset{(q-impl)}{=} (1 \to y) \to y \overset{(comm)}{=} (y \to 1) \to 1 \overset{(L)}{=} 1 \to 1 \overset{(L)}{=} 1$. Hence, $y \to (y \to x) \le y \to x$.

But we also have that $y \to x \overset{(K)}{\le} y \to (y \to x)$.
Then, by (qAn), it follows that $1 \to (y \to x) = 1 \to [y \to (y \to x)]$, hence, by (qM), $y \to x = y \to (y \to x)$, i.e. (pi) holds.

(qBIM2): Firstly, by (qB12), (comm) $\Longrightarrow$ (qAn). Then,
$((x \to y) \to x) \to x \overset{(comm)}{=} [x \to (x \to y)] \to (x \to y) \overset{(pi)}{=} (x \to y) \to (x \to y) \overset{(Re)}{=} 1$. Hence, $(x \to y) \to x \le x$.

But, we also have that $x \overset{(K)}{\le} (x \to y) \to x$.
Then, by (qAn), it follows that $1 \to x = 1 \to [(x \to y) \to x] \overset{(qM)}{=} (x \to y) \to x$, i.e. (q-impl) holds.

(qBIM2'): Firstly, by (qB12), (comm) $\Longrightarrow$ (qAn). Then,
$T \overset{notation}{=} [(x \to y) \to x] \to x \overset{(comm)}{=} [x \to (x \to y)] \to (x \to y)$
$\overset{(qM)}{=} [x \to (x \to y)] \to [1 \to (x \to y)]$
$\overset{(q-pi)}{=} [x \to (x \to y)] \to [(1 \to x) \to (x \to y)]$
$\overset{(qM)}{=} 1 \to ([x \to (x \to y)] \to [(1 \to x) \to (x \to y)])$
$\overset{(qR1)}{=} [(1 \to x) \to x] \to ([x \to (x \to y)] \to [(1 \to x) \to (x \to y)]) \overset{(BB)}{=} 1.$

Hence, $(x \to y) \to x \le x$. But we also have $x \overset{(K)}{\le} (x \to y) \to x$.
Finally, $(x \to y) \to x \overset{(qM)}{=} 1 \to [(x \to y) \to x] \overset{(qAn)}{=} 1 \to x$. i.e. (q-impl) holds.

(qBIM2"): By (qBIM1"), (qBIM2).

(qBIM3): $[(x \to y) \to y] \to [(y \to x) \to x]$
$\overset{(qM)}{=} 1 \to ([(x \to y) \to y] \to [(y \to x) \to x])$
$\overset{(Ex)}{=} [(x \to y) \to y] \to (1 \to [(y \to x) \to x])$
$\overset{(Ex)}{=} [(x \to y) \to y] \to [(y \to x) \to (1 \to x)]$
$\overset{(q-impl)}{=} [(x \to y) \to y] \to [(y \to x) \to ((x \to y) \to x)]$
$\overset{(Ex)}{=} (y \to x) \to ([(x \to y) \to y] \to [(x \to y) \to x]) \overset{(B)}{=} 1.$

Thus, we obtained that

$$[(x \to y) \to y] \to [(y \to x) \to x] = 1. \tag{12}$$

Similarly,

$$[(y \to x) \to x] \to [(x \to y) \to y] = 1. \tag{13}$$

From (12) and (13), by (qAn), we obtain $1 \to [(x \to y) \to y] = 1 \to [(y \to x) \to x]$, hence, by (qM), $(x \to y) \to y = (y \to x) \to x$, i.e. (comm) holds.

(qBIM5): Take $x = 1$ in (q-impl) ($(x \to y) \to x = 1 \to x$); we obtain: $(1 \to y) \to 1 = 1 \to 1 = \overset{(11-1)}{=} 1$, i.e. (qL($1 \to y$)) holds.

(qBIM6): $x \to (1 \to x) \overset{(q-impl)}{=} x \to [(x \to y) \to x] \overset{(K)}{=} 1$, i.e. (qR2) holds.

(qBIM7): By (qBIM3), (q-impl) + (Ex) + (B) + (qM) + (qAn) $\Longrightarrow$ (comm), and by (qBIM1'), (q-impl) + (BB) + (K) + (qM) + (qAn) $\Longrightarrow$ (pi). Conversely, by (qBIM2), (comm) + (pi) + (L) + (K) + (qM) $\Longrightarrow$ (q-impl). □

4.3.1 Quasi-join-semilattices from commutative quasi-algebras

Proposition 4.22 *Let $(A, \rightarrow, 1)$ $((A, \leq, \rightarrow, 1))$ be an algebra (a structure).*
Define a new binary operation $\vee$ by (dfV) ($x \vee y \stackrel{df.}{=} (x \rightarrow y) \rightarrow y$, for all $x, y \in A$).
Then, we have (with an independent numbering):
(qBBdfV1) (dfV) + (qM) $\Longrightarrow$ (qM($\vee$));
(qBBdfV2) (dfV) + (qI) + (qM) $\Longrightarrow$ (qVI);
(qBBdfV2') (dfV) + (qI1) $\Longrightarrow$ (qVI1);
(qBBdfV2") (dfV) + (qI2) $\Longrightarrow$ (qVI2).

Proof.
(qBBdfV1): $1 \rightarrow (x \vee y) \stackrel{(dfV)}{=} 1 \rightarrow ((x \rightarrow y) \rightarrow y) \stackrel{(qM)}{=} (x \rightarrow y) \rightarrow y \stackrel{(dfV)}{=} x \vee y$, i.e. (qM($\vee$)) holds.
(qBBdfV2): $(1 \rightarrow x) \vee (1 \rightarrow y) \stackrel{(dfV)}{=} ((1 \rightarrow x) \rightarrow (1 \rightarrow y)) \rightarrow (1 \rightarrow y) \stackrel{(qI)}{=} (x \rightarrow y) \rightarrow (1 \rightarrow y) \stackrel{(qM)}{=} (1 \rightarrow (x \rightarrow y)) \rightarrow (1 \rightarrow y) \stackrel{(qI)}{=} (x \rightarrow y) \rightarrow y \stackrel{(dfV)}{=} x \vee y$, i.e. (qVI) holds.
(qBBdfV2'): $(1 \rightarrow x) \vee y \stackrel{(dfV)}{=} ((1 \rightarrow x) \rightarrow y) \rightarrow y \stackrel{(qI1)}{=} (x \rightarrow y) \rightarrow y \stackrel{(dfV)}{=} x \vee y$, i.e. (qVI1) holds.
(qBBdfV2"): $x \vee (1 \rightarrow y) \stackrel{(dfV)}{=} (x \rightarrow (1 \rightarrow y)) \rightarrow (1 \rightarrow y) \stackrel{(qI2)}{=} (x \rightarrow y) \rightarrow (1 \rightarrow y) \stackrel{(qI2)}{=} (x \rightarrow y) \rightarrow y \stackrel{(dfV)}{=} x \vee y$, i.e. (qVI2) holds. □

Proposition 4.23 *(See Proposition 3.24 in the regular case)*
Let $(A, \rightarrow, 1)$ $((A, \leq, \rightarrow, 1))$ be an algebra (a structure).
Define a new binary operation $\vee$ by (dfV) ($x \vee y \stackrel{df.}{=} (x \rightarrow y) \rightarrow y$, for all $x, y \in A$).
Then, we have (following the numbering from Proposition 3.24):
(qBdfV2) (dfV) + (Re) $\Longrightarrow$ (qVid);
(qBdfV4) (dfV) + (L) + (qR1) $\Longrightarrow$ (V1-1);
(qBdfV5) (dfV) + (qI2) + (Ex) + (Re) $\Longrightarrow$ (qEqV);
(qBdfV7) (dfV) + (qEqV) + (Vcomm) + (VV) + (qM) $\Longrightarrow$ (VVV);
(qBdfV10) (dfV) + (q-impl) $\Longrightarrow$ (qVid).

Proof.
(qBdfV2): $x \vee x \stackrel{(dfV)}{=} (x \rightarrow x) \rightarrow x \stackrel{(Re)}{=} 1 \rightarrow x$, thus (qVid) holds.
(qBdfV4): $x \vee 1 \stackrel{(dfV)}{=} (x \rightarrow 1) \rightarrow 1 \stackrel{(L)}{=} 1 \rightarrow 1 \stackrel{(L)}{=} 1$;
$1 \vee x \stackrel{(dfV)}{=} (1 \rightarrow x) \rightarrow x \stackrel{(qR1)}{=} 1$; thus (V1-1) holds.
(qBdfV5): If $x \leq y$, i.e. $x \rightarrow y = 1$, then: $x \vee y \stackrel{(dfV)}{=} (x \rightarrow y) \rightarrow y = 1 \rightarrow y$.
If $1 \rightarrow y = x \vee y \stackrel{(dfV)}{=} (x \rightarrow y) \rightarrow y$, then: $x \rightarrow y \stackrel{(qI2)}{=} x \rightarrow (1 \rightarrow y) = x \rightarrow [(x \rightarrow y) \rightarrow y] \stackrel{(Ex)}{=} (x \rightarrow y) \rightarrow (x \rightarrow y) \stackrel{(Re)}{=} 1$, i.e. $x \leq y$. Thus, (qEqV) holds.
(qBdfV7): $y \stackrel{(K)}{\leq} z \rightarrow y \stackrel{(qEqV)}{\Leftrightarrow} y \vee (z \rightarrow y) = 1 \rightarrow (z \rightarrow y) \stackrel{(qM)}{=} z \rightarrow y$, hence
$(x \rightarrow y) \rightarrow (z \rightarrow y) = (x \rightarrow y) \rightarrow [y \vee (z \rightarrow y)] \stackrel{(Vcomm)}{=} (x \rightarrow y) \rightarrow [(z \rightarrow y) \vee y]$
$\stackrel{(VV)}{=} [(z \rightarrow y) \rightarrow y] \rightarrow [(x \rightarrow y) \rightarrow y] \stackrel{(dfV)}{=} (z \vee y) \rightarrow (x \vee y)$; thus (VVV) holds.
(qBdfV10): $x \vee x \stackrel{(dfV)}{=} (x \rightarrow x) \rightarrow x \stackrel{(q-impl)}{=} 1 \rightarrow x$, thus (qVid) holds. □

Then, we have the following theorem:

Theorem 4.24 *(See Theorem 3.25 in the regular case)*
Let $\mathcal{A} = (A, \rightarrow, 1)$ $(\mathcal{A} = (A, \leq, \rightarrow, 1))$ be an algebra (structure) verifying the properties (comm), (Re), (K), (Ex), (qAn), (qM).
Define a new operation $\vee$ by (dfV) ($x \vee y \stackrel{df.}{=} (x \rightarrow y) \rightarrow y$).
Then, $(A, \vee, \rightarrow, 1)$ $((A, \vee, \leq, \rightarrow, 1))$ is a quasi-$\vee$-semilattice with last element 1.

Proof. Firstly, (Re) $\Longrightarrow$ (11-1) and since (qM) also holds, then $\mathcal{A}$ is a quasi-algebra (quasi-structure). Then:
- by (qBBdfV1), (dfV) + (qM) $\Longrightarrow$ (qM($\vee$)) •
- by (qA7'), (qM) + (K) $\Longrightarrow$ (L);
- by (qBdfV2), (dfV) + (Re) $\Longrightarrow$ (qVid) •
- by (BdfV1), (dfV) + (comm) $\Longrightarrow$ (Vcomm) •
- since (Re) $\Longrightarrow$ (qRe($1 \to x$)), then by (qAA2), (Ex) + (qRe($1 \to x$)) + (qM) $\Longrightarrow$(qR1); by (qBdfV4), (dfV) + (L) + (qR1) $\Longrightarrow$ (V1-1) •
- by (qAA8), (Ex) + (qM) $\Longrightarrow$ (qI2); then, by (qBdfV5), (dfV) + (qI2) + (Ex) + (Re) $\Longrightarrow$ (qEqV) •
- by (BdfV6), (dfV) + (Ex) $\Longrightarrow$ (VV);
- by (qBdfV7) (dfV) + (qEqV) + (Vcomm) + (VV) + (qM) $\Longrightarrow$ (VVV);
- by (BdfV8), (dfV) + (Vcomm) + (Ex) + (VVV) $\Longrightarrow$ (Vassoc) • □

4.4 Quasi-algebras (quasi-structures) with product. Commutative quasi-monoids and quasi-residoids

Note that any algebra with (P) (or with (RP)) and any X-algebra with (R) (or with (PR)) that verify the property (qM) is a quasi-algebra (quasi-X-algebra) (because (Re) implies (11-1)).

Note also that the basic general equivalences from Figure 2 are valid for quasi-algebras (quasi-X-algebras).

4.4.1 Commutative quasi-monoids and quasi-residoids

We introduce the following new definitions.

Definitions 4.25 (See Definitions 3.40 and 3.39 (1) in the regular case)

(1) An algebra $(A, \to, \odot, 1)$ or $(A, \odot, \to, 1)$ is a *commutative (or abelian) quasi-monoid* if:
- the reduct $(A, \to, 1)$ is a quasi-algebra (i.e. (qM) and (11-1) hold) and
- the following properties hold: (Pcomm), (Passoc), (qP1-1), (qM($\odot$)), where: for all $x \in A$,

(qP1-1) $x \odot 1 = 1 \odot x = 1 \to x$: (qP-1) $x \odot 1 = 1 \to x$, (qP1-) $1 \odot x = 1 \to x$,
(qM($\odot$)) $1 \to (x \odot y) = x \odot y$.

(2) An algebra $(A, \to, 1)$ (a structure $(A, \leq, \to, 1)$) is a *commutative (or abelian) quasi-residoid* if the properties (qM) and (BB) hold.

Note that a commutative quasi-monoid is a quasi-algebra. By (qAA1), (qM) + (BB) $\Longrightarrow$ (11-1), hence a commutative quasi-residoid is also a quasi-algebra.

• Let us introduce the new quasi-properties:
(qG) (quasi-Gödel) $x \odot x = 1 \to x$, for all $x \in A$;

(qEqP) $x \to y = 1 \Leftrightarrow x \odot y = 1 \to x$,
(qEqP') $x \leq y \Leftrightarrow x \odot y = 1 \to x$,
(qdfrelP) $x \leq y \overset{df.}{\Leftrightarrow} x \odot y = 1 \to x$;
(qP=) $z \to x = 1,\ z \to y = 1 \Longrightarrow (1 \to z) \to (x \odot y) = 1$,
(qP=') $z \leq x,\ z \leq y \Longrightarrow 1 \to z \leq x \odot y$,

and the following special quasi-properties:
(qPI) $x \odot y = (1 \to x) \odot (1 \to y)$;
(qPI1) $x \odot y = (1 \to x) \odot y$;
(qPI2) $x \odot y = x \odot (1 \to y)$.

Remarks 4.26 (See Remarks 3.32 in the regular case) We have, obviously, the following connections:

(i) (EqrelR) $\Longrightarrow$ ((qEqP) $\Leftrightarrow$ (qEqP')); (qEqP) + (qEqP') $\Longrightarrow$ (EqrelR);
(ii) (qEqP) $\Longrightarrow$ ((dfrelR) $\Leftrightarrow$ (qdfrelP)); (dfrelR) + (qdfrelP) $\Longrightarrow$ (qEqP).

Some other connections are presented in the next five Propositions 4.27, 4.28, 4.29, 4.30, 4.31.

Proposition 4.27 *(See the similar Proposition 4.3)*
Let $(A, \odot, \to, 1)$ *or* $(A, \to, \odot, 1)$ *be an algebra of type* $(2, 2, 0)$ *(or, equivalently, by (EqrelR) and (dfrelR), let* $(A, \leq, \odot, \to, 1)$ *or* $(A, \leq, \to, \odot, 1)$ *be a structure).*
Then, we have (with an independent numbering):
(qBBP1) (Pcomm) + (qPI1) $\Longrightarrow$ *(qPI2),*
(qBBP1') (Pcomm) + (qPI2) $\Longrightarrow$ *(qP11),*
(qBBP1") (Pcomm) $\Longrightarrow$ *((qPI1)* $\Leftrightarrow$ *(qPI2));*
(qBBP2) (Pcomm) + (qPI1) $\Longrightarrow$ *(qPI),*
(qBBP2') (Pcomm) + (qPI2) $\Longrightarrow$ *(qPI);*
(qBBP3) (qG) + (Pcomm) + (Passoc) + (qM($\odot$ *))* $\Longrightarrow$ *(qPI);*
(qBBP4) (qPI) + (qM) $\Longrightarrow$ *(qPI1), (qPI2).*

Proof.
(qBBP1): $x \odot (1 \to y) \overset{(Pcomm)}{=} (1 \to y) \odot x \overset{(qPI1)}{=} y \odot x \overset{(Pcomm)}{=} x \odot y$; thus, (qPI2) holds.
(qBBP1'): $(1 \to x) \odot y \overset{(Pcomm)}{=} y \odot (1 \to x) \overset{(qPI2)}{=} y \odot x \overset{(Pcomm)}{=} x \odot y$; thus, (qPI1) holds.
(qBBP1"): By (qBBP1) and (qBBP1').
(qBBP2): $(1 \to x) \odot (1 \to y) \overset{(qPI1)}{=} x \odot (1 \to y) \overset{(Pcomm)}{=} (1 \to y) \odot x \overset{(qPI1)}{=} y \odot x \overset{(Pcomm)}{=} x \odot y$; thus, (qPI) holds.
(qBBP2'): Obviously.
(qBBP3): $(1 \to x) \odot (1 \to y) \overset{(qG)}{=} (x \odot x) \odot (y \odot y) \overset{(Passoc)}{=} x \odot (x \odot y) \odot y \overset{(Pcomm)}{=} x \odot (y \odot (x \odot y)) \overset{(Passoc)}{=} (x \odot y) \odot (x \odot y) \overset{(qG)}{=} 1 \to (x \odot y) \overset{(qM(\odot))}{=} x \odot y$; thus, (qPI) holds.
(qBBP4): $(1 \to x) \odot y \overset{(qPI)}{=} (1 \to (1 \to x)) \odot (1 \to y) \overset{(qM)}{=} (1 \to x) \odot (1 \to y) \overset{(qPI)}{=} x \odot y$; thus, (qPI1) holds.
$x \odot (1 \to y) \overset{(qPI)}{=} (1 \to x) \odot (1 \to (1 \to y)) \overset{(qM)}{=} (1 \to x) \odot (1 \to y) \overset{(qPI)}{=} x \odot y$; thus, (qPI2) holds. □

Proposition 4.28 *Let* $\mathcal{A} = (A, \to, \odot, 1)$ *be an algebra (or, equivalently,* $\mathcal{A} = (A, \leq, \to, \odot, 1)$ *be a structure) or, Galois dually, let* $\mathcal{A} = (A, \odot, \to, 1)$ *be an X-algebra (or* $\mathcal{A} = (A, \leq, \odot, \to, 1)$ *be an X-structure) such that the reduct* $(A, \to, 1)$ *(*$(A, \leq, \to, 1)$*) is an algebra (structure) verifying the properties (Re) and (Tr) (i.e.* $\leq$ *is a pre-order).*
Then, we have (with an independent numbering):
(qBBPR1) (PR) + (Re) + (Eq#) + (qI1) + (qAn) + (qM($\odot$ *))* $\Longrightarrow$ *(qPI);*
(qBBPR1') (PR) + (Re) + (Eq#) + (qI1) + (qAn) + (qM($\odot$ *))* $\Longrightarrow$ *(qPI1);*
(qBBPR1") (PR) + (Re) + (qI1) + (qAn) + (qM($\odot$ *))* $\Longrightarrow$ *(qPI2).*

Proof.
(qBBPR1): $(1 \to x) \odot (1 \to y) \leq a \overset{(PR)}{\Leftrightarrow} 1 \to x \leq (1 \to y) \to a \overset{(qI1)}{=} y \to a \overset{(\#\#)}{\Leftrightarrow} y \leq (1 \to x) \to a \overset{(qI1)}{=} x \to a \overset{(\#\#)}{\Leftrightarrow} x \leq y \to a \overset{(PR)}{\Leftrightarrow} x \odot y \leq a$;
then, by (Re), (qAn), (qM($\odot$)), we obtain that $(1 \to x) \odot (1 \to y) = x \odot y$, i.e. (qPI) holds.
(qBBPR1'): $(1 \to x) \odot y \leq a \overset{(PR)}{\Leftrightarrow} 1 \to x \leq y \to a \overset{(\#\#)}{\Leftrightarrow} y \leq (1 \to x) \to a \overset{(qI1)}{=} x \to a \overset{(\#\#)}{\Leftrightarrow} x \leq y \to a \overset{(PR)}{\Leftrightarrow} x \odot y \leq a$; thus (qPI1) holds.
(qBBPR1"): $x \odot (1 \to y) \leq a \overset{(PR)}{\Leftrightarrow} x \leq (1 \to y) \to a \overset{(qI1)}{=} y \to a \overset{(PR)}{\Leftrightarrow} x \odot y \leq a$; thus (qPI2) holds. □

Proposition 4.29 *(See the Proposition 3.42 in the regular case)*
Under the hypothesis from Proposition 4.28, we have (following the numbering from Proposition 3.42):
(qBP9) (qEqP) $\Longrightarrow$ *((L)* $\Leftrightarrow$ *(qP-1)).*

Proof.
(qBP9): $x \odot 1 \overset{(qP-1)}{=} 1 \to x \overset{(qEqP)}{\Leftrightarrow} x \to 1 \overset{(L)}{=} 1$. □

Proposition 4.30 *(See Proposition 3.43 in the regular case)*
Under the hypothesis from Proposition 4.28, we have (following the numbering from Proposition 3.43):
(qBPR2) (RP) + (Re) + (Ex) + (qAn) + (qM) $\Longrightarrow$ (PEx).

Proof. (qBPR2): Firstly, by (A28), (Ex) $\Longrightarrow$ (##); then $a \leq (x \odot y) \to z \overset{(\#\#)}{\Leftrightarrow} x \odot y \leq a \to z$ and $a \leq x \to (y \to z) \overset{(\#\#)}{\Leftrightarrow} x \leq a \to (y \to z) \overset{(Ex)}{\Leftrightarrow} x \leq y \to (a \to z) \overset{(RP)}{\Leftrightarrow} x \odot y \leq a \to z$; hence $(x \odot y) \to z = x \to (y \to z)$, by (Re), (qAn) and (qM), i.e. (PEx) holds. □

Proposition 4.31 *(See Proposition 3.44 in the regular case)*
Under the hypothesis from Proposition 4.28, we have (following the numbering from Proposition 3.44):
(qBG1) (qG) + (P- -) $\Longrightarrow$ (qP=);
(qBG2) (qG) + (P-) + (Pcomm) + (Pleq) + (K) + (qAn) + (qM) + (qM($\odot$)) + (qI1) $\Longrightarrow$ (qEqP);
(qBG3) (qEqP) $\Longrightarrow$ ((Re) $\Leftrightarrow$ (qG));
(qBG4) (pi) + (Pleq) + (PEx) + (Re) + (qM) + (qM($\odot$)) + (K) + (Tr) + (qAn) + (qPI) $\Longrightarrow$ (qG);
(qBG5) (pimpl-1) + (Pleq) + (PEx) + (Re) + (qM) + (qM($\odot$)) + (K) + (Tr) + (qAn) + (qPI) $\Longrightarrow$ (qG);
(qBG6) (q-impl) + (Pleq) + (PEx) + (Re) + (qM) + (qM($\odot$)) + (**) + (K) + (Tr) + (qAn) + (qPI) $\Longrightarrow$ (qG).

Proof.
(qBG1): $z \leq x$ and $z \leq y$ imply, by (P- -), that $1 \to z \overset{(qG)}{=} z \odot z \leq x \odot y$, i.e. (qP=) holds.
(qBG2): If $x \leq y$, then by (P-), $x \odot x \leq y \odot x$, hence $1 \to x \leq x \odot y$, by (qG) and (Pcomm). We also have that $x \odot y \overset{(Pleq)}{\leq} x \overset{(K)}{\leq} 1 \to x$. Then, by (qAn), $1 \to (x \odot y) = 1 \to (1 \to x)$, hence $x \odot y = 1 \to x$, by (qM) and (qM($\odot$)).
Conversely, if $x \odot y = 1 \to x$, then, $1 \to x \leq y$, by (Pleq), i.e. $(1 \to x) \to y = 1$; hence $x \to y = 1$, by (qI1), i.e. $x \leq y$.
(qBG3): Obviously.
(qBG4): (a) $x \odot x \leq 1 \to x$. Indeed, $x \odot x \overset{(Pleq)}{\leq} x \overset{(K)}{\leq} 1 \to x$, hence $x \odot x \leq 1 \to x$, by (Tr'); thus (a) holds.
(b) $1 \to x \leq x \odot x$. Indeed, $H \overset{notation}{=} (x \odot x) \to y \overset{(qPI)}{=} ((1 \to x) \odot (1 \to x)) \to y \overset{(PEx)}{=} (1 \to x) \to ((1 \to x) \to y) \overset{(pi)}{=} (1 \to x) \to y$; take now $y = x \odot x$ in H; we obtain: $1 \overset{(Re)}{=} (x \odot x) \to (x \odot x) = (1 \to x) \to (x \odot x)$, hence (b) holds.
Finally, (a) + (b) + (qAn) + (qM) + (qM($\odot$)) imply $x \odot x = 1 \to x$, i.e. (G) holds.
(qBG5): (a) $x \odot x \leq x$. Indeed, Indeed, $x \odot x \overset{(Pleq)}{\leq} x \overset{(K)}{\leq} 1 \to x$, hence $x \odot x \leq 1 \to x$, by (Tr'); thus (a) holds.
(b) $1 \to x \leq x \odot x$. Indeed, $(x \odot x) \to z \overset{(qPI)}{=} ((1 \to x) \odot (1 \to x)) \to z \overset{(PEx)}{=} (1 \to x) \to ((1 \to x) \to z) \overset{(pimpl-1)}{\leq} ((1 \to x) \to (1 \to x)) \to ((1 \to x) \to z) \overset{(Re)}{=} 1 \to ((1 \to x) \to z) \overset{(qM)}{=} (1 \to x) \to z$, hence $(x \odot x) \to z \leq (1 \to x) \to z$, which for $z = x \odot x$, gives:
$1 \overset{(Re)}{=} (x \odot x) \to (x \odot x) \leq (1 \to x) \to (x \odot x)$, i.e. $1 \to x \leq x \odot x$, by (qM); thus (b) holds.
Finally, (a) + (b) + (qAn) + (qM) + (qM($\odot$)) imply $x \odot x = 1 \to x$, i.e. (G) holds.
(qBG6): (a) $x \odot x \leq 1 \to x$. Indeed, $x \odot x \overset{(Pleq)}{\leq} x \overset{(K)}{\leq} 1 \to x$, hence $x \odot x \leq 1 \to x$, by (Tr'); thus (a) holds.
(b) $1 \to x \leq x \odot x$. Indeed, by (A0), (Re) $\Longrightarrow$ (S) and since
$[(1 \to x) \to (x \odot x)] \to (1 \to x) \overset{(q-impl)}{=} 1 \to (1 \to x) \overset{(qM)}{=} 1 \to x,$
then, by (S'), we obtain: $[(1 \to x) \to (x \odot x)] \to (1 \to x) \leq 1 \to x$; then, by (**'), we obtain:
$H \overset{notation}{=} (1 \to x) \to [(1 \to x) \to (x \odot x)] \leq$
$([(1 \to x) \to (x \odot x)] \to (1 \to x)) \to [(1 \to x) \to (x \odot x)] \overset{(q-impl)}{=}$
$1 \to [(1 \to x) \to (x \odot x)] \overset{(qM)}{=} (1 \to x) \to (x \odot x)$

and since $H \overset{(PEx)}{=} ((1 \to x) \odot (1 \to x)) \to (x \odot x) \overset{(qPI)}{=} (x \odot x) \to (x \odot x) \overset{(Re)}{=} 1$,
it follows that $1 \leq (1 \to x) \to (x \odot x)$, i.e. $1 \to [(1 \to x) \to (x \odot x)] = 1$, hence
$(1 \to x) \to (x \odot x) = 1$, by (qM), i.e. (b) holds.
Finally, (a) + (b) + (qAn) + (qM) + (qM($\odot$)) imply $x \odot x = 1 \to x$, i.e. (qG) holds. □

We now obtain the following Galois dual Propositions 4.32 and 4.34 and their corollaries.

Proposition 4.32 *(See Proposition 3.45 in the regular case)*
Let $\mathcal{A} = (A, \to, \odot, 1)$ ($\mathcal{A} = (A, \leq, \to, \odot, 1)$) be an algebra (structure) with (RP) (i.e. (Re), (Tr) and (RP) hold), verifying also the properties (qAn), (qM), (qM($\odot$)), (Ex).
Then $\mathcal{A}$ is an abelian quasi-monoid verifying (PEx).

Proof. Firstly, since (qM) holds and (Re) $\Longrightarrow$ (11-1), then $(A, \to, 1)$ is a quasi-algebra. Then, by (A28), (Ex) $\Longrightarrow$ (Eq#) and we have:

(Pcomm): $x \odot y \leq a \overset{(RP)}{\Leftrightarrow} x \leq y \to a \overset{(\#\#)}{\Leftrightarrow} y \leq x \to a \overset{(RP)}{\Leftrightarrow} y \odot x \leq a$. Then $1 \to (x \odot y) = 1 \to (y \odot x)$, by (Re) and (qAn); hence $x \odot y = y \odot x$, by (qM($\odot$)).

(Passoc): $(x \odot y) \odot z \leq a \overset{(RP)}{\Leftrightarrow} x \odot y \leq z \to a \overset{(RP)}{\Leftrightarrow} x \leq y \to (z \to a) \overset{(\#\#)}{\Leftrightarrow} y \leq x \to (z \to a) \overset{(Ex)}{\Leftrightarrow} y \leq z \to (x \to a)$ and
$x \odot (y \odot z) \leq a \overset{(RP)}{\Leftrightarrow} x \leq (y \odot z) \to a \overset{(\#\#)}{\Leftrightarrow} y \odot z \leq x \to a \overset{(RP)}{\Leftrightarrow} y \leq z \to (x \to a)$.
Thus, $(x \odot y) \odot z = x \odot (y \odot z)$, by (Re), (qAn), (qM($\odot$)).

(qP1-1): $x \odot 1 \leq a \overset{(RP)}{\Leftrightarrow} x \leq 1 \to a \overset{(\#\#)}{\Leftrightarrow} 1 \leq x \to a \overset{(qM)}{\Leftrightarrow} x \leq a$. Then, $1 \to (x \odot 1) = 1 \to x$, by (Re) and (qAn); then $x \odot 1 = 1 \to x$, by (qM($\odot$)).

$1 \odot x \leq a \overset{(RP)}{\Leftrightarrow} 1 \leq x \to a \overset{(qM)}{\Leftrightarrow} x \leq a$. Thus, $1 \odot x = 1 \to x$, by (Re), (qAn), (qM($\odot$)).

Thus, $\mathcal{A}$ is an abelian quasi-monoid. Finally, by (qBPR2), (RP) + (Re) + (Ex) + (qAn) + (qM) $\Longrightarrow$ (PEx); thus, (PEx) holds. □

Corollary 4.33 *(See Corollary 3.46 in the regular case)*
Let $\mathcal{A} = (A, \to, \odot, 1)$ ($\mathcal{A} = (A, \leq, \to, \odot, 1)$) be an algebra (structure) with (RP) (i.e. (Re), (Tr) and (RP) hold), such that the reduct $(A, \to, 1)$ is an abelian quasi-residoid (i.e. (qM), (BB) hold) verifying also (qAn), (D), (qM($\odot$)).
Then, $\mathcal{A}$ is an abelian quasi-monoid.

Proof. By (qA21), (BB) + (D) + (qM) + (qAn) $\Longrightarrow$ (Ex). Then apply the above Proposition 4.32. □

Proposition 4.34 *(See Proposition 3.47 in the regular case)*
Let $\mathcal{A} = (A, \odot, \to, 1)$ ($\mathcal{A} = (A, \leq, \odot, \to, 1)$) be an X-algebra (X-structure) with (PR) (i.e. (Re), (Tr), (PR)=(RP) hold), verifying also the properties (qR1) and (Pcomm), (Passoc), (qP1-1).
Then, the following properties hold: (qM), (PEx), (PBB).

Proof. By Lemma 3.35, properties (R), (RR), (P-) hold.

(qM): $1 \to (x \to y) = x \to y \overset{(R)}{\Longleftrightarrow} \max\{y \mid y \odot 1 \leq x \to y\} = x \to y \overset{(qP1-1)}{\Longleftrightarrow} \max\{y \mid 1 \to y \leq x \to y\} = x \to y$, which is true, by (qR1), since $(1 \to (x \to y)) \to (x \to y) = 1$. Thus (qM) holds.
Since (Re) implies (11-1), then $(A, \to, 1)$ is a quasi-algebra.

The rest of the proof is like in the regular case. □

Corollary 4.35 *(See Corollary 3.48 in the regular case)*
Let $\mathcal{A} = (A, \odot, \to, 1)$ ($\mathcal{A} = (A, \leq, \odot, \to, 1)$) be an X-algebra (X-structure) with (PR) (i.e. (Re), (Tr) and (PR) hold), verifying also the properties (qR1) and (Pcomm), (Passoc), (qP1-1). Then, the reduct $(A, \to, 1)$ is an abelian quasi-residoid (i.e. (qM) and (BB) hold) verifying (Ex), (B), (D).

Proof. By Proposition 4.34, the properties (qM), (PEx), (PBB) hold. Since (qM) holds and (Re) implies (11-1), it follows that $\mathcal{A}$ is a quasi-algebra.

By (BP2), (PEx) $\Longrightarrow$ ((PBB) $\Leftrightarrow$ (BB)), hence (BB) holds. Thus, the reduct $(A, \rightarrow, 1)$ ($(A, \leq, \rightarrow, 1)$) is an abelian quasi-residoid.

The rest of the proof is like in the regular case. □

• The equivalence between some quasi-algebras with (RP) and some quasi-X-algebras with (PR)

The announced equivalence is the following:

Theorem 4.36 *(See Theorem 3.49 in the regular case)*

(1) Let $\mathcal{A} = (A, \rightarrow, \odot, 1)$ be an algebra (or $\mathcal{A} = (A, \leq, \rightarrow, \odot, 1)$ be a structure) with (RP) (i.e. (Re), (Tr) and (RP) hold), verifying also the properties (qAn), (D) and (qM($\odot$)), and such that the reduct $(A, \rightarrow, 1)$ ($(A, \leq, \rightarrow, 1)$) is an abelian quasi-residoid (i.e. (qM) and (BB) hold). Define

$$\alpha(\mathcal{A}) \stackrel{df.}{=} (A, \odot, \rightarrow, 1) \quad (\alpha(\mathcal{A}) \stackrel{df.}{=} (A, \leq, \odot, \rightarrow, 1)).$$

Then, $\alpha(\mathcal{A})$ is an X-algebra (X-structure) with (PR) (i.e. (Re), (Tr), (PR)=(RP) hold) that is a commutative quasi-monoid.

(1') Conversely, let $\mathcal{A} = (A, \odot, \rightarrow, 1)$ be an X-algebra (or $\mathcal{A} = (A, \leq, \odot, \rightarrow, 1)$) be an X-structure) with (PR) (i.e. (Re), (Tr) and (PR) hold) that is a commutative quasi-monoid verifying (qR1). Define

$$\beta(\mathcal{A}) \stackrel{df.}{=} (A, \rightarrow, \odot, 1) \quad (\beta(\mathcal{A}) \stackrel{df.}{=} (A, \leq, \rightarrow, \odot, 1)).$$

Then, $\beta(\mathcal{A})$ is an algebra (structure) with (RP) (i.e. (Re), (Tr) and (RP) hold) such that the reduct $(A, \rightarrow, 1)$ is an abelian quasi-residoid verifying (Ex), (B), (D).

(2) The above defined mappings are mutually inverse.

Proof.

(1): By Corollary 4.33.

(1'): By Corollary 4.35.

(2): Obviously. □

Note that $\leq$ in Theorem 4.36 is a quasi-order relation and that the algebras and the X-algebras involved are quasi.

4.4.2 Quasi-meet-semilattices from quasi-algebras with product

We obtained the following two Galois dual theorems 4.37 and 4.39:

Theorem 4.37 *(See Theorem 3.50 in the regular case)*

Let $\mathcal{A} = (A, \rightarrow, \odot, 1)$ ($\mathcal{A} = (A, \leq, \rightarrow, \odot, 1)$) be an algebra (structure) with (PR) (i.e. (Re), (Tr) and (RP) hold), verifying also the properties (qAn), (qM), (qM($\odot$)), (K), (Ex) and (qG).
Define $\wedge \stackrel{df.}{=} \odot$.
Then, $(A, \rightarrow, \wedge, 1)$ ($(A, \leq, \rightarrow, \wedge, 1)$ is a quasi-$\wedge$-semilattice with top element 1.

Proof. We must prove that (qM), (11-1), (qM($\wedge$)), (qEqW), (qWid), (Wcomm), (Wassoc), (qW1-1) hold. Indeed,
(qM) holds and (11-1) follows by (Re), hence $\mathcal{A}$ is a quasi-algebra. (qM($\wedge$)) is (qM($\odot$)).
Then, since (qAn), (qM), (qM($\odot$)), (Ex) hold, by Proposition 4.32, $(A, \rightarrow, \odot, 1)$ is an abelian quasi-monoid, i.e. properties (Pcomm), (Passoc), (qP1-1) hold. Then,
- by (BPR1'), (RP) + (Pcomm) + (K) $\Longrightarrow$ (Pleq);
- by Lemma 3.35, (Re) + (Tr) + (PR) $\Longrightarrow$ (P-);
- by (A24), (Re) + (Ex) + (Tr) $\Longrightarrow$ (**); (Re) $\Longrightarrow$ (qRe($1 \rightarrow x$)); by (qAA7), (qRe($1 \rightarrow x$)) + (Ex) + (K) + (**) + (qAn) + (qM) $\Longrightarrow$ (qI1); and by (qBG2), (qG) + (P-) + (Pcomm) + (Pleq) + (K) + (qAn)+ (qM) + (qM($\odot$)) + (qI1) $\Longrightarrow$ (qEqP).
Now, (qEqW) is (qEqP), (qWid) is (qG), (Wcomm) is (Pcomm), (Wassoc) is (Passoc), (qW1-1) is (qP1-1).
□

Corollary 4.38 *Let $\mathcal{A} = (A, \to, \odot, 1)$ ($\mathcal{A} = (A, \leq, \to, \odot, 1)$) be an algebra (structure) with (RP) (i.e. (Re), (Tr) and (RP) hold), such that the reduct $(A, \to, 1)$ is an abelian quasi-residoid (i.e. (qM), (BB) hold) and the properties (qAn), (L), (D), (qI1), (qM($\odot$)) and (qG) hold.*
Define $\wedge \stackrel{df.}{=} \odot$.
Then, $(A, \to, \wedge, 1)$ ($(A, \leq, \to, \wedge, 1)$ is a quasi-$\wedge$-semilattice with top element 1.

Proof. By (qA21), (qM) + (BB) + (D) + (qAn) $\Longrightarrow$ (Ex) and by (qAA14), (qM) + (L) + (BB) + (qI1) $\Longrightarrow$ (K). Since (qAn), (qM), (qM($\odot$)), (K), (Ex), (qG) hold, now apply Theorem 4.37. □

Theorem 4.39 *(See Theorem 3.52 in the regular case)*
Let $\mathcal{A} = (A, \odot, \to, 1)$ ($\mathcal{A} = (A, \leq, \odot, \to, 1)$) be an X-algebra (X-structure) with (PR) (i.e. (Re), (Tr) and (PR) hold), verifying also the properties (qAn), (qM), (qM($\odot$)), (K), (qI1) and (Pcomm), (Passoc), (qP1-1) and (qG).
Define $\wedge \stackrel{df.}{=} \odot$.
Then, $(A, \wedge, \to, 1)$ ($(A, \leq, \wedge, \to, 1)$) is a quasi-$\wedge$-semilattice with top element 1.

Proof. Indeed, if we denote $\wedge \stackrel{df.}{=} \odot$, then note that: (qWid) is (qG), (Wcomm) is (Pcomm), (Wassoc) is (Passoc), (qW1-1) is (qP1-1) and (qM($\wedge$)) is (qM($\odot$)), hence $\wedge$ verifies the properties (qWid), (Wcomm), (Wassoc), (qW1-1), (qM($\wedge$)).
Then, since the property (qM) holds and the property (11-1) follows by (Re), $\mathcal{A}$ is a quasi-algebra.
By (BPR1'), (RP) + (Pcomm) + (K) $\Longrightarrow$ (Pleq); by Lemma 3.35, the property (P-) holds. Now, by (qBP2), (qG) + (P-) + (Pcomm) + (Pleq) + (K) + (qAn) + (qM) + (qM($\odot$)) + (qI1) $\Longrightarrow$ (qEqP). Thus, by the above definition of $\wedge$, it follows that (qEqW) holds too. □

4.5 Commutative quasi-algebras (quasi-structures) with product

In the commutative quasi-algebras (quasi-structures) (i.e. (qM) and (11-1) hold) having product, we can define a new operation, the join $\vee$, by (dfV); then, by (BdfV1), (dfV) + (comm) $\Longrightarrow$ (Vcomm), and by (qBBdfV1), (dfV) + (qM) $\Longrightarrow$ (qM($\vee$)); thus, (Vcomm) and (qM($\vee$)) hold.
In certain conditions, we can obtain a quasi-$\vee$-semilattice with top element (see Theorem 4.24).

Open problem 4.40 *It is an open problem to find conditions (if they exist) in which, by defining $\wedge \stackrel{df.}{=} \odot$, or otherwise, we can obtain a quasi-$\wedge$-semilattice with top element from a commutative quasi-algebra with product.*

• Let us introduce the new quasi-properties:
(qPabs1) (qP-absorption-1) $x \odot (x \vee y) = 1 \to x$,
(qPabs2) (qP-absorption-2) $x \vee (x \odot y) = 1 \to x$.

Then, we have the follwing Propositions 4.41, similar to Proposition 4.14.

Proposition 4.41 *(See Proposition 3.54 in the regular case)*
Let $\mathcal{A} = (A, \vee, \to, \odot, 1)$ be an algebra of type $(2, 2, 2, 0)$ (or, equivalently, $\mathcal{A} = (A, \leq, \vee, \to, \odot, 1)$ be a structure) or, Galois dually, let $\mathcal{A} = (A, \odot, \to, \vee, 1)$ be an algebra of type $(2, 2, 2, 0)$ (or $\mathcal{A} = (A, \leq, \odot, \to, \vee, 1)$ be a structure) such that the reduct $(A, \to, 1)$ ($(A, \leq, \to, 1)$) is an algebra (structure) verifying the properties (Re) and (Tr) (i.e. $\leq$ is a pre-order).
Then, we have (following the numbering from Proposition 3.54):
(qBVP1) (Pdis1-p) + (Pdis1-pp) + (qAn) + (qM($\vee$)) + (qM($\odot$)) $\Longrightarrow$ (Pdis1);
(qBVP1') (Vgeq) + (qV=) + (P-) + (qM($\odot$)) $\Longrightarrow$ (Pdis1-p);
(Pleq) + (qP=) + (qPI2) + (V- -) + (qV=) + (qM($\vee$)) $\Longrightarrow$ (Pdis1-p);
(qBVP1") (RP) + (Vgeq) + (qV=) + (qM) $\Longrightarrow$ (Pdis1-pp);

(qBVP2) (Pdis2-p) + (Pdis2-pp) + (qAn) + (qM($\vee$)) + (qM($\odot$)) $\Longrightarrow$ (Pdis2);

(qBVP2') (Pleq) + (qP=) + (V-) + (qM(∨)) $\Longrightarrow$ *(Pdis2-p);*
(Vgeq) + (qV=) + (qVI2) + (P- -) + (qP=) + (qM(⊙)) $\Longrightarrow$ *(Pdis2-p);*
(qBVP2") (Pdis1) + (Pleq) + (Pcomm) + (Vcomm) + (Vassoc) + (V-) + (qV=) + (qVI2) $\Longrightarrow$ *(Pdis2-pp);*

(qBVP3) (Vgeq) + (qEqP) $\Longrightarrow$ *(qPabs1);*
(qBVP4) (Pleq) + (qEqV) + (Vcomm) $\Longrightarrow$ *(qPabs2).*

Proof.
(qBVP1): Obviously.
(qBVP1'): First proof: $x, y \le x \vee y$, by (Vgeq'); then, $x \odot z,\ y \odot z \le (x \vee y) \odot z$, by (P-), hence $(x \odot z) \vee (y \odot z) \overset{(qV=)}{\le} 1 \to [(x \vee y) \odot z] \overset{(qM(\odot))}{=} (x \vee y) \odot z$; thus, (Pdis1-p) holds.
Second proof: On the one hand, we have $x \odot z \le x$ and $y \odot z \le y$, by (Pleq); then, $(x \odot z) \vee (y \odot z) \le x \vee y$, by (V- -). On the other hand, we have $x \odot z \le z$ and $y \odot z \le z$, by (Pleq); then, $(x \odot z) \vee (y \odot z) \le 1 \to z$, by (qV=). Consequently, $x \odot z) \vee (y \odot z) \overset{(qM(\vee))}{=} 1 \to [x \odot z) \vee (y \odot z)] \overset{(qP=)}{\le} (x \vee y) \odot (1 \to z) \overset{(qPI2)}{=} (x \vee y) \odot z$; thus, (Pdis1-p) holds.
(qBVP1"): Denote $Z \overset{notation}{=} (x \odot z) \vee (y \odot z)$; then $x \odot z, y \odot z \le Z$, by (Vgeq'); then, $x \le z \to Z$ and $y \le z \to Z$, by (RP); then, $x \vee y \overset{(qV=)}{\le} 1 \to [z \to Z] \overset{(qM)}{=} z \to Z$; hence $(x \vee y) \odot z \le Z$, by (RP), i.e. (Pdis1-pp) holds.
(qBVP2): Obviously.
(qBVP2'): First proof: $x \odot y \le x,\ y$, by (Pleq); then, $(x \odot y) \vee z \le x \vee z,\ y \vee z$, by (V-), hence $(x \odot y) \vee z \overset{(qM(\vee))}{=} 1 \to [(x \odot y) \vee z] \overset{(qP=)}{\le} (x \vee z) \odot (y \vee z)$; thus, (Pdis2-p) holds.
Second proof: On the one hand, we have: $x \le x \vee z$ and $y \le y \vee z$, by (Vgeq); then, $x \odot y \le (x \vee z) \odot (y \vee z)$, by (P- -). On the other hand, we have $z \le x \vee z$ and $z \le y \vee z$, by (Vgeq); then, $1 \to z \le (x \vee z) \odot (y \vee z)$, by (qP=). Consequently, $x \odot y) \vee z \overset{(qVI2)}{=} (x \odot y) \vee (1 \to z) \overset{(qV=)}{\le} 1 \to [(x \vee z) \odot (y \vee z)] \overset{(qM(\odot))}{=} (x \vee z) \odot (y \vee z)$; thus, (Pdis2-p) holds.
(qBVP2"): Denote $Z \overset{notation}{=} (x \vee z) \odot (y \vee z)$; then
$Z \overset{(Pdis1)}{=} (x \odot (y \vee z)) \vee (z \odot (y \vee z))$
$\overset{(Pcomm)}{=} ((y \vee z) \odot x) \vee ((y \vee z) \odot z)$
$\overset{(Pdis1)}{=} ((y \odot x) \vee (z \odot x)) \vee ((y \odot z) \vee (z \odot z))$
$\overset{(Vassoc)}{=} (y \odot x) \vee [(z \odot x) \vee (y \odot z) \vee (z \odot z)]$
$\overset{(Vcomm)}{=} [(z \odot x) \vee (y \odot z) \vee (z \odot z)] \vee (y \odot x)$.
But $z \odot x \le z$, $y \odot z \le z$, $z \odot z \le z$, by (Pleq);
hence, $(z \odot x) \vee (y \odot z) \vee (z \odot z) \le 1 \to z$, by (qV=);
hence, $Z \overset{(V-)}{\le} (1 \to z) \vee (y \odot x) \overset{(Pcomm),(Vcomm)}{=} (x \odot y) \vee (1 \to z) \overset{(qVI2)}{=} (x \odot y) \vee z$; thus (Pdis2-pp) holds.
(qBVP3): $x \overset{(Vgeq)}{\le} x \vee y \overset{(qEqP')}{\Leftrightarrow} x \odot (x \vee y) = 1 \to x$; thus (qPabs1) holds.
(qBVP4): $x \odot y \overset{(Pleq)}{\le} x \overset{(qEqV)}{\Leftrightarrow} (x \odot y) \vee x = 1 \to x$, hence $x \vee (x \odot y) = 1 \to x$, by (Vcomm); thus, (qPabs2) holds. □

Note that the two Propositions 4.14 and 4.41 are quite identique, if $\odot = \wedge$, and are only similar, if $\odot \neq \wedge$.

4.6 Positive implicative quasi-algebras (quasi-structures) with product.

In this subsection, we shall study the quasi-algebras (quasi-structures) having product and being positive implicative.

We have obtained the following two Galois dual Theorems 4.42 and 4.43.

Theorem 4.42 *(See Theorem 3.56 in the regular case)*

Let $\mathcal{A} = (A, \to, \odot, 1)$ ($\mathcal{A} = (A, \leq, \to, \odot, 1)$) be an algebra (structure) with (PR) (i.e. (Re), (Tr) and (RP) hold), verifying also the properties (qAn), (qM), (qM($\odot$)), (qPI), (K), (Ex) and (pimpl-1). Define $\wedge \stackrel{df.}{=} \odot$.
Then, $(A, \wedge, \to, 1)$ ($(A, \wedge, \leq, \to, 1)$) is a quasi-$\wedge$-semilattice with top element 1.

Proof. Since (qM) holds and (Re) implies (11-1), it follows that $(A, \to, 1)$ is a quasi-algebra.
By (A6), (Re) + (K) $\Longrightarrow$ (L).
By (A24), (Re) + (Ex) + (Tr) $\Longrightarrow$ (**); by (B13), (Re) + (L) + (Ex) + (**) $\Longrightarrow$ (pimpl-2) and by (qB0), (pimpl-1) + (pimpl-2) + (qAn) + (qM) $\Longrightarrow$ (pimpl), hence $\mathcal{A}$ is positive implicative.
Now,
- by (BPR1), (RP) + (K) + (Re) + (L) $\Longrightarrow$ (Pleq);
- by (qBPR2), (RP) + (Re) + (Ex) + (qAn) + (qM) $\Longrightarrow$ (PEx);
- by (qBG5), (pimpl-1) + (Pleq) + (PEx) + (Re) + (qM) + (qM($\odot$)) + (K) + (Tr) + (qAn) + (qPI) $\Longrightarrow$ (qG).
Since (qAn), (qM), (qM($\odot$)), K, (Ex), (qG) hold, now apply Theorem 4.37. $\Box$

Theorem 4.43 *(See Theorem 3.57 in the regular case)*

Let $\mathcal{A} = (A, \odot, \to, 1)$ ($\mathcal{A} = (A, \odot, \leq, \to, 1)$) be an X-algebra (X-structure) with (RP) (i.e. (Re), (TR) and (PR)=(RP) hold), such that the properties (Pcomm), (Passoc), (qP1-1), (qAn), (qM($\odot$)), (qR1) and (pimpl) (or (pi)) also hold.
Define $\wedge \stackrel{df.}{=} \odot$.
Then, $(A, \wedge, \to, 1)$ ($(A, \wedge, \leq, \to, 1)$) is a quasi-$\wedge$-semilattice with 1.

Proof. By Proposition 4.34, since (qR1), (Pcomm), (Passoc), (qP1-1) hold, then the properties (qM), (PEx), (PBB) hold. Since (Re) implies (11-1) and (qM) holds, it follows that $\mathcal{A}$ is a quasi-X-algebra.
By Corollary 4.35, the reduct $(A, \to, 1)$ is an abelian quasi-residoid (i.e. (qM) and (BB) hold) verifying (Ex), (B), (D).
By (qB35), (Re) + (Ex) + (B) + (qM) + (qAn) $\Longrightarrow$ ((pimpl) $\Leftrightarrow$ (pi)).
Now, by (B1), (pimpl) + (Re) $\Longrightarrow$ (L); by (A8), (Re) + (Ex) + (L) $\Longrightarrow$ (K).
By (BPR1'), (RP) + (Pcomm) + (K) $\Longrightarrow$ (Pleq).
By (A0), (Re) $\Longrightarrow$ (S) and (pimpl) + (S) $\Longrightarrow$ (pimpl-1).
By (A24), (Re) + (Ex) + (Tr) $\Longrightarrow$ (**), and by (qAA6), (qR1) + (K) + (**) + (qM) + (qAn) $\Longrightarrow$ (qI1).
By (A28), (Ex) $\Longrightarrow$ (Eq#), and by (qBBPR1), (PR) + (Re) + (qI1) + (qAn) + (qM($\odot$)) + (Eq#) $\Longrightarrow$ (qPI).
Now, by (qBG5), (pimpl-1) + (Pleq) + (PEx) + (Re) + (K) + (Tr) + (qM) + (qAn) + (qM($\odot$)) + (qPI) $\Longrightarrow$ (qG).
Finally, since (qAn), (qM), (qM($\odot$)), (K), (qI1), (Pcomm), (Passoc), (qP1-1) and (qG) hold, now apply Theorem 4.39. $\Box$

4.7 Quasi-implicative algebras (structures) with product

In this section, we shall study the algebras (structures) having product and being implicative.

We have obtained the following two Galois dual Theorems 4.44 and 4.45.

Theorem 4.44 *(See Theorem 3.58 in the regular case)*

Let $\mathcal{A} = (A, \to, \odot, 1)$ ($\mathcal{A} = (A, \leq, \to, \odot, 1)$) be an algebra (structure) with (RP) (i.e. (Re), (Tr) and (RP) hold), such that the reduct $(A, \to, 1)$ is an abelian quasi-residoid (i.e. (qM), (BB) hold) verifying (qAn), (qM($\odot$)), (D), (qI1) and (q-impl).
Define $x \vee y \stackrel{(dfV)}{=} (x \to y) \to y$ and $\wedge \stackrel{df.}{=} \odot$.
Then, $(A, \vee, \to, \wedge, 1)$ ($(A, \leq, \vee, \to, \wedge, 1)$) is a distributive Dedekind quasi-$\vee\wedge$-lattice with last element 1.

Proof. We must prove that (qM), (11-1); (qM($\wedge$)), (qEqW), (qWid), (Wcomm), (Wassoc), (qW1-1); (qM($\vee$)), (qEqV), (qVid), (Vcomm), (Vassoc), (V1-1); (qWabs1), (qWabs2); (Wdis1), (Wdis2) hold.

(qM) holds and (Re) implies (11-1); hence, $\mathcal{A}$ is a quasi-algebra. We must prove the rest. Indeed, we have:
- by (qA21), (qM) + (BB) + (D) + (qAn) $\Longrightarrow$ (Ex);
- by (A10"), (Ex) + (BB) $\Longrightarrow$ (B);
- by (qBIM3), (q-impl) + (Ex) + (B) + (qAn) + (qM) $\Longrightarrow$ (comm);
- by (qBIM5), (q-impl) + (11-1) $\Longrightarrow$ (qL($1 \to x$));
- by (qAA24), (qI1) $\Longrightarrow$ ((qL($1 \to x$)) $\Leftrightarrow$ (L)), hence (L) holds; - by (A8), (Re) + (L) + (Ex) $\Longrightarrow$ (K).
Then, by Theorem 4.24, since (comm), (Re), (qM), (K), (Ex), (qAn) hold, then $(A, \vee, \to, 1)$ $((A, \leq, \vee, \to, 1))$ is a quasi-$\vee$-semilattice with 1, i.e. (qM), (11-1), (qM($\vee$)), (qEqV), (qVid), (Vcomm), (Vassoc), (V1-1) hold. Then, by Theorem 4.5, (V-), (V- -), (Vgeq), (qV=) also hold.

By (BPR1), (PR) + (K) + (Re) + (L) $\Longrightarrow$ (Pleq);
- by (qBPR2), (PR) + (Re) + (Ex) + (qAn) + (qM) $\Longrightarrow$ (PEx);
- by (qA15'), (qM) + (BB) $\Longrightarrow$ (**);
- by (A28), (Ex) $\Longrightarrow$ (Eq#), and by (qBBPR1), (PR) + (Re) + (Eq#) + (qI1) + (qAn) + (qM($\odot$)) $\Longrightarrow$ (qPI);
- by (qBG6), (q-impl) + (Pleq) + (PEx) + (Re) + (qM) + (**) + (qM($\odot$)) + (K) + (Tr) + (qAn) + (qPI) $\Longrightarrow$ (qG).
Since (qM), (BB), (qAn), (qM($\odot$)), (L), (D), (qI1) and (qG) hold, then, by Corollary 4.38, $(A, \to, \wedge, 1)$ $((A, \leq, \to, \wedge, 1))$ is a quasi-$\wedge$-semilattice with top element 1, i.e. (qM), (11-1), (qM($\wedge$)) $\equiv$ (qM($\odot$)), (qEqW) $\equiv$ (qEqP), (qWid) $\equiv$ (qG), (Wcomm) $\equiv$ (Pcomm), (Wassoc) $\equiv$ (Passoc), (qW1-1) $\equiv$ (qP1-1) hold. Then, by Theorem 4.12, (W-) $\equiv$ (P-), (W- -) $\equiv$ (P- -), (Wleq) $\equiv$ (Pleq), (qW=) $\equiv$ (qP=) also hold.

Resuming, $(A, \vee, \to, \wedge, 1)$ $((A, \leq, \vee, \to, \wedge, 1))$ is both a quasi-$\vee$-semilattice and a quasi-$\wedge$-semilattice with last element 1.

Now, by (qBVP3), (Vgeq) + (qEqP) $\Longrightarrow$ (qPabs1) and by (qBVP4), (Pleq) + (qEqV) + (Vcomm) $\Longrightarrow$ (qPabs2); hence, (qWabs1) and (qWabs2) hold. Thus, $\mathcal{A}$ is a Dedekind quasi-$\vee\wedge$-lattice with 1.

Finally, we prove the distributivity:
- By (qBVP1'), (Vgeq) + (qV=) + (P-) + (qM($\odot$)) $\Longrightarrow$ (Pdis1-p);
- by (qBVP1"), (RP) + (Vgeq) + (qV=) + (qM) $\Longrightarrow$ (Pdis1-pp);
then, by (qBVP1), (Pdis1-p) + (Pdis1-pp) + (qAn) + (qM($\vee$)) + (qM($\odot$)) $\Longrightarrow$ (Pdis1); thus (Wdis1) holds.
- By (qBVP2'), (Pleq) + (qP=) + (V-) + (qM($\vee$)) $\Longrightarrow$ (Pdis2-p);
- by (qBBV3), (qVid) + (Vcomm) + (Vassoc) + (qM($\vee$)) $\Longrightarrow$ (qVI); by (qBBV4), (qVI) + (qM) $\Longrightarrow$ (qVI2);
- by (qBVP2"), (Pdis1) + (Pleq) + (Pcomm) + (Vcomm) + (Vassoc) + (V-) + (qV=) + (qVI2) $\Longrightarrow$ (Pdis2-pp);
then, by (qBVP2), (Pdis2-p) + (Pdis2-pp) + (qAn) + (qM($\vee$)) + (qM($\odot$)) $\Longrightarrow$ (Pdis2), i.e. (Wdis2) also holds. □

Theorem 4.45 *(See Theorem 3.59)*

Let $\mathcal{A} = (A, \odot, \to, 1)$ ($\mathcal{A} = (A, \leq, \odot, \to, 1)$ be an X-algebra (X-structure) with (RP) (i.e. (Re), (Tr), (RP)=(PR) hold), such that (Pcomm), (Passoc), (qP1-1) and also (K), (qAn), (qM($\odot$)), (qR1) and (q-impl) hold.
Define $\wedge \overset{df.}{=} \odot$ and $x \vee y \overset{(dfV)}{=} (x \to y) \to y$.
Then $\mathcal{A} = (A, \wedge, \to, \vee, 1)$ ($\mathcal{A} = (A, \leq, \wedge, \to, \vee, 1)$) is a distributive Dedekind quasi-$\wedge\vee$-lattice with 1.

Proof. We must prove that (qM), (11-1); (qM($\wedge$)), (qEqW), (qWid), (Wcomm), (Wassoc), (qW1-1); (qM($\vee$)), (qEqV), (qVid), (Vcomm), (Vassoc), (V1-1); (qWabs1), (qWabs2); (Wdis1), (Wdis2) hold.

By Proposition 4.34, the properties (qM), (PEx), (PBB) hold. Since (qM) holds and (Re) implies (11-1), then $(A, \to, 1)$ is a quasi-algebra, hence $\mathcal{A}$ is an abelian quasi-monoid.

By Corollary 4.35, the properties (BB), (Ex), (B), (D) hold.
- By (BPR1'), (PR) + (Pcomm) + (K) $\Longrightarrow$ (Pleq);
- by (qA15'), (qM) + (BB) $\Longrightarrow$ (**);
- by (qAA6), (qR1) + (K) + (**) + (qM) + (qAn) $\Longrightarrow$ (qI1);

- by (A28), (Ex) $\Longrightarrow$ (Eq#), and by (qBBPR1), (PR) + (Re) + (Eq#) + (qI1) + (qAn) + (qM($\odot$)) $\Longrightarrow$ (qPI);
- by (qBG6), (q-impl) + (Pleq) + (PEx) + (Re) + (qM) + (**) + (qM($\odot$)) + (K) + (Tr) + (qAn) + (qPI) $\Longrightarrow$ (qG).
Since the properties (K), (qAn), (qM), (qM($\odot$)), (qI1), (Pcomm), (Passoc), (qP1-1) and (qG) hold, then, by Theorem 4.39, $(A, \wedge, \rightarrow, 1)$ is a quasi-$\wedge$-semilattice with 1, i.e. (qM), (11-1), (qM($\wedge$)) $\equiv$ (qM($\odot$)), (qEqW) $\equiv$ (qEqP), (qWid) $\equiv$ (qG), (Wcomm) $\equiv$ (Pcomm), (Wassoc) $\equiv$ (Passoc), (qW1-1) $\equiv$ (qP1-1) hold. Then, by Theorem 4.12, (W-) $\equiv$ (P-), (W- -) $\equiv$ (P- -), (Wleq) $\equiv$ (Pleq), (qW=) $\equiv$ (qP=) also hold.

By (qBIM3), (q-impl) + (Ex) + (B) + (qAn) + (qM) $\Longrightarrow$ (comm).
Then, by Theorem 4.24, since (comm), (Re), (qM), (K), (Ex), (qAn) hold, then $(A, \vee, \rightarrow, 1)$ ($(A, \leq, \vee, \rightarrow, 1)$) is a quasi-$\vee$-semilattice with 1, i.e. (qM), (11-1), (qM($\vee$)), (qEqV), (qVid), (Vcomm), (Vassoc), (V1-1) hold. Then, by Theorem 4.5, (V-), (V- -), (Vgeq), (qV=) also hold.

The rest of the proof is the same as the proof of Theorem 4.44. □

4.8 Other new quasi-algebras

We shall introduce the positive implicative, commutative, quasi-implicative quasi-BCK algebras and the quasi-Hilbert algebras.

4.8.1 Positive implicative, commutative, q-implicative quasi-BCK algebras

Definition 4.46 Let $\mathcal{A} = (A, \rightarrow, 1)$ be a quasi-BCK algebra. We say that $\mathcal{A}$ is
- *positive implicative*, if property (pimpl) is satisfied;
- *commutative*, if property (comm) is satisfied;
- *quasi-implicative*, if property (q-impl) is satisfied.

Hence we obtain the following results.

Theorem 4.47 *(See Theorem 3.62 in the regular case)*
A quasi-BCK algebra is positive implicative if and only if the property (pi) holds if and only if the property (q-pi) holds (or, in a quasi-BCK algebra, we have $(pimpl) \iff (pi) \iff (q-pi)$*).*

Proof: By (qB35), (Re) + (Ex) + (B) + (qM) + (qAn) $\Longrightarrow$ ((pimpl) $\Leftrightarrow$ (pi)).
By (qB35'), (Re) + (Ex) + (B) + (qM) + (qI1) + (qAn) $\Longrightarrow$ ((pimpl) $\Leftrightarrow$ (q-pi)). □

Note that, since (comm) $\Longrightarrow$ (qAn), by (qB12), it follows that a commutative quasi-BCK algebra can be defined equivalently as an algebra $(A, \rightarrow, 1)$ verifying the properties (comm), (qM), (B), (C), (K). Hence, we have the following "quasi-result", that generalizes Yutani's result [28] from the the regular case:

Theorem 4.48 *The class of commutative quasi-BCK algebras is a variety.*

Open problem 4.49 Find if Yutani's system of axioms [29] ((comm), (M), (Re), (Ex)) for the variety of commutative BCK algebras can be extended to the variety of commutative quasi-BCK algebras, i.e. if the shorter system of axioms: (comm), (qM), (Re), (Ex) defines also the variety of commutative quasi-BCK algebras.

Theorem 4.50 *(See Theorem 3.64 in the regular case)*
In a commutative quasi-BCK algebra, properties (pi) and (q-impl) are equivalent.

Proof. By (qBIM2"), (comm) + (L) + (K) + (qM) $\Longrightarrow$ ((pi) $\Leftrightarrow$ (q-impl)). □

Theorem 4.51 *(See Theorem 3.65 in the regular case)*
Any quasi-implicative quasi-BCK algebra is commutative and positive implicative.

Proof. By (qBIM3), (q-impl) + (Ex) + (B) + (qM) + (qAn) imply (comm). By (qBIM1'), (q-impl) + (BB) + (K) + (qM) + (qAn) imply (pi), and by (qB15), (pi) + (Re) + (Ex) + (B) + (**) + (*) + (L) + (qAn) + (qM) imply (pimpl). □

Theorem 4.52 *(See Theorem 3.66 in the regular case)*
Any commutative and positive implicative quasi-BCK algebra is quasi-implicative.

Proof. By Theorem 4.47, a quasi-BCK algebra is positive implicative if and only if the property (pi) holds. Then, by (qBIM2), (comm) + (pi) + (L) + (K) + (qM) $\Longrightarrow$ (q-impl). □

Corollary 4.53 *(See Corollary 3.67 in the regular case) In a quasi-BCK algebra we have:*

$$(q-impl) \;\Leftrightarrow\; ((comm) \;+\; (pi)\,(\Leftrightarrow\; (pimpl)))$$

Proof. By Theorems 4.51 and 4.52. □

4.8.2 Quasi-Hilbert algebras

We introduce the following

Definition 4.54
A *quasi-Hilbert algebra* (or a *qHilbert algebra* for short) is an algebra $\mathcal{A} = (A, \rightarrow, 1)$ of type $(2, 0)$ satisfying: for all $x, y, z \in A$,
(K) $\quad x \rightarrow (y \rightarrow x) = 1$,
(pimpl-1) $[x \rightarrow (y \rightarrow z)] \rightarrow [(x \rightarrow y) \rightarrow (x \rightarrow z)] = 1$,
(qM) $\quad 1 \rightarrow (x \rightarrow y) = x \rightarrow y$,
(qAn) $\quad x \rightarrow y = 1 = y \rightarrow x \implies 1 \rightarrow x = 1 \rightarrow y$.

We shall prove the following generalizing "quasi-result":

Theorem 4.55 *(See Theorem 3.69 in the regular case)*
Quasi-Hilbert algebras are categorically equivalent to positive implicative quasi-BCK algebras.

Proof:
$\Longrightarrow$: Let $\mathcal{A}$ be a quasi-Hilbert algebra, i.e. (K), (pimpl-1), (qM) and (qAn) hold. We must prove that $\mathcal{A}$ is a positive implicative quasi-BCK algebra. Using the group (qBCK-2) of quasi-properties to define a quasi-BCK algebra, we must prove that (B), (C), (K), (qM), (qAn), (pimpl) hold. Note that it remains to prove that (B), (C) and (pimpl) hold. Indeed, by (qA7'), (qM) + (K) imply (L); by (qB20'), (pimpl-1) + (L) + (qM) imply (*); by (qA13'), (qM) + (*) imply (Tr); by (B22), (pimpl-1) + (K) + (Tr) imply (B); thus (B) holds. Then, by (qB24), (pimpl-1) + (*) + (K) + (qM) imply (**); by (B25), (pimpl-1) + (K) + (Tr) + (**) imply (C); thus (C) holds. Finally, by (qB26), (pimpl-1) + (K) + (qM) + (qAn) + (C) + (**) + (Tr) imply (pimpl).
$\Longleftarrow$: Let $\mathcal{A}$ be a positive implicative quasi-BCK algebra, i.e. (B), (C), (K), (qM), (qAN) and (pimpl) hold. We must prove that $\mathcal{A}$ is a quasi-Hilbert algebra, i.e. that (K), (pimpl-1), (qM), (qAn) hold. Note that it remains to prove that (pimpl-1) holds. Indeed, since (Re) holds in any quasi-BCK algebra, and by (A0), (Re) $\Longrightarrow$ (S), it follows that (S) holds. Then by (S), (pimpl) implies (pimpl-1). □

Corollary 4.56 *(see Corollary 3.70 in the regular case)*
Any commutative quasi-Hilbert algebra is quasi-implicative.

Proof. By above Theorem 4.55 and by Theorem 4.52. □

Note that:
- any quasi-Hilbert algebra is a quasi-algebra, since (qM) and (11-1) hold, by (qA3");
- a quasi-Hilbert algebra is quasi-subtractive [16], by (qA35);
- any quasi-Hilbert algebra satisfying (M) is a Hilbert algebra.

Denote by **qRM**, **qRML**, **qBCI**, **qBCK**, **qHilbert** the classes of quasi-RM algebras, of quasi-RML algebras, of quasi-BCI algebras, of quasi-BCK algebras and of quasi-Hilbert algebras, respectively. We have then the expected Hierarchy 2 from Figure 8.

The theory of quasi-algebras (quasi-structures) will be continued with the bounded case and with the study of the negation, in part III.

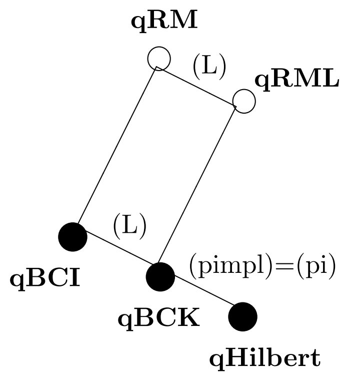
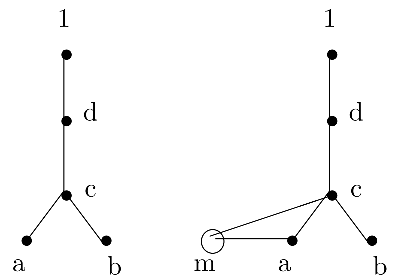


Figure 8: Hierarchy 2

5 Exemples of finite quasi-algebras

Example 5.1 Let us reconsider the algebra $\mathcal{A}_1 = (A_1 = \{a, b, c, d, 1\}, \rightarrow, 1)$ from ([16], Example 6.6), represented by the Hasse diagram in Figure 9, with the table of $\rightarrow$ presented below. $\mathcal{A}_1$ is a regular BCK algebra which does not verify: (pimpl) for d, d, c; (comm) for a, c; (impl) for c, a. It has not the property (P), because, for example, $a \odot b = \min\{z \mid a \leq b \rightarrow z\} = \min\{a, b, c, d, 1\}$ does not exist.

The associated quasi-BCK algebra $\mathcal{A}_1^1 = (A_1^1 = \{a, b, c, d, m, 1\}, \rightarrow, 1)$, obtained from A_1 by adding a paralel element m to a, represented by the quasi-Hasse diagram also in Figure 9, with the table of $\rightarrow$ presented below, does not verify: (pimpl) for d, d, c; (comm) for a, c; (q-impl) for c, a. $R(A_1^1) = A_1$.



$\mathcal{A}_1$ BCK algebra $\quad$ $\mathcal{A}_1^1$ quasi-BCK algebra

Figure 9: A regular BCK algebra $\mathcal{A}_1$ and an associated quasi-BCK algebra $\mathcal{A}_1^1$ which are not: positive implicative, commutative, (q-)implicative

$\mathcal{A}_1$

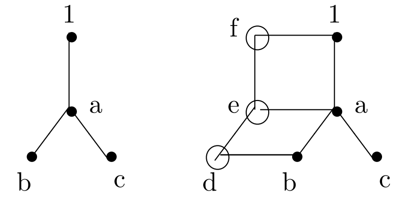
$\rightarrow$	**a**	**b**	**c**	**d**	**1**
a	**1**	**b**	**1**	**1**	**1**
b	**a**	**1**	**1**	**1**	**1**
c	**a**	**b**	**1**	**1**	**1**
d	**a**	**b**	**d**	**1**	**1**
1	**a**	**b**	**c**	**d**	**1**

$\mathcal{A}_1^1$

$\rightarrow$	**a**	**b**	**c**	**d**	m	**1**
a	**1**	**b**	**1**	**1**	1	**1**
b	**a**	**1**	**1**	**1**	a	**1**
c	**a**	**b**	**1**	**1**	a	**1**
d	**a**	**b**	**d**	**1**	a	**1**
m	1	b	1	1	1	1
1	**a**	**b**	**c**	**d**	a	**1**

Example 5.2 Let us consider the algebra $\mathcal{A}_2 = (A_2 = \{a, b, c, 1\}, \rightarrow, 1)$, represented by the Hasse diagram in Figure 10, with the table of $\rightarrow$ presented below. $\mathcal{A}_2$ is a commutative regular BCK algebra, which does not verify: (pimpl) for b, a, c; (impl) for a, b. It has not the property (P), because, for examples, $a \odot a = \min\{z \mid a \leq a \rightarrow z\} = \min\{a, b, c, 1\}$ does not exist.

The associated quasi-BCK algebra $\mathcal{A}_2^1 = (A_2^1 = \{a, b, c, d, e, f, 1\}, \rightarrow, 1)$, represented by the quasi-Hasse diagram also in Figure 10, with the corresponding table of $\rightarrow$, is commutative and does not verify: (pimpl) for b, a, c; (q-impl) for a, b. $R(A_2^1) = A_2$.



$\mathcal{A}_2$ comm. BCK algebra $\mathcal{A}_2^1$ comm. quasi-BCK algebra

Figure 10: The commutative regular BCK algebra $\mathcal{A}_2$ and the associated commutative quasi-BCK algebra $\mathcal{A}_2^1$

$\mathcal{A}_2$

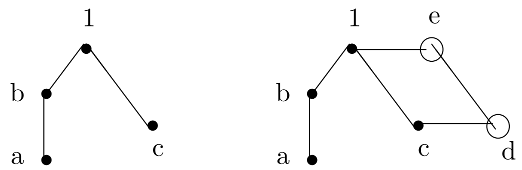
$\rightarrow$	**a**	**b**	**c**	**1**
a	**1**	**a**	**a**	**1**
b	**1**	**1**	**a**	**1**
c	**1**	**a**	**1**	**1**
1	**a**	**b**	**c**	**1**

$\mathcal{A}_2^1$

$\rightarrow$	**a**	**b**	**c**	d	e	f	**1**
a	**1**	**a**	**a**	a	1	1	**1**
b	**1**	**1**	**a**	1	1	1	**1**
c	**1**	**a**	**1**	a	1	1	**1**
d	1	1	a	1	1	1	1
e	1	a	a	a	1	1	1
f	a	b	c	b	a	1	1
1	**a**	**b**	**c**	b	a	1	**1**

Example 5.3 Let us consider the algebra $\mathcal{A}_3 = (A_3 = \{a, b, c, 1\}, \rightarrow, 1)$, represented by the Hasse diagram in Figure 11, with the table of $\rightarrow$ presented below. $\mathcal{A}_3$ is a positive implicative regular BCK algebra - hence a **Hilbert algebra** - which does not verify: (comm) for a, b; (impl) for b, a. It has not the property (P), because, for example, $a \odot c = \min\{z \mid a \leq c \rightarrow z\} = \min\{a, b, c, 1\}$ does not exist.

The associated quasi-BCK algebra $\mathcal{A}_3^1 = (A_3^1 = \{a, b, c, d, e, 1\}, \rightarrow, 1)$, represented by the quasi-Hasse diagram also in Figure 11, with the corresponding table of $\rightarrow$, is a quasi-Hilbert algebra, which does not verify: (comm) for a, b; (q-impl) for b, a. $R(A_3^1) = A_3$.



$\mathcal{A}_3$ Hilbert algebra $\mathcal{A}_3^1$ quasi-Hilbert algebra

Figure 11: The Hilbert algebra $\mathcal{A}_3$ and the associated quasi-Hilbert algebra $\mathcal{A}_3^1$

$\mathcal{A}_3$

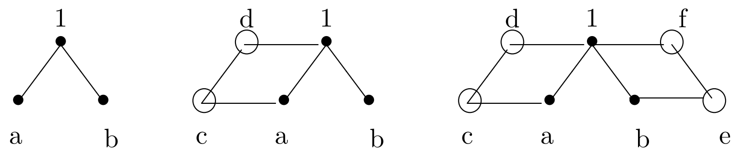
$\to$	a	b	c	1
a	1	1	c	1
b	a	1	c	1
c	a	b	1	1
1	a	b	c	1

$\mathcal{A}_3^1$

$\to$	a	b	c	d	e	1
a	1	1	c	c	1	1
b	a	1	c	c	1	1
c	a	b	1	1	1	1
d	a	b	1	1	1	1
e	a	b	c	c	1	1
1	a	b	c	c	1	1

Example 5.4 Let us consider the algebra $\mathcal{A}_4 = (A_4 = \{a, b, 1\}, \to, 1)$, represented by the Hasse diagram in Figure 12, with the table of $\to$ presented below. $\mathcal{A}_4$ is an implicative regular BCK algebra - hence commutative and positive implicative also. It has not the property (P), because, for example, $a \odot b = \min\{z \mid a \leq b \to z\} = \min\{a, b, 1\}$ does not exist.

The associated quasi-BCK algebras $\mathcal{A}_4^1 = (A_4^1 = \{a, b, c, d, 1\}, \to, 1)$ and $\mathcal{A}_4^2 = (A_4^2 = \{a, b, c, d, e, f, 1\}, \to, 1)$, represented by the quasi-Hasse diagrams also in Figure 12, with the corresponding tables of $\to$, are q-implicative quasi-BCK algebras, hence commutative and positive implicative also. $R(A_4^1) = R(A_4^2) = A_4$.



$\mathcal{A}_4$ impl. BCK algebra — $\mathcal{A}_4^1$ q-impl. qBCK algebra — $\mathcal{A}_4^2$ q-impl. qBCK algebra

Figure 12: The implicative regular BCK algebra $\mathcal{A}_4$ and the q-implicative quasi-BCK algebras $\mathcal{A}_4^1$, $\mathcal{A}_4^2$

$\mathcal{A}_4$

$\to$	a	b	1
a	1	b	1
b	a	1	1
1	a	b	1

$\mathcal{A}_4^1$

$\to$	a	b	c	d	1
a	1	b	1	1	1
b	a	1	a	1	1
c	1	b	1	1	1
d	a	b	a	1	1
1	a	b	a	1	1

$\mathcal{A}_4^2$

$\to$	a	b	c	d	e	f	1
a	1	b	1	1	b	1	1
b	a	1	a	1	1	1	1
c	1	b	1	1	b	1	1
d	a	b	a	1	b	1	1
e	a	1	a	1	1	1	1
f	a	b	a	1	b	1	1
1	a	b	a	1	b	1	1

Example 5.5 Let us consider the set $A = \mathbf{Z}^- \cup \{1\}$, i.e. $A = \{\ldots, -3, -2, -1, 0, 1\}$, where the poset $(A, \leq, 1)$ is a **lattice with last element** 1. Consider on A the **Lukasiewicz implication** (residuum) $\to_L$ given by: for all $x, y \in A$,

$$x \to_L y = \begin{cases} 1, & \text{if} \quad x \leq y \\ y - x + 1, & \text{if} \quad x > y, \end{cases}$$

hence the table of $\to_L$ is the following:

$\to_L$	...	-3	-2	-1	0	1
$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	...	1	1	1	1	1
-2	...	0	1	1	1	1
-1	...	-1	0	1	1	1
0	...	-2	-1	0	1	1
1	...	-3	-2	-1	0	1

Then, $\mathcal{A} = (A, \to_L, 1)$ $(\mathcal{A} = (A, \leq, \to_L, 1))$ is a BCK algebra (structure).

$\mathcal{A}$ is commutative: indeed, by (K) ($y \leq x \to_L y$), for all $x, y \in A$,

$Left \overset{notation}{=} (x \to_L y) \to_L y = \begin{cases} 1 \to_L y = y, & \text{if} \quad x \leq y \\ (y - x + 1) \to_L y = y - (y - x + 1) + 1 = x, & \text{if} \quad x > y, \end{cases}$ and

$Right \overset{notation}{=} (y \to_L x) \to_L x = \begin{cases} 1 \to_L x = x, & \text{if} \quad y \leq x \\ (x - y + 1) \to_L x = x - (x - y + 1) + 1 = y, & \text{if} \quad y > x, \end{cases}$

hence $Left = Right = \begin{cases} y, & \text{if} \quad x < y, \\ x, & \text{if} \quad x \geq y. \end{cases}$

$\mathcal{A}$ is with (P): indeed, for all $x, y \in A$,

$x \odot y \overset{(dfP)}{=} \min\{z \mid x \leq y \to_L z = z - y + 1\} = \min\{z \mid z \geq x + y - 1\} = x + y - 1$. It follows that the table of $\odot$ is the following:

$\odot$	...	-3	-2	-1	0	1
$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
-3	...	-7	-6	-5	-4	-3
-2	...	-6	-5	-4	-3	-2
-1	...	-5	-4	-3	-2	-1
0	...	-4	-3	-2	-1	0
1	...	-3	-2	-1	0	1

Consequently, $\mathcal{A}$ is a **commutative BCK(P) algebra**. Hence, we can apply Theorem 3.79 to find more on it (note that $\odot \neq \wedge$). Remark also that, by taking finite segments (subalgebras) $[m, 1] = \{n \mid m \leq n \leq 1\} \subset A$, for all $m \in \mathbf{Z}$, then we obtain the finite (bounded) Weisberg algebras $\mathcal{L}_2$, $\mathcal{L}_3$, ... described in ([14], paragraph 4.1.1).

If, for example, we add one parallel element a to 1, i.e. we have the following quasi-lattice with 1:

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