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by

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Abstract

Starting from quasi-Wajsberg algebras (which are generalizations of Wajsberg algebras), whose regular sets are Wajsberg algebras, we introduce a theory of quasi-algebras versus, in parallel, a theory of regular algebras.

In Part III, the third part of the theory of quasi-algebras is presented, including the list qC of some quasi-properties of the quasi-negation and many connections, versus the third part of a theory of regular algebras, including the list C of some properties of the negation and many connections. We prove that the bounded, commutative quasi-BCK algebras with an involutive quasi-negation are categorically equivalent to quasi-Wajsberg algebras. We introduce the quasi-Boolean algebras and prove that the bounded, quasi-implicative quasi-BCK algebras with an involutive quasi-negation are categorically equivalent to quasi-Boolean algebras.

Keywords: quasi-MV algebra, quasi-Wajsberg algebra, MV algebra, Wajsberg algebra, BCK algebra, quasi-BCK, Boolean algebra, quasi-Boolean algebra

AMS classification (2010): 06F35, 03G25, 06A06

1 Introduction

The quasi-MV algebras were introduced in 2006 [24], as generalizations of MV algebras introduced in 1958 [5], following an investigation into the foundations of quantum computing. Since then, many papers investigated them [1], [29], [13], [23].

The quasi-Wajsberg algebras were introduced in 2010 [2], as generalizations of Wajsberg algebras introduced in 1984 [7]; they are term-equivalent to quasi-MV algebras, just as Wajsberg algebras are term equivalent to MV algebras. The regular set $R(A)$ of any quasi-Wajsberg algebra A is a Wajsberg algebra. Remark that any Wajsberg algebra A has in the signature an implication $\to$ and a constant 1 that verify the following two properties, among many others: for all $x \in A$,

(Re) $x \to x = 1$, (M) $1 \to x = x$,

while any quasi-Wajsberg algebra A has in the signature an implication $\to$ and a constant 1 that verify the following two properties, among many others: for all $x, y \in A$,

(Re) $x \to x = 1$, (qM) $1 \to (x \to y) = x \to y$.

Note that (M) implies (qM) and this is the most important reason why the quasi-Wajsberg algebras are generalizations of Wajsberg algebras.

We have introduced in 2013 [18] many new generalizations of BCI, of BCK and of Hilbert algebras, in a general investigation of algebras $(A, \to, 1)$ of type $(2, 0)$ that can verify properties in a given list of properties. Among the new generalizations, the most general one is the RM algebra, i.e. an algebra $(A, \to, 1)$ verifying the properties (Re), (M).

Based mainly on the results in [2] and in [18] and on the above remarks, we have developed a theory of quasi-algebras (including the lists qA, qB, qC of properties, with many connections) versus, in parallel, a theory of regular algebras (including the lists A, B, C of properties, with many connections). We have introduced new quasi-algebras: the quasi-RM, quasi-RML, quasi-BCI, quasi-BCK, quasi-Hilbert algebras and the quasi-Boolean algebras, as generalizations of the corresponding regular algebras: RM, RML, BCI, BCK, Hilbert and Boolean algebras. We have made the connection with the quasi-Wajsberg algebras.

In Part I, the first part of the theory of quasi-algebras is presented [19], including the list qA of basic properties and many connections - versus the first part of a theory of regular algebras, including the list A of basic properties and many connections. We introduce the quasi-order and the quasi-Hasse diagram - versus the regular order and the Hasse diagram - and we study the quasi-ordered algebras (structures). We introduce the quasi-RM and the quasi-RML algebras and we present two equivalent definitions of quasi-BCI and of quasi-BCK algebras.

In Part II, the second part of the theory of quasi-algebras is presented [20], including the list qB of particular properties and many connections - versus the second part of a theory of regular algebras, including the list B of particular properties and many connections. We introduce the positive implicative, commutative and quasi-implicative quasi-BCK algebras and the quasi-Hilbert algebras and we prove that the quasi-Hilbert algebras coincide with positive implicative quasi-BCK algebras.

In Part III, the third part of the theory of quasi-algebras is presented, including the list qC of some quasi-properties of the quasi-negation and many connections, versus the third part of a theory of regular algebras, including the list C of some properties of the negation and many connections. We prove that the bounded, commutative quasi-BCK algebras with an involutive quasi-negation are categorically equivalent to quasi-Wajsberg algebras. We also introduce the quasi-Boolean algebras and prove that the bounded, quasi-implicative quasi-BCK algebras with involutive quasi-negation are categorically equivalent to quasi-Boolean algebras.

2 Introduction to the theory of regular algebras (structures) - Part III

In the first subsection, we shall recall from [19] the Part I of the theory of regular algebras (structures). In the second subsection, we shall recall from [20] the Part II of the theory of regular algebras (structures). In the third subsection, we shall present the Part III of the theory.

Let $\mathcal{A} = (A, \to, 1)$ be an *algebra* of type $(2, 0)$ through this section, where a binary relation $\leq$ can be *defined* by: for all x, y,

$$x \leq y \overset{def.}{\Longleftrightarrow} x \to y = 1. \tag{1}$$

Equivalently,
let $\mathcal{A} = (A, \leq, \to, 1)$ be a *structure* where $\leq$ is a binary relation on A, $\to$ is a binary operation (an implication) on A and $1 \in A$, all *connected* by:

$$x \leq y \Longleftrightarrow x \to y = 1. \tag{2}$$

Recall first the definitions:

Definitions 2.1 [19]

(1) The algebra $(A, \to, 1)$ (or, equivalently, the structure $(A, \leq, \to, 1)$) is called *regular*, if it satisfies the property (M).

(1') Any algebra (structure) $\mathcal{A}' = (A, \sigma)$ whose signature σ contains $\to$, 1 ($\leq$, $\to$, 1, respectively) is also called *regular*, if it satisfies the property (M).

(1") Any algebra (structure) $\mathcal{A}'' = (A, \tau)$ which is term equivalent to a regular algebra (structure) $\mathcal{A}' = (A, \sigma)$, is also called *regular*.

(2) The implication $\to$ from a regular algebra (structure) is called *regular implication*.

(3) The binary relation $\leq$ of a regular algebra (structure) is called *binary regular relation*.

Remark 2.2 [19] By (M), we have that: $V_M = V = U = A$, and this is the basic, definable property of regular algebras (structures), where

$$U \overset{df.}{=} \{x \to y \mid x, y \in A\},\ V \overset{df.}{=} \{1 \to x \mid x \in A\},\ V_M \overset{df.}{=} \{x \in A \mid x \overset{(M)}{=} 1 \to x\}.$$

2.1 The list A of basic properties and connections [19]

2.1.1 The List A of basic properties

Consider the following list A of properties (those from [18] plus two new properties: (#), (##)) that can be satisfied by $\mathcal{A}$ (in fact, the properties in the list A are the most important properties satisfied by a BCK algebra (see [18])). We divided the list into two parts: the properties in Part 1 are those that will be generalized, when considering the quasi-algebras (quasi-structures).

List A, Part 1

(An) (Antisymmetry) $x \to y = 1 = y \to x \implies x = y$,
(M) $1 \to x = x$;
(N) $1 \to x = 1 \implies x = 1$,
(Re) (Reflexivity) $x \to x = 1$ (we prefer here notation (Re) instead of (I) in the theory of BCI algebras),
(L) (Last element) $x \to 1 = 1$.

List A, Part 2

(11-1) $1 \to 1 = 1$,
(B) $(y \to z) \to [(x \to y) \to (x \to z)] = 1$,
(BB) $(y \to z) \to [(z \to x) \to (y \to x)] = 1$,
(*) $y \to z = 1 \implies (x \to y) \to (x \to z) = 1$,
(**) $y \to z = 1 \implies (z \to x) \to (y \to x) = 1$,
(C) $[x \to (y \to z)] \to [y \to (x \to z)] = 1$,
(D) $y \to [(y \to x) \to x] = 1$,
(Ex) (Exchange) $x \to (y \to z) = y \to (x \to z)$;
(K) $x \to (y \to x) = 1$,
(S) $x = y \implies x \to y = 1$,
(Tr) (Transitivity) $x \to y = 1 = y \to z \implies x \to z = 1$,
(#) $x \to (y \to z) = 1 \Longrightarrow y \to (x \to z) = 1$,
(##) $x \to (y \to z) = 1 \Longleftrightarrow y \to (x \to z) = 1$.

Remarks 2.3 [19]
(i) (M) $\Longrightarrow$ (11-1); (Re) $\Longrightarrow$ (11-1); (L) $\Longrightarrow$ (11-1).
(ii) The central role of property (M) in the study of regular algebras (structures) [18] is given by the fact that it determines that $V_M = V = U = A$, i.e. all the elements of A appear compulsory inside the table of $\to$ ($\to$: $A \times A \longrightarrow A$).

2.1.2 Connections between the properties in the list A [19]

Proposition 2.4 *[18] [19] Let* $(A, \to, 1)$ *be an algebra of type* $(2, 0)$*. Then the following are true:*
(A0) (Re) $\Longrightarrow$ *(S);*
(A00) (M) $\Longrightarrow$ *(N);*
(A1) (L) + (An) $\Longrightarrow$ *(N);*
(A2) (K) + (An) $\Longrightarrow$ *(N);*
(A3) (C) + (An) $\Longrightarrow$ *(Ex);* *(A3') (Ex) + (Re)* $\Longrightarrow$ *(C);*
(A4) (Re) + (Ex) $\Longrightarrow$ *(D);* *(A4') (D) + (Re) + (An)* $\Longrightarrow$ *(N);*
(A5) (Re) + (Ex) + (An) $\Longrightarrow$ *(M);*
(A6) (Re) + (K) $\Longrightarrow$ *(L);*
(A7) (N) + (K) $\Longrightarrow$ *(L);* *(A7') (M) + (K)* $\Longrightarrow$ *(L);*
(A8) (Re) + (L) + (Ex) $\Longrightarrow$ *(K);*
(A9) (M) + (L) + (B) $\Longrightarrow$ *(K);* *(A9') (M) + (L) + (**)* $\Longrightarrow$ *(K);*
(A10) (Ex) $\Longrightarrow$ *(B)* $\Leftrightarrow$ *(BB);*

(A10') (Ex) + (B) $\Longrightarrow$ (BB);   (A10") (Ex) + (BB) $\Longrightarrow$ (B);
(A11) (Re) + (Ex) + (*) $\Longrightarrow$ (BB);
(A12) (N) + (B) $\Longrightarrow$ (*);   (A12') (M) + (B) $\Longrightarrow$ (*);
(A13) (N) + (*) $\Longrightarrow$ (Tr);   (A13') (M) + (*) $\Longrightarrow$ (Tr);
(A14) (N) + (B) $\Longrightarrow$ (Tr);   (A14') (M) + (B) $\Longrightarrow$ (Tr);
(A15) (N) + (BB) $\Longrightarrow$ (**);   (A15') (M) + (BB) $\Longrightarrow$ (**);
(A16) (N) + (**) $\Longrightarrow$ (Tr);   (A16') (M) + (**) $\Longrightarrow$ (Tr);
(A17) (N) + (BB) $\Longrightarrow$ (Tr);   (A17') (M) + (BB) $\Longrightarrow$ (Tr);
(A18) (M) + (BB) $\Longrightarrow$ (Re);   (A18') (M) + (BB) $\Longrightarrow$ (D);
(A19) (M) + (B) $\Longrightarrow$ (Re);
(A20) (BB) + (D) + (N) $\Longrightarrow$ (C);   (A20') (M) + (BB) $\Longrightarrow$ (C);
(A21) (BB) + (D) + (N) + (An) $\Longrightarrow$ (Ex);
(A21') (BB) + (D) + (L) + (An) $\Longrightarrow$ (Ex);
(A21") (M) + (BB) + (An) $\Longrightarrow$ (Ex);
(A22) (K) + (Ex) + (M) $\Longrightarrow$ (Re);
(A23) (C) + (K) + (An) $\Longrightarrow$ (Re);
(A24) (Re) + (Ex) + (Tr) $\Longrightarrow$ (**).

Proposition 2.5 *[19] Let $(A, \to, 1)$ be an algebra of type $(2, 0)$. Then the following are true:*
(A9") (M) + (L) + (BB) $\Longrightarrow$ (K);
(A18") (M) + (D) $\Longrightarrow$ (Re);
(A25) (D) + (K) + (N) + (An) $\Longrightarrow$ (M);
(A26) (#) $\Longleftrightarrow$ (##) ;
(A27) (M) + (C) $\Longrightarrow$ (#);
(A28) (Ex) $\Longrightarrow$ (##);
(A29) (BB) + (#) $\Longrightarrow$ (B);   (A29') (B) + (#) $\Longrightarrow$ (BB);
(A30) (Re) + (B) + (Tr) + (#) $\Longrightarrow$ (C);
(A31) (Re) + (#) $\Longrightarrow$ (D) (see (A4));
(A32) (Re) + (#) + (An) $\Longrightarrow$ (M) (see (A5)).

Theorem 2.6 *[18], [19] (Generalization of ([3], Lemma 1.2 and Proposition 1.3))*
If properties (Re), (M), (Ex) hold, then: $(BB) \Leftrightarrow (B) \Leftrightarrow (*)$.

Theorem 2.7 *[18], [19]*
If properties (Re), (M), (Ex) hold, then: $(**) \Leftrightarrow (Tr)$.

Theorem 2.8 *[18], [19]*
If properties (M), (B), (An) hold, then: $(Ex) \Leftrightarrow (BB)$.

Theorem 2.9 *[18], [19] (Michael Kinyon) In any algebra $(A, \to, 1)$, we have:*
(i) (M) + (BB) $\Longrightarrow$ (B),
(ii) (M) + (B) $\Longrightarrow$ (**).

By Kinyon's Theorem 2.9(i) and (A12'), we obtained immediately that:

Corollary 2.10 [18], [19] (M) + (BB) $\Longrightarrow$ (*).

Concluding, by above Kinyon's Theorem 2.9 and (A12'), (A13'), (A16'), we have obtained:

Corollary 2.11 [18], [19] In any algebra $(A, \to, 1)$ verifying (M), we have:

$$(BB) \Longrightarrow (B) \Longrightarrow (*), (**) \Longrightarrow (Tr).$$

2.2 The list B of particular properties. Connections [20]

2.2.1 The List B of some particular properties

Consider the following list of particular properties that can be satisfied by $\mathcal{A}$, also divided into two parts:

List B, Part 1

(impl) (implicative) $(x \to y) \to x = x$;
(pi) $x \to (x \to y) = x \to y$;

(Vid) $x \vee x = x$;
(Ve) $x \leq y \Leftrightarrow x \vee y = y$;
(V=) $x \leq z,\ y \leq z \implies x \vee y \leq z$;

(Wid) $x \wedge x = x$,
(W1-1) $x \wedge 1 = 1 \wedge x = x$;
(We) $x \leq y \Leftrightarrow x \wedge y = x$;
(W=) $x \geq z,\ y \geq z \implies x \wedge y \geq z$;
(Wab1) (absorbtion-1) $x \wedge (x \vee y) = x$,
(Wab2) (absorbtion-2) $x \vee (x \wedge y) = x$.

List B, Part 2

(pimpl) (positive implicative) $x \to (y \to z) = (x \to y) \to (x \to z)$,
(pimpl-1) $[x \to (y \to z)] \to [(x \to y) \to (x \to z)] = 1$,
(pimpl-2) $[(x \to y) \to (x \to z)] \to [x \to (y \to z)] = 1$;
($) $x \to (y \to z) = 1 \implies (x \to y) \to (x \to z) = 1$,
($') $x \leq y \to z \implies x \to y \leq x \to z$;

(comm) (commutative) $(x \to y) \to y = (y \to x) \to x$;
(comm-1) $[(x \to y) \to y] \to x = (x \to (y \to x)) \to (y \to x)$;

(dfP) (def. of $\odot$ (Product)) $\exists\ x \odot y = \min\{z \mid x \leq y \to z\}$;
(dual) (dfS) (def. of Sum) $\exists\ x \oplus y = \max\{z \mid x \geq y \to^R z\}$;
(RP) $x \odot y \leq z \Leftrightarrow x \leq y \to z$,
(Pne) $x \odot 1 = x$,
(Pcomm) (commutativity of $\odot$ (Product)) $x \odot y = y \odot x$,
(Passoc) (associativity of $\odot$ (Product)) $(x \odot y) \odot z = x \odot (y \odot z)$,
(P-) $x \leq y \implies a \odot x \leq a \odot y$,
(P- -) $x \leq y,\ a \leq b \implies x \odot a \leq y \odot b$,
(Prel) $x \odot y \leq x, y$;

(dfV) (def. of $\vee$ (vee)) $x \vee y = (x \to y) \to y$,
(VP) $z \odot (x \vee y) = (z \odot x) \vee (z \odot y)$,
(Vcomm) (commutativity of $\vee$) $x \vee y = y \vee x$,
(Vassoc) (associativity of $\vee$) $(x \vee y) \vee z = x \vee (y \vee z)$,
(V1-1) $x \vee 1 = 1 \vee x = 1$,
(V-) $x \leq y \implies a \vee x \leq a \vee y$,
(V- -) $x \leq y,\ a \leq b \implies x \vee a \leq y \vee b$,
(Vgeq) $x, y \leq x \vee y$,
(V-V) $z \to (x \vee y) = (x \to y) \to (z \to y)$,
(VVV) $(x \to y) \to (z \to y) = (z \vee y) \to (x \vee y)$.

2.2.2 Connections between the properties in lists A and B

Recall the following connections from [18]:

Proposition 2.12 *(See [18] Remark 6.2, Propositions 6.4, 6.9, 6.8, Theorems 6.13, 6.16, Remarks 6.20 (ii), Proposition 6.21, Theorems 6.23, 6.25)*
Let $(A, \to, 1)$ be an algebra of type $(2, 0)$. Then, we have:
(B0) (pimpl-1) + (pimpl-2) + (An) $\Longrightarrow$ (pimpl);
(B1) (pimpl) + (Re) $\Longrightarrow$ (L);
(B2) (pi) + (Re) $\Longrightarrow$ (L);
(B3) (pimpl) + (Re) +(M) $\Longrightarrow$ (pi);
(B4) (pimpl) + (Re) + (L) $\Longrightarrow$ (K);
(B5) (pimpl) + (K) $\Longrightarrow$ (B);
(B6) (pimpl) + (Re) $\Longrightarrow$ (B);
(B7) (pimpl) + (Re) + (M) $\Longrightarrow$ (*), (**);
(B8) (pimpl) + (L) $\Longrightarrow$ (*) (Michael Kinyon);
(B9) (pimpl) + (Re) + (M) $\Longrightarrow$ (BB) (Michael Kinyon);
(B10) (pimpl) + (Re) + (M) $\Longrightarrow$ (C) (Michael Kinyon);
(B11) (pimpl) + (Re) + (M) + (An) $\Longrightarrow$ (Ex) (Michael Kinyon);

(B12) (comm) + (M) $\Longrightarrow$ (An);

(B13) (Re) + (L) + (Ex) + (**) $\Longrightarrow$ (pimpl-2);
(B14) (pi) + (Ex) + (B) + (*) $\Longrightarrow$ (pimpl-1);
(B15) (pi) + (Re) + (Ex) + (B) + (**) + (*) + (L) + (An) $\Longrightarrow$ (pimpl);
(B16) (pi) + (Re) + (M) + (B) + (An) $\Longrightarrow$ (Ex) $\Leftrightarrow$ (BB) $\Leftrightarrow$ (pimpl);
(B17) (pi) + (Re) + (M) + (Ex) + (An) $\Longrightarrow$ (BB) $\Leftrightarrow$ (B) $\Leftrightarrow$ (*) $\Leftrightarrow$ (pimpl).

We now added the following new connections between the properties in lists A and B [20]:

Proposition 2.13 *[20] Let $(A, \to, 1)$ be an algebra of type $(2, 0)$. Then, the following hold:*
(B18) (K) + (**) + (C) + (Tr) $\Longrightarrow$ (pimpl-2);

(B19) (pimpl-1) + (K) + (N) $\Longrightarrow$ (Re); (B19') (pimpl-1) + (K) + (M) $\Longrightarrow$ (Re);
(B20) (pimpl-1) + (L) + (N) $\Longrightarrow$ (*); (B20') (pimpl-1) + (L) + (M) $\Longrightarrow$ (*);
(B21) (pimpl-1) + (L) + (N) $\Longrightarrow$ (Tr); (B21') (pimpl-1) + (L) + (M) $\Longrightarrow$ (Tr);
(B22) (pimpl-1) + (K) + (Tr) $\Longrightarrow$ (B);
(B23) (pimpl-1) + (Re) + (*) + (K) + (M) $\Longrightarrow$ (D);
(B24) (pimpl-1) + (*) + (K) + (M) $\Longrightarrow$ (**);
(B25) (pimpl-1) + (K) + (**) + (Tr) $\Longrightarrow$ (C);
(B26) (pimpl-1) + (K) + (An) + (C) + (**) + (Tr) $\Longrightarrow$ (pimpl);
(B27) (pimpl-1) + (N) $\Longrightarrow$ ($); (B27') (pimpl-1) + (M) $\Longrightarrow$ ($);
(B28) ($) + (K) + (Tr) $\Longrightarrow$ (#);

(B29) (comm) + (L) + (M) $\Longrightarrow$ (Re);
(B30) (comm) + (Re) + (K) + (M) $\Longrightarrow$ (L);
(B31) (comm) + (K) + (BB) + (Tr) + (M) $\Longrightarrow$ (#);
(B32) (comm) + (Re) + (BB) + (M) $\Longrightarrow$ (L);
(B33) (comm) $\Longrightarrow$ (comm-1);
(B34) (Re) + (M) + (Ex) + (B) + (*) + (**) + (L) + (An) $\Longrightarrow$ ((pimpl) $\Leftrightarrow$ (pi)).

Proposition 2.14 *Let* $(A, \rightarrow, 1)$ *be an algebra of type* $(2,0)$. *Then, the following hold:*
(BIM1) (impl) $\Longrightarrow$ *(pi);*
(BIM2) (comm) + (pi) + (L) + (K) + (M) $\Longrightarrow$ *(impl);*
(BIM2') (comm) + (L) + (K) + (M) $\Longrightarrow$ *((pi)* $\Leftrightarrow$ *(impl));*
(BIM3) (impl) + (Ex) + (B) + (M) + (An) $\Longrightarrow$ *(comm);*
(BIM4) (impl) + (Re) $\Longrightarrow$ *(M);*   *(BIM4') (impl) + (L)* $\Longrightarrow$ *(M);*
(BIM5) (impl) + (M) $\Longrightarrow$ *(L);*
(BIM6) (impl) + (K) $\Longrightarrow$ *(Re);*
(BIM7) (L) + (K) + (M) + (Ex) + (B) + (An) $\Longrightarrow$ *((impl)* $\Leftrightarrow$ *((comm) + (pi))).*

Proposition 2.15 *Let* $(A, \rightarrow, 1)$ *be an algebra of type* $(2,0)$. *Then, we have:*
(BV1) (dfV) + (Re) + (M) $\Longrightarrow$ *(Vid);*
(BV2) (dfV) + (comm) $\Longrightarrow$ *(Vcomm);*
(BV3) (dfV) + (D) + (K) $\Longrightarrow$ *(Vgeq);*
(BV4) (dfV) + (L) + (M) + (Re) $\Longrightarrow$ *(V1-1);*
(BV5) (dfV) + (M) + (Ex) + (Re) $\Longrightarrow$ *(Ve);*
(BV6) (dfV) + (Ex) $\Longrightarrow$ *(V-V);*
(BV7) (dfV) + (Vcomm) + (Ve) + (V-V) + (K) $\Longrightarrow$ *(VVV);*
(BV8) (dfV) + (Vcomm) + (Ex) + (VVV) $\Longrightarrow$ *(Vassoc);*
(BV9) (dfV) + (**) $\Longrightarrow$ *(V-);*
(BV10) (V-) + (Vcomm) + (Tr) $\Longrightarrow$ *(V- -);*
(BV11) (dfV) + (Ex) + (Re) + (D) + (L) + (**) $\Longrightarrow$ *(V=).*

2.3 Bounded regular algebras (structures). Regular algebras (structures) with negation. The List C of some properties of the negation. Connections

As we know, in bounded BCK algebras we can *define* a negation $^-$ by: $x^- = x \rightarrow 0$, while in the Wajsberg algebras (MV algebras) and in the Boolean algebras, there exists a negation $^-$ and an implication $\rightarrow$ and an element 1 and, if we denote 1^- by 0, then the negation is *connected* to $\rightarrow$ and 1 by: $x^- = x \rightarrow 0$.

Consequently, just as *the equivalence:*

$$(Equ) \quad x \leq y \quad \Longleftrightarrow \quad x \rightarrow y = 1$$

can be used either
- as *the definition* of the binary regular relation $\leq$ in the regular algebra $(A, \rightarrow, 1)$ or
- as *the connection* between the binary regular relation $\leq$ and $\rightarrow$, 1 in the regular structure $(A, \leq, \rightarrow, 1)$, the same, *the equality:*

$$(Neg) \qquad x^- \quad = \quad x \rightarrow 0$$

can be used either
- as *the definition* of the negation $^-$ in the *bounded regular algebra* $(A, \rightarrow, 1)$ *(structure* $(A, \leq, \rightarrow, 1)$*)*, and in this case $1^- \stackrel{def.}{=} 1 \rightarrow 0 = 0$, by (M), and $0^- \stackrel{def.}{=} 0 \rightarrow 0 = 1$, by (Re) - see the next subsubsection 2.3.1 - or
- as *the connection* between the negation $^-$ and $\rightarrow$, 1 in the *regular algebra* $(A, \rightarrow, ^-, 1)$ *(structure* $(A, \leq, \rightarrow, ^-, 1)$*) with a negation*, and in this case $0 \stackrel{notation}{=} 1^-$ and 0^- must equal $0 \rightarrow 0 = 0$, by (Re) - see the subsubsection 2.3.2.

Note that, in both cases, the negation is *unique*.

We shall then present a List C of some properties of the negation - in the subsubsection 2.3.3 - and some connections between the properties in Lists A, B, C - in the subsubsection 2.3.4.

2.3.1 Bounded regular algebras (structures)

Let $\mathcal{A} = (A, \to, 1)$ be an ordered regular algebra (or, equivalently, let $\mathcal{A} = (A, \leq, \to, 1)$ be an ordered regular structure) and $\leq$ be the order of $\mathcal{A}$ (i.e. (M), (Re), (An), (Tr) hold) through this subsubsection.

Definitions 2.16

(i) An element $l \in A$ is called *last regular element* or *the greatest regular element* of $\mathcal{A}$ if $x \to l = 1$ (or $x \leq l$), for every $x \in A$.

(ii) An element $f \in A$ is called *first regular element* or *the smallest regular element* of $\mathcal{A}$ if $f \to x = 1$ (or $f \leq x$), for every $x \in A$.

Hence, the notions of *first regular element* and *last regular element* are *dual to each other*. Note that both the first regular element and the last regular element of $\mathcal{A}$ are unique, by (An).

The special element $1 \in A$ can be the last regular element. In this case, we have the property:
(L) (Last) $x \to 1 = 1$, for all $x \in A$, or, equivalently,
(L') (Last) $x \leq 1$, for all $x \in A$. Recall that if (K) holds, then (L) holds, by (A6), (A7').
An ordered regular algebra with property (L) is an ordered RML algebra [19].

The first regular element will be denoted by 0; hence we have the property:
(F) (First) $0 \to x = 1$, for all $x \in A$, or, equivalently,
(F') (First) $0 \leq x$, for all $x \in A$.

Definition 2.17 An ordered regular algebra (structure) with first regular element 0 and last regular element 1 is called *bounded* and is denoted by $(A, \to, 0, 1)$ ($(A, \leq, \to, 0, 1)$, respectively).

Note that a bounded ordered regular algebra is a bounded ordered RML algebra.

Let us introduce the following new property:
(10-0) $1 \to 0 = 0$.
Note that: (M) $\Longrightarrow$ (10-0).
Let $\mathcal{A} = (A, \to, 0, 1)$ be a bounded ordered regular algebra (or, equivalently, let $\mathcal{A} = (A, \leq, \to, 0, 1)$ be a bounded ordered regular structure) (i.e. properties (M), (Re), (An), (Tr) and (F), (L) hold).
We define a *negation* $^- : A \longrightarrow A$ by using (Neg): for all $x \in A$,

$$x^- \overset{def.}{=} x \to 0.$$

Hence, the negation is *unique* and we have: $1^- \overset{def.}{=} 1 \to 0 \overset{(M)}{=} 0$ and $0^- \overset{def.}{=} 0 \to 0 \overset{(Re)}{=} 1$.

The negation is called *involutive* if the property (DN) holds: $(x^-)^- = x$, for all $x \in A$.

Remark 2.18 If the regular algebra (structure) is not ordered, then one can extend the above notions.

2.3.2 Regular algebras (structures) with negation

Definition 2.19 A *regular algebra (structure) with negation* is an algebra $(A, \to, ^-, 1)$ (a structure $(A, \leq, \to, ^-, 1)$) such that $(A, \to, 1)$ ($(A, \leq, \to, 1)$ respectively) is a regular algebra (structure) (i.e. (M) holds) and $^-$ is an unary operation on A, called *regular negation*, which is connected with $\to$, 1 by (Neg): for all $x \in A$,

$$x^- = x \to 0,$$

where 0 is the notation for the element 1^-.

Hence, the negation is *unique* and we have: $0 \overset{notation}{=} 1^-$ and $0^- = 0 \to 0 = 1$, if (Re) holds.
The negation is called *involutive* if the property (DN) holds: $(x^-)^- = x$, for all $x \in A$.
We shall prove later on that the Wajsberg algebra is an example of regular algebra with negation, namely with involutive negation.

2.3.3 The List C of some properties of the negation

Consider the following list C of the properties of the negation, divided into two parts:

List C, Part 1

($\overline{M}$) $1 \to x^- = x^-$;
(F) (First) $0 \to x = 1$,
(F') (First) $0 \leq x$;
(Neg) $x^- = x \to 0$;

(DN) (Double negation) $(x^-)^- = x$;
(TN) (Triple Negation) $((x^-)^-)^- = x^-$;

(Neg4) $x \to (x^-)^- = 1$,
(Neg4') $x \leq (x^-)^-$;
(Neg5) $(x \to y)^- \to x = 1$,
(Neg5') $(x \to y)^- \leq x$;
(Neg6) $x \to x^- = x^-$;

(DN5) $x^- \to (x \to y) = 1$,
(DN5') $x^- \leq x \to y$;
(DN6) $x^- \to x = x$;

(V-0) $x \vee 0 = (x^-)^-$,
(V0-) $0 \vee x = x$.

List C, Part 2

(10-0) $1 \to 0 = 0$;
(Neg1-0) $1^- = 0$,
(Neg0-1) $0^- = 1$;

(Neg1) $(x \to y) \to (y^- \to x^-) = 1$,
(Neg1') $x \to y \leq y^- \to x^-$;
(Neg2) $x \to y = 1 \Longrightarrow y^- \to x^- = 1$,
(Neg2') $x \leq y \Longrightarrow y^- \leq x^-$;
(Neg3) $y \to x^- = x \to y^-$;

(Neg7) $x \wedge x^- = 0$;
(Neg8) $(x \oplus y =) x^- \to y \leq x \vee y$;

(DN1) $(y^- \to x^-) \to (x \to y) = 1$,
(DN1') $y^- \to x^- \leq x \to y$;
(DN2) $x \to y = y^- \to x^-$;
(DN3) $y^- \to x = x^- \to y$;
(DN4) $x \leq y \Longleftrightarrow y^- \leq x^-$,
(DN4') $x \to y = 1 \Longleftrightarrow y^- \to x^- = 1$;

(DN7) $x \vee x^- = 1$;
(DN8) $x \wedge y \leq (x \to y^-)^- (= x \odot y)$;

(Px) $x \odot x^- = 0$;

($P_{(DN)}$) $x \odot y = (x \to y^-)^-$,
($VP_{(DN)}$) $x \vee y = y^- \to (y^- \odot x)$ or $x \vee y^- = y \to (y \odot x)$;

(dfW) (def. of $\wedge$ (wedge)) $x \wedge y = (x^- \vee y^-)^-$,
(WP) $z \odot (x \wedge y) = (z \odot x) \wedge (z \odot y)$,
(div) (divisibility) $x \wedge y = x \odot (x \to y)$,

(Wcomm) (commutativity of $\wedge$) $x \wedge y = y \wedge x$,
(Wassoc) (associativity of $\wedge$) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$,
(W0-0) $x \wedge 0 = 0 \wedge x = 0$,
(W-) $x \leq y \Longrightarrow a \wedge x \leq a \wedge y$,
(W- -) $x \leq y,\ a \leq b \Longrightarrow x \wedge a \leq y \wedge b$,
(Wleq) $x, y \geq x \wedge y$,
(WW-) $(x \wedge y) \to z = (y \to x) \to (y \to z)$,
(W-W) $z \to (x \wedge y) = (z \to x) \wedge (z \to y)$,
(WV-) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$;
(WWW) $(y \to x) \to (y \to z) = (x \wedge y) \to (z \wedge y)$;

($V_{(DN)}$) $x \vee y = x^- \to y$ $(= (x \to^R y^-)^-)$,
($W_{(DN)}$) $x \wedge y = (x \to y^-)^-$,
(M1) (de Morgan law 1) $(x \vee y)^- = x^- \wedge y^-$,
(M2) (de Morgan law 2) $(x \wedge y)^- = x^- \vee y^-$.

2.3.4 Connections between the properties in Lists A, B, C

• **Connections between the properties in Part 1 of Lists A,C**

Theorem 2.20
(i) (Neg) + (M) $\Longrightarrow$ ($\overline{M}$);
(ii) ($\overline{M}$) + (DN) $\Longrightarrow$ (M).

Proof. (i): $1 \to x^- \stackrel{(Neg)}{=} 1 \to (x \to 0) \stackrel{(M)}{=} x \to 0 \stackrel{(Neg)}{=} x^-$, i.e. ($\overline{M}$) holds.
(ii): $1 \to x \stackrel{(DN)}{=} 1 \to (x^-)^- \stackrel{(\overline{M})}{=} (x^-)^- \stackrel{(DN)}{=} x$, i.e. (M) holds. □

• **Connections between the properties in List C**

Proposition 2.21
(C1) (DN) + (Neg1) $\Longrightarrow$ (DN1), (C1') (DN) + (DN1) $\Longrightarrow$ (Neg1),
(C1") (DN) $\Longrightarrow$ ((Neg1) $\Leftrightarrow$ (DN1));

(C2) (DN) + (DN2) $\Longrightarrow$ (DN3);

(C3) (DN) + (Neg3) $\Longrightarrow$ (DN3), (C3') (DN) + (DN3) $\Longrightarrow$ (Neg3),
(C3") (DN) $\Longrightarrow$ ((Neg3) $\Leftrightarrow$ (DN3));

(C4) (DN2) $\Longrightarrow$ (DN4);

(C5) (DN) + (DN2) + (Neg5) $\Longrightarrow$ (DN5), (C5') (DN) + (DN2) + (DN5) $\Longrightarrow$ (Neg5),
(C5") (DN) + (DN2) $\Longrightarrow$ ((Neg5) $\Leftrightarrow$ (DN5));

(C6) (DN) + (Neg6) $\Longrightarrow$ (DN6), (C6') (DN) + (DN6) $\Longrightarrow$ (Neg6),
(C6") (DN) $\Longrightarrow$ ((Neg6) $\Leftrightarrow$ (DN6)).

Proof.

(C1): $y^- \to x^- \overset{(Neg1)}{\leq} (x^-)^- \to (y^-)^- \overset{(DN)}{=} x \to y$, i.e. (DN1) holds.

(C1'): $x \to y \overset{(DN)}{=} (x^-)^- \to (y^-)^- \overset{(DN1)}{\leq} y^- \to x^-$, i.e. (Neg1) holds.

(C1"): By (C1) and (C1').

(C2): $x^- \to y \overset{(DN)}{=} x^- \to (y^-)^- \overset{(DN2)}{=} y^- \to x$, i.e. (DN3) holds.

(C3): $y^- \to x \overset{(DN)}{=} y^- \to (x^-)^- \overset{(Neg3)}{=} x^- \to (y^-)^- \overset{(DN)}{=} x^- \to y$, i.e. (DN3) holds.

(C3'): $y \to x^- \overset{(DN)}{=} (y^-)^- \to x^- \overset{(DN3)}{=} (x^-)^- \to y^- \overset{(DN)}{=} x \to y^-$, i.e. (Neg3) holds.

(C3"): By (C3) and (C3').

(C4): By (DN2), $x \to y = y^- \to x^-$, hence $x \to y = 1 \Longleftrightarrow y^- \to x^- = 1$, i.e. (DN4) holds.

(C5): $x^- \to (x \to y) \overset{(DN)}{=} x^- \to ((x \to y)^-)^- \overset{(DN2)}{=} (x \to y)^- \to x \overset{(Neg5)}{=} 1$, i.e. (DN5) holds.

(C5'): $(x \to y)^- \to x \overset{(DN)}{=} (x \to y)^- \to (x^-)^- \overset{(DN2)}{=} x^- \to (x \to y) \overset{(DN5)}{=} 1$, i.e. (Neg5) holds.

(C5"): By (C5) and (C5').

(C6): $x^- \to x \overset{(DN)}{=} x^- \to (x^-)^- \overset{(Neg6)}{=} (x^-)^- \overset{(DN)}{=} x$, i.e. (DN6) holds.

(C6'): $x \to x^- \overset{(DN)}{=} (x^-)^- \to x^- \overset{(DN6)}{=} x^-$, i.e. (Neg6) holds.

(C6"): By (C6) and (C6'). □

Proposition 2.22

(CN1) (Neg1-0) + (DN2) + (DN) + ($\overline{M}$) $\Longrightarrow$ (Neg);

(CN2) (Neg1-0) + (DN) $\Longrightarrow$ (Neg0-1).

Proof.

(CN1): $x \to 0 \overset{(Neg1-0)}{=} x \to 1^- \overset{(DN)}{=} (x^-)^- \to 1^- \overset{(DN2)}{=} 1 \to x^- \overset{(\overline{M})}{=} x^-$, i.e. (Neg) holds.

(CN2): $0^- \overset{(Neg1-0)}{=} (1^-)^- \overset{(DN)}{=} 1$, i.e. (Neg0-1) holds. □

• **Connections between the properties in list A and List C**

Proposition 2.23

(CA0) (Neg) + (M) $\Longrightarrow$ (Neg1-0); (Neg) + (Re) $\Longrightarrow$ (Neg0-1);

(CA1) (Neg) + (BB) $\Longrightarrow$ (Neg1);

(CA2) (M) + (Neg1) $\Longrightarrow$ (Neg2);

(CA3) (Neg) + (Ex) $\Longrightarrow$ (Neg3);

(CA4) (Neg) + (D) $\Longrightarrow$ (Neg4);

(CA5) (Neg2) + (Neg4) + (An) $\Longrightarrow$ (TN);

(CA6) (Neg1) + (DN1) + (An) $\Longrightarrow$ (DN2);

(CA7) (BB) + (An) + (DN) $\Longrightarrow$ (DN4);

(CA8) (DN1) + (K) + (Tr) $\Longrightarrow$ (DN5);

(CA9) (DN1) + (K) + (Tr) + (\$) + (Re) + (M) + (An) $\Longrightarrow$ (DN).

Proof.

(CA0): $1^- \overset{(Neg)}{=} 1 \to 0 \overset{(M)}{=} 0$ and $0^- \overset{(Neg)}{=} 0 \to 0 \overset{(Re)}{=} 1$.

(CA1): $y^- \to x^- \overset{(Neg)}{=} (y \to 0) \to (x \to 0) \overset{(BB')}{\geq} x \to y$, i.e. (Neg1) holds.

(CA2): If $x \leq y$, i.e. $x \to y = 1$, then $1 \overset{(Neg1)}{=} (x \to y) \to (y^- \to x^-) = 1 \to (y^- \to x^-) \overset{(M)}{=} y^- \to x^-$, i.e. $y^- \leq x^-$; thus, (Neg2) holds.

(CA3): $y \to x^- \overset{(Neg)}{=} y \to (x \to 0) \overset{(Ex)}{=} x \to (y \to 0) \overset{(Neg)}{=} x \to y^-$, i.e. (Neg3) holds.

(CA4): $x \overset{(D')}{\leq} (x \to 0) \to 0 \overset{(Neg)}{=} (x^-)^-$, i.e. (Neg4) holds.

(CA5): $x \overset{(Neg4)}{\leq} (x^-)^- \Longrightarrow ((x^-)^-)^- \leq x^-$, by (Neg2). On the other hand, by (Neg4), $x^- \leq ((x^-)^-)^-$. Then, by (An), $x^- = ((x^-)^-)^-$, i.e. (TN) holds.

(CA6): By (Neg1), $x \to y \le y^- \to x^-$. On the other hand, by (DN1), $y^- \to x^- \le x \to y$. Then, by (An), we obtain that $x \to y = y^- \to x^-$, i.e. (DN2) holds.

(CA7): By (CA1), (BB) $\Longrightarrow$ (Neg1); by (C1), (Neg1) + (DN) $\Longrightarrow$ (DN1); by (CA6), (Neg1) + (DN1) + (An) $\Longrightarrow$ (DN2); by (C4), (DN2) $\Longrightarrow$ (DN4).

(CA8): $x^- \overset{(K')}{\le} y^- \to x^- \overset{(DN1')}{\le} x \to y$, hence by (Tr), $x^- \le x \to y$, i.e. (DN5') holds.

(CA9): First, we prove that: (a) $(x^-)^- \le x$. Indeed, $(x^-)^- \overset{(K')}{\le} (((x^-)^-)^-)^- \to (x^-)^- \overset{(DN1')}{\le} x^- \to ((x^-)^-)^- \overset{(DN1)}{\le} (x^-)^- \to x$, hence $(x^-)^- \le (x^-)^- \to x$, by (Tr); then, by ($'), $(x^-)^- \to (x^-)^- \le (x^-)^- \to x$, hence $1 \le (x^-)^- \to x$, by (Re), i.e. $1 \to ((x^-)^- \to x) = 1$; then, by (M), $(x^-)^- \to x = 1$, i.e. (a) holds.

Then, we prove that: (b) $x \le (x^-)^-$. Indeed, $((x^-)^-)^- \overset{(a)}{\le} x^-$; then, $1 = ((x^-)^-)^- \to x^- \overset{(DN1)}{\le} x \to (x^-)^-$, hence $1 \to (x \to (x^-)^-) = 1$; then, by (M), $x \to (x^-)^- = 1$, i.e. (b) holds.

Now, (a) + (b) + (An) imply $(x^-)^- = x$, i.e. (DN) holds. □

Proposition 2.24

(CAN1) (Neg1-0) + (DN2) + (DN) + (M) $\Longrightarrow$ (Neg);
(CAN2) (Neg1-0) + (11-1) + (Neg) $\Longrightarrow$ (10-0);
(CAN3) (Neg1-0) + (DN2) + (F) $\Longrightarrow$ (L);
(CAN4) (Neg1-0) + (DN1) + (K) + (BB) + (M) $\Longrightarrow$ (F);
(CAN5) (Neg1-0) + (DN) + (DN2) + (L) $\Longrightarrow$ (F).

Proof.

(CAN1): $x \to 0 \overset{(M)}{=} (1 \to x) \to 0 \overset{(Neg1-0)}{=} (1 \to x) \to 1^- \overset{(DN2)}{=} (1^-)^- \to (1 \to x)^- \overset{(DN)}{=} 1 \to (1 \to x)^- \overset{(M)}{=} x^-$, i.e. (Neg) holds.

(CAN2): $1 \to 0 \overset{(11-1)}{=} (1 \to 1) \to 0 \overset{(Neg)}{=} (1 \to 1)^- \overset{(11-1)}{=} 1^- \overset{(Neg1-0)}{=} 0$, i.e. (10-0) holds.

(CAN3): $x \to 1 \overset{(DN2)}{=} 1^- \to x^- \overset{(Neg1-0)}{=} 0 \to x^- \overset{(F)}{=} 1$, i.e. (L) holds.

(CAN4): $0 \to x \overset{(Neg1-0)}{=} 1^- \to x \overset{(M)}{=} 1 \to (1^- \to x)$
$\overset{(DN1)}{=} [(x^- \to 1^-) \to (1 \to x)] \to (1^- \to x)$
$\overset{(M)}{=} [(x^- \to 1^-) \to x] \to (1^- \to x)$
$\overset{(M)}{=} 1 \to ([(x^- \to 1^-) \to x] \to (1^- \to x))$
$\overset{(K)}{=} [1^- \to (x^- \to 1^-)] \to ([(x^- \to 1^-) \to x] \to (1^- \to x)) \overset{(BB)}{=} 1$, i.e. (F) holds.

(CAN5): $0 \to x \overset{(Neg1-0)}{=} 1^- \to x \overset{(DN)}{=} 1^- \to (x^-)^- \overset{(DN2)}{=} x^- \to 1 \overset{(L)}{=} 1$, i.e. (F) holds. □

- **Connections between the properties in list B and List C**

Proposition 2.25

(CB1) (comm) + (Neg) + (F) + (M) $\Longrightarrow$ (DN);
(CB2) (impl) + (Neg) $\Longrightarrow$ (DN6);
(CB3) (pimpl) + (Neg) + (Re) + (M) $\Longrightarrow$ (Neg6);
(CB4) (pimpl-1) + (Neg) + (Re) + (M) + (K) + (An) $\Longrightarrow$ (Neg6);
(CB5) (DN) + (DN2) + (DN6) + (DN5) + (K) + (*) + (An) $\Longrightarrow$ (impl).

Proof.

(CB1): $(x^-)^- \overset{(Neg)}{=} (x \to 0) \to 0 \overset{(comm)}{=} (0 \to x) \to x \overset{(F)}{=} 1 \to x \overset{(M)}{=} x$, i.e. (DN) holds.

(CB2): Take $y = 0$ in (impl) ($(x \to y) \to x = x$); we obtain: $(x \to 0) \to x = x$, hence $x \overset{(Neg)}{=} x^- \to x$, i.e. (DN6) holds.

(CB3): In (pimpl) ($x \to (y \to z) = (x \to y) \to (x \to z)$), take $y = x$ and $z = 0$; we then obtain: $x \to (x \to 0) = (x \to x) \to (x \to 0)$, hence $x \to x^- = 1 \to x^- = x^-$, by (Neg), (Re), (M), i.e. (Neg6) holds.

(CB4): On the one hand, $x^- \leq x \to x^-$, by (K'). On the other hand,
$x \to x^- \overset{(Neg)}{=} x \to (x \to 0) \overset{(pimpl-1')}{\leq} (x \to x) \to (x \to 0) \overset{(Re)}{=} 1 \to (x \to 0) \overset{(M)}{=} x \to 0 \overset{(Neg)}{=} x^-$.
Now, by (An), we obtain that $x \to x^- = x^-$, i.e. (Neg6) holds.

(CB5): First, note that we have: (a) $x \leq (x \to y) \to x$, by (K').
Then, we prove: (b) $(x \to y) \to x \leq x$. Indeed, $(x \to y) \to x \overset{(DN2)}{=} x^- \to (x \to y)^-$. But (DN) + (DN2) + (DN5) imply (Neg5), by (C5'), i.e. $(x \to y)^- \leq x$; then, by (*'), $x^- \to (x \to y)^- \leq x^- \to x \overset{(DN6)}{=} x$, i.e. $x^- \to (x \to y)^- \leq x$. Hence, (b) holds.
Now, (a) + (b) + (An) imply $(x \to y) \to x = x$, i.e. (impl) holds. □

Proposition 2.26
(CBN1) (comm-1) + (Neg1-0) + (DN2) + (DN) + (Neg) + (Re) + (M) + (L) $\Longrightarrow$ (F);
(CBN2) (comm) + (Neg1-0) + (DN1) + (DN5) + (**) + (F) + (M) + (An) $\Longrightarrow$ (Neg).

Proof.
(CBN1): First, we prove:

$$(a) \quad (x^- \to 1^-) \to x = 1.$$

Indeed, $(x^- \to 1^-) \to x \overset{(DN2)}{=} (1 \to x) \to x \overset{(M)}{=} x \to x \overset{(Re)}{=} 1$.
Then, we prove:

$$(b) \quad [x \to (1^- \to x)] \to (1^- \to x) = 1.$$

Indeed, $[x \to (1^- \to x)] \to (1^- \to x) \overset{(comm-1)}{=} [(x \to 1^-) \to 1^-] \to x \overset{(Neg1-0)}{=} [(x \to 0) \to 1^-] \to x \overset{(Neg)}{=} (x^- \to 1^-) \to x \overset{(a)}{=} 1$.
Now, $0 \to x \overset{(Neg1-0)}{=} 1^- \to x \overset{(M)}{=} 1 \to [1^- \to x] \overset{(L)}{=} (x \to 1) \to [1^- \to x]$
$\overset{(L)}{=} (x \to [(x \to 0) \to 1]) \to [1^- \to x]$
$\overset{(Neg)}{=} (x \to (x^- \to 1)) \to (1^- \to x)$
$\overset{(DN)}{=} (x \to (x^- \to (1^-)^-)) \to (1^- \to x)$
$\overset{(DN2)}{=} (x \to (1^- \to x)) \to (1^- \to x) \overset{(b)}{=} 1$, i.e. (F) holds.

(CBN2): First, we prove that: (a) $x \to 0 \leq x^-$.
Indeed, $x^- \to 0 \overset{(Neg1-0)}{=} x^- \to 1^- \overset{(DN1)}{\leq} 1 \to x \overset{(M)}{=} x$. Hence, by (**), we obtain: $x \to 0 \leq (x^- \to 0) \to 0 \overset{(comm)}{=} (0 \to x^-) \to x^- \overset{(F)}{=} 1 \to x^- \overset{(M)}{=} x^-$, i.e. (a) holds.
Then, we prove that: (b) $x^- \leq x \to 0$.
Indeed, by (DN6), $x^- \leq x \to 0$, i.e. (b) holds.
Now, from (a) and (b), by (An), we obtain $x^- = x \to 0$, i.e. (Neg) holds. □

- **The product $\odot$ in a bounded algebra (structure)**

Proposition 2.27
(CP1) (RP) + (Neg4) + (Neg) + (F) + (An) $\Longrightarrow$ (Px).

Proof.
(CP1): $x \odot x^- \overset{(RP)}{\Leftrightarrow} x \leq x^- \to 0 \overset{(Neg)}{=} x^=$; since $x \leq x^=$, by (Neg4'), it follows that $x \odot x^- \leq 0$. But we also have $0 \leq x \odot x^-$, by (F). It follows, by (An), that $x \odot x^- = 0$, i.e. (Px) holds. □

- **The product $\odot$ in a bounded algebra (structure) with (DN)**

Proposition 2.28
(CV1) (dfV) + (Pcomm) + ($P_{(DN)}$) + (DN2) + (DN) $\Longrightarrow$ ($VP_{(DN)}$).

Proof.

(CV1): $x \vee y \overset{(dfV)}{=} (x \to y) \to y \overset{(DN2)}{=} y^- \to (x \to y)^- \overset{(DN)}{=} y^- \to (x \to y^=)^- \overset{(P_{(DN)})}{=} y^- \to (x \odot y^-) \overset{(Pcomm)}{=} y^- \to (y^- \odot x)$. □

- **The meet (wedge) $\wedge$. The regular lattice.**

Let us introduce the following binary operation:

$$(dfW) \qquad x \wedge y = (x^- \vee y^-)^-.$$

Proposition 2.29

(CWP0) (Pcomm) + ($P_{(DN)}$) + (DN2) + (dfV) + (Vcomm) + (dfW) $\Longrightarrow$ (div);

(CWP1) (Wleq) + (P-) + (Px) + (W=) + (Pleq) + (VP) + (V-0) + (div) + (We) + ($P_{(DN)}$) + ($VP_{(DN)}$) + (Pcomm) + (Wd2) + (DN2) + (DN) + (Tr) + (*) + (An) $\Longrightarrow$ (WP).

Proof.

(CWP0): $x \odot (x \to y) \overset{(Pcomm)}{=} (x \to y) \odot x \overset{(P_{(DN)})}{=} ((x \to y) \to x^-)^- \overset{(DN2)}{=} ((y^- \to x^-) \to x^-)^- \overset{(dfV)}{=} (y^- \vee x^-)^- \overset{(Vcomm)}{=} (x^- \vee y^-)^- \overset{(dfW)}{=} x \wedge y$; thus (div) holds.

(CWP1): We must prove that: $a \odot (x \wedge y) = (a \odot x) \wedge (a \odot y)$.

(a) $a \odot (x \wedge y) \leq (a \odot x) \wedge (a \odot y)$. Indeed,
$x \wedge y \overset{(Wleq)}{\leq} x, y \overset{(P-)}{\Longrightarrow} a \odot (x \wedge y) \leq a \odot x, a \odot y \overset{(W=)}{\Longrightarrow}$
$a \odot (x \wedge y) \leq (a \odot x) \wedge (a \odot y)$, i.e. (a) holds.

(b) $Z \overset{notation}{=} (a \odot x) \wedge (a \odot y) \leq a \odot (x \wedge y)$. Indeed,
first, $Z \overset{(Wleq)}{\leq} a \odot x \overset{(Pleq)}{\leq} a$, hence $Z \leq a$, by (Tr), i.e. by (We) we obtain:

$$Z \wedge a = Z. \tag{3}$$

Then, $Z \overset{(Wleq)}{\leq} a \odot x, a \odot y$ imply, by (*):

$$a \to Z \leq a \to (a \odot x), a \to (a \odot y). \tag{4}$$

But, by ($VP_{(DN)}$), we have that:

$$a \to (a \odot x) = x \vee a^-. \tag{5}$$

Then, by (4), (5), we obtain:
$a \to Z \leq a \to (a \odot x) = x \vee a^-, a \to (a \odot y) = y \vee a^- \overset{(W=)}{\Longrightarrow}$
$a \to Z \leq (x \vee a^-) \wedge (y \vee a^-) \overset{(Wd2)}{=} (x \wedge y) \vee a^- \overset{(P-)}{\Longrightarrow}$
$a \odot (a \to Z) \leq a \odot [(x \wedge y) \vee a^-] \overset{(VP)}{=} (a \odot (x \wedge y)) \vee (a \odot a^-) \overset{(Px)}{=} (a \odot (x \wedge y)) \vee 0 \overset{(V-0)}{=} a \odot (x \wedge y)$, hence $a \odot (a \to Z) \leq a \odot (x \wedge y)$; but $a \odot (a \to Z) \overset{(div)}{=} a \wedge Z \overset{(3)}{=} Z$; hence $Z \leq a \odot (x \wedge y)$, i.e. (b) holds.

Finally, (a) + (b) + (An) $\Longrightarrow$ (WP), thus (WP) holds. □

Denote $(x^-)^-$ by $x^=$.

Proposition 2.30

(CV0) (dfV) + (F) + (M) $\Longrightarrow$ (V0-); (dfV) + (Neg) $\Longrightarrow$ (V-0);
(CW0) (dfW) + (Neg0-1) + (Neg1-0) + (V1-1) $\Longrightarrow$ (W0-0);

(CW1) (dfW) + (Vid) + (DN) $\Longrightarrow$ (Wid);
(CW2) (dfW) + (Vcomm) $\Longrightarrow$ (Wcomm);
(CW3) (dfW) + (Vgeq) + (Neg2) + (DN) $\Longrightarrow$ (Wleq);
(CW4) (dfW) + (Neg1-0) + (V-0) + (V0-) + (DN) $\Longrightarrow$ (W1-1);
(CW5) (dfW) + (DN4) + (Ve) + (DN) + (Vcomm) $\Longrightarrow$ (We);

(CW6) (dfW) + (V-V) + (DN2) + (DN) $\Longrightarrow$ (WW-);
(CW6') (dfW) + (W=) + (V=) + (Wleq) + (VP) + (DN2) + (DN) + (*) + (An) $\Longrightarrow$ (W-W);
(CW6") (W-W) + (M1) + (DN2) $\Longrightarrow$ (WV-);
(CW7) (dfW) + (VVV) + (DN2) $\Longrightarrow$ (WWW);
(CW8) (dfW) + (Vassoc) + (DN) $\Longrightarrow$ (Wassoc);
(CW9) (dfW) + (V-) + (Neg2) $\Longrightarrow$ (W-);
(CW10) (dfW) + (V- -) + (Neg2) $\Longrightarrow$ (W- -);
(CW11) (dfW) + (V=) + (Neg2) + (DN) $\Longrightarrow$ (W=);

(CW12) (dfW) + (DN) $\Longrightarrow$ (M1) + (M2);
(CW13) (Vgeq) + (We) $\Longrightarrow$ (Wab1); (Wleq) + (Ve) + (Vcomm) $\Longrightarrow$ (Wab2);
(CW14) (Wleq) + (V-) + (W=) + (VP$_{(DN)}$) + (M2) + (WV-) + (P-) + (Pcomm) + (W- -) + (Neg2) + (*) + (An) $\Longrightarrow$ (Wd2);
(CW14') (Vgeq) + (W-) + (V=) + (Wcomm) + (div) + (VP) + (**) + (P) + (V- -) + (An) $\Longrightarrow$ (Wd1).*

Proof.
(CV0): $0 \vee x \overset{(dfV)}{=} (0 \to x) \to x \overset{(F)}{=} 1 \to x \overset{(M)}{=} x$, i.e. (V0-) holds.
$x \vee 0 \overset{(dfV)}{=} (x \to 0) \to 0 \overset{(Neg)}{=} (x^-)^-$, i.e. (V-0) holds.
(CW0): $x \wedge 0 \overset{(dfW)}{=} (x^- \vee 0^-)^- \overset{(Neg0-1)}{=} (x^- \vee 1)^- \overset{(V1-1)}{=} 1^- \overset{(Neg1-0)}{=} 0$ and $0 \wedge x \overset{(dfW)}{=} (0^- \vee x^-)^- \overset{(Neg0-1)}{=} (1 \vee x^-)^- \overset{(V1-1)}{=} 1^- = 0$, i.e. (W0-0) holds.
(CW1): $x \wedge x \overset{(dfW)}{=} (x^- \vee x^-)^- \overset{(Vid)}{=} (x^-)^- \overset{(DN)}{=} x$, thus (Wid) holds.
(CW2): $x \wedge y \overset{(dfW)}{=} (x^- \vee y^-)^- \overset{(Vcomm)}{=} (y^- \vee x^-)^- \overset{(dfW)}{=} y \wedge x$, thus (Wcomm) holds.
(CW3): By (Vgeq), $x^-, y^- \leq x^- \vee y^-$, then by (Neg2'), $(x^- \vee y^-)^- \leq (x^-)^-, (y^-)^-$, i.e. $x \wedge y \leq x, y$, by (dfW) and (DN); thus, (Wleq) holds.
(CW4): $x \wedge 1 \overset{(dfW)}{=} (x^- \vee 1^-)^- \overset{(Neg1-0)}{=} (x^- \vee 0)^- \overset{(V-0)}{=} (x^-)^- \overset{(DN)}{=} x$ and $1 \wedge x \overset{(dfW)}{=} (1^- \vee x^-)^- \overset{(Neg1-0)}{=} (0 \vee x^-)^- \overset{(V0-)}{=} (x^-)^- \overset{(DN)}{=} x$; thus (W1-1) holds.
(CW5): $x \leq y \overset{(DN4)}{\Leftrightarrow} y^- \leq x^- \overset{(Ve)}{\Leftrightarrow} y^- \vee x^- = x^- \overset{(DN)}{\Leftrightarrow} (y^- \vee x^-)^- = (x^-)^-$, i.e. $x \wedge y = x$, by (Vcomm) and (DN); thus (We) holds.
(CW6): $(x \wedge y) \to z \overset{(dfW)}{=} (x^- \vee y^-)^- \to z \overset{(DN)}{=} (x^- \vee y^-)^- \to (z^-)^- \overset{(DN2)}{=} z^- \to (x^- \vee y^-) \overset{(V-V)}{=} (x^- \to y^-) \to (z^- \to y^-) \overset{(DN2)}{=} (y \to x) \to (y \to z)$; thus (WW-) holds.
(CW6'): We must prove that $a \to (x \wedge y) = (a \to x) \wedge (a \to y)$. Indeed,
(a) $a \to (x \wedge y) \leq (a \to x) \wedge (a \to y)$. Indeed,
$x \wedge y \overset{(Wleq)}{\leq} x,\ y \overset{(*)}{\Longrightarrow} a \to (x \wedge y) \leq a \to x,\ a \to y \overset{(W=)}{\Longrightarrow} a \to (x \wedge y) \leq (a \to x) \wedge (a \to y)$, i.e. (a) holds.
(b) $Z \overset{notation}{=} (a \to x) \wedge (a \to y) \leq a \to (x \wedge y)$. Indeed,
$Z \overset{(Wleq)}{\leq} a \to x,\ a \to y \overset{(DN2)}{\Leftrightarrow} Z \leq x^- \to a^-,\ y^- \to a^- \overset{(RP)}{\Leftrightarrow} Z \odot x^- \leq a^-,\ Z \odot y^- \leq a^- \overset{(V=)}{\Leftrightarrow} Z \odot x^- \vee Z \odot y^- \leq a^- \overset{(VP)}{\Leftrightarrow} Z \odot (x^- \vee y^-) \leq a^- \overset{(RP)}{\Leftrightarrow} Z \leq (x^- \vee y^-) \to a^- \overset{(DN2)}{\Leftrightarrow} Z \leq a^= \to (x^- \vee y^-)^- \overset{(DN),(dfW)}{\Leftrightarrow} Z \leq a \to (x \wedge y)$, i.e. (b) holds.
Finally, (a) + (b) + (An) $\Longrightarrow$ (W-W), i.e. (W-W) holds.
(CW6"): $(x \vee y) \to a \overset{(DN2)}{=} a^- \to (x \vee y)^- \overset{(M1)}{=} a^- \to (x^- \wedge y^-) \overset{(W-W)}{=} (a^- \to x^-) \wedge (a^- \to y^-) \overset{(DN2)}{=} (x \to a) \wedge (y \to a)$, i.e. (WV-) holds.
(CW7): $(y \to x) \to (y \to z) \overset{(DN2)}{=} (x^- \to y^-) \to (z^- \to y^-) \overset{(VVV)}{=} (z^- \vee y^-) \to (x^- \vee y^-) \overset{(DN2)}{=} (x^- \vee y^-)^- \to (z^- \vee y^-)^- \overset{(dfW)}{=} (x \wedge y) \to (z \to y)$; thus (WWW) holds.
(CW8): $(x \wedge y) \wedge z \overset{(dfW)}{=} (x^- \vee y^-)^- \wedge z \overset{(dfW)}{=} ((x^- \vee y^-)^= \vee z^-)^- \overset{(DN)}{=} ((x^- \vee y^-) \vee z^-)^- \overset{(Vassoc)}{=} (x^- \vee (y^- \vee z^-))^- \overset{(DN)}{=} (x^- \vee (y^- \vee z^-)^=)^- \overset{(dfW)}{=} (x^- \vee (y \wedge z)^-)^- \overset{(dfW)}{=} x \wedge (y \wedge z)$; thus (Wassoc) holds.

(CW9): $x \leq y \overset{(Neg2')}{\Longrightarrow} y^- \leq x^- \overset{(V-)}{\Longrightarrow} y^- \vee a^- \leq x^- \vee a^- \overset{(Neg2')}{\Longrightarrow} (x^- \vee a^-)^- \leq (y^- \vee a^-)^-$, i.e. $x \wedge a \leq y \wedge a$, by (dfW); thus (W-) holds.

(CW10): $(x \leq y$ and $a \leq b) \overset{(Neg2')}{\Longrightarrow} (y^- \leq x^-$ and $b^- \leq a^-) \overset{(V--)}{\Longrightarrow} y^- \vee b^- \leq x^- \vee a^- \overset{(Neg2')}{\Longrightarrow} (x^- \vee a^-)^- \leq (y^- \vee b^-)^-$, i.e. $x \wedge a \leq y \wedge b$; thus (W- -) holds.

(CW11): $(z \leq x$ and $z \leq y \overset{(Neg2')}{\Longrightarrow} (x^- \leq z^-, y^- \leq z^-) \overset{(V=)}{\Longrightarrow} x^- \vee y^- \leq z^- \overset{(Neg2')}{\Longrightarrow} z^= \leq (x^- \vee y^-)^-$, i.e. $z \leq x \wedge y$, by (DN) and (dfW); thus (W=) holds.

(CW12): $x^- \wedge y^- \overset{(dfW)}{=} (x^= \vee y^=)^- \overset{(DN)}{=} (x \vee y)^-$, thus (M1) holds.
$(x \wedge y)^- \overset{(dfW)}{=} (x^- \vee y^-)^= \overset{(DN)}{=} x^- \vee y^-$, thus (M2) holds.

(CW13): $x \overset{(Vgeq)}{\leq} x \vee y \overset{(We)}{\Leftrightarrow} x \wedge (x \vee y) = x$, thus (Wab1) holds.
$x \wedge y \overset{(Wleq)}{\leq} x \overset{(Ve)}{\Leftrightarrow} (x \wedge y) \vee x = x$, thus (Wab2) holds.

(CW14): We must prove that: $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$. Indeed:

(a) $a \vee (x \wedge y) \leq (a \vee x) \wedge (a \vee y)$. Indeed,
$x \wedge y \overset{(Wleq)}{\leq} x,\ y \overset{(V-)}{\Longrightarrow} a \vee (x \wedge y) \leq a \vee x,\ a \vee y \overset{(W=)}{\Longrightarrow}$
$a \vee (x \wedge y) \leq (a \vee x) \wedge (a \vee y)$; thus (a) holds.

(b) $(a \vee x) \wedge (a \vee y) \leq a \vee (x \wedge y)$. Indeed,
$a \vee (x \wedge y) \overset{(VP_{(DN)})}{=} (x \wedge y)^- \to ((x \wedge y)^- \odot a) \overset{(M2)}{=}$
$(x^- \vee y^-) \to ((x \wedge y)^- \odot a) \overset{(WV-)}{=}$
$(x^- \to ((x \wedge y)^- \odot a)) \wedge (y^- \to ((x \wedge y)^- \odot a))$, hence

$$a \vee (x \wedge y) = (x^- \to ((x \wedge y)^- \odot a)) \wedge (y^- \to ((x \wedge y)^- \odot a)). \quad (6)$$

But $x \wedge y \overset{(Wleq)}{\leq} x,\ y \overset{(Neg2')}{\Longrightarrow} x^-,\ y^- \leq (x \wedge y)^- \overset{((P-),(Pcomm)}{\Longrightarrow}$
$x^- \odot a,\ y^- \odot a \leq (x \wedge y)^- \odot a \overset{(*)}{\Longrightarrow}$
$[x^- \to (x^- \odot a) \leq x^- \to ((x \wedge y)^- \odot a)$ and $y^- \to (y^- \odot a) \leq y^- \to ((x \wedge y)^- \odot a)] \overset{(W--)}{\Longrightarrow}$
$(x^- \to (x^- \odot a)) \wedge (y^- \to (y^- \odot a)) \leq (x^- \to ((x \wedge y)^- \odot a)) \wedge (y^- \to ((x \wedge y)^- \odot a))$;
but $x^- \to (x^- \odot a) \overset{(VP_{(DN)})}{=} a \vee x$ and $y^- \to (y^- \odot a) \overset{(VP_{(DN)})}{=} a \vee y$, hence

$$(a \vee x) \wedge (a \vee y) \leq (x^- \to ((x \wedge y)^- \odot a)) \wedge (y^- \to ((x \wedge y)^- \odot a)). \quad (7)$$

By (7) and (6), we obtain: $(a \vee x) \wedge (a \vee y) \leq a \vee (x \wedge y)$, i.e. (b) holds.

Finally, (a) + (b) + (An) $\Longrightarrow$ (Wd2), i.e. (Wd2) holds.

(CW14'): We must prove that: $a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y)$. Indeed:

(a) $(a \wedge x) \vee (a \wedge y) \leq a \wedge (x \vee y)$. Indeed,
$x, y \overset{(Vgeq)}{\leq} x \vee y \overset{(W-)}{\Longrightarrow} a \wedge x, a \wedge y \leq a \wedge (x \vee y) \overset{(V=)}{\Longrightarrow}$
$(a \wedge x) \vee (a \wedge y) \leq a \wedge (x \vee y)$; thus (a) holds.

(b) $a \wedge (x \vee y) \leq (a \wedge x) \vee (a \wedge y)$. Indeed,
$a \wedge (x \vee y) \overset{(Wcomm)}{=} (x \vee y) \wedge a \overset{(div)}{=} (x \vee y) \odot [(x \vee y) \to a] \overset{(VP)}{=}$
$(x \odot ((x \vee y) \to a)) \vee (y \odot ((x \vee y) \to a))$, hence

$$a \wedge (x \vee y) = (x \odot ((x \vee y) \to a)) \vee (y \odot ((x \vee y) \to a)). \quad (8)$$

But $x, y \overset{(Vgeq)}{\leq} x \vee y \overset{(**)}{\Longrightarrow} (x \vee y) \to a \leq x \to a, y \to a \overset{(P-)}{\Longrightarrow}$
$(x \odot ((x \vee y) \to a) \leq x \odot (x \to a) \overset{(div)}{=} x \wedge a \overset{(Wcomm)}{=} a \wedge x$ and
$y \odot ((x \vee y) \to a) \leq y \odot (y \to a) \overset{(div)}{=} y \wedge a \overset{(Wcomm)}{=} a \wedge y)$; then, by (V- -), we obtain:

$$(x \odot ((x \vee y) \to a)) \vee (y \odot ((x \vee y) \to a)) \leq (a \wedge x) \vee (a \wedge y). \quad (9)$$

By (8) and (9), we obtain: $a \wedge (x \vee y) \leq (a \wedge x) \vee (a \wedge y)$, i.e. (b) holds.

Finally, (a) + (b) + (An) $\Longrightarrow$ (Wd1), i.e. (Wd1) holds. $\Box$

Examples 2.31

Consider the following **distributive bounded Dedekind lattice**: $(A = \{0, a, b, 1\}, \wedge, \vee, 0, 1)$, given by the following tables:

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

Hence, the corresponding distributive bounded Ore lattice ($x \leq y \Leftrightarrow x \vee y = y \Leftrightarrow x \wedge y = x$) is given by the Hasse diagram from Figure 1.

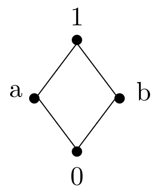

Figure 1: The Hasse diagram of the given distributive bounded lattice

Since $x \leq y \Leftrightarrow x \to y = 1$ (EqrelR) and (M) ($1 \to x = x$) holds, it follows that the corresponding *general table of implication* $\to_g$ (residuum, in certain cases) is as follows:

$\to_g$	0	a	b	1
0	1	1	1	1
a	r_{21}	1	r_{23}	1
b	r_{31}	r_{32}	1	1
1	0	a	b	1

Note that the properties (Re), (M), (L) and (An) hold, hence all the algebras $(A = \{0, a, b, 1\}, \to_g, 0, 1)$ are **bounded aRML algebras**, as defined in [18]. Since there are four free spaces $(r_{21}, r_{23}, r_{31}, r_{32})$ that can be filled with one element of $\{0, a, b\}$, it follows that there are $3^4 = 81$ bounded aRML algebras, where the elements $(r_{21}, r_{23}, r_{31}, r_{32})$ are (in increasing lexicographical order):
(1) (0,0,0,0); (2) (0,0,0,a); (3) (0,0,0,b);
(4) (0,0,a,0); (5) (0,0,a,a); (6) (0,0,a,b);
(7) (0,0,b,0); (8) (0,0,b,a); (9) (0,0,b,b);

(10) (0,a,0,0); (11) (0,a,0,a); (12) (0,a,0,b);
(13) (0,a,a,0); (14) (0,a,a,a); (15) (0,a,a,b);
(16) (0,a,b,0); (17) (0,a,b,a); (18) (0,a,b,b);

(19) (0,b,0,0); (20) (0,b,0,a); (21) (0,b,0,b);
(22) (0,b,a,0); (23) (0,b,a,a); (24) (0,b,a,b);
(25) (0,b,b,0); (26) (0,b,b,a); (27) (0,b,b,b);

(28) (a,0,0,0); (29) (a,0,0,a); (30) (a,0,0,b);
(31) (a,0,a,0); (32) (a,0,a,a); (33) (a,0,a,b);
(34) (a,0,b,0); (35) (a,0,b,a); (36) (a,0,b,b);

(37) (a,a,0,0); (38) (a,a,0,a); (39) (a,a,0,b);
(40) (a,a,a,0); (41) (a,a,a,a); (42) (a,a,a,b);
(43) (a,a,b,0); (44) (a,a,b,a); (45) (a,a,b,b);

(46) (a,b,0,0); (47) (a,b,0,a); (48) (a,b,0,b);

(49) (a,b,a,0); (50) (a,b,a,a); (51) (a,b,a,b);
(52) (a,b,b,0); (53) (a,b,b,a); (54) (a,b,b,b);

(55) (b,0,0,0); (56) (b,0,0,a); (57) (b,0,0,b);
(58) (b,0,a,0); (59) (b,0,a,a); (60) (b,0,a,b);
(61) (b,0,b,0); (62) (b,0,b,a); (63) (b,0,b,b);

(64) (b,a,0,0); (65) (b,a,0,a); (66) (b,a,0,b);
(67) (b,a,a,0); (68) (b,a,a,a); (69) (b,a,a,b);
(70) (b,a,b,0); (71) (b,a,b,a); (72) (b,a,b,b);

(73) (b,b,0,0); (74) (b,b,0,a); (75) (b,b,0,b);
(76) (b,b,a,0); (77) (b,b,a,a); (78) (b,b,a,b);
(79) (b,b,b,0); (80) (0,b,b,a); (81) (b,b,b,b).

Consequently, there exist 81 distributive bounded Dedekind regular-lattices ($(A = \{0, a, b, 1\}, \wedge, \vee, \rightarrow_1, 0, 1)$ **-** $(A = \{0, a, b, 1\}, \wedge, \vee, \rightarrow_{81}, 0, 1)$**) corresponding to the given distributive bounded Dedekind lattice** $(A = \{0, a, b, 1\}, \wedge, \vee, 0, 1)$.

Note that, by Theorem **??**, the properties (Re), (M), (L), (An) and (Tr) hold, hence all the 81 reduct algebras $(A = \{0, a, b, 1\}, \rightarrow_1, 0, 1)$ - $(A = \{0, a, b, 1\}, \rightarrow_{81}, 0, 1)$ are bounded **oRML algebras** (in fact bounded oRML lattices), as defined in [18].

By checking (with a PASCAL programm) the other possible considered properties ((Ex), (pimpl), (pi), (BB), (**), (B), (*), (D), (comm), (impl)) that can be verified by these 81 implications $\rightarrow_1$ - $\rightarrow_{81}$, we have obtained that there are:

- 26 proper **bounded oRML algebras without (D)** (i.e. only (Tr) also holds), namely: 4, 6, 13, 15, 22, 24, 31, 32, 33, 40, 42, 49, 51, 55, 56, 57, 61, 62, 63, 64, 65, 66, 70, 71, 72, 79;
- 19 proper **bounded oRML algebras with (D)** (i.e. only (Tr), (D) also hold), namely: 7, 8, 16, 17, 25, 28, 29, 30, 36, 43, 44, 46, 48, 54, 60, 67, 68, 69, 78;
- 6 proper **commutative bounded oRML algebras with (D)** (i.e. only (Tr), (D), (comm) also hold), namely: 34, 35, 52, 58, 59, 76;

- 4 proper **bounded *aRML algebras without (D)** (i.e. only (*), (Tr) also hold): 14, 41, 75, 81;
- 12 proper **bounded *aRML algebras with (D)** (i.e. only (*), (Tr), (D) also hold), namely: 3, 9, 10, 11, 12, 18, 21, 27, 37, 38, 39, 45;
- 2 proper **bounded pi-*aRML algebras without (D)** (i.e. only (pi), (*), (Tr) also hold: 5,73;
- 3 proper **bounded pi-*aRML algebras with (D)** (i.e. only (pi), (*), (Tr), (D) also hold), namely: 1, 2, 19;
- 2 proper **bounded aRML** algebras without (D)** (i.e. only (**), (Tr) also hold), namely: 50, 80;
- 1 proper **commutative bounded aRML** algebra with (D)** (i.e. only (**), (Tr), (D), (comm) also hold), namely: 53;
- 2 proper **bounded pi-BCC algebras** (i.e. only (pi), (**), (B), (*), (Tr) also hold), namely: 23, 74;

- 2 proper **bounded aBE** algebras** (i.e. only (Ex), (**), (Tr), (D) also hold), namely: 26, 47;
- 1 proper **bounded positive implicative BCK algebra = bounded Hilbert algebra** (i.e. only (Ex), (pimpl), (pi), (BB), (**), (B), (*), (Tr), (D) also hold), namely: 20;
- 1 **bounded implicative BCK algebra = Boole algebra** (i.e. (Ex), (pimpl), (pi), (BB), (**), (B), (*), (Tr), (D), (comm), (impl) also hold), namely: 77.

Note that the most reach in properties is number 77, which is a regular-Boolean algebra.

Note that the commutative ones (34, 35; 52,53; 58, 59; 76, 77) verify: $x \vee y = (x \rightarrow y) \rightarrow y$, for all $x, y \in A$.

Looking for the property (P) ($x \odot y = \min\{z \mid x \leq y \rightarrow z\}$, when the minimum does exist), hence for the existence of the product $\odot$ and if $\odot$ equals $\wedge$, note first that, for the table of *general implication* $\rightarrow_g$, we obtain the table of *general product* $\odot_g$ as follows:

$\odot_g$	0	a	b	1
0	0	0	0	0
a	0	p_{22}	p_{23}	a
b	0	p_{32}	p_{33}	b
1	0	a	b	1

Then, we have obtained the following results:
- numbers 58, 59, 76, 77 have (P) and $\odot = \wedge$;
- number 1 has (P), but $\odot \neq \wedge$, namely: $p_{22} = a \quad p_{23} = b$, $p_{32} = a \quad p_{33} = b$
- numbers 4, 5 have (P), but $\odot \neq \wedge$, namely $p_{22} = a \quad p_{23} = 0$, $p_{32} = a \quad p_{33} = b$
- numbers 7, 9 have (P), but $\odot \neq \wedge$, namely $p_{22} = a \quad p_{23} = b$, $p_{32} = a \quad p_{33} = 0$
- numbers 28, 37, 40 have (P), but $\odot \neq \wedge$, namely $p_{22} = 0 \quad p_{23} = b$, $p_{32} = a \quad p_{33} = b$
- numbers 31, 32, 41 have (P), but $\odot \neq \wedge$, namely $p_{22} = 0 \quad p_{23} = 0$, $p_{32} = a \quad p_{33} = b$
- numbers 34, 36, 43, 45 have (P), but $\odot \neq \wedge$, namely $p_{22} = 0 \quad p_{23} = b$, $p_{32} = a \quad p_{33} = 0$
- numbers 55, 73 have (P), but $\odot \neq \wedge$, namely $p_{22} = a \quad p_{23} = b$, $p_{32} = 0 \quad p_{33} = b$
- numbers 61, 63, 79, 81 have (P), but $\odot \neq \wedge$, namely $p_{22} = a \quad p_{23} = b$, $p_{32} = 0 \quad p_{33} = 0$
- the following 56 numbers have no (P), since $\min\{a, b, 1\}$ does not exist:
2, 3, 6, 8; 10 - 18; 19 - 27; 29, 30, 33, 35, 38, 39, 42, 44; 46 - 54; 56, 57, 60, 62; 64 - 72; 74, 75, 78, 80.

Note that since all the 81 aRML algebras (lattices) are bounded, a regular negation $^-$ does exist: for all $x \in A$, $x^- \overset{df.}{=} x \to 0$. The negation $^-$ verifies in some cases the Double Negation property (DN): $(x^-)^- = x$, for all $x \in A$, namely in the two cases: (i) $r_{21} = a$, $r_{31} = b$; (ii) $r_{21} = b$, $r_{31} = a$. Hence, we have $2 \times 9 = 18$ bounded aRML algebras with (DN), namely:
(i) 34, 35, 36, 43, 44, 45, 52, 53, 54;
(ii) 58, 59, 60, 67, 68, 69, 76, 77, 78.

Finally, note that there are 4 special distributive bounded Dedekind regular-lattices, which satisfy (DN) - with the same negation, which have the property (P) and $\odot = \wedge$, which are commutative and $x \vee y = (x \to y) \to y$; these are the numbers 58, 59, 76, 77. Namely, we have:

$\mathcal{A}_{58} = (A = \{0, a, b, 1\}, \wedge, \vee, \to_{58}, 0, 1)$ is a bounded commutative oRML algebra (lattice).

$\mathcal{A}_{59} = (A = \{0, a, b, 1\}, \wedge, \vee, \to_{59}, 0, 1)$ is a bounded commutative oRML algebra (lattice).

The tables of $\to_{58}$ and $\to_{59}$ are, respectively:

$\to_{58}$	0	a	b	1
0	1	1	1	1
a	**b**	1	**0**	1
b	**a**	**0**	1	1
1	0	a	b	1

$\to_{59}$	0	a	b	1
0	1	1	1	1
a	**b**	1	**0**	1
b	**a**	**a**	1	1
1	0	a	b	1

$\mathcal{A}_{76} = (A = \{0, a, b, 1\}, \wedge, \vee, \to_{76}, 0, 1)$ is a bounded commutative oRML algebra (lattice).

$\mathcal{A}_{77} = (A = \{0, a, b, 1\}, \wedge, \vee, \to_{77}, 0, 1)$ is a Boolean algebra.

The tables of $\to_{76}$ and $\to_{77}$ are, respectively:

$\to_{76}$	0	a	b	1
0	1	1	1	1
a	**b**	1	**b**	1
b	**a**	**0**	1	1
1	0	a	b	1

$\to_{77}$	0	a	b	1
0	1	1	1	1
a	**b**	1	**b**	1
b	**a**	**a**	1	1
1	0	a	b	1

2.4 Other regular algebras

We shall recall here definitions and results concerning the bounded BCK algebras, the Hilbert algebras, the Wajsberg algebras and the Boolean algebras. Some new results are proved.

Definition 2.32 [19]
An algebra $(A, \rightarrow, 1)$ is a *BCK algebra*, if it verifies the properties:
(BCK1) (BB), (D), (Re), (L), (An) [21], or, equivalently
(BCK2) [18], (B), (C), (K), (An).

Recall that a BCK algebra verifies all the properties in List A.

Definition 2.33 [6] [20]
An algebra $(A, \rightarrow, 1)$ is a *Hilbert algebra*, if it verifies the axioms (K), (pimpl-1), (An).

See more about BCK algebras in the books [27], [16].

Definition 2.34 [22] Let $\mathcal{A} = (A, \rightarrow, 1)$ be a BCK algebra. We say that $\mathcal{A}$ is
- *positive implicative*, if property (pimpl) is satisfied;
- *commutative*, if property (comm) is satisfied;
- *implicative*, if property (impl) is satisfied.

Theorem 2.35 *([22], Theorem 8) [20]*
A BCK algebra is positive implicative if and only if the property (pi) holds (or, in a BCK algebra the properties (pimpl) and (pi) are equivalent).

Theorem 2.36 *([16], Remarks 2.1.32 (1)) (see [20] for a direct proof)*
Hilbert algebras are categorically equivalent to positive implicative BCK algebras.

Theorem 2.37 *([22], Theorem 9) [20]*
In a commutative BCK algebra, the properties (pi) and (impl) are equivalent.

Theorem 2.38 *([22], Theorem 10) [20]*
Any implicative BCK algebra is commutative and positive implicative.

Theorem 2.39 *[20]*
Any commutative and positive implicative BCK algebra is implicative.

Corollary 2.40 *[20] In a BCK algebra we have:*

$$(impl) \Leftrightarrow ((comm) + (pi) (\Leftrightarrow (pimpl)))$$

Corollary 2.41 *[20] Any commutative Hilbert algebra is implicative.*

2.4.1 Bounded BCK algebras

Resuming, we have the following two cases.

• **The bounded BCK algebra**
A bounded BCK algebra verifies (following the second definition, (BCK2)): (B), (C), (K), (An); (F), (Neg). It also verifies (M) and (11-1), among others.

It is denoted by $(A, \rightarrow, 0, 1)$.

Property (Neg) introduces a unique negation $^- : A \longrightarrow A$, so a bounded BCK algebra is in fact a *bounded BCK algebra with negation*; hence, it could be denoted also by $(A, \rightarrow, ^-, 0, 1)$.

Note that any bounded BCK algebra is a bounded ordered regular algebra.

Note also that a bounded BCK algebra verifies: $(\overline{M})$, by Theorem 2.20 (i); (Neg1-0) and (Neg0-1), by (CA0); (Neg1), by (CA1); (Neg2), by (CA2); (Neg3), by (CA3); (Neg4), by (CA4); (TN), by (CA5).

• **The bounded BCK algebra with involutive negation**
A bounded BCK algebra with involutive negation $^- : A \longrightarrow A$ verifies the axioms (following the second definition, (qBCK2)): (B), (C), (K), (An); (F), (Neg); (DN).

It is denoted by $(A, \rightarrow, 0, 1)$.

Note that a bounded BCK algebra with involutive negation verifies (besides $(\overline{M})$, (Neg1-0), (Neg0-1), (Neg1)-(Neg3), (TN)) also: (DN1), by (C1); (DN2), by (CA6); (DN4), by (CA7); (DN3), by (C3); (DN5), by (CA8); (Neg5), by (C5').

Note also that the involutive negation is unique.

2.4.2 Wajsberg algebras. Connection

Recall from [7] the definition of Wajsberg algebras:

Definition 2.42 An algebra $(A, \rightarrow, ^-, 1)$ of type $(2, 1, 0)$ is a *Wajsberg algebra* if the properties (M), (BB), (comm), (DN1) hold.

Note that a Wajsberg algebra is a regular algebra, since (M) holds.

• **The Wajsberg algebras as regular algebras with involutive negation**

We shall prove here that the Wajsberg algebra [7] is an example of *regular algebra with negation*, namely with involutive negation.

Theorem 2.43 *A Wajsberg algebra is a regular algebra with involutive negation.*

Proof. Let $\mathcal{A} = (A, \rightarrow, ^-, 1)$ be a Wajsberg algebra, i.e. (M), (BB), (comm), (DN1) hold. Denote 1^- by 0, hence (Neg1-0) holds too.
We must prove that $\mathcal{A}$ is a regular algebra with negation and that the negation is involutive. By Definition 2.19, we must prove that:
(1) $(A, \rightarrow, 1)$ is a regular algebra, i.e. (M) holds;
(2) the property (Neg) holds.
Finally, we must prove that (DN) holds. Indeed,
(1): By definition.
(2): By (B12), (comm) + (M) $\Longrightarrow$ (An);
- by (A17'), (M) + (BB) $\Longrightarrow$ (Tr);
- by (A18), (M) + (BB) $\Longrightarrow$ (Re);
- by (A18'), (M) + (BB) $\Longrightarrow$ (D);
- by (A20'), (M) + (BB) $\Longrightarrow$ (C);
- by (A3), (C) + (An) $\Longrightarrow$ (Ex);
- by (B32), (comm) + (Re) + (BB) + (M) $\Longrightarrow$ (L);
- by (A8), (L) + (Re) + (Ex) $\Longrightarrow$ (K);
- by (CAN4), (Neg1-0) + (DN1) + (K) + (BB) + (M) $\Longrightarrow$ (F);
- by (CA8), (DN1) + (K) + (Tr) $\Longrightarrow$(DN5);
- by (A15'), (M) + (BB) $\Longrightarrow$ (**);
- by (CBN2), (comm) + (DN1) + (Neg1-0) + (DN5) + (**) + (F) + (M) + (An) $\Longrightarrow$ (Neg), thus (Neg) holds.
Finally, by (CB1), (comm) + (Neg) + (F) + (M) $\Longrightarrow$ (DN), thus (DN) holds. □

• **The connection with the Wajsberg algebras**
Recall that any bounded commutative BCK algebra is categorically equivalent to MV algebras ([28]) and that MV algebras are categorically equivalent (in fact are term equivalent) to Wajsberg algebras ([7]). Hence we have:

Theorem 2.44 *Bounded commutative BCK algebras are categorically equivalent to Wajsberg algebras.*

We shall present here a direct proof of this theorem, for its importance.

The direct proof of Theorem 2.44

(1) Let $(A, \to, 0, 1)$ be a bounded, commutative BCK algebra, i.e. properties (BB), (D), (Re), (L), (An); (F); (comm) hold, following the first definition of BCK algebras. Denote $x^- = x \to 0$, i.e. (Neg) holds too.
We must prove that $(A, \to, ^-, 1)$ is a Wajsberg algebra, i.e. properties (M), (BB), (comm), (DN1) hold.
Note that it remains to prove that (M) and (DN1) hold. We shall also prove that (DN) holds too. Indeed,
- by (A21'), (BB) + (D) + (L) + (An) $\Longrightarrow$ (Ex);
- by (A5), (Ex) + (Re) + (An) $\Longrightarrow$ (M), hence (M) holds •
- By (CB1), (comm) + (Neg) + (F) + (M) $\Longrightarrow$ (DN), thus (DN) holds •
- By (CA1), (Neg) + (BB) $\Longrightarrow$ (Neg1);
- by (C1), (DN) + (Neg1) $\Longrightarrow$ (DN1), hence (DN1) holds •

(1') Let now $(A, \to, ^-, 1)$ be a Wajsberg algebra and denote 1^- by 0, hence the properties (M), (BB), (comm), (DN1); (Neg1-0) hold.
We must prove that $(A, \to, 0, 1)$ is a bounded, commutative BCK algebra (with involutive negation), i.e. the properties (BB), (D), (Re), (L), (An); (F); (comm) hold.
Note that it remains to prove that (D), (Re), (L), (An); (F), (Neg) hold. Indeed,
- by (B12), (comm) + (M) $\Longrightarrow$ (An), thus (An) holds •
- By (A17'), (M) + (BB) $\Longrightarrow$ (Tr);
- by (A18), (M) + (BB) $\Longrightarrow$ (Re), thus (Re) holds •
- By (A18'), (M) + (BB) $\Longrightarrow$ (D), thus (D) holds •
- By (A20'), (M) + (BB) $\Longrightarrow$ (C);
- by (A3), (C) + (An) $\Longrightarrow$ (Ex);
- by (B32), (comm) + (Re) + (BB) + (M) $\Longrightarrow$ (L), thus (L) holds •
- By (A8), (L) + (Re) + (Ex) $\Longrightarrow$ (K);
- by (CAN4), (Neg1-0) + (DN1) + (K) + (BB) + (M) $\Longrightarrow$ (F), thus (F) holds •
- By (CA8), (DN1) + (K) + (Tr) $\Longrightarrow$(DN5);
- by (A15'), (M) + (BB) $\Longrightarrow$ (**);
- by (CBN2), (comm) + (DN1) + (Neg1-0) + (DN5) + (**) + (F) + (M) + (An) $\Longrightarrow$ (Neg); thus (Neg) holds •
- By (CB1), (comm) + (Neg) + (F) + (M) $\Longrightarrow$ (DN), thus (DN) holds • □

Remark 2.45 Theorem 2.43 is a corollary of above Theorem 2.44.

Note that, by above Theorem 2.43, a Wajsberg algebra verifies:
- ($\overline{M}$), (Neg1-0), (Neg0-1), (Neg1)-(Neg3) and (TN), since it is a bounded BCK algebra, hence it has a negation;
- (DN1)-(DN3), (DN5) and (Neg5), since its negation is involutive.

2.4.3 Boolean algebras. Connections

There are many definitions of Boolean algebras. We shall use here the definition introduced in 2009 [17] and presented also in [8], because it is most connected to the classical logic:

Definition 2.46 A *Boolean algebra* is an algebra $\mathcal{A} = (A, \to, ^-, 1)$ of type $(2, 1, 0)$ verifying the axioms: (K), (pimpl-1), (DN1), (An).

Remark 2.47 Note that a Boolean algebra is a regular algebra, i.e. (M) holds. Indeed:
- by (A2), (K) + (An) $\Longrightarrow$ (N);
- by (A7), (N) + (K) $\Longrightarrow$ (L);
- by (B19), (pimpl-1) + (K) + (N) $\Longrightarrow$ (Re);
- by (B27), (pimpl-1) + (N) $\Longrightarrow$ ($);
- by (B21), (pimpl-1) + (L) + (N) $\Longrightarrow$ (Tr);
- By (B28), ($) + (K) + (Tr) $\Longrightarrow$ (#);

- by (A31), (Re) + (#) $\Longrightarrow$ (D);
- by (A25), (D) + (K) + (N) + (An) $\Longrightarrow$ (M), thus (M) holds.

• **Boolean algebras as regular algebras with involutive negation**

We shall prove here that the Boolean algebra is another example of *regular algebra with negation*, namely with involutive negation.

Theorem 2.48
The Boolean algebras are regular algebras with involutive negation.

Proof. Let $\mathcal{A} = (A, \rightarrow, ^-, 1)$ be a Boolean algebra, i.e. properties (K), (pimpl-1), (DN1), (An) hold. Denote 1^- by 0, hence (Neg1-0) holds too.
We must prove that $\mathcal{A}$ is a regular algebra with negation and that the negation is involutive. By Definition 2.19, we must prove that:
(1) $(A, \rightarrow, 1)$ is a regular algebra, i.e. (M) holds;
(2) the property (Neg) holds.
Finally, we must prove that (DN) holds. Indeed,
(1): by the proof of Remark 2.47, (N), (L), (Re), ($), (Tr), (#), (D) hold, hence (M) holds;
(2): further,
- by (CA9), (DN1) + (K) + (Tr) + ($) + (Re) + (M) + (An) $\Longrightarrow$ (DN); thus (DN) holds •
- By (C1'), (DN1) + (DN) $\Longrightarrow$ (Neg1);
- by (CA6), (Neg1) + (DN1) + (An) $\Longrightarrow$ (DN2);
- by (CAN1), (Neg1-0) + (DN2) + (DN) + (M) $\Longrightarrow$ (Neg); thus (Neg) holds • □

• **Connections with the Boolean algebras**
Then we have the following connections.

Theorem 2.49 *Bounded implicative BCK algebras are categorically equivalent to Boolean algebras.*

Proof.
(1) (See [22], Theorem 12)
Let $\mathcal{A} = (A, \rightarrow, 0, 1)$ be a bounded implicative BCK algebra, i.e. (B), (C), (K), (An); (F); (impl) hold.
Define a negation $^-$ on A by: $x^- = x \rightarrow 0$, i.e. (Neg) holds too.
We must prove that $(A, \rightarrow, ^-, 1)$ is a Boolean algebra, i.e. (K), (pimpl-1), (DN1), (An) hold.
Note that it remains to prove (pimpl-1), (DN1) and also (DN). Indeed,
- by Theorem 2.38, $\mathcal{A}$ is commutative and positive implicative, i.e. (comm) and (pimpl) hold;
- by (A23), (C) + (K) + (An) $\Longrightarrow$ (Re);
- by (A0), (Re) $\Longrightarrow$ (S);
- by (S), (pimpl) $\Longrightarrow$ (pimpl-1); thus (pimpl-1) holds •
- By (A3), (C) + (An) $\Longrightarrow$ (Ex);
- by (A5), (Re) + (Ex) + (An) $\Longrightarrow$ (M);
- by (CB1), (comm) + (Neg) + (F) + (M) $\Longrightarrow$ (DN); thus (DN) holds •
- By (A10'), (Ex) + (B) $\Longrightarrow$ (BB);
- by (CA1), (Neg) + (BB) $\Longrightarrow$ (Neg1);
- by (C1), (DN) + (Neg1) $\Longrightarrow$ (DN1); thus (DN1) holds •

(1') Let $\mathcal{A} = (A, \rightarrow, ^-, 1)$ be a Boolean algebra, i.e. properties (K), (pimpl-1), (DN1), (An) hold. Denote 1^- by 0, hence (Neg1-0) holds too.
We must prove that $(A, \rightarrow, 0, 1)$ is a bounded implicative BCK algebra, i.e. (B), (C), (K), (An); (F), (Neg); (DN); (impl) hold.
Note that it remains to prove: (B), (C); (F), (Neg); (DN); (impl). Indeed,
- by (A2), (K) + (An) $\Longrightarrow$ (N);
- by (A7), (N) + (K) $\Longrightarrow$ (L);
- by (B19), (pimpl-1) + (K) + (N) $\Longrightarrow$ (Re);
- by (B27), (pimpl-1) + (N) $\Longrightarrow$ ($);

- by (B21), (pimpl-1) + (L) + (N) $\Longrightarrow$ (Tr);
- by (B22), (pimpl-1) + (K) + (Tr) $\Longrightarrow$ (B); thus (B) holds •
- By (B28), ($) + (K) + (Tr) $\Longrightarrow$ (#);
- by (A31), (Re) + (#) $\Longrightarrow$ (D);
- by (A25), (D) + (K) + (N) + (An) $\Longrightarrow$ (M);
- by (A29'), (B) + (#) $\Longrightarrow$ (BB);
- by (A21), (BB) + (D) + (N) + (An) $\Longrightarrow$ (Ex);
- by (A3'), (Ex) + (Re) $\Longrightarrow$ (C); thus (C) holds •
- By (CA9), (DN1) + (K) + (Tr) + ($) + (Re) + (M) + (An) $\Longrightarrow$ (DN); thus (DN) holds •
- By (C1'), (DN1) + (DN) $\Longrightarrow$ (Neg1);
- by (CA6), (Neg1) + (DN1) + (An) $\Longrightarrow$ (DN2);
- by (CAN1), (Neg1-0) + (DN2) + (DN) + (M) $\Longrightarrow$ (Neg); thus (Neg) holds •
- By (CAN5), (Neg1-0) + (DN) + (DN2) + (L) $\Longrightarrow$ (F); thus (F) holds •
- By (CB4), (pimpl-1) + (Neg) + (Re) + (M) + (K) + (An) $\Longrightarrow$ (Neg6);
- by (C6), (DN) + (Neg6) $\Longrightarrow$ (DN6);
- by (CA8), (DN1) + (K) + (Tr) $\Longrightarrow$ (DN5);
- by (A12'), (M) + (B) $\Longrightarrow$ (*);
- by (CB5), (DN) + (DN2) + (DN6) + (DN5) + (K) + (*) + (An) $\Longrightarrow$ (impl); thus (impl) holds • □

Remark 2.50 Theorem 2.48 is a corollary of above Theorem 2.49.

Note that, by the proof of above Theorem 2.48, a Boolean algebra verifies:
- ($\overline{M}$), (Neg1-0), (Neg0-1), (Neg1)-(Neg3) and (TN), since it is a bounded BCK algebra, hence it has a negation;
- (DN1)-(DN3), (DN5) and (Neg5), since its negation is involutive;
- (Neg6) and (DN6), by the proof.

Theorem 2.51 *Bounded positive implicative BCK algebras with involutive negation are categorically equivalent to Boolean algebras.*

Proof.

(1) Let $\mathcal{A} = (A, \rightarrow, 0, 1)$ be a bounded positive implicative BCK algebra with involutive negation; hence (B), (C), (K), (An); (F); (pimpl) hold; define a negation $^-$ on A by: $x^- = x \rightarrow 0$, i.e. (Neg) holds too and also (DN).
We must prove that $(A, \rightarrow, ^-, 1)$ is a Boolean algebra, i.e. (K), (pimpl-1), (DN1), (An) hold.
Note that it remains to prove (pimpl-1) and (DN1). Indeed,
- by (A23), (C) + (K) + (An) $\Longrightarrow$ (Re);
- by (A0), (Re) $\Longrightarrow$ (S);
- by (S), (pimpl) $\Longrightarrow$ (pimpl-1), thus (pimpl-1)holds •
- By (A3), (C) + (An) $\Longrightarrow$ (Ex);
- by (A10'), (Ex) + (B) $\Longrightarrow$ (BB);
- by (CA1), (Neg) + (BB) $\Longrightarrow$ (Neg1);
- by (C1), (DN) + (Neg1) $\Longrightarrow$ (DN1) •

(1') Let now $(A, \rightarrow, ^-, 1)$ be a Boolean algebra, i.e. (K), (pimpl-1), (DN1), (An) hold; put $0 = 1^-$, i.e. (Neg1-0) holds too.
We must prove that $(A, \rightarrow, 0, 1)$ is a bounded positive implicative BCK algebra with involutive negation, i.e. (B), (C), (K), (An); (F), (Neg); (DN); (pimpl) hold.
It remains to prove: (B), (C), (F), (Neg), (DN), (pimpl). Indeed, as in the proof of Theorem 2.49
- by (A2), (K) + (An) $\Longrightarrow$ (N);
- by (A7), (N) + (K) $\Longrightarrow$ (L);
- by (B19), (pimpl-1) + (K) + (N) $\Longrightarrow$ (Re);
- by (B27), (pimpl-1) + (N) $\Longrightarrow$ ($);
- by (B21), (pimpl-1) + (L) + (N) $\Longrightarrow$ (Tr);
- by (B22), (pimpl-1) + (K) + (Tr) $\Longrightarrow$ (B); thus (B) holds •

- By (B28), ($) + (K) + (Tr) $\Longrightarrow$ (#);
- by (A31), (Re) + (#) $\Longrightarrow$ (D);
- by (A25), (D) + (K) + (N) + (An) $\Longrightarrow$ (M);
- by (A29'), (B) + (#) $\Longrightarrow$ (BB);
- by (A21), (BB) + (D) + (N) + (An) $\Longrightarrow$ (Ex);
- by (A3'), (Ex) + (Re) $\Longrightarrow$ (C); thus (C) holds •
- By (CA9), (DN1) + (K) + (Tr) + ($) + (Re) + (M) + (An) $\Longrightarrow$ (DN); thus (DN) holds •
- By (C1'), (DN1) + (DN) $\Longrightarrow$ (Neg1);
- by (CA6), (Neg1) + (DN1) + (An) $\Longrightarrow$ (DN2);
- by (CAN1), (Neg1-0) + (DN2) + (DN) + (M) $\Longrightarrow$ (Neg); thus (Neg) holds •
- By (CAN5), (Neg1-0) + (DN) + (DN2) + (L) $\Longrightarrow$ (F); thus (F) holds •
- By (A12), (N) + (B) $\Longrightarrow$ (*);
- by (B24), (pimpl-1) + (*) + (K) + (M) $\Longrightarrow$ (**);
- by (B26), (pimpl-1) + (K) + (C) + (An) + (**) + (Tr) $\Longrightarrow$ (pimpl), thus (pimpl) holds • □

It is known that any bounded Hilbert algebra with involutive negation is a Boolean algebra [4]. Moreover, we have the following result.

Corollary 2.52 *Bounded Hilbert algebras with involutive negation are categorically equivalent to Boolean algebras.*

Proof. By Theorem 2.36, Hilbert algebras are categorically equivalent to positive implicative BCK algebras; hence bounded Hilbert algebras with involutive negation are categorically equivalent to bounded positive implicative BCK algebras with involutive negation. Then apply Theorem 2.51. □

2.4.4 Hierarchies of bounded BCK algebras

By using the hierarchies of different classes of BCK algebras from Part II, we obtain the following hierarchies of different classes of bounded BCK algebras in next Figure 2.

3 Introduction to the theory of quasi-algebras (quasi-structures) - Part III

In the first subsection, we shall recall from [19] the Part I of the theory of quasi-algebras (quasi-structures). In the second subsection, we shall recall from [20] the Part II of the theory of quasi-algebras (quasi-structures). In the third subsection, we shall present the Part III of the theory.

Let $\mathcal{A} = (A, \rightarrow, 1)$ be an *algebra* of type $(2, 0)$ through this section, where a binary relation $\leq$ can be *defined* by: for all x, y,

$$x \leq y \overset{def.}{\Longleftrightarrow} x \rightarrow y = 1. \tag{10}$$

Equivalently,
let $\mathcal{A} = (A, \leq, \rightarrow, 1)$ be a *structure* where $\leq$ is a binary relation on A, $\rightarrow$ is a binary operation (an implication) on A and $1 \in A$, all *connected* by:

$$x \leq y \Longleftrightarrow x \rightarrow y = 1. \tag{11}$$

Recall first the followings:

Definitions 3.1 [19]

(q1) The algebra $(A, \Rightarrow, 1)$ (or, equivalently, the structure $(A, \preceq, \Rightarrow, 1)$) is called *quasi-algebra* (*quasi-structure*, respectively) if it satisfies the properties (qM) and (11-1).

(q1') Any algebra (structure) $\mathcal{A}' = (A, \sigma)$ whose signature σ contains $\rightarrow$, 1 ($\leq$, $\rightarrow$, 1, respectively) is also called *quasi-algebra* (*quasi-structure*), if it satisfies the properties (qM) and (11-1).

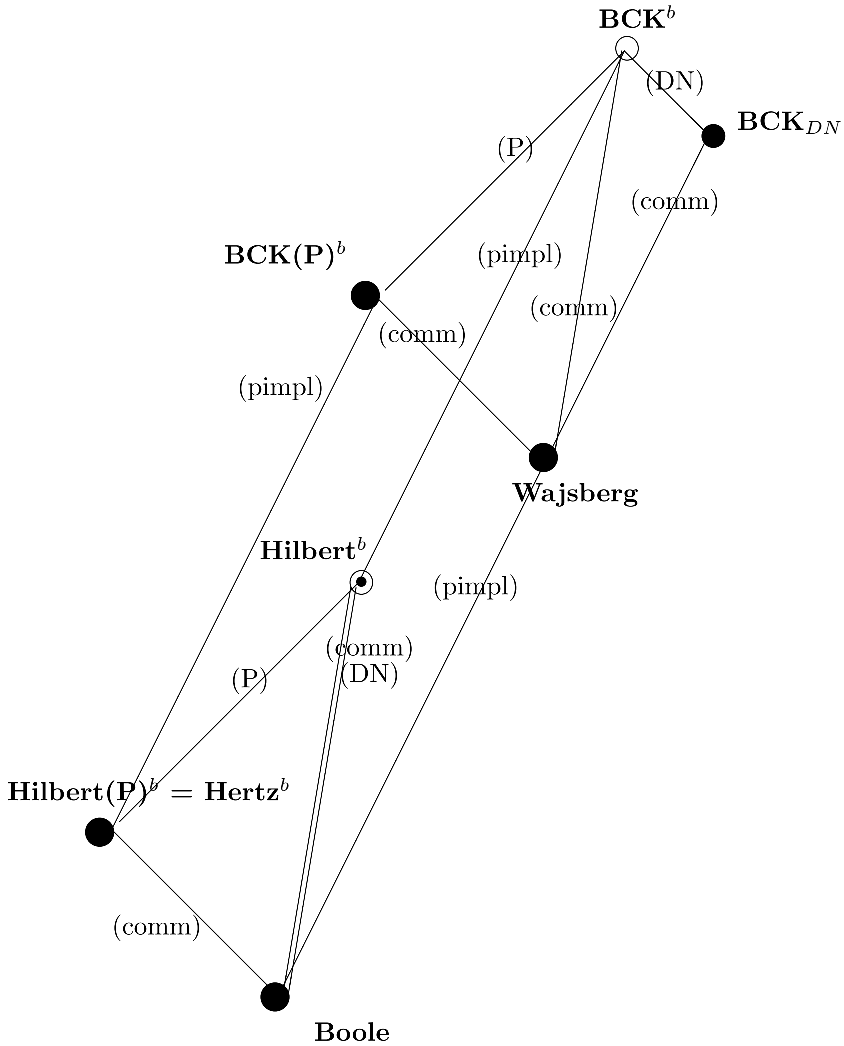

Figure 2: Hierarchies of classes of bounded BCK algebras

(q1") Any algebra (structure) $\mathcal{A}'' = (A, \tau)$ which is term equivalent to a quasi-algebra (structure) $\mathcal{A}' = (A, \sigma)$, is also called *quasi-algebra* (*quasi-structure*).

(q2) The implication $\Rightarrow$ from a quasi-algebra (quasi-structure) is called *quasi-implication*.

(q3) The binary relation $\preceq$ of a quasi-algebra (quasi-structure) is called *binary quasi-relation*.

Remark 3.2 [19] (qM) is different of (M) if and only if $V_M = V = U \subset A$, and this is the basic, definable property of quasi-algebras (quasi-structures), where

$$U \overset{df.}{=} \{x \to y \mid x, y \in A\},\ V \overset{df.}{=} \{1 \to x \mid x \in A\},\ V_M \overset{df.}{=} \{x \in A \mid x \overset{(M)}{=} 1 \to x\}.$$

Definitions 3.3 [19]

(1) For every quasi-algebra (quasi-structure) $\mathcal{A}$, the subset $V_M = V = U$ of A will be called the *regular* set of $\mathcal{A}$ and will be denoted by $R(A)$:

$$R(A) \overset{def.}{=} V_M = V = U.$$

The elements of $R(A)$ are called the *regular elements* of A.

(2) The quasi-algebra (quasi-structure) $\mathcal{A}$ is called *proper* if $R(A) \neq A$ (i.e. (M) $\not\Longleftrightarrow$ (qM)); otherwise, $\mathcal{A}$ is a regular algebra (structure).

Theorem 3.4 [19] Let $\mathcal{A} = (A, \to, 1)$ be a proper quasi-algebra (or, equivalently, let $\mathcal{A} = (A, \leq, \to, 1)$ be a proper quasi-structure). Then, $\mathcal{R}(\mathcal{A}) = (R(A), \to, 1)$ is a regular algebra (or, equivalently, $\mathcal{R}(\mathcal{A}) = (R(A), \leq, \to, 1)$ is a regular structure, respectively).

Definition 3.5 [19] We call *proper quasi-properties* the following nine: (qAn), (qM), (qM($1 \to y$)), (qN), (qN($1 \to y$)), (qRe), (qRe($1 \to y$)), (qL), (qL($1 \to y$)) which form the Part 1 of List qA (corresponding to the five properties (An), (M), (N), (Re), (L) respectively, which form the Part 1 of List A).

3.1 The list qA of basic quasi-properties. Connections [19]

3.1.1 The List qA of basic quasi-properties

The list qA of "quasi-properties" that can be satisfied by $\mathcal{A}$ has also two parts, and follows closely the list A of properties. The proper quasi-properties in Part 1 of List qA are generalizations of the properties in Part 1 of List A, while the "quasi-properties" in Part 2 of List qA are both the properties in Part 2 of List A and seven new specific properties. We have understood now which was the criterion by which a property was written in Part 1 or in Part 2 of the list.

List qA, Part 1

(qAn) (quasi-Antisymmetry) $x \to y = 1 = y \to x \implies 1 \to x = 1 \to y$,
(qM) $1 \to (x \to y) = x \to y$;
(qM($1 \to x$)) $1 \to (1 \to x) = 1 \to x$;
(qN) $1 \to (x \to y) = 1 \implies x \to y = 1$,
(qN($1 \to x$)) $1 \to (1 \to x) = 1 \implies 1 \to x = 1$,
(qRe) (quasi-Reflexivity) $(x \to y) \to (x \to y) = 1$,
(qRe($1 \to x$)) $(1 \to x) \to (1 \to x) = 1$,
(qL) $(x \to y) \to 1 = 1$,
(qL($1 \to x$)) $(1 \to x) \to 1 = 1$.

List qA, Part 2

(11-1), (B), (BB), (*), (**), (C), (D), (Ex), (K), (S), (Tr); (#), (##);

(qR) $(x \to y) \to ((1 \to x) \to (1 \to y)) = 1$,

(qR1) $(1 \to x) \to x = 1$,
(qR2) $x \to (1 \to x) = 1$,
(qR3) $(x \to (1 \to y)) \to (1 \to (x \to y)) = 1$;

(qI) $x \to y = (1 \to x) \to (1 \to y)$,
(qI1) $x \to y = (1 \to x) \to y$,
(qI2) $x \to y = x \to (1 \to y)$,
(qI3) $(1 \to x) \to (1 \to y) = (1 \to x) \to y$;

(qrelI) $x \leq y \Leftrightarrow 1 \to x \leq 1 \to y$,
(qrelI1) $x \leq y \Leftrightarrow 1 \to x \leq y$,
(qrelI2) $x \leq y \Leftrightarrow x \leq 1 \to y$.

3.1.2 Connections between the properties in Lists A, qA

• Connections between the properties in List A, Part 1 and the proper quasi-properties in List qA, Part 1 [19]

Theorem 3.6 *[19] Let $(A, \to, 1)$ be an algebra of type $(2, 0)$. Then the following are true:*
(i) (An) $\Longrightarrow$ (qAn);
(ii) (M) $\Longrightarrow$ (qM) $\Longrightarrow$ (qM$(1 \to x)$);
(iii) (N) $\Longrightarrow$ (qN) $\Longrightarrow$ (qN$(1 \to x)$);
(iv) (Re) $\Longrightarrow$ (qRe) $\Longrightarrow$ (qRe$(1 \to x)$);
(v) (L) $\Longrightarrow$ (qL) $\Longrightarrow$ (qL$(1 \to x)$);
(vi) (M) + (qAn) $\Longrightarrow$ (An);
(vii) (M) + (qRe$(1 \to x)$) $\Longrightarrow$ (Re);
(viii) (M) + (qL$(1 \to x)$) $\Longrightarrow$ (L);
(ix) (qM) + (qRe$(1 \to x)$) $\Longrightarrow$ (qRe);
(x) (qM) + (qL$(1 \to x)$) $\Longrightarrow$ (qL).

• Connections between the quasi-properties in List qA [19]

Proposition 3.7 *[19] (See Proposition 2.4)*
Let $(A, \to, 1)$ be an algebra of type $(2, 0)$. Then the following are true (following the numbering from Proposition 2.4):
(qA00) (qM) $\Longrightarrow$ (qN); (qA00') (qM$(1 \to x)$) $\Longrightarrow$ (qN $(1 \to x)$);
(qA3) (C) + (qM) + (qAn) $\Longrightarrow$ (Ex);
(qA4) (Ex) + (qRe) $\Longrightarrow$ (D);
(qA7) (qN) + (K) $\Longrightarrow$ (L); (qA7') (qM) + (K) $\Longrightarrow$ (L);
(qA12) (qN) + (B) $\Longrightarrow$ (*); (qA12') (qM) + (B) $\Longrightarrow$ (*);
(qA13) (qN) + (*) $\Longrightarrow$ (Tr); (qA13') (qM) + (*) $\Longrightarrow$ (Tr);
(qA14) (qN) + (B) $\Longrightarrow$ (Tr); (qA14') (qM) + (B) $\Longrightarrow$ (Tr);
(qA15) (qN) + (BB) $\Longrightarrow$ (**); (qA15') (qM) + (BB) $\Longrightarrow$ (**);
(qA16) (qN) + (**) $\Longrightarrow$ (Tr); (qA16') (qM) + (**) $\Longrightarrow$ (Tr);
(qA17) (qN) + (BB) $\Longrightarrow$ (Tr); (qA17') (qM) + (BB) $\Longrightarrow$ (Tr);
(qA18) (qM) + (BB) $\Longrightarrow$ (qRe$(1 \to y)$);
(qA19) (qM) + (B) $\Longrightarrow$ (qRe$(1 \to x)$);
(qA20) (BB) + (D) + (qN) $\Longrightarrow$ (C);
(qA20') (BB) + (D) + (qM) $\Longrightarrow$ (C);
(qA21) (BB) + (D) + (qM) + (qAn) $\Longrightarrow$ (Ex);
(qA22) (K) + (Ex) + (qM) $\Longrightarrow$ (Re);
(qA23) (C) + (K) + (qM) + (qAn) $\Longrightarrow$ (Re).

Proposition 3.8 *[19] (See Proposition 2.5)*
Let $(A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. We have the additional properties (following the numbering from Proposition 2.5):
(qA18") (qM) + (D) $\Longrightarrow$ (qR1);
(qA25) (D) + (K) + (qN) + (qAn) $\Longrightarrow$ (qM$(1 \rightarrow y)$);
(qA27) (qM) + (C) $\Longrightarrow$ (#).

Now we recall from [19] the corresponding theorems of (above recalled from [18]) Theorems 2.6, 2.7, 2.8, 2.9.

Theorem 3.9 *[19]*
If properties (Re), (qM), (Ex) hold, then: $(BB) \Leftrightarrow (B) \Leftrightarrow (*)$.

Theorem 3.10 *[19]*
If properties (Re), (qM), (Ex) hold, then: $(**) \Leftrightarrow (Tr)$.

Theorem 3.11 *[19]*
If properties (B), (D), (qM), (qAn) hold, then: $(Ex) \Leftrightarrow (BB)$.

Theorem 3.12 *[19] In any algebra $(A, \rightarrow, 1)$ we have:*
(i) (qM) + (BB) + (D) imply (B),
(ii) (qM) + (B) imply (**).

Concluding, by above Theorem 3.12 and (qA12'), (qA13'), (qA16'), we have immediately obtained:

Corollary 3.13 [19] In any algebra $(A, \rightarrow, 1)$ verifying (qM), we have:

$$(BB) + D \Longrightarrow (B) \Longrightarrow (*), (**) \Longrightarrow (Tr).$$

Proposition 3.14 *[19] Let $(A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. Then we have the following additional quasi-properties, with an independent numbering:*
(qAA1) (qM) + (BB) $\Longrightarrow$ (11-1); (qAA1') (qM) + (B) $\Longrightarrow$ (11-1);
(qAA1") (qM) + (K) $\Longrightarrow$ (11-1);
(qAA2) (Ex) + (qRe$(1 \rightarrow y)$) + (qM) $\Longrightarrow$ (qR1);
(qAA3) (B) $\Longrightarrow$ (qR);
(qAA4) (K) $\Longrightarrow$ (qR2);
(qAA5) (qR1) + (BB) $\Longrightarrow$ (qR3);
(qAA6) (qR1) + (K) + (**) + (qM) + (qAn) $\Longrightarrow$ (qI1);
(qAA7) (qRe$(1 \rightarrow y)$) + (Ex) + (K) + (**) + (qM) + (qAn) $\Longrightarrow$ (qI1);
(qAA7') (D) + (K) + (**) + (qM) + (qAn) $\Longrightarrow$ (qI1);
(qAA8) (Ex) + (qM) $\Longrightarrow$ (qI2) + (qI3);
(qAA9) (qI1) + (qI3) $\Longrightarrow$ (qI);
(qAA10) (qR1) + (qR2) + (BB) + (qM) + (qAn) $\Longrightarrow$ (qI1) + (qI2);
(qAA11) (qR3) + (K) + (*) + (qM) + (qAn) $\Longrightarrow$ (qI2);
(qAA12) (qI1) + (qI2) $\Longrightarrow$ (qI);
(qAA13) (qI) + (BB) + (qM) $\Longrightarrow$ (Re); (see (A18), (qA18))
(qAA14) (qI1) + (BB) + (L) + (qM) $\Longrightarrow$ (K); (see (A9"))
(qAA15) (B) + (Ex) + (K) + (**) + (qM) + (qAn) $\Longrightarrow$ (qI);
(qAA15') (qRe$(1 \rightarrow y)$) + (Ex) + (K) + (**) + (qM) + (qAn) $\Longrightarrow$ (qI);
(qAA15") (Re) + (Tr) + (Ex) + (L) + (qM) + (qAn) $\Leftrightarrow$ (qRe$(1 \rightarrow y)$) + (Ex) + (K) + (**) + (qM) + (qAn);
(qAA15"') (Re) + (Tr) + (Ex) + (L) + (qM) + (qAn) $\Longrightarrow$ (qI);
(qAA16) (qI) + (qRe$(1 \rightarrow y)$) $\Longrightarrow$ (Re);
(qAA17) (qI) $\Longrightarrow$ ((qRe$(1 \rightarrow y)$) $\Leftrightarrow$ (Re));

(qAA18) (#) + (qM) + (qR1) $\Longrightarrow$ *(qRe(*$1 \to x$*));*
(qAA18') (#) + (qM) + (qRe($1 \to x$*))* $\Longrightarrow$ *(qR1);*
(qAA18") (#) + (qM) $\Longrightarrow$ *((qR1)* $\Leftrightarrow$ *(qRe(*$1 \to x$*)));*
(qAA19) (qI) $\Longrightarrow$ *(qrelI);*
(qAA19') (qI1) $\Longrightarrow$ *(qrelI1);*
(qAA19") (qI2) $\Longrightarrow$ *(qrelI2).*

3.2 The list qB of particular quasi-properties. Connections [20]

3.2.1 The List qB of particular quasi-properties

The list qB of "quasi-properties" that can be satisfied by $\mathcal{A}$ has also two parts, and follows closely the list B of properties. The "quasi-property" in Part 1 of List qB is a generalization of the property in Part 1 of List B, while the "quasi-properies" in Part 2 of List qB are the properties in Part 2 of List B. We shall understand now which was the criterion by which a property was written in Part 1 or in Part 2 of the list.

List qB, Part 1

(q-impl) (quasi-implicative) $(x \to y) \to x = 1 \to x$;
(q-pi) $(1 \to x) \to (x \to y) = 1 \to (x \to y)$;

(qM($x \odot y$)) $1 \to (x \odot y) = x \odot y$;

(qVid) $x \vee x = 1 \to x$;
(qVe) $x \leq y \Leftrightarrow x \vee y = 1 \to y$;
(qV=) $x \leq z,\ y \leq z \implies x \vee y \leq 1 \to z$;

(qWid) $x \wedge x = 1 \to x$,
(qW1-1) $x \wedge 1 = 1 \wedge x = 1 \to x$,
(qWe) $x \leq y \Leftrightarrow x \wedge y = 1 \to x$,
(qW=) $x \geq z,\ y \geq z \implies x \wedge y \geq 1 \to z$;
(Wab1) (absorbtion-1) $x \wedge (x \vee y) = 1 \to x$,
(Wab2) (absorbtion-2) $x \vee (x \wedge y) = 1 \to x$.

List qB, Part 2

(pimpl), (pimpl-1), (pimpl-2), ($); (comm), (comm-1);
(dfP), (dfS), (RP), (Pne), (Pcomm), (Passoc), (P-), (P- -), (Pleq);
(dfV), (VP); (Vcomm), (Vassoc), (V1-1), (V-), (V- -), (Vgeq), (V-V), (VVV).

3.2.2 Connections between the Lists A, qA; B, qB

- **Connections between Part 1 in Lists B, qB [20]**

Proposition 3.15 *[20] Let* $(A, \to, 1)$ *be an algebra of type* $(2, 0)$. *Then we have:*
(i) (q-impl) +(M) $\Longrightarrow$ *(impl);*
(ii) (q-pi) + (M) $\Longrightarrow$ *(pi).*

- **Connections between the quasi-properties in Lists qA, qB [20]**

We recall the corresponding "quasi-results" from Proposition 2.12 and Proposition 2.13 and new "quasi-results".

Proposition 3.16 *(See Proposition 2.12) [20]*
Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. Then, we have (following the numbering from Proposition 2.12):
(qB0) (pimpl-1) + (pimpl-2) + (qM) + (qAn) $\Longrightarrow$ (pimpl);
(qB3) (pimpl) + (Re) +(qM) $\Longrightarrow$ (pi);
(qB7) (pimpl) + (Re) + (qM) $\Longrightarrow$ (*), (**);
(qB9) (pimpl) + (Re) + (qM) $\Longrightarrow$ (BB);
(qB10) (pimpl) + (Re) + (qM) $\Longrightarrow$ (C);
(qB11) (pimpl) + (Re) + (qM) + (qAn) $\Longrightarrow$ (Ex);
(qB12) (comm) $\Longrightarrow$ (qAn);
(qB15) (pi) + (Re) + (Ex) + (B) + (**) + (*) + (L) + (qAn) + (qM) $\Longrightarrow$ (pimpl);
(qB16) (pi) + (Re) + (qM) + (B) + (D) +(qAn) $\Longrightarrow$ (Ex) $\Leftrightarrow$ (BB) $\Leftrightarrow$ (pimpl);
(qB17) (pi) + (Re) + (qM) + (Ex) + (qAn) $\Longrightarrow$ (BB) $\Leftrightarrow$ (B) $\Leftrightarrow$ (*) $\Leftrightarrow$ (pimpl).

Proposition 3.17 *(See Proposition 2.13) [20]*
Let $\mathcal{A}$ be an algebra of type $(2, 0)$. Then, we have (following the numbering from Proposition 2.13):
(qB19) (pimpl-1) + (K) + (qN) $\Longrightarrow$ (Re);
(qB19') (pimpl-1) + (K) + (qM) $\Longrightarrow$ (Re);
(qB20) (pimpl-1) + (L) + (qN) $\Longrightarrow$ (*);
(qB20') (pimpl-1) + (L) + (qM) $\Longrightarrow$ (*);
(qB21) (pimpl-1) + (L) + (qN) $\Longrightarrow$ (Tr);
(qB21') (pimpl-1) + (L) + (qM) $\Longrightarrow$ (Tr);
(qB23) (pimpl-1) + (Re) + (*) + (K) + (qM) $\Longrightarrow$ (D);
(qB24) (pimpl-1) + (*) + (K) + (qM) $\Longrightarrow$ (**);
(qB26) (pimpl-1) + (K) + (qM) + (qAn) + (C) + (**) + (Tr) $\Longrightarrow$ (pimpl);
(qB27) (pimpl-1) + (qN) $\Longrightarrow$ (\$);
(qB27') (pimpl-1) + (qM) $\Longrightarrow$ (\$);
(qB31) (comm) + (K) +(BB) + (Tr) + (qM) $\Longrightarrow$ (#);
(qB32) (comm) + (qR1) + (BB) + (qM) $\Longrightarrow$ (qL($1 \rightarrow x$));
(qB34) (Re) + (qM) + (Ex) + (B) + (*) + (**) + (L) + (qAn) $\Longrightarrow$ ((pimpl) $\Leftrightarrow$ (pi)).

Proposition 3.18 *Let $\mathcal{A}$ be an algebra of type $(2, 0)$. Then the additional quasi-property holds (with an independent numbering):*
(qBB1) (comm) + (qRe) + (BB) + (qM) $\Longrightarrow$ (qR1).

Proposition 3.19 *(See Proposition 2.14)*
Let $(A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. Then, the following hold:
(qBIM1) (q-impl) $\Longrightarrow$ (q-pi);
(qBIM1') (q-impl) + (BB) + (K) + (qM) + (qAn) $\Longrightarrow$ (pi);
(qBIM1") (comm) + (q-impl) + (L) + (K) + (qM) $\Longrightarrow$ (pi);
(qBIM2) (comm) + (pi) + (L) + (K) + (qM) $\Longrightarrow$ (q-impl);
(qBIM2') (comm) + (L) + (K) + (qM) $\Longrightarrow$ ((pi) $\Leftrightarrow$ (q-impl));
(qBIM3) (q-impl) + (Ex) + (B) + (qM) + (qAn) $\Longrightarrow$ (comm);
(qBIM5) (q-impl) + (qM) + (11-1) $\Longrightarrow$ (qL($1 \rightarrow y$));
(qBIM6) (q-impl) + (K) $\Longrightarrow$ (qR2);
(qBIM7) (K) + (L) + (Ex) + (B) ($\Leftrightarrow$ (BB)) + (qM) + (qAn) $\Longrightarrow$ ((q-impl) $\Leftrightarrow$ ((comm) + (pi))).

Proposition 3.20 *(See Proposition 2.15)*
Let $(A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. Then, the following hold (following the numbering from Proposition 2.15):

(qBV1) (dfV) + (Re) $\Longrightarrow$ *(qVid);*
(qBV4) (dfV) + (L) + (qR1) $\Longrightarrow$ *(V1-1);*
(qBV5) (dfV) + (qI2) + (Ex) + (Re) $\Longrightarrow$ *(qVe);*
(qBV7) (dfV) + (qVe) + (Vcomm) + (V-V) + (qM) $\Longrightarrow$ *(VVV);*
(qBV11) (V- -) + (qVid) $\Longrightarrow$ *(qV=).*

3.3 Bounded quasi-algebras (quasi-structures). Quasi-algebras (quasi-structures) with quasi-negation. The List qC of some properties of the quasi-negation. Connections

As we know, in quasi-Wajsberg algebras (quasi-MV algebras) there exists a negation $^-$ and an implication $\rightarrow$ and an element 1 and, if we denote 1^- by 0, then the quasi-negation is *connected* to $\rightarrow$ and 1 by: $x \rightarrow 0 = 1 \rightarrow x^- = (1 \rightarrow x)^-$.

Consequently, just as *the equivalence:*

$$(Equ) \quad x \leq y \quad \Longleftrightarrow \quad x \rightarrow y = 1$$

can be used either
- as *the definition* of the binary quasi-relation $\leq$ in the quasi-algebra $(A, \rightarrow, 1)$ or
- as *the connection* between the binary quasi-relation $\leq$ and $\rightarrow$, 1 in the quasi-structure $(A, \leq, \rightarrow, 1)$,
the same, *the equalities:*

$$(qNeg) \qquad x \rightarrow 0 = 1 \rightarrow x^- = (1 \rightarrow x)^-$$

will be used either
- as *the definition* of the quasi-negation $^-$ in the *bounded quasi-algebra* $(A, \rightarrow, 1)$ *(quasi-structure* $(A, \leq, \rightarrow, 1)$*)*, and in this case $1^- \overset{def.}{=} 1 \rightarrow 0 = 0$, by (M), and $0^- \overset{def.}{=} 0 \rightarrow 0 = 1$, by (Re) - see the next subsubsection 3.3.1 - or
- as *the connection* between the quasi-negation $^-$ and $\rightarrow$, 1 in the *quasi-algebra* $(A, \rightarrow, ^-, 1)$ *(quasi-structure* $(A, \leq, \rightarrow, ^-, 1)$*) with a negation*, and in this case $0 \overset{notation}{=} 1^-$ and 0^- must equal $0 \rightarrow 0 = 0$, by (Re) - see the subsubsection 3.3.2.

Note that, in the first case, the negation will be *not unique*.

We shall then present a List qC of some quasi-properties of the quasi-negation - in the subsubsection 3.3.3 - and some connections between the quasi-properties in Lists qA, qB, qC - in the subsubsection 3.3.4.

3.3.1 Bounded quasi-algebras (quasi-structures)

Let $\mathcal{A} = (A, \rightarrow, 1)$ be a quasi-ordered algebra (or, equivalently, let $\mathcal{A} = (A, \leq, \rightarrow, 1)$ be a quasi-ordered structure) [19] and $\leq$ be the quasi-order of $\mathcal{A}$ (i.e. (qM), (Re) (hence (11-1)), (qAn), (Tr) hold) through this subsection.

Then, by Theorem 3.4, $\mathcal{R}(\mathcal{A}) = (R(A), \rightarrow, 1)$ is a regular ordered algebra (or, equivalently, $\mathcal{R}(\mathcal{A}) = (R(A), \leq, \rightarrow, 1)$ is a regular ordered structure) and $\leq$ is a regular order of $\mathcal{R}(\mathcal{A})$ (i.e. (M), (Re), (An), (Tr) hold) [19].

Definitions 3.21

(i) An element $l \in A$ is called *last quasi-element* or *the greatest quasi-element* of $\mathcal{A}$ if $(1 \rightarrow x) \rightarrow (1 \rightarrow l) = 1$ (or $1 \rightarrow x \leq 1 \rightarrow l$), for every $x \in A$.

(ii) An element $f \in A$ is called *first quasi-element* or *the smallest quasi-element*) of $\mathcal{A}$ if $(1 \rightarrow f) \rightarrow (1 \rightarrow x) = 1$ (or $1 \rightarrow f \leq 1 \rightarrow x$), for every $x \in A$.

Hence, *the notions of first quasi-element and last quasi-element are dual to each other.*

Note that in a quasi-ordered algebra (structure) there can be more first quasi-elements and more last quasi-elements, i.e. the first quasi-element and the last quasi-element can be not unique (when they exist) - see Examples 3.24. Anyway, they form clouds: the cloud of first elements and the cloud of last elements.

Recall now ([19], Lemma 4.22) that "*if properties (Ex), (L) also hold, then every cloud in a quasi-ordered algebra (structure) $\mathcal{A}$ contains exactly one regular element*".

The special element $1 \in A$ is a regular element (i.e. $1 \in R(A)$) and it can be the last quasi-element; in this case, we have the property: for every $x \in A$,
(L) (Last) $x \to 1 = 1$ or, equivalently, (L') (Last) $x \leq 1$.

Note that there can be other last quasi-elements too, but not as regular elements, by ([19], Lemma 4.22), if properties (Ex), (L) hold; in this case, all the last quasi-elements of $\mathcal{A}$ form the cloud of 1, denoted by $C(1) =\mid 1 \mid$.

In the cloud of the first quasi-elements, only one is a regular element, by ([19], Lemma 4.22), if properties (Ex), (L) hold; the first quasi-element which is a regular element also will be denoted by 0. Consequently,
-(1) $0 \in R(A)$ and, consequently, $1 \to 0 = 0$, by (M);
-(2) we have the property: for every $x \in A$,
(F) (First) $0 \to x = 1$ or, equivalently, (F') (First) $0 \leq x$;
-(3) all the first quasi-elements of $\mathcal{A}$ form the cloud of 0, denoted by $C(0) =\mid 0 \mid$.

Definition 3.22 A quasi-ordered algebra (quasi-ordered structure) with first quasi-element 0 and last quasi-element 1 is called *bounded* and is denoted by $\mathcal{A} = (A, \to, 0, 1)$ ($\mathcal{A} = (A, \leq, \to, 0, 1)$, respectively).

Let us introduce the new property (10-0):
(10-0) $1 \to 0 = 0$.

Remarks 3.23 Let $\mathcal{A} = (A, \to, 0, 1)$ be a bounded quasi-algebra. Then:
(1) $\mathcal{R}(\mathcal{A}) = (R(A), \to, 0, 1)$ is a bounded regular algebra;
(2) the property (10-0) holds.

Example 3.24 Let us consider from [19] the following two quasi-algebras $\mathcal{A}_1 = (A_1 = \{0, x, 1\}, \to, 1)$ and $\mathcal{A}_2 = (A_2 = \{0, x, y, 1\}, \to, 1)$ given by the following tables of $\to$:

$\mathcal{A}_1$

$\to$	**0**	x	**1**
0	**1**	1	**1**
x	1	1	1
1	**0**	0	**1**

$\mathcal{A}_2$

$\to$	**0**	x	y	**1**
0	**1**	1	1	**1**
x	1	1	1	1
y	0	0	1	1
1	**0**	0	1	**1**

and represented in two ways by the quasi-Hasse diagrams from Figure 3.

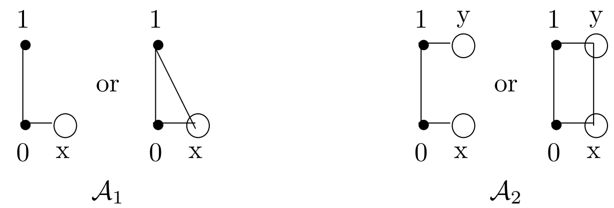

Figure 3: The quasi-Hasse diagrams of quasi-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$

In Figure 3, both quasi-structures $\mathcal{A}_1$ and $\mathcal{A}_2$ are bounded: $\mathcal{A}_1$ has 1 as quasi-last element and 0 and x as quasi-first elements, i.e. $C(0) =\mid 0 \mid 0 = \{0, x\}$ and $C(1) =\mid 1 \mid= \{1\}$; $\mathcal{A}_2$ has 1 and y as quasi-last elements and 0 and x as quasi-first elements, i.e. $C(0) =\mid 0 \mid= \{0, x\}$ and $C(1) =\mid 1 \mid= \{1, y\}$. Note that $R(A_1) = R(A_2) = \{0, 1\}$.

Remark 3.25 If the quasi-algebra (quasi-structure) is not ordered, then we can extend the above notions.

• **The quasi-negation: property (qNeg)**
Let $\mathcal{A} = (A, \to, 0, 1)$ be a bounded proper quasi-ordered algebra (or, equivalently, let $\mathcal{A} = (A, \leq, \to, 0, 1)$ be a bounded proper quasi-ordered structure) (i.e. properties (qM), (Re), (qAn), (Tr), (F), (L), (11-1) and (10-0) hold) through this subsubsection.

Then, its subalgebra $\mathcal{R}(\mathcal{A}) = (R(A), \to, 0, 1)$ is a bounded regular ordered algebra (or, equivalently, $\mathcal{R}(\mathcal{A}) = (R(A), \leq, \to, 0, 1)$ is a bounded regular structure) (i.e. properties (M), (Re), (An), (Tr), (F), (L) hold in $R(A) = V_M = V = U$).

Consequently, we can define a *unique regular negation* $^- : R(A) \longrightarrow R(A)$ by using (Neg):

$$(Neg_R) \quad r^- = r \to 0, \quad \forall\, r \in R(A)$$

or, equivalently,

$$(Neg_V) \quad (1 \to x)^- = (1 \to x) \to 0, \quad \forall\, x \in A,$$

or, equivalently,

$$(Neg_U) \quad (x \to y)^- = (x \to y) \to 0, \quad \forall\, x, y \in A.$$

Note that:
$1^- \overset{(11-1)}{=} (1 \to 1)^- \overset{(Neg_V)}{=} (1 \to 1) \to 0 \overset{(11-1)}{=} 1 \to 0 \overset{(10-0)}{=} 0$, i.e. (Neg1-0) holds, and
$0^- \overset{(10-0)}{=} (1 \to 0)^- \overset{(Neg_V)}{=} (1 \to 0) \to 0 \overset{(10-0)}{=} 0 \to 0 \overset{(Re)}{=} 1$, i.e. (Neg0-1) holds.

We can extend the regular negation $^- : R(A) \longrightarrow R(A)$ to a *quasi-negation* $\neg : A \longrightarrow A$, which is connected to $\to$ and 1 by:

$$(qNeg) \quad x \to 0 = 1 \to \neg(x) = \neg(1 \to x), \quad \textit{for all } x \in A.$$

Proposition 3.26 *The quasi-negation $\neg$ on A is indeed an extension of the negation $^-$ on $R(A)$:*

$$\neg\,|_{R(A)} = {}^-.$$

Proof. For all $r \in R(A)$, by (qNeg), $r \to 0 = 1 \to \neg(r) = \neg(1 \to r)$, hence $r \to 0 = \neg(r)$, by (M); but, by (Neg$_R$), $r \to 0 = r^-$. Hence, $\neg(r) = r^-$. □

Remark 3.27 For the same bounded quasi-ordered algebra (structure), there can be more quasi-negations verifying (qNeg), i.e. the quasi-negation is not unique (see the examples in a subsequent section).

A quasi-negation is called *involutive* if the property (DN) holds: $\neg(\neg(x)) = x$, for all $x \in A$.

Convention: In order to simplify the writing, the quasi-negation $\neg$ on A and the regular negation $^-$ on $R(A)$ will be denoted the same in the sequel, namely by $^-$. Hence (qNeg) becomes:

$$x \to 0 \overset{(\alpha)}{=} 1 \to x^- \overset{(\beta)}{=} (1 \to x)^-.$$

3.3.2 Quasi-algebras (quasi-structures) with quasi-negation

Definition 3.28 (See Definition 2.19 in the regular case)
A *quasi-algebra (quasi-structure) with quasi-negation* is an algebra $(A, \to, ^-, 1)$ (a structure $(A, \leq, \to, ^-, 1)$) such that $(A, \to, 1)$ ($(A, \leq, \to, 1)$ respectively) is a quasi-algebra (quasi-structure) (i.e. (qM) and (11-1) hold) and $^-$ is an unary operation on A, called *quasi-negation*, which is connected with $\to$, 1 by (qNeg) (i.e. (α) + (β)): for all $x \in A$,

$$x \to 0 = 1 \to x^- = (1 \to x)^-,$$

where 0 is the notation for the element 1^-.

We shall say that a *quasi-algebra (quasi-structure) with quasi-negation* is associated to a quasi-algebra (quasi-structure).

Remark 3.29 For the same quasi-algebra $(A, \to, 1)$ (quasi-structure $(A, \leq, \to, 1)$ respectively), there can be more quasi-negations $^-$ verifying (qNeg), i.e. there can be more quasi-algebras (quasi-structures) with quasi-negation associated (see the examples in a subsequent section).

Note that we have: $0 \overset{notation}{=} 1^-$ and $0^- = 0 \to 0 = 1$, if (Re) holds.
The quasi-negation is called *involutive* if the property (DN) holds: $(x^-)^- = x$, for all $x \in A$.

We shall prove, in the next section, that the quasi-Wajsberg algebra [2] is an example of quasi-algebra with quasi-negation, namely with involutive quasi-negation.

3.3.3 The list qC of some quasi-properties of the quasi-negation

The list qC of some quasi-properties of the quasi-negation has also two parts:

List qC, Part 1

$(q\overline{M})$ $\quad 1 \to (x \to y)^- = (x \to y)^-$,
$(q\overline{M(1 \to x)})$ $\quad 1 \to (1 \to x)^- = (1 \to x)^-$;

(qF) $\quad 0 \to (x \to y) = 1$,
(qF') $\quad 0 \leq x \to y$;
(qF$(1 \to x)$) $\quad 0 \to (1 \to x) = 1$,
(qF$(1 \to x)$') $\quad 0 \leq 1 \to x$;

(qNeg) $\quad x \to 0 = 1 \to x^- = (1 \to x)^-$;
(α) $\quad x \to 0 = 1 \to x^-$,
(β) $\quad 1 \to x^- = (1 \to x)^-$,
(γ) $\quad x \to 0 = (1 \to x)^-$;
q(Neg$_U$) $(x \to y)^- = (x \to y) \to 0$,
(Neg$_V$) $(1 \to x)^- = (1 \to x) \to 0$;

(qDN) (quasi-Double Negation) $\quad ((x \to y)^-)^- = x \to y$,
(qDN$(1 \to x)$) (quasi-Double Negation) $((1 \to x)^-)^- = 1 \to x$;
(qTN) (quasi-Triple Negation) $(((x \to y)^-)^-)^- = (x \to y)^-$,
(qTN$(1 \to x)$) (quasi-Triple Negation) $(((1 \to x)^-)^-)^- = (1 \to x)^-$;

(qNeg4) $x \leq ((1 \to x)^-)^-$;
(qNeg4') $x \to ((1 \to x)^-)^- = 1$;
(qNeg5) $(x \to y)^- \to (1 \to x) = 1$,
(qNeg5') $(x \to y)^- \leq 1 \to x$,
(qNeg6) $x \to (1 \to x)^- = (1 \to x)^-$;

(qDN5) $(1 \to x)^- \to (x \to y) = 1$,
(qDN5') $(1 \to x)^- \leq x \to y$,
(qDN6) $(1 \to x)^- \to x = 1 \to x$;

(qV-0) $x \vee 0 = ((1 \to x)^-)^-$,
(qV0-) $0 \vee x = 1 \to x$.

List qC, Part 2

(10-0), (Neg1-0), (Neg0-1); (Neg1) - (Neg3); (Neg7), (Neg8); (DN1) - (DN4); (DN7), (DN8);
(Px); (P$_{(DN)}$), (VP$_{(DN)}$); (V-0), (V0-);
(dfW), (WP), (div); (Wcomm), (Wassoc), (Wab1), (Wab2), (Wd2), (Wd1), (W0-0), (W1-0), (W-), (W-

-), (Wleq), (WV-), (WWW); ($V_{(DN)}$), ($W_{(DN)}$), (M1), (M2).

3.3.4 Connections between the Lists A,qA; B, qB; C, qC

- **Connections between the quasi-properties in Part 1 of Lists qA, qC**

Theorem 3.30 *(See Theorem 2.20)*

(qi) (β) + $(qM(1\to x)) \Longrightarrow (q\overline{M(1\to x)})$;

(qii) $(q\overline{M(1\to x)})$ + (β) + $(qDN(1\to x)) \Longrightarrow (qM(1\to x))$.

Proof.

(qi): $1\to(1\to x)^- \overset{(\beta)}{=} 1\to(1\to x^-) \overset{(qM(1\to x))}{=} 1\to x^- \overset{(\beta)}{=} (1\to x)^-$, i.e. $(q\overline{M(1\to x)})$ holds.

(qii): $1\to(1\to x) \overset{(qDN(1\to x))}{=} 1\to((1\to x)^-)^- \overset{(\beta)}{=} 1\to(1\to x^-)^- \overset{(q\overline{M(1\to x)})}{=} (1\to x^-)^- \overset{(\beta)}{=} ((1\to x)^-)^- \overset{(qDN(1\to x))}{=} 1\to x$, i.e. (qM($1\to x$)) holds. □

Theorem 3.31

(i) $(\overline{M}) \Longrightarrow (q\overline{M}) \Longrightarrow (q\overline{M(1\to x)})$.

(ii) (F) $\Longrightarrow$ (qF) $\Longrightarrow$ (qF($1\to x$));

(iii) (Neg) $\Longrightarrow$ (qNeg);

(iv) (Neg) $\Longrightarrow$ (Neg_U) $\Longrightarrow$ (Neg_V);

(v) (DN) $\Longrightarrow$ (qDN) $\Longrightarrow$ (qDN($1\to x$));

(vi) (TN) $\Longrightarrow$ (qTN) $\Longrightarrow$ (qTN($1\to x$));

(vii) (M) + (qNeg) $\Longrightarrow$ (Neg);

(viii) (M) + (Neg_V) $\Longrightarrow$ (Neg);

(ix) (qNeg) + (qI1) $\Longrightarrow$ (Neg_V);

(x) (M) + (qDN($1\to x$)) $\Longrightarrow$ (DN);

(xi) (M) + (qTN($1\to x$)) $\Longrightarrow$ (TN);

(xii) (M) + (qNeg6) $\Longrightarrow$ (Neg6);

(xiii) (M) + (qNeg5) $\Longrightarrow$ (Neg5);

(xiv) (M) + (qDN6) $\Longrightarrow$ (DN6);

(xv) (M) + (qDN5) $\Longrightarrow$ (DN5).

Proof. Immediately. □

Proposition 3.32

(j) (α) + $(\beta) \Longrightarrow (\gamma)$,

(jj) (α) + $(\gamma) \Longrightarrow (\beta)$,

(jjj) $(\alpha) \Longrightarrow ((\beta) \Leftrightarrow (\gamma))$;

(j') (γ) + $(\beta) \Longrightarrow (\alpha)$,

(jj') (γ) + $(\alpha) \Longrightarrow (\beta)$,

(jjj') $(\gamma) \Longrightarrow ((\alpha) \Leftrightarrow (\beta))$;

(jv) (qNeg) $\Longleftrightarrow ((\alpha) + (\beta)) \Longleftrightarrow ((\gamma) + (\beta))$.

Proof. Immediately. □

Remarks 3.33

(i) Note that, if $(q\overline{M})$ or $(q\overline{M(1\to x)})$ hold, then $R(A)$ is closed under the negation $^-$ defined by (Neg_R) or, equivalently, by (Neg_V) or, equivalently, by (Neg_U).

(ii) If (qNeg) holds (i.e. (α) and (β) hold, for example), then $(q\overline{M(1\to x)})$ holds.

(iii) Note that the quasi-negation $^-$ is not uniquely determined on A verifying (qNeg) ((α) and (β)), while the regular negation $^-$ is uniquely determined on $R(A)$ by (Neg_V) or, equivalently, by (Neg_U).

Proposition 3.34 *(See Proposition 2.21)*
(qC5) (DN) + (DN2) + (qNeg5) $\Longrightarrow$ (qDN5),
(qC5') (DN) + (DN2) + (qDN5) $\Longrightarrow$ (qNeg5),
(qC5") (DN) + (DN2) $\Longrightarrow$ ((qNeg5) $\Leftrightarrow$ (qDN5));

(qC6) (DN) + (DN2) + (β) + (qNeg6) $\Longrightarrow$ (qDN6),
(qC6') (DN) + (DN2) + (β) + (qDN6) $\Longrightarrow$ (qNeg6),
(qC6") (DN) + (DN2) + (β) $\Longrightarrow$ ((qNeg6) $\Leftrightarrow$ (qDN6)).

Proof.
(qC5): $(1 \to x)^- \to (x \to y) \overset{(DN)}{=} (1 \to x)^- \to ((x \to y)^-)^-$
$\overset{(DN2)}{=} (x \to y)^- \to (1 \to x) \overset{(qNeg5)}{=} 1$, i.e. (qDN5) holds.
(qC5'): $(x \to y)^- \to (1 \to x) \overset{(DN)}{=} (x \to y)^- \to ((1 \to x)^-)^-$
$\overset{(DN2)}{=} (1 \to x)^- \to (x \to y) \overset{(qDN5)}{=} 1$, i.e. (qNeg5) holds.
(qC5"): By (qC5) and (qC5').
(qC6): $(1 \to x)^- \to x \overset{(DN)}{=} (1 \to x)^- \to (x^-)^-$
$\overset{(DN2)}{=} x^- \to (1 \to x) \overset{(DN)}{=} x^- \to ((1 \to x)^-)^-$
$\overset{(\beta)}{=} x^- \to (1 \to x^-)^- \overset{(qNeg6)}{=} (1 \to x^-)^-$
$\overset{(\beta)}{=} ((1 \to x)^-)^- \overset{(DN)}{=} 1 \to x$, i.e. (qDN6) holds.
(qC6') $x \to (1 \to x)^- \overset{(DN)}{=} (x^-)^- \to (1 \to x)^-$
$\overset{(DN2)}{=} (1 \to x) \to x^- \overset{(DN)}{=} ((1 \to x)^-)^- \to x^-$
$\overset{(\beta)}{=} (1 \to x^-)^- \to x^- \overset{(qDN6)}{=} 1 \to x^- \overset{(\beta)}{=} (1 \to x)^-$, i.e. (qNeg6) holds.
(qC6"): By (qC6) and (qC6'). □

Proposition 3.35 *(See Proposition 2.22)*
(qCN1) (Neg1-0) + (DN2) + (DN) $\Longrightarrow$ (α);
(qCN1') (α) + (qI1) + ($\overline{qM(1 \to x)}$) $\Longrightarrow$ (β).

Proof.
(qCN1): (see the proof of (qW35) from [2])
$1 \to x^- \overset{(DN)}{=} (1^-)^- \to x^- \overset{(DN2)}{=} x \to 1^- \overset{(Neg1-0)}{=} x \to 0$, i.e. ($\alpha$) holds.
(qCN1'): $1 \to x^- \overset{(\alpha)}{=} x \to 0 \overset{(qI1)}{=} (1 \to x) \to 0 \overset{(\alpha)}{=} 1 \to (1 \to x)^- \overset{(\overline{qM(1\to x)})}{=} (1 \to x)^-$, i.e. ($\beta$) holds. □

• **Connections between the properties in List qA and List qC**

Proposition 3.36 *(See Proposition 2.23)*
(qCA1) (α) + (qI) + (BB) $\Longrightarrow$ (Neg1);
(qCA2) (qM) + (Neg1) $\Longrightarrow$ (Neg2);
(qCA3) (Ex) + (qM) + (α) $\Longrightarrow$ (Neg3);
(qCA4) (qNeg) + (D) + ($\overline{qM(1 \to x)}$) $\Longrightarrow$ (qNeg4);
(qCA5) (Neg2) + (qNeg4) + (qAn) + (β) + (qM) $\Longrightarrow$ (qTN(1 $\to$ x));
(qCA6) (Neg1) + (DN1) + (qAn) + (qM) $\Longrightarrow$ (DN2);
(qCA7) (DN) + (α) + (qI) + (BB) + (qAn) + (qM) $\Longrightarrow$ (DN4);
(qCA8) (DN1) + (β) + (K) + (Tr) + (*) + (qM) $\Longrightarrow$ (qDN5);
(qCA9) (DN1) + (β) + (K) + (Tr) + ($) + (Re) + (qM) + (qAn) $\Longrightarrow$ (qDN(1 $\to$ x)).

Proof.
(qCA1): $y^- \to x^- \overset{(qI)}{=} (1 \to y^-) \to (1 \to x^-) \overset{(\alpha)}{=} (y \to 0) \to (x \to 0) \overset{(BB')}{\geq} x \to y$.

(qCA2): If $x \leq y$, i.e. $x \to y = 1$, then $1 \stackrel{(Neg1)}{=} (x \to y) \to (y^- \to x^-) = 1 \to (y^- \to x^-) \stackrel{(qM)}{=} y^- \to x^-$, i.e. $y^- \leq x^-$ and thus (Neg2) holds.

(qCA3): $y \to x^- \stackrel{(qM)}{=} 1 \to (y \to x^-) \stackrel{(Ex)}{=} y \to (1 \to x^-) \stackrel{(\alpha)}{=} y \to (x \to 0) \stackrel{(Ex)}{=} x \to (y \to 0) \stackrel{(\alpha)}{=} x \to (1 \to y^-) \stackrel{(Ex)}{=} 1 \to (x \to y^-) \stackrel{(qM)}{=} x \to y^-$, i.e. (Neg3) holds.

(qCA4): $x \stackrel{(D')}{\leq} (x \to 0) \to 0 \stackrel{(\alpha)}{=} (1 \to x^-) \to 0 \stackrel{(\alpha)}{=} 1 \to (1 \to x^-)^- \stackrel{(\overline{qM(1\to y)})}{=} (1 \to x^-)^- \stackrel{(\beta)}{=} ((1 \to x)^-)^-$, i.e. (qNeg4) holds.

(qCA5): By (qNeg4), $x \leq ((1 \to x)^-)^-$, hence by (Neg2), $(((1 \to x)^-)^-)^- \leq x^-$.
On the other hand, by (qNeg4), $x^- \leq ((1 \to x^-)^-)^- \stackrel{(\beta)}{=} (((1 \to x)^-)^-)^-$. Then, by (qAn), we obtain: $1 \to x^- = 1 \to (((1 \to x)^-)^-)^-$; but $1 \to x^- \stackrel{(\beta)}{=} (1 \to x)^-$ and $1 \to (((1 \to x)^-)^-)^- \stackrel{(\beta)}{=} 1 \to ((1 \to x^-)^-)^- \stackrel{(\beta)}{=} 1 \to (1 \to (x^-)^-)^- \stackrel{(\beta)}{=} 1 \to [1 \to ((x^-)^-)^-] \stackrel{(qM)}{=} 1 \to ((x^-)^-)^- \stackrel{(\beta)}{=} (((1 \to x)^-)^-)^-$. Hence, (qTN($1 \to x$)) holds.

(qCA6): By (Neg1), $x \to y \leq y^- \to x^-$. On the other hand, by (DN1), $y^- \to x^- \leq x \to y$. Then, by (qAn), $1 \to (x \to y) = 1 \to (y^- \to x^-)$, hence by (qM), $x \to y = y^- \to x^-$, i.e. (DN2) holds.

(qCA7): By (qCA1), (BB) + (α) + (qI) $\Longrightarrow$ (Neg1);
by (C1), (DN) + (Neg1) $\Longrightarrow$ (DN1);
by (qCA6), (DN1) + (Neg1) + (qAn) + (qM) $\Longrightarrow$ (DN2);
by (C4), (DN2) $\Longrightarrow$ (DN4); thus, (DN4) holds.

(qCA8): By (K'), $x^- \leq y^- \to x^-$, then by (*), $1 \to x^- \leq 1 \to (y^- \to x^-)$, hence $(1 \to x)^- \leq y^- \to x^-$, by ($\beta$) and (qM). But $y^- \to x^- \leq x \to y$, by (DN1). Hence, by (Tr), $(1 \to x)^- \leq x \to y$, i.e. (qDN5) holds.

(qCA9): First, we prove: (a) $((1 \to x)^-)^- \leq 1 \to x$.
Indeed, $((1 \to x)^-)^- \stackrel{(K')}{\leq} ((((1 \to x)^-)^-)^-)^- \to ((1 \to x)^-)^-$
$\stackrel{(DN1)}{\leq} (1 \to x)^- \to (((1 \to x)^-)^-)^- \stackrel{(DN1)}{\leq} ((1 \to x)^-)^- \to (1 \to x)$.
Hence, by (Tr), $((1 \to x)^-)^- \leq ((1 \to x)^-)^- \to (1 \to x)$, and by (\$), we obtain:
$1 \stackrel{(Re)}{=} ((1 \to x)^-)^- \to ((1 \to x)^-)^- \leq ((1 \to x)^-)^- \to (1 \to x)$; hence, by (qM), (a) holds.
Then, we prove: (b) $1 \to x \leq ((1 \to x)^-)^-$.
Indeed, by (a), $(((1 \to x)^-)^-)^- \leq (1 \to x)^-$, hence $(((1 \to x)^-)^-)^- \to (1 \to x)^- = 1$; then, by (DN1'), $1 \leq (1 \to x) \to ((1 \to x)^-)^-$, hence by (qM), $(1 \to x) \to ((1 \to x)^-)^- = 1$, i.e. (b) holds.
Now, (a) + (b) + (qAn) + (qM) + (β) $\Longrightarrow$ (qDN($1 \to x$)); indeed,
$1 \to x \stackrel{(qM)}{=} 1 \to (1 \to x) = 1 \to ((1 \to x)^-)^- \stackrel{(\beta)}{=} 1 \to (1 \to x^-)^- \stackrel{(\beta)}{=} 1 \to (1 \to (x^-)^-)) \stackrel{(qM)}{=} 1 \to (x^-)^- \stackrel{(\beta)}{=} (1 \to x^-)^- \stackrel{(\beta)}{=} ((1 \to x)^-)^-$, i.e. (qDN($1 \to x$)) holds. □

Proposition 3.37 *(See Proposition 2.24)*
(qCAN1) (Neg1-0) + (DN2) + (DN) + ($\overline{qM(1 \to x)}$) + (qM) $\Longrightarrow$ (Neg$_U$);
(qCAN2) (Neg1-0) + (11-1) + (Neg$_U$) $\Longrightarrow$ (10-0);
(qCAN3) (Neg1-0) + (DN2) + (qR3) + (qF) + (qM) $\Longrightarrow$ (L);
(qCAN4) (Neg1-0) + (DN1) + (qI2) + (K) + (BB) + (qM) $\Longrightarrow$ (F).

Proof.
(qCAN1): (see the proof of (qW18) from [2])
$(x \to y) \to 0 \stackrel{(Neg1-0)}{=} (x \to y) \to 1^- \stackrel{(qM)}{=} [1 \to (x \to y)] \to 1^-$
$\stackrel{(DN)}{=} ([1 \to (x \to y)]^-)^- \to 1^- \stackrel{(DN2)}{=} 1 \to [1 \to (x \to y)]^-$
$\stackrel{(\overline{qM(1\to x)})}{=} [1 \to (x \to y)]^- \stackrel{(qM)}{=} (x \to y)^-$, i.e. (Neg$_U$) is verified.

(qCAN2): (see the proof of (qW19) from [2])
$1 \to 0 \stackrel{(11-1)}{=} (1 \to 1) \to 0 \stackrel{(Neg_U)}{=} (1 \to 1)^- \stackrel{(11-1)}{=} 1^- = 0$, i.e. (10-0) holds.

(qCAN3): (see the proof of (qW24) from [2])
$x \to 1 \stackrel{(DN2)}{=} 1^- \to x^- \stackrel{(qM)}{=} 1 \to (1^- \to x^-)$

$\overset{(qF)}{=} (0 \to (1 \to x^-)) \to (1^- \to x^-)$
$\overset{(Neg1-0)}{=} (1^- \to (1 \to x^-)) \to (1^- \to x^-)$
$\overset{(qM)}{=} (1^- \to (1 \to x^-)) \to (1 \to (1^- \to x^-)) \overset{(qR3)}{=} 1$, i.e. (L) holds.

(qCAN4): (see the proof of (qW34) from [2])

$0 \to x \overset{(Neg1-0)}{=} 1^- \to x \overset{(qM)}{=} 1 \to (1^- \to x)$
$\overset{(DN1)}{=} [(x^- \to 1^-) \to (1 \to x)] \to (1^- \to x)$
$\overset{(qI2)}{=} [(x^- \to 1^-) \to x] \to (1^- \to x)$
$\overset{(qM)}{=} 1 \to ([(x^- \to 1^-) \to x] \to (1^- \to x))$
$\overset{(K)}{=} [1^- \to (x^- \to 1^-)] \to ([(x^- \to 1^-) \to x] \to (1^- \to x)) \overset{(BB)}{=} 1.$ □

• **Connections between List qB and List qC** Let $\mathcal{A}$ be a bounded proper quasi-ordered algebra (structure) with quasi-negation (i.e. properties (qM), (Re), (qAn), (Tr), (F), (L), (α) and (β) hold) through this subsubsection.

Proposition 3.38 *(See Proposition 2.25)*
(qCB1) (comm) + (qNeg) + (F) + (qM$(1 \to x)$) $\Longrightarrow$ (qDN$(1 \to x)$);
(qCB2) (q-impl) + (qNeg) $\Longrightarrow$ (qDN6);
(qCB3) (pimpl) + (qNeg) + (Re) + (qM) $\Longrightarrow$ (qNeg6);
(qCB4) (pimpl-1) + (qNeg) + (Re) + (K) + (qM) + (qAn) $\Longrightarrow$ (qNeg6);
(qCB5) (DN) + (DN2) + (qDN6) + (qDN5) + (K) + (*) + (qM) + (qAn) $\Longrightarrow$ (q-impl).

Proof.
(qCB1): $((1 \to x)^-)^- \overset{(qM(1\to x))}{=} ((1 \to (1 \to x))^-)^- \overset{(\beta)}{=} (1 \to (1 \to x)^-)^- \overset{(\alpha)}{=} (1 \to x)^- \to 0 \overset{(\alpha)}{=} (x \to 0) \to 0 \overset{(comm)}{=} (0 \to x) \to x \overset{(F)}{=} 1 \to x$, i.e. (qDN$(1 \to x)$) holds.

(qCB2): In (q-impl) ($(x \to y) \to x = 1 \to x$), we take $y = 0$; we obtain:
$1 \to x = (x \to 0) \to x \overset{(\alpha)}{=} (1 \to x^-) \to x \overset{(\beta)}{=} (1 \to x)^- \to x$, i.e. (qDN6) holds.

(qCB3): In (pimpl) ($x \to (y \to z) = (x \to y) \to (x \to z)$), we take $y = x$ and $z = 0$; we obtain:
$x \to (1 \to x)^- \overset{(\beta)}{=} x \to (1 \to x^-)$
$\overset{(\alpha)}{=} x \to (x \to 0) = (x \to x) \to (x \to 0)$
$\overset{(Re)}{=} 1 \to (x \to 0) \overset{(qM)}{=} x \to 0 \overset{(\alpha)}{=} 1 \to x^- \overset{(\beta)}{=} (1 \to x)^-$, i.e. (qNeg6) holds.

(qCB4): On the one hand, $(1 \to x)^- \overset{(K')}{\leq} x \to (1 \to x)^-$.
On the other hand, $x \to (1 \to x)^- \overset{(\beta)}{=} x \to (1 \to x^-)$
$\overset{(\alpha)}{=} x \to (x \to 0) \overset{(pimpl-1')}{\leq} (x \to x) \to (x \to 0)$
$\overset{(Re)}{=} 1 \to (x \to 0) \overset{(qM)}{=} x \to 0 \overset{(\alpha)+(\beta)}{=} (1 \to x)^-$.
Now, by (qAn) and (qM), we obtain:
$(1 \to x)^- \overset{(\beta)}{=} 1 \to x^- \overset{(qM)}{=} 1 \to (1 \to x^-)$
$\overset{(\beta)}{=} 1 \to (1 \to x)^- = 1 \to (x \to (1 \to x)^-)$
$\overset{(qM)}{=} x \to (1 \to x)^-$, i.e. (qNeg6) holds.

(qCB5): First, we prove: (a) $1 \to x \leq (x \to y) \to x$.
Indeed, $x \overset{(K')}{\leq} (x \to y) \to x$ implies, by (*'): $1 \to x \leq 1 \to [(x \to y) \to x] \overset{(qM)}{=} (x \to y) \to x$, i.e. (a) holds.
Then, we prove: (b) $(x \to y) \to x \leq 1 \to x$.
Indeed, $(x \to y) \to x \overset{(DN2)}{=} x^- \to (x \to y)^-$.
Then, by (qC5'), (DN) + (DN2) + (qDN5) $\Longrightarrow$ (qNeg5), hence: $(x \to y)^- \leq 1 \to x$; hence, by (*'):
$x^- \to (x \to y)^- \leq x^- \to (1 \to x) \overset{(DN)}{=} x^- \to ((1 \to x)^-)^-$

$\overset{(DN2)}{=} (1 \to x)^- \to x \overset{(qDN6)}{=} 1 \to x$, i.e. (b) holds.
Now, (a) + (b) + (qAn) + (qM) $\Longrightarrow$ (q-impl). □

Proposition 3.39 *(See Proposition 2.26)*
(qCBN1) (comm-1) + (Neg1-0) + (DN2) + (DN) + (Neg$_U$) + (qRe) + (qM) + (qL) $\Longrightarrow$ (qF);
(qCBN2) (comm) + (Neg1-0) + (DN1) + (DN5) + (*) + (**) + (F) + (β) + (qI1) + (qM) + (qAn) $\Longrightarrow$ (Neg$_V$);
(qCBN3) (comm) + (Neg1-0) + (DN2) + (DN) + (Neg$_U$) + (qR1) + (L) $\Longrightarrow$ (qR2).

Proof.
(qCBN1): (see the proof of (qW21), (qW22), (qW23) from [2])
First, we prove that:

$$(a) \quad [(x \to y)^- \to 1^-] \to (x \to y) = 1.$$

Indeed, $[(x \to y)^- \to 1^-] \to (x \to y) \overset{(DN2)}{=} [1 \to (x \to y)] \to (x \to y) \overset{(qM)}{=} (x \to y) \to (x \to y) \overset{(qRe)}{=} 1.$
Second, we prove that:

$$(b) \quad [(x \to y) \to (1^- \to (x \to y))] \to (1^- \to (x \to y)) = 1.$$

Indeed, $[(x \to y) \to (1^- \to (x \to y))] \to (1^- \to (x \to y))$
$\overset{(comm-1)}{=} [((x \to y) \to 1^-) \to 1^-] \to (x \to y)$
$\overset{(Neg_U)}{=} [(x \to y)^- \to 1^-] \to (x \to y) \overset{(a)}{=} 1.$
Now, $0 \to (x \to y) \overset{(qM)}{=} 1 \to [1^- \to (x \to y)]$
$\overset{(qL)}{=} ((x \to y) \to 1) \to [1^- \to (x \to y)]$
$\overset{(qL)}{=} ((x \to y) \to [((x \to y) \to 0) \to 1]) \to [1^- \to (x \to y)]$
$\overset{(Neg_U)}{=} ((x \to y) \to [(x \to y)^- \to 1]) \to [1^- \to (x \to y)]$
$\overset{(DN2)+(DN)}{=} ((x \to y) \to [1^- \to (x \to y)]) \to [1^- \to (x \to y)] \overset{(b)}{=} 1$, i.e. (qF) holds.
(qCBN2): On the one hand, $x^- \to 0 \overset{(Neg1-0)}{=} x^- \to 1^- \overset{(DN1)}{\leq} 1 \to x$; then,
$x \to 0 \overset{(qI1)}{=} (1 \to x) \to 0 \overset{(**')}{\leq} (x^- \to 0) \to 0 \overset{(comm)}{=} (0 \to x^-) \to x^- \overset{(F)}{=} 1 \to x^- \overset{(\beta)}{=} (1 \to x)^-.$
Hence, $x \to 0 = (1 \to x) \to 0 \leq (1 \to x)^- \overset{(\beta)}{=} 1 \to x^-.$
On the other hand, by (DN6), $x^- \leq x \to 0$; hence $1 \to x^- \overset{(*')}{\leq} 1 \to (x \to 0) \overset{(qM)}{=} x \to 0.$
Now, by (qAn), we obtain: $x \to 0 \overset{(qM)}{=} 1 \to (x \to 0) = 1 \to (1 \to x^-) \overset{(qM)}{=} 1 \to x^-.$
Thus, $x \to 0 \overset{(qI1)}{=} (1 \to x) \to 0 = (1 \to x)^-$, i.e. (Neg$_V$) holds.
(qCBN3): $x \to (1 \to x) \overset{(L)}{=} x \to ((x^- \to 1) \to x)$
$\overset{(DN)}{=} x \to ((x^- \to (1^-)^-) \to x) \overset{(DN2)}{=} x \to ((1^- \to x) \to x)$
$\overset{(comm)}{=} x \to ((x \to 1^-) \to 1^-) \overset{(Neg1-0)}{=} x \to ((x \to 1^-) \to 0)$
$\overset{(Neg_U)}{=} x \to (x \to 1^-)^- \overset{(DN)}{=} (x^-)^- \to (x \to 1^-)^-$
$\overset{(DN2)}{=} (x \to 1^-) \to x^- \overset{(DN)}{=} ((x^-)^- \to 1^-) \to x^-$
$\overset{(DN2)}{=} (1 \to x^-) \to x^- \overset{(qR1)}{=} 1$, i.e. (qR2) holds. □

• **The product $\odot$ in a bounded quasi-algebra (quasi-structure)**

Proposition 3.40
(qCP1) (RP) + (qM($x \odot y$)) + (qNeg4) + (qNeg) + (F) + (qAn) + (10-0) $\Longrightarrow$ (Px).

Proof.
(qCP1): $x \odot x^- \overset{(RP)}{\Leftrightarrow} x \leq x^- \to 0 \overset{(\alpha)}{=} 1 \to x^= \overset{(\beta)}{=} (1 \to x^-)^- \overset{(\beta)}{=} ((1 \to x)^-)^-$; since $x \leq (1 \to x)^=$, by (qNeg4'), it follows that $x \odot x^- \leq 0$. But we also have $0 \leq x \odot x^-$, by (F). It follows, by (qAn), that

$1 \to (x \odot x^-) = 1 \to 0$; but $1 \to (x \odot x^-) \overset{(qM(x\odot y))}{=} x \odot x^-$ and $1 \to 0 \overset{(10-0)}{=} 0$, hence $x \odot x^- = 0$, i.e. (Px) holds. □

• **The product $\odot$ in a bounded quasi-algebra (quasi-structure) with involutive quasi-negation**

Proposition 3.41
(CV1) ???? (dfV) + (Pcomm) + ($P_{(DN)}$) + (DN2) + (DN) $\Longrightarrow$ ($VP_{(DN)}$).

Proof.
(CV1): $x \vee y \overset{(dfV)}{=} (x \to y) \to y \overset{(DN2)}{=} y^- \to (x \to y)^- \overset{(DN)}{=} y^- \to (x \to y^=)^- \overset{(P_{(DN)})}{=} y^- \to (x \odot y^-) \overset{(Pcomm)}{=} y^- \to (y^- \odot x)$. □

• **The meet (wedge) $\wedge$. The quasi-lattice.**

Let us introduce the following binary operation:

$$(dfW) \qquad x \wedge y = (x^- \vee y^-)^-.$$

Proposition 3.42
(qCV0) (dfV) + (F) $\Longrightarrow$ (qV0-); (dfV) + (qNeg) + (qM($1 \to x$)) $\Longrightarrow$ (qV-0);
(qCW1) (dfW) + (qVid) + (β) + (DN) $\Longrightarrow$ (qWid);
(qCW4) (dfW) + (qV-0) + (qV0-) + (Neg1-0) + (β) + (DN) + (qM) $\Longrightarrow$ (qW1-1);
(qCW5) (dfW) + (Vcomm) + (qVe) + (β) + (DN4) + (DN) $\Longrightarrow$ (qWe);
(qCW13) (Vgeq) + (qWe) $\Longrightarrow$ (qWab1); (Wleq) + (qVe) + (Vcomm) $\Longrightarrow$ (qWab2).

Proof.
(qCV0): $0 \vee x \overset{(dfV)}{=} (0 \to x) \to x \overset{(F)}{=} 1 \to x$, thus (qV0-) holds.
$x \vee 0 \overset{(dfV)}{=} (x \to 0) \to 0 \overset{(\alpha)}{=} (1 \to x^-) \to 0 \overset{(\alpha)}{=} 1 \to (1 \to x^-)^- \overset{(\beta)}{=} (1 \to (1 \to x^-))^- \overset{(qM(1\to x))}{=} (1 \to x^-)^- \overset{(\beta)}{=} (1 \to x)^=$, thus (qV-0) holds.
(qCW1): $x \wedge x \overset{(dfW)}{=} (x^- \vee x^-)^- \overset{(qVid)}{=} (1 \to x^-)^- \overset{(\beta)}{=} (1 \to x)^= \overset{(DN)}{=} 1 \to x$, thus (qWid) holds.
(qCW4): $x \wedge 1 \overset{(dfW)}{=} (x^- \vee 1^-)^- \overset{(Neg1-0)}{=} (x^- \vee 0)^- \overset{(qV-0)}{=} ((1 \to x^-)^=)^- \overset{(DN)}{=} (1 \to x^-)^- \overset{(\beta)}{=} (1 \to x)^= \overset{(DN)}{=} 1 \to x$ and $1 \wedge x \overset{(dfW)}{=} (1^- \vee x^-)^- \overset{(Neg1-0)}{=} (0 \vee x^-)^- \overset{(qV0-)}{=} (1 \to x^-)^- \overset{(\beta)}{=} (1 \to x)^= \overset{(DN)}{=} 1 \to x$; thus (qW1-1) holds.
(qCW5): $x \leq y \overset{(DN4)}{\Leftrightarrow} y^- \leq x^- \overset{(qVe)}{=} y^- \vee x^- = 1 \to x^- \overset{(\beta)}{=} (1 \to x)^- \overset{(DN)}{\Leftrightarrow} (y^- \vee x^-)^- = (1 \to x)^= \overset{(DN)}{=} 1 \to x$ and since $(y^- \vee x^-)^- \overset{(Vcomm)}{=} (x^- \vee y^-)^- \overset{(dfW)}{=} x \wedge y$, it follows that $x \leq y \Leftrightarrow x \wedge y = 1 \to x$, i.e. (qWe) holds.
(qCW13): $x \overset{(Vgeq)}{\leq} x \vee y \overset{(qWe)}{\Leftrightarrow} x \wedge (x \vee y) = 1 \to x$, thus (qWab1) holds.
$x \wedge y \overset{(Wleq)}{\leq} x \overset{(qVe)}{\Leftrightarrow} (x \wedge y) \vee x = 1 \to x$, hence $x \vee (x \wedge y) = 1 \to x$, by (Vcomm), i.e. (qWab2) holds. □

4 Other quasi-algebras

We shall recall here definitions and results concerning the quasi-BCK algebras and quasi-Hilbert algebras, we shall establish the connection with quasi-Wajsberg algebras and we shall introduce the quasi-Boolean algebras and we shall establish connections with quasi-Boolean algebras.

We have presented in Part I [19] two equivalent definitions of quasi-BCK algebras:

Definition 4.1 [19] Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2, 0)$ (or, equivalently, let $\mathcal{A} = (A, \leq, \rightarrow, 1)$ be a structure), as before. $\mathcal{A}$ is called a *quasi-BCK algebra* (or a *qBCK algebra*, for short) if one of the following two equivalent groups of quasi-properties is satisfied:
(qBCK-1) (BB), (D), (Re), (L), (qM), (qAn) and
(qBCK-2) (B), (C), (K), (qM), (qAn).

Recall that a quasi-BCK algebra verifies all the quasi-properties in List qA.

Definition 4.2 A *quasi-Hilbert algebra* (or a *qHilbert algebra* for short) is an algebra $\mathcal{A} = (A, \rightarrow, 1)$ of type $(2, 0)$ satisfying (K), (pimpl-1), (qM), (qAn).

Definition 4.3 [20] Let $\mathcal{A} = (A, \rightarrow, 1)$ be a quasi-BCK algebra. We say that $\mathcal{A}$ is
- *positive implicative*, if property (pimpl) is satisfied;
- *commutative*, if property (comm) is satisfied;
- *quasi-implicative*, if property (q-impl) is satisfied.

Theorem 4.4 *[20] (See Theorem 2.35 in regular case)*
A quasi-BCK algebra is positive implicative if and only if the property (pi) holds (or, in a quasi-BCK algebra, we have $(pimpl) \iff (pi)$*).*

Theorem 4.5 *[20] (See Theorem 2.36 in regular case)*
Quasi-Hilbert algebras are categorically equivalent to positive implicative quasi-BCK algebras.

Theorem 4.6 *[20] (See Theorem 2.37 in regular case)*
In a commutative quasi-BCK algebra, properties (pi) and (q-impl) are equivalent.

Theorem 4.7 *[20] (See Theorem 2.38 in regular case)*
Any quasi-implicative quasi-BCK algebra is commutative and positive implicative.

Theorem 4.8 *[20] (See Theorem 2.39 in the regular case)*
Any commutative and positive implicative quasi-BCK algebra is quasi-implicative.

Corollary 4.9 *[20] (See Corollary 2.40 in the regular case) In a quasi-BCK algebra we have:*

$$(q - impl) \Leftrightarrow ((comm) + (pi) (\Leftrightarrow (pimpl)))$$

Corollary 4.10 *[20] (see Corollary 2.41 in the regular case)*
Any commutative quasi-Hilbert algebra is quasi-implicative.

4.1 The bounded quasi-BCK algebras

Resuming, we have three cases.

4.1.1 The bounded quasi-BCK algebra

A bounded quasi-BCK algebra verifies the axioms (following the second definition, (qBCK2)): (B), (C), (K), (qM), (qAn); (F), (10-0), (Neg_U). It also verifies (11-1), by (qAA1'), among others.

It is denoted by $(A, \rightarrow, 0, 1)$.

Note that any bounded quasi-BCK algebra is a bounded quasi-ordered algebra.

Note also that property (Neg_U) introduces an unique negation $^- : R(A) \longrightarrow R(A)$.

4.1.2 The bounded quasi-BCK algebra with a quasi-negation

A bounded quasi-BCK algebra with a quasi-negation $^- : A \longrightarrow A$ verifies the axioms (following the second definition, (qBCK2)): (B), (C), (K), (qM), (qAn); (F), (10-0), (Neg_U); (qNeg). Consequently, it also verifies ($q\overline{M}(1 \rightarrow x)$), by Theorem 3.30 (qi).

It is denoted by $(A, \rightarrow, ^-, 0, 1)$.

Note that the quasi-negation $^-$ is not unique, while $^- |_{R(A)}$, denoted also by $^-$, is unique.

Note also that a bounded quasi-BCK algebra with a quasi-negation verifies also: (Neg1), by (qCA1); (Neg2), by (qCA2); (Neg3), by (qCA3); (qNeg4), by (qCA4); (qTN$(1 \rightarrow x)$), by (qCA5).

4.1.3 The bounded quasi-BCK algebra with an involutive quasi-negation

A bounded quasi-BCK algebra with an involutive quasi-negation $^- : A \longrightarrow A$ verifies the axioms (following the second definition, (qBCK2)): (B), (C), (K), (qM), (qAn); (F), (10-0), (Neg_U); (qNeg) (hence $(\underline{qM(1 \to x)})$), (DN).

It is denoted also by $(A, \to, ^-, 0, 1)$.

Note that the involutive quasi-negation $^-$ is not unique, while $^- \mid_{R(A)}$, denoted also by $^-$, is involutive and unique.

Note also that bounded quasi-BCK algebra with an involutive quasi-negation verifies also: (DN1), by (C1); (DN2), by (qCA6); (DN3), by (C3); (DN4), by (qCA7); (qDN5), by (qCA8); (qNeg5), by (qC5').

4.2 Quasi-Wajsberg algebras. Connection

Recall from [2] the definition of quasi-Wajsberg algebras:

Definition 4.11 [2] An algebra $A, \to, ^-, 1)$ of type $(2, 1, 0)$ is a *quasi-Wajsberg algebra* if the following properties are verified: for all $x, y, z \in A$,
(qM) $1 \to (x \to y) = x \to y$,
(BB) $(x \to y) \to [(y \to z) \to (x \to z)] = 1$,
(comm) $(x \to y) \to y = (y \to x) \to x$,
(DN1) $(x^- \to y^-) \to (y \to x) = 1$,
(DN) $(x^-)^- = x$,
$(\underline{qM(1 \to x)})$ $1 \to (1 \to x)^- = (1 \to x)^-$.

Note that a quasi-Wajsberg algebra is a quasi-algebra, i.e. (qM) and (11-1) hold. Indeed, by (qAA1), (qM) + (BB) $\Longrightarrow$ (11-1).

Note also that if (M) holds, then a quasi-Wajsberg algebra becomes a Wajsberg algebra.

4.2.1 The quasi-Wajsberg algebras as quasi-algebras with involutive quasi-negation

We shall prove that the quasi-Wajsberg algebra is an example of quasi-algebra with quasi-negation, namely with involutive quasi-negation.

Theorem 4.12 *(See Theorem 2.43 in regular case)*
The quasi-Wajsberg algebra is a quasi-algebra with involutive quasi-negation.

Proof. Let $\mathcal{A} = (A, \to, ^-, 1)$ be a quasi-Wajsberg algebra, i.e. (qM), (BB), (comm), (DN1), (DN), $(\underline{qM(1 \to x)})$ hold. Denote 1^- by 0, i.e. (Neg1-0) holds too.
We must prove that $\mathcal{A}$ is a quasi-algebra with quasi-negation and that the quasi-negation is involutive (which is obvious). Hence, we must prove, by Definition 3.28, that:
(1) $(A, \to, 1)$ is a quasi-algebra, i.e. (qM) and (11-1) hold;
(2) the property (qNeg) holds.

(1): obviously.

(2): We must prove that (α) and (β) hold. Indeed,
- by (qB12), (comm) $\Longrightarrow$ (qAn);
- by (qC1'), (DN1) + (DN) $\Longrightarrow$ (Neg1);
- by (qCA6), (Neg1) + (DN1) + (qAn) + (qM) $\Longrightarrow$ (DN2);
- by (qCN1), (Neg1-0) + (DN2) + (DN) $\Longrightarrow$ (α), thus (α) holds •
- By (qA18), (qM) +(BB) $\Longrightarrow$ (qRe$(1 \to x)$);
- by Theorem 3.6 (ix), (qM) + (qRe$(1 \to x)$) $\Longrightarrow$ (qRe);
- by (qBB1), (comm) + (qRe) + (BB) + (qM) $\Longrightarrow$ (qR1);
- by (qB32), (comm) + (qR1) + (BB) + (qM) $\Longrightarrow$ (qL$(1 \to x)$);
- by Theorem 3.6 (x), (qM) + (qL$(1 \to x)$) $\Longrightarrow$ (qL);
- by (qAA5), (qR1) + (BB) $\Longrightarrow$ (qR3);

- by (qCAN1), (Neg1-0) + (DN) + (DN2) + (qM) + ($\overline{qM(1 \to x)}$) $\Longrightarrow$ (Neg$_U$);
- by (B33), (comm) $\Longrightarrow$ (comm-1);
- by (qCBN1), (comm-1) + (Neg1-0) + (DN2) + (DN) + (Neg$_U$) + (qL) + (qRe) + (qM) $\Longrightarrow$ (qF);
- by (qCAN3), (qF) + (Neg1-0) + (DN2) + (qR3) + (qM) $\Longrightarrow$ (L);
- by (qCBN3), (comm) + (Neg1-0) + (L) + (DN2) + (DN) + (Neg$_U$) + (qR1) $\Longrightarrow$ (qR2);
- by (qAA10), (qR1) + (qR2) + (BB) + (qM) + (qAn) $\Longrightarrow$ (qI1);
- by (qCN1'), (α) + ($\overline{qM(1 \to x)}$) + (qI1) $\Longrightarrow$ (β), thus (β) holds, and hence (qNeg) holds • □

4.2.2 The connection with the quasi-Wajsberg algebras

We shall prove the following theorem.

Theorem 4.13 *(See Theorem 2.44 in the regular case)*
Bounded, commutative quasi-BCK algebras with an involutive quasi-negation are categorically equivalent to quasi-Wajsberg algebras.

Proof.
(1) Let $(A, \to, ^-, 0, 1)$ be a bounded, commutative quasi-BCK algebra with an involutive quasi-negation, i.e. properties (BB), (D), (Re), (L), (qM), (qAn); (F), (10-0), (Neg$_U$); (α), (β) (hence ($\overline{qM(1 \to x)}$)); (DN); (comm) hold.
We shall prove that $(A, \to, ^-, 1)$ is a quasi-Wajsberg algebra, i.e. (qM), (BB), (comm), (DN1), ($\overline{qM(1 \to x)}$) hold.
Note that it remains to prove that (DN1) and ($\overline{qM(1 \to x)}$) hold. Indeed,
- by (qA21), (BB) + (D) + (qM) + (qAn) $\Longrightarrow$ (Ex);
- by (qA17'), (BB) + (qM) $\Longrightarrow$ (Tr);
- by (qAA15"'), (Ex) + (Tr) + (Re) + (L) + (qM) + (qAn) $\Longrightarrow$ (qI);
- by (qCA1), (qI) + (α) + (BB) $\Longrightarrow$ (Neg1);
- by (C1), (Neg1) + (DN) $\Longrightarrow$ (DN1); thus (DN1) holds •
- By Theorem 3.6 (ii), (qM) $\Longrightarrow$ (qM($1 \to x$));
- by Theorem 3.30 (qi), (β) + (qM($1 \to x$)) $\Longrightarrow$ ($\overline{qM(1 \to x)}$), thus ($\overline{qM(1 \to x)}$) holds •

(1') Let now $(A, \to, ^-, 1)$ be a quasi-Wajsberg algebra, i.e. (qM), (BB), (comm), (DN1), (DN), ($\overline{qM(1 \to x)}$) hold. Denote 1^- by 0, i.e. (Neg1-0) holds.
We must prove that $(A, \to, ^-, 0, 1)$ is a bounded, commutative quasi-BCK algebra with involutive quasi-negation, i.e. properties (BB), (D), (Re), (L), (qM), (qAn); (F), (10-0), (Neg$_U$); (qNeg) (i.e. (α), (β)); (DN); (comm) hold.
Note that it remains to prove that (D), (Re), (L), (qAn); (F), (10-0), (Neg$_U$); (α), (β) hold. Indeed,
- by (qB12), (comm) $\Longrightarrow$ (qAn), thus (qAn) holds •
- By (qA17'), (qM) + (BB) $\Longrightarrow$ (Tr);
- by (qAA1), (qM) + (BB) $\Longrightarrow$ (11-1);
- by (qA18), (qM) +(BB) $\Longrightarrow$ (qRe($1 \to x$));
- by Theorem 3.6 (ix), (qM) + (qRe($1 \to x$)) $\Longrightarrow$ (qRe);
- by (qBB1), (comm) + (qRe) + (BB) + (qM) $\Longrightarrow$ (qR1);
- by (qB32), (comm) + (qR1) + (BB) + (qM) $\Longrightarrow$ (qL($1 \to x$));
- by Theorem 3.6 (x), (qM) + (qL($1 \to x$)) $\Longrightarrow$ (qL);
- by (qAA5), (qR1) + (BB) $\Longrightarrow$ (qR3);
- by (qC1'), (DN1) + (DN) $\Longrightarrow$ (Neg1);
- by (qCA6), (Neg1) + (DN1) + (qAn) + (qM) $\Longrightarrow$ (DN2);
- by (qCAN1), (Neg1-0) + (DN) + (DN2) + (qM) + ($\overline{qM(1 \to x)}$) $\Longrightarrow$ (Neg$_U$), thus (Neg$_U$) holds •
- By (qCAN2), (Neg1-0) + (11-1) + (Neg$_U$) $\Longrightarrow$ (10-0), thus (10-0) holds •
- By (B33), (comm) $\Longrightarrow$ (comm-1);
- by (qCBN1), (comm-1) + (Neg1-0) + (DN2) + (DN) + (Neg$_U$) + (qL) + (qRe) + (qM) $\Longrightarrow$ (qF);
- by (qCAN3), (qF) + (Neg1-0) + (DN2) + (qR3) + (qM) $\Longrightarrow$ (L), thus (L) holds •
- By (qCBN3), (comm) + (Neg1-0) + (L) + (DN2) + (DN) + (Neg$_U$) + (qR1) $\Longrightarrow$ (qR2);
- by (qAA10), (qR1) + (qR2) + (BB) + (qM) + (qAn) $\Longrightarrow$ (qI1) + (qI2);

- by (qAA12), (qI1) + (qI2) $\Longrightarrow$ (qI);
- by (qAA13), (qI) + (BB) + (qM) $\Longrightarrow$ (Re), thus (Re) holds •
- By (qAA14), (qI1) + (L) + (BB) + (qM) $\Longrightarrow$(K);
- by (qB31), (comm) + (K) + (BB) + (Tr) + (qM) $\Longrightarrow$ (#);
- by (A29), (#) + (BB) $\Longrightarrow$ (B);
- by (A30), (B) + (#) + (Re) + (Tr) $\Longrightarrow$ (C);
- by (qA3), (C) + (qM) + (qAn) $\Longrightarrow$ (Ex);
- by (A4), (Ex) + (Re) $\Longrightarrow$ (D), thus (D) holds •
- By (qCAN4), (Neg1-0) + (DN1) + (qI2) + (K) + (BB) + (qM) $\Longrightarrow$ (F), thus (F) holds •
- By (qCN1), (Neg1-0) + (DN2) + (DN) $\Longrightarrow$ (α), thus (α) holds •
- By (qCN1'), (α) + ($qM(1 \to x)$) + (qI1) $\Longrightarrow$ (β), thus (β) holds (and hence (qNeg) holds) • □

Remark 4.14 Theorem 4.12 is a corollary of above Theorem 4.13.

Note that, by above Theorem 4.13, a quasi-Wajsberg algebra verifies:
- (Neg1)-(Neg3) and (qTN($1 \to x$)), since it is a bounded quasi-BCK algebra with quasi-negation;
- (DN1)-(DN3) and (qDN5), (qNeg5), since the quasi-negation is involutive.

4.3 Quasi-Boolean algebras. Connections

We are able now to introduce the notion of quasi-Boole algebra (quasi-Boolean algebra).

Definition 4.15 A *quasi-Boole algebra*, or *quasi-Boolean algebra*, is an algebra $\mathcal{A} = (A, \to, ^-, 1)$ of type $(2, 1, 0)$ verifying the axioms: for all $x, y, z \in A$,
(K) $x \to (y \to x) = 1$,
(pimpl-1) $[x \to (y \to z)] \to [(x \to y) \to (x \to z)] = 1$,
(DN1) $(y^- \to x^-) \to (x \to y) = 1$,
(qAn) $x \to y = 1 = y \to x \implies 1 \to x = 1 \to y$,
(qM) $1 \to (x \to y) = x \to y$,
($qM(1 \to x)$) $1 \to (1 \to x)^- = (1 \to x)^-$,
(DN) $(x^-)^- = x$.

Note that a quasi-Boole algebra is a quasi-algebra, i.e. (qM) and (11-1) hold. Indeed, by (qA7'), (qM) + (K) $\Longrightarrow$ (L), and by Remarks 2.3 (i), (L) $\Longrightarrow$ (11-1).

4.3.1 Quasi-Boolean algebras as quasi-algebras with involutive quasi-negation

Theorem 4.16 *(See Theorem 2.48 in the regular case)*
The quasi-Boole algebras are quasi-algebras with involutive quasi-negation.

Proof. Let $\mathcal{A} = (A, \to, ^-, 1)$ be a quasi-Boolean algebra, i.e. the quasi-properties (K), (pimpl-1), (DN1), (qAn), (qM), ($qM(1 \to x)$), (DN) hold. Denote 1^- by 0, i.e. (Neg1-0) holds too.
We must prove that $\mathcal{A}$ is a quasi-algebra with quasi-negation and that the quasi-negation is involutive (which is obvious). Hence, we must prove, by Definition 3.28, that:
(1) $(A, \to, 1)$ is a quasi-algebra, i.e. (qM) and (11-1) hold;
(2) the property (qNeg) holds.
(1): obviously.
(2): by (qA7'), (qM) + (K) $\Longrightarrow$ (L);
- by (qB19'), (pimpl-1) + (K) + (qM) $\Longrightarrow$ (Re);
- by (qB20'), (pimpl-1) + (L) + (qM) $\Longrightarrow$ (*);
- by (qB23), (pimpl-1) + (Re) + (*) + (K) + (qM) $\Longrightarrow$ (D);
- by (qB24), (pimpl-1) + (*) + (K) + (qM) $\Longrightarrow$ (**);
- by (C1'), (DN) + (DN1) $\Longrightarrow$ (Neg1);
- by (qCA6), (Neg1) + (DN1) + (qAn) + (qM) $\Longrightarrow$ (DN2);
- by (qCN1), (Neg1-0) + (DN2) + (DN) $\Longrightarrow$ (α), thus (α) holds •
- By (qAA7'), (K) + (**) + (D) + (qM) + (qAn) $\Longrightarrow$ (qI1);
- by (qCN1'), (α) + (qI1) + ($qM(1 \to x)$) $\Longrightarrow$ (β), thus (β) holds (and hence (qNeg) holds) • □

4.3.2 Connections with the quasi-Boolean algebras

We have the following connections.

Theorem 4.17 *(See Theorem 2.49 in the regular case)*
Bounded, quasi-implicative quasi-BCK algebras with involutive quasi-negation are categorically equivalent to quasi-Boolean algebras.

Proof.
(1) Let $(A, \to, ^-, 0, 1)$ be a bounded, quasi-implicative quasi-BCK algebra with involutive quasi-negation, i.e. (B), (C), (K), (qM), (qAn); (F), (10-0), (Neg_U); (qNeg) (i.e. (α), (β)); (DN); (q-impl) hold.
We must prove that $(A, \to, ^-, 1)$ is a quasi-Boolean algebra, i.e. (K), (pimpl-1), (DN1), (qAn), (qM), $(\overline{qM(1 \to x)})$, (DN) hold.
Note that it remains to prove that (pimpl-1), (DN1), $(\overline{qM(1 \to x)})$ hold. Indeed,
- by Theorem 4.7, (q-impl) $\Longrightarrow$ (comm) + (pimpl);
- by (qA23), (C) + (K) + (qM) + (qAn) $\Longrightarrow$ (Re);
- by (A0), (Re) $\Longrightarrow$ (S);
- by (S), (pimpl) $\Longrightarrow$ (pimpl-1), thus (pimpl-1) holds •
- Any quasi-BCK algebra verifies all the properties in list qA, hence (BB) and (qI) hold;
- by (qCA1), (α) + (qI) + (BB) $\Longrightarrow$ (Neg1);
- by (C1), (DN) + (Neg1) $\Longrightarrow$ (DN1), thus (DN1) holds •
- By Theorem 3.6 (ii), (qM) $\Longrightarrow$ $(qM(1 \to x))$;
- by Theorem 3.30 (qi), (β) + $(qM(1 \to x))$ $\Longrightarrow$ $(\overline{qM(1 \to x)})$, thus $(\overline{qM(1 \to x)})$ holds •

(1') Conversely, let $(A, \to, ^-, 1)$ be a quasi-Boole algebra, i.e. (K), (pimpl-1), (DN1), (qAn), (qM), $(\overline{qM(1 \to x)})$, (DN) hold. Denote 1^- by 0, i.e. (Neg1-0) holds too.
We must prove that $(A, \to, ^-, 0, 1)$ is a bounded, quasi-implicative quasi-BCK algebra with involutive quasi-negation, i.e. (B), (C), (K), (qM), (qAn); (F), (10-0), (Neg_U); (qNeg) (i.e. (α), (β)); (DN); (q-impl) hold.
Note that it remains to prove that (B), (C); (F), (10-0), (Neg_U); (qNeg); (q-impl) hold. Indeed,
- by (qA7'), (qM) + (K) $\Longrightarrow$ (L);
- by (qB19'), (pimpl-1) + (K) + (qM) $\Longrightarrow$ (Re);
- by (qB20'), (pimpl-1) + (L) + (qM) $\Longrightarrow$ (*);
- by (qB21'), (pimpl-1) + (L) + (qM) $\Longrightarrow$ (Tr);
- by (qB23), (pimpl-1) + (Re) + (*) + (K) + (qM) $\Longrightarrow$ (D);
- by (qB24), (pimpl-1) + (*) + (K) + (qM) $\Longrightarrow$ (**);
- by (B25), (pimpl-1) + (K) + (**) + (Tr) $\Longrightarrow$ (C), thus (C) holds •
- By (B18), (K) + (**) + (C) + (Tr) $\Longrightarrow$ (pimpl-2);
- by (qB0), (pimpl-1) + (pimpl-2) + (qM) + (qAn) $\Longrightarrow$ (pimpl);
- by (qB9), (pimpl) + (Re) + (qM) $\Longrightarrow$ (BB);
- by (qA3), (C) + (qM) + (qAn) $\Longrightarrow$ (Ex);
- by (A10"), (Ex) + (BB) $\Longrightarrow$ (B), thus (B) hold •
- By (qAA8), (Ex) + (qM) $\Longrightarrow$ (qI2);
- by (qCAN4), (Neg1-0) + (DN1) + (qI2) + (K) + (BB) + (qM) $\Longrightarrow$ (F), thus (F) holds •
- By (C1'), (DN) + (DN1) $\Longrightarrow$ (Neg1);
- by (qCA6), (Neg1) + (DN1) + (qAn) + (qM) $\Longrightarrow$ (DN2);
- by (qCAN1), (Neg1-0) + (DN2) + (DN) + $(\overline{qM(1 \to x)})$ + (qM) $\Longrightarrow$ (Neg_U), thus (Neg_U) holds •
- By Remarks 2.3 (i), (L) $\Longrightarrow$ (11-1);
- by (qCAN2), (Neg1-0) + (11-1) + (Neg_U) $\Longrightarrow$ (10-0), thus (10-0) holds •
- By (qCN1), (Neg1-0) + (DN2) + (DN) $\Longrightarrow$ (α), thus (α) holds •
- By (qAA7'), (K) + (**) + (D) + (qM) + (qAn) $\Longrightarrow$ (qI1);
- by (qCN1'), (α) + (qI1) + $(\overline{qM(1 \to x)})$ $\Longrightarrow$ (β), thus (β) holds (and hence (qNeg) holds) •
- By (qCB3), (pimpl) + (qNeg) + (Re) + (qM) $\Longrightarrow$ (qNeg6);
- by (qC6), (DN) + (DN2) + (β) + (qNeg6) $\Longrightarrow$ (qDN6);

- by (qCA8), (DN1) + (β) + (K) + (Tr) + (*) + (qM) $\Longrightarrow$ (qDN5);
- by (qCB5), (DN) + (DN2) + (qDN6) + (qDN5) + (K) + (*) + (qM) + (qAn) $\Longrightarrow$ (q-impl) $\bullet$ $\Box$

Remark 4.18 Theorem 4.16 is a corollary of above Theorem 4.17.

Note that, by the proof of Theorem 4.17, a quasi-Boole algebra verifies:
- (Neg1)-(Neg3) and (qTN($1 \to x$)), since it is a bounded quasi-BCK algebra with quasi-negation;
- (DN1)-(DN3) and (qDN5), (qNeg5), since the quasi-negation is involutive;
- (qNeg6) and (qDN6), by the proof.

Theorem 4.19 *(See Theorem 2.51 in the regular case)*
Bounded, positive implicative quasi-BCK algebras with involutive quasi-negation are categorically equivalent to quasi-Boolean algebras.

Proof. (follows closely the proof of Theorem 4.17)

(1) Let $(A, \to, ^-, 0, 1)$ be a bounded, positive implicative quasi-BCK algebra with involutive quasi-negation, i.e. (B), (C), (K), (qM), (qAn); (F), (10-0), (Neg$_U$); (qNeg) (i.e. (α), (β)); (DN); (pimpl) hold. We must prove that $(A, \to, ^-, 1)$ is a quasi-Boolean algebra, i.e. (K), (pimpl-1), (DN1), (qAn), (qM), ($\overline{qM(1 \to x)}$), (DN) hold.
Note that it remains to prove that (pimpl-1), (DN1), ($\overline{qM(1 \to x)}$) hold. Indeed,
- by (qA23), (C) + (K) + (qM) + (qAn) $\Longrightarrow$ (Re);
- by (A0), (Re) $\Longrightarrow$ (S);
- by (S), (pimpl) $\Longrightarrow$ (pimpl-1), thus (pimpl-1) holds $\bullet$
- Any quasi-BCK algebra verifies all the properties in list qA, hence (BB) and (qI) hold;
- by (qCA1), (α) + (qI) + (BB) $\Longrightarrow$ (Neg1);
- by (C1), (DN) + (Neg1) $\Longrightarrow$ (DN1), thus (DN1) holds $\bullet$
- By Theorem 3.6 (ii), (qM) $\Longrightarrow$ (qM($1 \to x$));
- by Theorem 3.30 (qi), (β) + (qM($1 \to x$)) $\Longrightarrow$ ($\overline{qM(1 \to x)}$), thus ($\overline{qM(1 \to x)}$) holds $\bullet$

(1') Conversely, let $(A, \to, ^-, 1)$ be a quasi-Boole algebra, i.e. (K), (pimpl-1), (DN1), (qAn), (qM), ($\overline{qM(1 \to x)}$), (DN) hold. Denote 1^- by 0, i.e. (Neg1-0) holds too.
We must prove that $(A, \to, ^-, 0, 1)$ is a bounded, positive implicative quasi-BCK algebra with involutive quasi-negation, i.e. (B), (C), (K), (qM), (qAn); (F), (10-0), (Neg$_U$); (qNeg) (i.e. (α), (β)); (DN); (pimpl) hold.
Note that it remains to prove that (B), (C); (F), (10-0), (Neg$_U$); (qNeg); (pimpl) hold. Indeed,
- by (qA7'), (qM) + (K) $\Longrightarrow$ (L);
- by (qB19'), (pimpl-1) + (K) + (qM) $\Longrightarrow$ (Re);
- by (qB20'), (pimpl-1) + (L) + (qM) $\Longrightarrow$ (*);
- by (qB21'), (pimpl-1) + (L) + (qM) $\Longrightarrow$ (Tr);
- by (qB23), (pimpl-1) + (Re) + (*) + (K) + (qM) $\Longrightarrow$ (D);
- by (qB24), (pimpl-1) + (*) + (K) + (qM) $\Longrightarrow$ (**);
- by (B25), (pimpl-1) + (K) + (**) + (Tr) $\Longrightarrow$ (C), thus (C) holds $\bullet$
- By (B18), (K) + (**) + (C) + (Tr) $\Longrightarrow$ (pimpl-2);
- by (qB0), (pimpl-1) + (pimpl-2) + (qM) + (qAn) $\Longrightarrow$ (pimpl), thus (pimpl) holds $\bullet$
- By (qB9), (pimpl) + (Re) + (qM) $\Longrightarrow$ (BB);
- by (qA3), (C) + (qM) + (qAn) $\Longrightarrow$ (Ex);
- by (A10"), (Ex) + (BB) $\Longrightarrow$ (B), thus (B) hold $\bullet$
- By (qAA8), (Ex) + (qM) $\Longrightarrow$ (qI2);
- by (qCAN4), (Neg1-0) + (DN1) + (qI2) + (K) + (BB) + (qM) $\Longrightarrow$ (F), thus (F) holds $\bullet$
- By (C1'), (DN) + (DN1) $\Longrightarrow$ (Neg1);
- by (qCA6), (Neg1) + (DN1) + (qAn) + (qM) $\Longrightarrow$ (DN2);
- by (qCAN1), (Neg1-0) + (DN2) + (DN) + ($\overline{qM(1 \to x)}$) + (qM) $\Longrightarrow$ (Neg$_U$), thus (Neg$_U$) holds $\bullet$
- By Remarks 2.3 (i), (L) $\Longrightarrow$ (11-1);
- by (qCAN2), (Neg1-0) + (11-1) + (Neg$_U$) $\Longrightarrow$ (10-0), thus (10-0) holds $\bullet$
- By (qCN1), (Neg1-0) + (DN2) + (DN) $\Longrightarrow$ (α), thus (α) holds $\bullet$

- By (qAA7'), (K) + (**) + (D) + (qM) + (qAn) $\Longrightarrow$ (qI1);
- by (qCN1'), (α) + (qI1) + ($qM(1 \to x)$) $\Longrightarrow$ (β), thus (β) holds (and hence (qNeg) holds) • □

Corollary 4.20 *(See Corollary 2.52 in the regular case)*
Bounded quasi-Hilbert algebras with involutive quasi-negation are categorically equivalent to quasi-Boolean algebras.

Proof. By Theorem 4.5, quasi-Hilbert algebras are categorically equivalent to positive implicative quasi-BCK algebras. Hence, bounded quasi-Hilbert algebras with involutive quasi-negation are categorically equivalent to bounded positive implicative quasi-BCK algebras with involutive quasi-negation. Then apply Theorem 4.19. □

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