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VASILE I. URSU

Abstract: It is proved that the elementary theory of the class of all finite commutative loops is undecidability. As a consequence, we establish the undecidability of elementary theories of certain classes of finite commutative loops, as well as certain classes of commutative quasigroups, and the groupoid.

Key words: loupe class, closed formula, elementary theory, nilpotent loop, metaabel loop, associator.

1. **INTRODUCTION**

Let $\boldsymbol{K}$ be a class of models of finite signature σ. As usual, by the *elementary theories* $T(\boldsymbol{K})$ we mean the collection of all formulas of signature σ true on all models from $\boldsymbol{K}$. Since the signature σ is a finite set, then under this assumption all the formulas of the narrow predicate calculus of the signature can be numbered in a natural way. It is said that the class $\boldsymbol{K}$, or that the elementary theory $T(\boldsymbol{K})$ of class $\boldsymbol{K}$ is *recursively undecidable* (or *unsolvable*) if the collection of numbers of all formulas of the narrow predicate calculus that are true on $\boldsymbol{K}$ is a recursive set of natural numbers. Otherwise, the elementary theory $T(\boldsymbol{K})$ is said to be recursively undecidable (or unsolvable). An important question connected with the study of elementary theories of various classes of algebras is the question of the algorithmic decidability of the class of all finite algebras. At present, many classes of finite classical algebras with an undecidability theory are known (see [1]). The author in [3] pointed out that among such classes with undecidability theories is the class of all finite quasigroups of any nonassociative variety of commutative Moufang loops (resp., distributive quasigroups or CH-quasigroups).

In this paper we establish a correspondence between commutative rings with unity and automorphic loops. In this correspondence to a class of commutative rings with unity and characteristic p, where p is any prime number, there corresponds an certainly axiomatizable class J_p of commutative automorphic loops with identity $x^p = 1$. An effective method is given, that allows for each formula, a narrow predicate calculus related to commutative rings with unity, to obtain a formula corresponding to loops such that the truth of the first formula on the ring is equivalent to the truth of the second formula in the corresponding commutative automorphic loop. Then from the fact that the elementary theory of all finite commutative rings with unity simple characteristic is undecidable (see [2, 5]), from which the following theorem follows directly: *an elementary theory of the class J_p of commutative automorphic loops is undecidable.* The class J_p of commutative automorphic loops indicated in this theorem is a finitely axiomatizable subclass of the class of all finite commutative loops (respectively Jordan loops or automorphic commutative loops), the class of all finite 2-nilpotent commutative loops (respectively Jordan loops or automorphic commutative loops), the class of all commutative groupoids etc. Therefore, the elementary theories of all the mentioned classes are undecidable.

The basic concepts of loupes can be found in the monograph Bruck R.H. [6] or V.D. Belousova [7]. The results of [1] are also used in the proof.

2. DIRECT MAPPING φ

Let $\boldsymbol{K}$ the class of all commutative rings and K be an arbitrary ring from $\boldsymbol{K}$. In the set $\varphi(K)$ of quaternions (a,b,c,d) of elements of K, which we denote by L, we introduce two operations - multiplication and division, denoted by symbols $\cdot$ and $/$, formulas:

$$(2.1)\qquad (a,b,c,d)\cdot(x,y,z,t) = \big(a+x,\ b+y,\ c+z+ax\cdot(b+y),\ z+t+by\cdot(a+x)\big);$$

$$(2.2)\qquad (a,b,c,d)/(x,y,z,t) = \big(a-x,\ b-y,\ c-z+b\cdot x(x-a),\ d-t+a\cdot y(y-b)\big).$$

It is easy to verify that the set L together with certain operations of multiplication and division by the definite formulas (2.1) and (2.2) is a commutative loop with unit element quaternion of the form $(0,0,0,0)$, which we denote by e. It follows from (2.1) that quaternion's of the form $(0,0,c,d)$ and only they are elements of the center $Z(L)$ of the loop, i.e.

$$Z(L) = \{(0,0,c,d) \mid c,d \in K\}.$$

From formulas (2.1) and (2.2) it is easy to obtain that for elements $(a,b,c,d),(m,n,p,q)$ and (x,y,z,t) from L, the associator of these elements

$$(2.3)\qquad \big((a,b,c,d),(m,n,p,q),(x,y,z,t)\big) = \big(0,\ 0,\ mx\cdot b - ma\cdot y,\ ny\cdot a - nb\cdot x\big)$$

is contained in the center $Z(L)$. Therefore, the commutative loop L is nilpotent of class 2. According to formulas (2.1) and (2.3)

$$\big((a,b,c,d)^2,\ (m,n,p,q),\ (a,b,c,d)\big)$$

$$= \big((2a,2b,2c+2a^2b,2d+2b^2a),\ (m,n,p,q),\ (a,b,c,d)\big) = (0,0,0,0),$$

i.e. in L, the identity $x^2\cdot yx = x^2y\cdot x$ is true, but this means that the commutative loop L of Jordan. Hence, $L=\varphi(K)$ - the Jordan metabelian loop.

Suppose K_1, K_2 that there are two classes of models. We shall say from Maltsev, which K_1 is syntactically contained in K_2 if there is an algorithm that allows for each formula Φ_1 of the signature K_1 to construct some formula Φ_2 of a signature K_2 so that the truth of the formula Φ_2 on all K_2 - models implies the truth of the formula Φ_2 on all K_2 - models. Classes K_1, K_2 are syntactically equivalent if each of them is syntactically contained in the other.

In the case of syntactic equivalence, we have that each closed formula $\Phi_i \in T(K_i)$ if and only if the corresponding closed formula $\Phi_1 \in T(K_1)$. It is immediately seen that if the elementary theory K_1 is undecidability and K_1, K_2 syntactically equivalent, then the elementary theory K_2 is also decidability.

We now consider some closed formula of the narrow predicate calculus

$$\Phi = (Q_1x_1)\dots(Q_nx_n)\,\Phi_0(x_1,\dots,x_n)\quad (Q_i = \exists,\forall),$$

the open part Φ_0 of which contains only two extra-logical symbols - the signs of multiplication and division. The requirement that on $\varphi(K)$ the formula Φ be true is equivalent to some requirement stated on the ring K. This last requirement can be rewritten in the form of a formula $\varphi(\Phi)$ of the narrow predicate calculus. For this it is sufficient to replace in Φ four quantifiers $(Q_ix_i')(Q_ix_i'')(Q_ix_i''')(Q_ix_i'^v)$ in each quantifier (Q_ix_i), replace each expression $x_ix_j = x_k$ by the formula

$$x_i' + x_j' = x_k' \ \&\ x_i'' + x_j'' = x_k'' \ \&\ x_i''' + x_j''' + x_i'x_j'\cdot(x_i''+x_j'') = x_k'''$$

$$\&\ x_i'^v + x_j'^v + x_i''x_j''\cdot(x_i'+x_j') = x_k'^v,$$

and each expression $x_i/x_j = x_k$ is replaced by the formula

$$x_i' - x_j' = x_k' \ \&\ x_i'' - x_j'' = x_k'' \ \&\ x_i''' - x_j''' + x_i''\cdot x_j\,(x_j' - x_i') = x_k'''$$

$$\&\ x_i'^v - x_j'^v + x_i'\cdot x_j''\,(x_j'' - x_i'') = x_k'^v.$$

Therefore, the mapping $\Phi \to \varphi(\Phi)$ is a syntactic embedding of a class $\varphi(\boldsymbol{K})$ in the class $\boldsymbol{K}$. In particular, if any given class of rings K has a decidable elementary theory, then the undecidability elementary theory will also have a corresponding class of loops $\varphi(K)$.

3. THE CLASS OF JORDAN LOOPS WITH SELECTED ELEMENTS

Further we assume that the ring K under consideration has unity *1*. Then the metabelian Jordan loop $L = \varphi(K)$ is naturally identified with elements

$$e_1 = (1, 0, 0, 0), \quad e_2 = (0, 1, 0, 0).$$

Therefore, a Jordan loupe L will be considered as a signature $< \cdot, /, e_1, e_2 >$ algebra consisting of functional binary $\cdot, /$ and zero operations e_1, e_2, which we sometimes call a Jordan loupe with selected elements e_1, e_2 or simply an enriched Jordan loupe. We denote the class of all such enriched Jordan loops $L = \varphi(K)$ by $\boldsymbol{L}$, and denote by $\boldsymbol{K_1}$ the class of all commutative rings with unity *1*. So, if the ring K is from a class $\boldsymbol{K_1}$, then $L = \varphi(K)$ it is a Jordan loupe with the selected elements e_1, e_2 from the class $\boldsymbol{L}$. Under the sub-loop, center, or internal substitution of the Jordan enriched loupe L, we mean the sub-loop, center, or internal substitution of the Jordan loop L. It is easy to verify that any enriched Jordan loop $L \in \boldsymbol{L}$ has the following properties:

P1) subsets

$$L_1 = \{x \in L \mid xe_1 \cdot e_1 = x \cdot e_1e_1 \, (\Leftrightarrow \; e_1e_2 \cdot x = e_1 \cdot e_2x)\},$$

$$L_2 = \{x \in L \mid xe_2 \cdot e_2 = x \cdot e_2e_2 (\Leftrightarrow \; e_2e_1 \cdot x = e_2 \cdot e_1x)\}$$

with respect to the multiplication of a loop L are abelian groups, are normal in L, for which the equations $L_1 \cap L_2 = Z(L)$, $L_1L_2 = L$ are true and

$$(x, y, z) = (x, z, y), \quad (y, x, z) = e \tag{3.4}$$

for any $x \in L$ and any $y, z \in L_i \quad (i = 1, 2)$;

P2) subsets

$$Z_1 = \{x \in Z(L) \mid (\exists y \in L_1)\, (e_2, e_1, y) = x\}, \quad Z_2 = \{x \in \; Z \,|\, (\exists y \in L_2)(e_1, e_2, y) = x\}$$

are the central sub-loop L for which

$$Z_1 = 1x \in Z(L) \;\mid\; (\exists y \in L_2)(y, e_1, e_1) = x\}, \quad Z_2 = \{x \in \; Z \;\mid\; (\exists y \in L_1)(y, e_2, e_2) = x\},$$

$$Z(L) = Z_1Z_2, \quad Z_1 \cap Z_2 = \{e\};$$

P3) into L a true quasi-identity:

$$\begin{gathered}(x_1, e_1, e_1) = e \;\&\; (y_1, e_1, e_1) = e \;\&\; (e_2, e_1, x_2) = e \;\&\; (e_2, e_1, y_2) = e \\ \&\; (e_2, e_1, x_1) = z_1 \;\&\; (e_2, e_1, y_1) = z_1 \;\&\; (x_2, e_1, e_1) = z_2 \\ \&\; (y_2, e_1, e_1) = z_2 \;\Rightarrow\; (x_2, e_1, x_1) = \; (y_2, e_1, y_1);\end{gathered} \tag{3.5}$$

P4) for each pair of elements $z_1, z_2 \in Z_1$ there L exist elements $x_1 \in L_1$, $x_2 \in L_2$ satisfying the relations

$$(e_2, e_1, x_1) = z_2, \quad (x_2, e_1, e_1) = z_2; \tag{3.6}$$

P5) there are homeomorphisms $\lambda : Z_1 \to L_2$, $\mu : Z_1 \to L_1$ and an isomorphism $\nu : Z_1 \to Z_2$ such that $\lambda(c) = e_2$, $\mu(c) = e_1$ where $c = (e_2, e_1, e_1)$ the following equalities hold:

$$(3.7) \qquad \big(\lambda(z), e_1, e_1\big) = z, \quad \big(e_2, e_1, \mu(z)\big) = z;$$

$$(3.8) \qquad \nu\big(\lambda(z_1z_2),\, \mu(z_3), \mu(z_4)\big) = \big(\mu(z_1z_2),\, \lambda(z_3),\, \lambda(z_4)\big);$$

$$(3.9) \qquad \big(\lambda(z_1),\, e_1,\, \mu([\lambda(z_2),\, e_1,\, \mu(z_3)])\big) = \big(\lambda(z_1),\, \mu(z_2),\, \mu(z_3)\big).$$

4. REVERSE MAPPING ψ

We denote by $\boldsymbol{L_4}$ the subclass of the class $\boldsymbol{L}$ of all Jordan loops from the properties $P1) - P4)$ considered in the signature $< \cdot, /, e_1, e_2 >$. Since properties $P1) - P4)$ can easily be written in the formulas of a narrow predicate calculus, $\boldsymbol{L_4}$ is a finitely axiomatizable class. Let L there be some Jordan loop from class $\boldsymbol{L_4}$. Let $Z = Z_1Z_2$ the center of the loop L. We define new binary operations in the central subloop $\oplus$ and $\times$ suppose for $z_1, z_2 \in Z_1$

$$(4.10) \qquad z_1 \oplus z_2 = z_1 \cdot z_2,$$

$$(4.11) \qquad z_1 \times z_2 = (x_2, e_1, x_1),$$

where $x_1,\ x_2 \in L$ the following conditions:

$$(4.12) \qquad (e_2, e_1, x_1) = z_1, \quad x_1e_1 \cdot e_1 = x_1 \cdot e_1e_1,$$

$$(4.13) \qquad (x_2, e_1, e_1) = z_2, \quad x_2e_2\, e_2 = x_2 \cdot e_2e_2.$$

Let us prove that

set Z_1 together with certain operations of addition $\oplus$ and multiplication $\times$ by means of formulas (4.12) *and* (4.13) *is a commutative ring with unit $e = (e_2, e_1, e_1)$.*

Proof. First we note that the addition of Z_1 coincides with the loop multiplication, and therefore Z_1 with respect to addition it is an abelian group. Further, according to condition $P4)$, there exist elements $x_1 \in L_1, x_2 \in L_2$ satisfying conditions (4.12) and (4.13) exist. Let y_1, y_2 some other solution for (4.12) and (4.13). According to $P1)$, from the equality $y_1e_1 \cdot e_1 = y_1 \cdot e_1e_1$, $y_2e_2 \cdot e_2 = y_2 \cdot e_2e_2$ follows $y_1 \in L_1, y_2 \in L_2$ and $y_1/x_1 \in L_1$, $y_2/x_2 \in L_2$. Because the

$$(e_2, e_1, y_1) = (e_2, e_1, x_1), \quad (y_2, e_1, e_1) = (x_2, e_1, e_1),$$

then

$$(e_2, e_1, y_1) \cdot (e_2, e_1, x_1)^{-1} = e, \quad (y_2, e_1, e_1) \cdot (x_2, e_1, e_1)^{-1} = e,$$

from where follows

$$(e_2, e_1, y_1/x_1) = e, \quad (y_2/x_2, e_1, e_1) = e.$$

But the first of the last two equalities, according to the construction L_2, is equivalent to $(y_1/x_1, e_2, e_2) = e$. So, we got

$$(y_1/x_1, e_2, e_2) = e, \quad (y_2/x_2, e_1, e_1) = e,$$

but this means, according to $P1)$ that $y_1/x_1 \in L_2$, $y_2/x_2 \in L_1$. But since $y_1/x_1 \in L_1$, $y_2/x_2 \in L_2$, then $y_1/x_1, y_2/x_2 \in L_1 \cap L_2 = Z(L)$. Hence $y_1 = x_1z_1$, $y_2 = x_2z_2$ for some $z_1, z_2 \in Z$ and, consequently,

$$(e_2, y_1, e_1) = (e_2, x_1z, e_1) = (e_2, x_1, e_1) = z_1,$$

$$(y_2, e_1, e_1) = (x_2 z_2, e_1, e_1) = (x_2, e_1, e_1) = z_2.$$

Thus, the multiplication operation $\times$ is always feasible and unambiguous.

Let us prove that $\times$ a commutative operation. Let

$$z_1 \times z_2 = (x_2, e_1, x_1) \text{ and } z_2 \times z_1 = (y_2, e_1, y_1),$$

where the elements $x_1, y_1 \in L_1$ and $x_2, y_2 \in L_2$ satisfy the requirements (4.12) and (4.13). But then, according to (3.5), we obtain the equality $(x_1, e_1, x_2) = (y_1, e_1, y_2)$, i.e. $z_1 \times z_2 = z_2 \times z_1$.

According to the definition of the operation of multiplication by direct calculation, it is verified that $c = (e_2, e_1, e_1)$ is a neutral element with respect to the operation of multiplication: $c \times z = z \times c = z$ for any $z \in Z_1$.

To prove the distributive relations

$$(u \otimes v) \times w = u \times w \otimes v \times w, \quad w \times (u \otimes v) = w \times u \otimes w \times v, \tag{4.14}$$

we put

$$u = (e_2, e_1, x), \quad v = (e_2, e_1, y), \quad w = (z, e_1, e_1), \quad x, y \in L_1,\ z \in L_2.$$

We have

$$u \oplus v = (e_2, e_1, x) \cdot (e_2, e_1, y) = (e_2, e_1, xy),$$

but then,

$$(u \oplus v) \times w = (z, e_1, e_1) \times (e_2, e_1, xy)$$

$$= (z, e_1, xy) = (z, e_1, x) \cdot (z, e_1, y) = u \times w \oplus v \times w.$$

The second equality in (4.14) is proved similarly.

A mapping associating with each Jordan loop $L \in \boldsymbol{L_4}$ a commutative ring Z_1 defined in this way will be denoted by ψ.

If $\Psi = (Q_1 x_1) \ldots (Q_n x_n) \Psi_0(x_1, \ldots, x_n)$ - any closed formula of the elementary theory of a commutative ring with unity, then $\psi(\Psi)$ we agree to denote by the symbol the elementary theory of commutative loops, which are obtained from Ψ the replacement of quantors $Q_1 x_1$ by specialized quantifiers $Q_1^{\rho} x_1$, symbol *1* in Ψ - by the expression (e_2, e_1, e), each expression $x_i + x_j = x_k$ - by the formula $x_i \oplus x_j = x_k$ and each expression $x_i \times x_j = x_k$ - by the formula

$$(\exists u)(\exists v)((v, e_1, u) = x_k) \;\&\; ue_1 \cdot e_1 = u \cdot e_1 e_1 \;\&\; ve_2 \cdot e_2 = v \cdot e_2 e_2 \;\&\; (e_2, e_1, x_i) = u$$

$$\&\; (x_j, e_1, e_1) = v).$$

The predicate $\rho(x)$ in the quantifier $Q_i^{\rho} x$ is defined by the formula

$$\rho(x) = (\forall u)(\forall v)(\exists w)\big(x \cdot uv = xu \cdot v \;\&\; we_1 \cdot e = w \cdot e_1 e_1 \;\&\; (e_2, e_2, w) = x\big).$$

It is clear that Ψ is true on the ring $\psi(L)$ if and only if $\psi(\Psi)$ is true on the loop L, and therefore the elementary theory of the ring $\psi(L)$ is syntactically contained in the elementary theory of the Jordan rich enrichment loop L. From this it follows, in particular, that if the elementary theory of the ring $\psi(L)$ is undecidability, then the elementary theory of the loop L is also undecidability.

5. RECIPROCITY OF MAPPINGS $\boldsymbol{\varphi}$ AND $\boldsymbol{\psi}$

From the preceding results it follows that for any commutative ring K with unity

$$\psi(\varphi(K)) \cong K.$$

Let us show that for the corresponding syntactic transformations there is an isomorphism

$$\varphi(\psi(L)) \cong L.$$

Indeed, let L loop of the class $\boldsymbol{L_5}$ and Z_1 its central subloop. Using formulas (2.1) and (2.2), in L we introduce operations $\oplus$, $\times$ and for the obtained commutative ring $Z_1 = \psi(L)$ we construct a loop $\varphi(Z_1)$, formed by quaternions of elements of Z_1. We need to find an isomorphism of $\varphi(Z_1)$ on L.

By hypothesis, there exists a homomorphism $\lambda : Z_1 \to L_2$, $\mu : Z_1 \to L_1$ and an isomorphism $\nu : Z_1 \to Z_2$ having properties $P5$). We affix each element $k = (k_1, k_2, k_3, k_4)$ of the loop $\varphi(Z_1)$, to the corresponding the element $\tau(k) = \mu(k_1)\lambda(k_2)k_3\nu(k_4)$ of the loop L. We show that the mapping τ is a homomorphism $\varphi(Z_1)$ on L. First we note that if $z_1, z_2 \in Z_1$ then according to (3.7) $(\lambda(z_2), e_1, e_1) = z_2$ and $(e_2, e_1, \mu(z_1)) = z_1$, and then we have

$$z_1 \times z_2 = \big(\lambda(z_2), e_1, \mu(z_1)\big). \tag{5.15}$$

Now let $k = (k_1, k_2, k_3, k_4)$ and $l = (l, l_2, l_3, l_4)$ be an arbitrary elements of L. According to (5.15) and (3.9),

$$(k_1 \times l_1) \times (k_2 l_2) = \big(\lambda(k_2 l_2), e_1, \mu(k_1 \times l_1)\big)$$

$$= \big(\lambda(k_2 l_2), e_1, \mu\big(\lambda(l_1), e_1, \mu(k_1)\big)\big) = \big(\lambda(k_2 l_2), \mu(l_1), \mu(k_1)\big),$$

i.e.

$$(k_1 \times l_1) \times (k_2 l_2) = \big(\lambda(k_2 l_2), \mu(l_1), \mu(k_1)\big). \tag{5.16}$$

Similarly it is proved and the equality

$$(k_2 \times l_2) \times (k_1 l_1) = \big(\lambda(k_1 l_1), \mu(l_2), \mu(k_2)\big). \tag{5.17}$$

Then, according to (2.1), (5.15)–(5.17) we have

$$kl = \big(k_1 \oplus l_1,\ k_2 \oplus l_2,\ k_3 \oplus l_3 \oplus (k_1 \times l_1) \times (k_2 \oplus l_2),\ k_4 \oplus l_4 \oplus (k_2 \times l_2) \times (k_1 \oplus l_1)\big)$$

$$= \big(k_1 l_1,\ k_2 l_2,\ k_3 l_3 \cdot [(k_1 \times l_1) \times (k_2 l_2)],\ k_4 l_4 \cdot [(k_2 \times l_2) \times (k_1 l_1)]\big)$$

$$= \big(k_1 l_1,\ k_2 l_2,\ k_3 l_3 \cdot \big(\lambda(k_2 l_2),\ \mu(l_1),\ \mu(k_1)\big),\ k_4 l_4 \cdot \big(\lambda(k_1 l_1), \mu(l_2),\ \mu(k_2)\big)\big),$$

i.e.

$$kl = \big(k_1 l_1,\ k_2 l_2,\ k_3 l_3 \cdot \big(\lambda(k_2 l_2),\ \mu(l_1),\ \mu(k_1)\big),\ k_4 l_4 \cdot \big(\lambda(k_1 l_1),\ \mu(l_2),\ \mu(k_2)\big)\big).$$

Hence, according to the definition and relations (3.4) and (3.8), we have

$$\tau(kl) = \mu(k_1 l_1)\lambda(k_2 l_2) \cdot (k_3 l_3)\big(\lambda(k_2 l_2), \mu(l_1), \mu(k_1)\big)\nu(k_4 l_4 \cdot \big(\lambda(k_1 l_1),\ \mu(l_2),\ \mu(k_2)\big)\big)$$

$$\mu(k_1 l_1)\lambda(k_2 l_2) \cdot k_3 l_3\big(\lambda(k_2 l_2),\ \mu(k_1),\ \mu(l_1)\big)\nu(k_4 l_4)\nu\big(\lambda(k_1 l_1),\ \mu(k_2), \mu(l_2)\big)$$

$$= \mu(k_1 l_1)\lambda(k_2 l_2) \cdot k_3 l_3\big(\lambda(k_2 l_2),\ \mu(k_1),\ \mu(l_1)\big)\nu(k_4 l_4)\big(\mu(k_1 l_1),\ \lambda(k_2),\ \lambda(l_2)\big)$$

i.e.

$$\tau(kl) = \mu(k_1 l_1)\lambda(k_2 l_2) \cdot k_3 l_3\big(\lambda(k_2 l_2),\ \mu(k_1),\ \mu(l_1)\big)\nu(k_4 l_4)\big(\mu(k_1 l_1),\ \lambda(k_2), \lambda(l_2)\big).$$

On the other hand, according to (3.4),

$$\tau(k)\tau(l) = [\mu(k_1)\lambda(k_2)\cdot k_3\nu(k_4)]\cdot[\mu(l_1)\lambda(l_2)\cdot l_3\nu(l_4)]$$

$$= [\mu(k_1)\lambda(k_2)\cdot \mu(l_1)\lambda(l_2)]\cdot k_3l_3\nu(k_4l_4)$$

$$= [\mu(k_1)\cdot\big(\lambda(k_2)\cdot\mu(l_1)\lambda(l_2)\big)]\cdot k_3l_3\nu(k_4l_4)\big(\mu(k_1),\lambda(k_2),\mu(l_1)\lambda(l_2)\big)$$

$$= [\mu(k_1)\cdot\big(\mu(l_1)\lambda(l_2)\cdot\lambda(k_2)\big)]\cdot k_3l_3\nu(k_4l_4)\big(\mu(k_1),\lambda(k_2),\,\lambda(l_2)\big)$$

$$= [\mu(k_1)\cdot\mu(l_1)\lambda(k_2l_2)]\cdot k_3l_3\nu(k_4l_4)\big(\mu(l_1),\,\lambda(l_2),\,\lambda(k_2)\big)\big(\mu(k_1),\,\lambda(k_2),\,\lambda(l_2)\big)$$

$$= [\mu(k_1)\cdot\mu(l_1)\lambda(k_2l_2)]\cdot k_3l_3\nu(k_4l_4)\big(\mu(l_1),\,\lambda(k_2),\,\lambda(l_2)\big)\big(\mu(k_1),\,\lambda(k_2),\,\lambda(l_2)\big)$$

$$= [\lambda(k_2l_2)\mu(l_1)\cdot\mu(k_1)]\cdot k_3l_3\nu(k_4l_4)\big(\mu(l_1k_1),\lambda(k_2),\lambda(l_2)\big)$$

$$= \lambda(l_2k_2)\mu(l_1k_1)\cdot k_3l_3\big(\lambda(l_2k_2),\,\mu(l_1),\,\mu(k_1)\big)\nu(k_4l_4)\big(\mu(k_1l_1),\,\lambda(k_2),\,\lambda(l_2)\big)$$

$$= \mu(k_1l_1)\lambda(k_2l_2)\cdot k_3l_3\big(\lambda(l_2k_2),\,\mu(k_1),\,\mu(l_1)\big)\nu(k_4l_4)\big(\mu(k_1l_1),\,\lambda(k_2),\lambda(l_2)\big)$$

i.e. $\tau(kl) = \tau(k)\tau(l)$. Since in the loop $\varphi(Z_1)$ the equality $(k/l)t = k$ implies equality $\tau(k/l)\tau(t) = \tau(k)$ in the loop L, and so is equality $\tau(k/l) = \tau(k)/\tau(l)$. In the loop $\varphi(Z_1)$ the selected elements are

$$e_1' = (c,e,e,e), \quad e_2' = (e,c,e,e),$$

where e is the unit of the loop L. According to $P5$), $\mu(c) = e_1$ and $\lambda(c) = e_2$ then

$$\tau(e_1') = \mu(c)\lambda(e)\,e\nu(e) = e_1, \quad \tau(e_2') = \mu(e)\lambda(c)\,e\nu(e) = e_2.$$

Thus, τ it is a homomorphism of the enriched loop $\varphi(Z_1)$ in the enriched loop L.

Let us find the kernel of the homomorphism τ. Let $\tau(k) = e$ where $k = (k_1,k_2,k_3,k_4)\in\varphi(Z_1)$. Then $\mu(k_1)\lambda(k_2)k_3\nu(k_4) = e$ and

$$\mu(k_1) = e/\lambda(k_2)k_3\nu(k_4) = \big(\lambda(k_2)k_3\nu(k_4)\big)^{-1} = \lambda(k_2^{-1})k_3^{-1}\nu(k_4^{-1}),$$

$$\lambda(k_2) = e/\mu(k_1)k_3\nu(k_4) = \big(\mu(k_1)k_3\nu(k_4)\big)^{-1} = \mu(k_1^{-1})k_3^{-1}\nu(k_4^{-1}),$$

means $\mu(k_1)\in L_1\cap L_2$, $\lambda(k_2)\in L_1\cap L_2$. But according to $P1$) $L_1\cap L_2 = Z(L)$ therefore, $\mu(k_1),\lambda(k_2)\in Z(L)$. Hence, by virtue of the equalities (3.7)

$$e = \big(e_2,e_1,\mu(k_1)\big) = k_1, \quad e = \big(\lambda(k_2),e_1,e_1\big) = k_2.$$

Then we have $k_3 = \nu(k_4) = e$ and $k_3 = \nu(k_4^{-1})$. Whence follows, belongs to the intersection of the central subloop. But, according to, $P2$), therefore, hence and. Hence k is a unit in a loop and the homomorphism is an isomorphism

$$e = \big(e_2,e_1,\mu(k_1)\big) = k_1, \quad e = \big(\lambda(k_2),e_1,e_1\big) = k_2.$$

Then we have $k_3 = \nu(k_4) = e$ and $k_3 = \nu(k_4^{-1})$. Then we have $k_3 = Z_1\cap Z_2$. But according to $P2$) $Z_1\cap Z_2 = \{e\}$, so $k_3 = e$, consequently $k_4 = e$. Hence k is a unit in a loop $\varphi(Z_1)$ and the homomorphism τ is an isomorphism.

It remains to show that τ it maps the loop $\varphi(Z_1)$ to the entire loop L. Indeed, let $f\in L$. we put

$$f_1 = (e_2,e_1,f), \quad f_2 = (f,e_1,e_1), \quad \mu(f_1) = k_1, \quad \lambda(f_2) = k_2$$

and show that $f_1,f_2\in Z_1(L)$ and $f/(k_1k_2)\in Z(L)$. According to $P1$) $L = L_1L_2$, $f_1,f_2\in Z_1(L)$ and $f/(k_1k_2)\in Z(L)$. According to $P1$), $L = L_1L_2$, so for $f = g_1g_2$, $g_1\in L_1$, $g_2\in L_2$.

Then, according to (3.4),

$$f_1 = (e_2, e_1, f) = (e_2, e_1, g_1g_2) = (e_2, e_1, g_1)\,(e_2, e_1, g_2) = (e_2, e_1, g_1),$$

$$f_2 = (f, e_1, e_1) = (g_1g_2, e_1, e_1) = (g_1, e_1, e_1)\,(g_2, e_1, e_1) = (g_2, e_1, e_1),$$

and hence, in view of $P2$), $f_1, f_2 \in Z_1(L)$.

By the first formula in (3.7), $\big(\lambda(g_2), e_1, e_1\big) = g_2$ and comparing with $(f, e_1, e_1) = f_2$, we get $\big(f/\lambda(g_2), e_1, e_1\big) = e$. From here $f/k_2 \in L_1$ and $(f/k_2)/k_1 \in L_1$. Then

$$e = \big((f/k_2)/k_1, e_1, e_1\big) = (f \cdot k_2^{-1}k_1^{-1}, e_1, e_1) = (f \cdot (k_1k_2)^{-1}, e_1, e_1) = (f/(k_1k_2), e_1, e_1),$$

i.e. $\big(f/(k_1k_2), e_1, e_1\big) = e$ and therefore $f/(k_1k_2) \in L_1$. Similarly, we prove that $f/(k_1k_2) \in L_2$. But then $f/(k_1k_2) \in Z(L)$. According to $P2$) and the definition of isomorphism ν from $P5$), we have $Z(L) = Z_1Z_2 = Z_1\nu(Z_1)$. Then $f/(k_1k_2) \in Z_1\nu(Z_1)$ and, therefore, exists in Z_1 such elements k_3, k_4, what $f = k_1k_2 \cdot k_3\nu(k_4)$. Denoting $f_3 = k_3,\ f_4 = k_4$, it is easy to verify that the quaternion $(f_1, f_2, f_3, f_4) \in \varphi(Z_1)$ is mapped by τ to the f, and hence τ maps $\varphi(Z_1)$ onto L, as and required. We have proved

Theorem 5.1. *The mapping φ is a one-to-one mapping, up to an isomorphism, of the class $\boldsymbol{K_1}$ of all commutative rings with unity to the class $\boldsymbol{L_5}$ of all enriched Jordan loops satisfying the requirements $P1) - P5)$. At the same time, if on some ring $K \in \boldsymbol{K_1}$, the closed formula Ψ of the narrow predicate calculus is true, then on the loop $\psi(K)$ the true formula $\psi(\Psi)$, and conversely, if on the loop $L \in \boldsymbol{L_5}$ any closed formula Φ of the narrow predicate calculus is true, then on the corresponding commutative ring $\psi(L)$ the formula $\varphi(\Phi)$ is true.*

From conditions $P1) - P5)$ defining the class, condition $P5$) is more complicated than the others. In the following sections we will indicate several more narrow classes of loops that leave Theorem 5.1 in force and admit a simple characteristic.

6. SOME SPECIAL CASES

Let L be an arbitrary non-trivial loop. If there exists a natural number n such that for each egalitat

$$x^n = e,$$

then the smallest of these numbers n is called the exponent of the loop L. If there are no such numbers n, then we call the loop L an exponent of zero or a loop without torsion. For any prime $p = 2, 3, ...$ we denote by $\boldsymbol{J_p}$ the class of all $\boldsymbol{L_4}$-lows with exponent p. It is easy to see that all conditions $P1) - P4)$ can be written in the form of closed formulas, and so the class $\boldsymbol{J_p}$ is certainly axiomatizable.

Theorem 6.1. *For any prime number p, the mapping φ is a one-to-one correspondence (up to isomorphism) between commutative rings of characteristic p with unity and $\boldsymbol{J_p}$-loops.*

Proof. Let K be a commutative ring of characteristic p with unity *1*. In view of Theorem 5.1, the ring K corresponds to the Jordan $\boldsymbol{L_5}$-loop $L = \varphi(R)$. Then for any element, according to (2.1), we have

$$r^p = (pr_1, pr_2, pr_3 + pr_1^p r_2, pr_4 + pr_2^p r_1),$$

hence it follows that there L is a loop with exponent p. Conversely, it is obvious. It only remains to prove that any $\boldsymbol{L_4}$-loop with exponent p satisfies condition $P5$).

The central subloop Z_1 of a loop L can be regarded as a linear space over a simple field P of characteristic p. We choose in Z_1 a basis $B = \{b_i \,|\, i \in I\}$ over P. Obviously, we can assume that $b_0 = c = (e_2, e_1, e_1) \in B$. Now we construct homomorphisms $\lambda : Z_1 \to L_2$ and $\mu : Z_1 \to L_1$. First we note that the correspondences:

(6.18) $$x \to (x, e_1, e_1)\ (x \in L_2);$$

and

$$x \to (e_2, x, e_1)\ (x \in L_1) \tag{6.19}$$

is a homomorphism L_2 on Z_1 and L_1 on Z_1. For each $b_i \in B$ we denote it by some preimage with respect to the homomorphism (6.18) and (6.19) across $c_i \in L_2$ and $d_i \in L_1$. Obviously, we can assume that the inverse images of the element b_0 in L_2 and in L_1 with respect to the indicated homomorphisms are $c_0 = e_2$ and $d_0 = e_1$. Now the homomorphisms λ and μ are constructed as follows: for each element $z = \sum n_i b_i\ (n_i \in P)$ of Z_1 assuming by definition $\lambda(z) = \prod c_i^{n_i}$ and $\mu(z) = \prod d_i^{n_i}$. Because the

$$(c_0, e_1, e_1) = c, \quad (e_2, d_0, e_1) = c,$$

then $\lambda(c) = \lambda(b_0) = c_0 = e_2$, $\mu(c) = \mu(b_0) = d_0 = e_1$ and

$$\big(\lambda(z), e_1, e_1\big) = \big(\prod c_i^{n_i}, e_1, e_1\big) = \prod (c_i, e_1, e_1)^{n_i} = \prod b_i^{n_i} = z,$$

$$\big(e_2, \mu(z), e_1\big) = \big(e_2, \prod d_i^{n_i}, e_1\big) = \prod (e_2, d_i, e_1)^{n_i} = \prod b_i^{n_i} = z.$$

The theorem is proved.

Similarly, as in groups, we agree to call the loop L complete, if equation $x^n = a$ for each a in L and every integer n has at least one solution in L. Loops without elements of finite order, also, we agree to call loops *without torsion*. A ring is said to be complete (resp., without torsion) if its additive group is complete (respectively, without torsion).

Using analogous arguments, as in the proof of Theorem 6.1, in result is we get

THEOREM 6.2. *The mapping φ is a one-to-one correspondence, up to an isomorphism, between commutative* (*respectively without torsion*) *torsion rings with unity and complete* (*respectively, without torsion*) *Jordan $\boldsymbol{L_4}$-loops.*

7. UNDECIDABILITY OF THEORIES OF CERTAIN CLASSES OF FINITE COMMUTATIVE LOOPS

Let L be any enriched Jordan loop satisfying conditions $P1) - P4)$, and $K \in \psi(L)$ let be the corresponding ring with unity. Then the formulas (4.10), (4.11) determine the exact interpretation K in L with the distinguishing predicate with a distinguishing predicate

$$\rho(x) = (\forall u)\,(\forall v)\,(\exists w)\,\big(x \cdot uv = xu \cdot v \ \ \&\ \ we_1 \cdot e = w \cdot e_1 e_1 \ \ \&\ \ (e_2, e_2, w) = x\big).$$

The same formulas give an interpretation of any class $\boldsymbol{K}$ of commutative rings with unity in the class of corresponding loops $\boldsymbol{L} = \psi(\boldsymbol{K})$. To get rid of the allocated elements it suffices to use the following well-known assertion: if the class of models $\boldsymbol{M}$ with allocated elements $a_1, \ldots, a_m$ is characterized by the axiom $\Phi(a_1, \ldots, a_m)$, then a class $\boldsymbol{M'}$ without the above mentioned elements, characterized by the axiom $(\exists a_1) \ldots (\exists a_m) \Phi(a_1, \ldots, a_m)$, syntactically is equivalent to $\boldsymbol{M}$. This is true, since any formula $\Psi(a_1, \ldots, a_m)$ belongs to the elementary theory $T(\boldsymbol{M})$ if and only if formula $(\exists a_1) \ldots (\exists a_m)\big(\Phi(a_1, \ldots, a_m) \to \Psi(a_1, \ldots, a_m)\big)$ belongs to the elementary theory $T(\boldsymbol{M'})$. Now taking for $\mathbf{K}$ an arbitrary axiomatizable undecidability class of commutative rings with unity and considering any class $\boldsymbol{M}$ of commutative loops with distinguished elements containing $\boldsymbol{L} = \varphi(\boldsymbol{K})$ we obtain that $\boldsymbol{M}$ is undecidable. Since for any prime $p = 2, 3, \ldots$ the elementary theory of commutative rings with unit prime characteristic p is undesirability (see [2, 5]), then (see Theorems 5.1 and 6.1) immediately implies the following more important proposition:

THEOREM 7.1. *The elementary theory of the class $\boldsymbol{J_p}$* (p *is a fixed prime number*) *of Jordan loops is undecidability.*

For any prime $p = 2, 3, \ldots$ every Jordanian loop class is a finitely axiomatizable subclass of the class of all finite commutative loops (respectively Jordan loops or automorphic commutative loops), the class of all finite n-nilpotent commutative loops (respectively Jordan loops or automorphic commutative loupe) for any $n \geq 2$, class of all finite commutative groupoids, and so on. Therefore, the elementary theories of all the above classes are undecidability.

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