Bisimulations in an arbitrary stratified institution*

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Within the general framework of logics as institutions, stratified institutions capture a wide range of formalisms whose satisfaction relations between models and sentences are parameterized by model states. In this paper, we lay the foundations of a theory of abstract bisimulations for logics formalized as stratified institutions, and we examine the relationship that they generate between the concepts of bisimilarity and elementary equivalence. We discuss institution-theoretic abstractions of bounded homomorphisms, frame extractions, zig-zag relations, and modal saturation, and we show how they could work together to bring forth several Hennessy-Milner results for stratified institutions.

Keywords: Stratified institutions, Bisimulation, Bounded homomorphism, Frame extraction, Modal saturation.

1 Introduction

Building on Burstall and Goguen's pioneering work on algebraic specification languages [BG79], the theory of *institutions* emerged in the '80s [GB83; GB92] as a means to cope with the "population explosion among the logical systems being used in computer science" at the time – a trend which has only continued to increase in the past decades. In a nutshell, institutions formalize the intuitive notion of logical system by treating the syntax (sentences) of logical languages, their semantics

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(models), and the satisfaction relations between models and sentences in a fully abstract manner. They do so through a careful use of category theory, which has enabled institutions to support two major and interrelated research directions: formal system development [ST11] and abstract model theory [Dia08]. Both strive to identify appropriate levels of abstraction where various logical phenomena occur. This process reduces the effort needed to derive similar results or properties for different logical systems, thus addressing redundancy, and at the same time it helps us develop a better understanding of the essence of those phenomena by discarding extraneous details of concrete logics; often, it also leads to extensions of the original notion of institution by introducing some additional structure or constraints that are relevant to the task under consideration. That is the case, for example, of general logics [Mes89], which provide support for syntactic entailment [see also Mos+07]; of substitution systems [ŢF17], which allow for an institution-independent form of logic programming to be developed; or of stratified institutions [AD07], which capture possible-worlds semantics and provide the context of the present study.

Refined in [Dia17] and further developed in [AB19; Găi20], the theory of stratified institutions caters for logics where the satisfaction of sentences by models is parameterized by a notion of *state* of a model. The most prominent example is that of conventional modal logic [BRV01], with its multitude of variants, for which (a) the models are Kripke structures having states given by possible worlds, and (b) the satisfaction of sentences is locally defined at a possible world. But this kind of dependence of satisfaction relations on model states is not limited to logics with Kripke semantics. Even first-order logic was initially formalized as an (ordinary) institution [GB83] in this manner – long before the introduction of stratified institutions. The satisfaction of a formula by a model is defined therein relative to some assignment of values in the model to the variables in the formula.

In this paper, we return to the close connection between formal system development (and analysis) and model theory by advancing a stratified institution-theoretic approach to bisimulations. The concept of bisimulation is famously one of the most important tools in both computational-process theory and the modal-logic literature. For the former, it is central to the study of transition systems, where it captures the observational equivalence of processes [HM80; HM85], whereas for the latter it leads to van Benthem's well-known semantic characterization of modal logic as the bisimulation-invariant fragment of first-order logic [Ben76; Ben01].

Following a series of previous categorical accounts of bisimulations [e.g., AM89; Gol06; and also JNW96], we develop bisimulation representations in arbitrary stratified institutions as spans of *bounded homomorphisms*. However, in contrast to the usual practice of modal logic – where the underlying Kripke frames of models allow for concrete bounded homomorphisms to be defined – in the abstract setting of stratified institutions we work with an axiomatic notion of boundedness that is subject to mild technical assumptions, and which can be regarded as a parameter of the concept of bisimulation. This means that different selections of bounded homomorphisms may lead to different kinds of bisimulations, with different properties. In this context, we examine the relationship between bisimilarity and elementary equivalence, and we identify additional conditions that a logical system and/or its selection of bounded homomorphisms should meet in order to obtain Hennessy-Milner theorems. All in all, we establish sufficient conditions under which bisimilarity and elementary equivalence coincide for a conceptual hierarchy that comprises three increasingly complex sets of hypotheses, each providing a richer theory than the previous one:

- At the most basic level, which applies to any stratified institution, we develop a characterization of the Hennessy-Milner property in terms of features of bounded homomorphisms such as being elementary;
- Next, we examine stratified institutions equipped with a frame extraction – i.e., institutions whose models have underlying Kripke frames. This enables us to introduce *frame-bounded homomorphisms*, which generalize concrete notions of boundedness from modal and hybrid logics, and allow for a Hennessy-Milner theorem that applies to *zig-zag* elementary-equivalence relations.
- At the richest level, we consider an institution-theoretic notion of modally saturated model, and we show that the zig-zag property of the elementary-equivalence relation holds for all such models, provided that the institution has certain logical connectives and maximally consistent state theories.

The paper is organized as follows. Section 2 reviews the concept of stratified institution, establishes the notations and terminology that we use in this work, and presents a few families of examples of stratified institutions that stem from the area of modal and hybrid logic, as well as automata theory and open first-order logic. In Section 3, we introduce abstract bounded homomorphisms and bisimulations, develop our first Hennessy-Milner result, and examine situations in which the Hennessy-Milner property can be obtained by translation along a signature morphism. Section 4 deals with frame-bounded homomorphisms in the context of institutions equipped with a binary frame extraction and discusses the role of zig-zag relations. Finally, in Section 5, we propose a generalization of the conventional notion of modal saturation and we show how it can be used to derive a more specific Hennessy-Milner theorem that is readily applicable to our benchmark examples.

2 Stratified institutions

To set the stage, in this section we recall from [Dia17] the main concept of stratified institution and introduce a few examples used throughout the paper. We generally assume readers to be familiar with basic notions of category theory, the most complex of which being lax natural transformations. The terminology and notations we use are primarily based on [Mac98], except for the composition of morphisms f

and g, which we prefer to write in diagrammatic order as $f \mathbin{;} g$, and for natural transformations, which we write using a double arrow. We say that a subcategory $\mathbb{C} \subseteq \mathbb{D}$ is broad when it contains all the objects in $\mathbb{D}$. We let $\mathbb{S}\mathrm{et}$ denote the category of sets and functions, and $\mathbb{C}\mathrm{at}$ denote the higher category of categories and functors. For every category $\mathbb{C}$, we also write $\mathrm{Set}_{\mathbb{C}}$, or just Set when the context $\mathbb{C}$ can be easily inferred, for the constant functor $\mathbb{C} \to \mathbb{C}\mathrm{at}$ that maps every object of $\mathbb{C}$ to $\mathbb{S}\mathrm{et}$, and every morphism in $\mathbb{C}$ to the identity functor $id_{\mathbb{S}\mathrm{et}}$. In complex expressions, to avoid notational clutter, we often denote the action of a function $f\colon X \to Y$ on an element $x \in X$ by fx, using plain juxtaposition, instead of $f(x)$.

Definition 2.1. A *stratified institution* $\mathcal{S}$ is a tuple $\langle \mathbb{S}\mathrm{ig}^{\mathcal{S}}, \mathrm{Sen}^{\mathcal{S}}, \mathrm{Mod}^{\mathcal{S}}, [\![\_]\!]^{\mathcal{S}}, \models^{\mathcal{S}} \rangle$ consisting of:

- a category $\mathbb{S}\mathrm{ig}^{\mathcal{S}}$ of *signatures* and *signature morphisms*,
- a *sentence functor* $\mathrm{Sen}^{\mathcal{S}}\colon \mathbb{S}\mathrm{ig}^{\mathcal{S}} \to \mathbb{S}\mathrm{et}$ defining, for every signature Σ, a set $\mathrm{Sen}^{\mathcal{S}}(\Sigma)$ of Σ-*sentences* and, for every signature morphism $\varphi\colon \Sigma \to \Sigma'$, a *sentence-translation map* $\mathrm{Sen}^{\mathcal{S}}(\varphi)\colon \mathrm{Sen}^{\mathcal{S}}(\Sigma) \to \mathrm{Sen}^{\mathcal{S}}(\Sigma')$,
- a *model functor* $\mathrm{Mod}^{\mathcal{S}}\colon (\mathbb{S}\mathrm{ig}^{\mathcal{S}})^{\mathrm{op}} \to \mathbb{C}\mathrm{at}$ defining, for every signature Σ, a category $\mathrm{Mod}^{\mathcal{S}}(\Sigma)$ of Σ-*models* and Σ-*homomorphisms* and, for every signature morphism $\varphi\colon \Sigma \to \Sigma'$, a *reduct functor* $\mathrm{Mod}^{\mathcal{S}}(\varphi)\colon \mathrm{Mod}^{\mathcal{S}}(\Sigma') \to \mathrm{Mod}^{\mathcal{S}}(\Sigma)$,
- a lax natural transformation $[\![\_]\!]^{\mathcal{S}}\colon \mathrm{Mod}^{\mathcal{S}} \Rightarrow \mathrm{Set}$ defining, for every signature Σ, a *state-space functor* $[\![\_]\!]^{\mathcal{S}}_{\Sigma}\colon \mathrm{Mod}^{\mathcal{S}}(\Sigma) \to \mathbb{S}\mathrm{et}$ and, for every morphism $\varphi\colon \Sigma \to \Sigma'$, a *state-reduction* natural transformation $[\![\_]\!]^{\mathcal{S}}_{\varphi}\colon [\![\_]\!]^{\mathcal{S}}_{\Sigma'} \Rightarrow \mathrm{Mod}^{\mathcal{S}}(\varphi) \mathbin{;} [\![\_]\!]^{\mathcal{S}}_{\Sigma}$, and
- a signature-indexed family of *satisfaction relations* $\models^{\mathcal{S}}_{\Sigma}$ between Σ-models M and Σ-sentences ρ, parameterized by *states* $w \in [\![M]\!]^{\mathcal{S}}_{\Sigma}$ and denoted $M \models^{\mathcal{S},w}_{\Sigma} \rho$

such that the following *satisfaction condition* holds for every signature morphism $\varphi\colon \Sigma \to \Sigma'$, every Σ'-model M', every state $w' \in [\![M']\!]^{\mathcal{S}}_{\Sigma'}$, and every Σ-sentence ρ:

$$M' \models^{\mathcal{S},w'}_{\Sigma'} \mathrm{Sen}^{\mathcal{S}}(\varphi)(\rho) \quad \text{if and only if} \quad \mathrm{Mod}^{\mathcal{S}}(\varphi)(M') \models^{\mathcal{S},[\![M']\!]^{\mathcal{S}}_{\varphi} w'}_{\Sigma} \rho.$$

As usual in institution theory, we may simplify the notations introduced in Definition 2.1 by omitting superscripts or subscripts that can be easily inferred from the context. We may, therefore, denote the state space of a Σ-model M simply by $[\![M]\!]$ instead of $[\![M]\!]^{\mathcal{S}}_{\Sigma}$; or, given a signature morphism $\varphi\colon \Sigma \to \Sigma'$ and a Σ'-model M', we may denote the action of the state-reduction function $[\![M']\!]^{\mathcal{S}}_{\varphi}$ on a state $w' \in [\![M']\!]$ by $[\![M']\!]_{\varphi} w'$ instead of $[\![M']\!]^{\mathcal{S}}_{\varphi} w'$. In addition, we typically denote the sentence-translation function $\mathrm{Sen}(\varphi)$ by $\varphi(\_)$ and the model-reduct functor $\mathrm{Mod}(\varphi)$ by $\_\!\upharpoonright_{\varphi}$. Using these notations, when $M = M'\!\upharpoonright_{\varphi}$, we may say that M is the φ-reduct of M', or that M' is a φ-expansion of M; the same applies to homomorphisms. For convenience, we also overload the notation of satisfaction relations to indicate the simultaneous satisfaction of multiple sentences at a given state: for every subset $\Gamma \subseteq \mathrm{Sen}(\Sigma)$, $M \vDash^w \Gamma$ when $M \vDash^w \rho$ for all sentences $\rho \in \Gamma$.

We say that a stratified institution is *strong* (following terminology from [JFS17]) when the lax natural transformation defining its stratification has this property – that is, for every signature morphism $\varphi\colon \Sigma \to \Sigma'$ and every Σ'-model M', the state-reduction function $[\![M']\!]_\varphi$ is a bijection. And we say that a stratified institution is *strict* when all state-reduction functions are identities.

Remark 2.2. Any ordinary institution – sans states – can be regarded as a stratified institution by defining $[\![M]\!]$ as a singleton for every model M. In the other direction, there are two common ways to derive ordinary institutions from stratified ones [Dia17]:

- by defining *pointed models* as pairs $\langle M, w \rangle$ consisting of a stratified model M and a state $w \in [\![M]\!]$, and by defining ordinary satisfaction relations between pointed models and sentences, where $\langle M, w \rangle \vDash \rho$ if and only if $M \vDash^w \rho$. This captures the conventional notion of *local satisfaction* from modal logic.

- by considering a *global* notion of satisfaction whereby $M \vDash \rho$ if and only if $M \vDash^w \rho$ for all states $w \in [\![M]\!]$. This is the route taken in early institution-theoretic formalizations of modal logics. Technically, this kind of 'flattening' requires the state-reduction functions $[\![M']\!]_\varphi$ to be surjective – a condition which holds trivially in concrete situations, as we will see below.

Example 2.3 (Modal logics)**.** The literature on modal logics [e.g., BRV01; Bla] is a prime source of examples of stratified institutions. Most of them are obtained by blending modal features – in particular, Kripke semantics – with features of other conventional logics in a process that is sometimes referred to as 'modalization' [DS07].

For example, the prototypical form of modal logic arises through the modalization of propositional logic, which yields the stratified institution MPL. Its signatures are plain sets whose elements we call *propositional symbols*, while signature morphisms are ordinary functions; so $\mathbb{S}\mathrm{ig}^{\mathrm{MPL}} = \mathbb{S}\mathrm{et}$. For every signature P, the sentences in $\mathrm{Sen}^{\mathrm{MPL}}(P)$ are defined according to the following grammar:

$$\rho \ ::= \ \pi \in P \ \mid \ \rho \wedge \rho \ \mid \ \neg \rho \ \mid \ \Diamond \rho$$

Other logical connectives such as the disjunction, implication, necessity, etc., can be introduced in the usual manner based on the ones listed above. The translation of sentences along a function $\varphi\colon P \to P'$ is defined using structural induction by replacing the symbols in P with symbols in P' according to φ.

The models in $\mathrm{Mod}^{\mathrm{MPL}}(P)$ are Kripke *structures* $\langle W, M \rangle$ where:

- W is a Kripke *frame* consisting of a set $|W|$ whose elements are called *possible worlds* and a binary *accessibility relation* W_λ on $|W|$, and

- M is a function assigning a subset $M(w) \subseteq P$ to every possible world $w \in |W|$; therefore, for every w, $M(w)$ is a propositional model for the signature P.

Model homomorphisms $\langle W_1, M_1 \rangle \to \langle W_2, M_2 \rangle$ are functions $h\colon |W_1| \to |W_2|$ such that $h(W_{1,\lambda}) \subseteq W_{2,\lambda}$, which ensures that h is a Kripke-frame homomorphism, and $M_1(w) \subseteq M_2(h(w))$ for every possible world w in W_1. The reduction of a P'-model $\langle W', M' \rangle$ along a function $\varphi\colon P \to P'$ is the Kripke structure $\langle W', M \rangle$ with the same possible worlds and accessibility relation as the original model and with $M(w) = \varphi^{-1}(M'(w))$ for all $w \in |W'|$. Similarly, for P'-homomorphisms h', based on the monotonicity of the inverse-image function, we have $h' {\restriction_{\varphi}} = h'$.

The stratification of MPL is strict and is defined by $[\![\langle W, M \rangle]\!]_P = |W|$ for Kripke structures of signature P and by $[\![h]\!]_P = h$ for P-homomorphisms.

Lastly, the satisfaction of MPL sentences in a Kripke structure $\langle W, M \rangle$ at a possible world $w \in |W|$ is defined by structural induction as follows:

- for propositional symbols, $\langle W, M \rangle \vDash^w \pi$ when $\pi \in M(w)$;
- for conjunctions, $\langle W, M \rangle \vDash^w \rho_1 \wedge \rho_2$ when $\langle W, M \rangle \vDash^w \rho_1$ and $\langle W, M \rangle \vDash^w \rho_2$;
- for negations, $\langle W, M \rangle \vDash^w \neg\rho$ when $\langle W, M \rangle \nvDash^w \rho$;
- for possibilities, $\langle W, M \rangle \vDash^w \Diamond\rho$ when $\langle W, M \rangle \vDash^s \rho$ for some $(w, s) \in W_\lambda$.

The construction outlined above for MPL has numerous variations in the modal-logic literature. It can be restricted, which is typically done by imposing constraints on the accessibility relations of Kripke structures; for example, they may be required to be serial, reflexive, preorders, or equivalences, leading to the D, T, S4, or S5 variants, respectively, of modal propositional logic. Or it can be extended by considering additional accessibility relations – and, consequently, additional modal operators – or by allowing the arities of the accessibility relations to be arbitrarily large instead of being binary. Both types of extensions are captured by means of a general notion of *modal similarity type* [see BRV01], which is one of the key ingredients in institution-theoretic formalizations of such logics [e.g., DS07; Mar+11].

The institutional process of modalization enables the role of propositional logic as a *base logic* to be easily filled by other, more elaborate, formalisms such as many-sorted equational logic, first-order logic, or partial algebra. These make possible additional semantic constraints on the structure of the base models that label possible worlds, leading to Kripke structures with constant domains, which may be subject to further information-sharing constraints as in [Dia16], or to Kripke structures with varying domains and different kinds of quantification schemes as in [ŢCF21].

Example 2.4 (Hybrid logics)**.** Hybrid languages [Bla00] extend the capabilities of modal languages by introducing new syntactic constructs, called *nominals*, which are used to refer to and reason about individual possible worlds within Kripke structures.

Similarly to modal logics, they can be obtained through an institution-independent process of 'hybridization' [Mar+11; DM16]; in essence, this refines modalization by adding nominals to the signatures of the resulting logic – with suitable subsequent upgrades to the sentences, models, and satisfaction relations of the logic.

To illustrate the approach, consider the following hybridization of propositional logic that gives rise to the stratified institution HPL. The signatures, in this case, are pairs (N, P) consisting of disjoint sets N and P of nominals and propositional symbols, respectively. Correspondingly, signature morphisms are pairs of functions, one acting on nominals and another acting on propositional symbols.

The sentences of HPL are generated by a grammar similar to the one defined in Example 2.3 that includes, in addition, the following two productions:

$$\rho \ ::= \ i \in N \ \mid \ @i \cdot \rho$$

This syntax is occasionally broadened [e.g. Gor96; Bra11] to accommodate quantifiers over possible worlds or a special *store* operator that binds a nominal to the possible world where the operator – and the sentence it builds – is evaluated.

The models of HPL augment the Kripke structures $\langle W, M \rangle$ defined for MPL by adding, for every nominal $i \in N$, a possible-world interpretation $W_i \in |W|$. As expected, HPL homomorphisms $h\colon \langle W_1, M_1 \rangle \to \langle W_2, M_2 \rangle$ preserve the interpretation of nominals, in the sense that $h(W_{1,i}) = W_{2,i}$ for all nominals $i \in N$.

Model states and the mapping of states along homomorphisms are defined exactly as in the non-hybrid case by letting $[\![\langle W, M \rangle]\!] = |W|$ for models and $[\![h]\!] = h$ for homomorphisms. Hence the stratification of HPL is strict as well.

The ingredients listed above are brought together by the satisfaction relations of HPL between Kripke structures and hybrid sentences, whose inductive definition expands the satisfaction of modal sentences with the following two cases:

- for nominal sentences, $\langle W, M \rangle \vDash^w i$ when $w = W_i$;
- for local-satisfaction operators, $\langle W, M \rangle \vDash^w @i \cdot \rho$ when $\langle W, M \rangle \vDash^{W_i} \rho$.

All the variations of the process of modalization described in Example 2.3 have hybrid counterparts, so the landscape of hybrid logics is at least as vast as that of traditional modal logics – without nominals. For simplicity of presentation, we only refer to HPL in the next sections of the paper; however, except for tweaking some of the parameters that are being used and checking prerequisites, the properties we examine in those sections generally apply to many other hybrid logics.

Example 2.5 (Automata)**.** The theory of finite state machines provides yet another class of examples of stratified institutions. We look at a simple representative based on non-deterministic automata, whose stratified institution we denote by NFA; for other institution-theoretic formalizations of automata, see, e.g., [Dia08; ŢF15; Dia22].

Similarly to MPL, the signatures of NFA are plain sets, in this case regarded as input alphabets or vocabularies of the automata. Therefore, $\mathbb{S}\mathrm{ig}^{\mathrm{NFA}} = \mathbb{S}\mathrm{et}$.

For every alphabet A, the sentences in $\mathrm{Sen}^{\mathrm{NFA}}(A)$ are finite sequences (words) built from symbols in A; so, using the Kleene operator, we can write $\mathrm{Sen}^{\mathrm{NFA}}(A) = A^*$. And for every function $\varphi\colon A \to A'$, the sentence translation $\mathrm{Sen}^{\mathrm{NFA}}(\varphi)$ is the unique homomorphic extension $\varphi^*\colon A^* \to (A')^*$ satisfying $\varphi^*(\varepsilon) = \varepsilon$ for the empty word ε, $\varphi^*(a) = \varphi(a)$ for symbols $a \in A$, and $\varphi^*(uv) = \varphi^*(u)\varphi^*(v)$ for concatenations.

The NFA models over A are non-deterministic finite automata $\langle Q, \Delta, F\rangle$ where Q is a finite set of *states*, Δ is a family of *transition relations* $\Delta_a \subseteq Q \times Q$ indexed by symbols $a \in A$, and $F \subseteq Q$ is a subset of *final*, or *accepting*, states. Their homomorphisms $\langle Q_1, \Delta_1, F_1\rangle \to \langle Q_2, \Delta_2, F_2\rangle$ are functions $h\colon Q_1 \to Q_2$ that preserve both transitions and final states – i.e., $h(\Delta_{1,a}) \subseteq \Delta_{2,a}$ for all $a \in A$ and $h(F_1) \subseteq F_2$. Given a signature morphism $\varphi\colon A \to A'$, the φ-reduct of an A'-automaton $\langle Q', \Delta', F'\rangle$ is the A-automaton $\langle Q', \Delta, F'\rangle$ with transitions given by $\Delta_a = \Delta'_{\varphi(a)}$ for all $a \in A$. The reducts of homomorphisms are defined, similarly to MPL, by $h'{\upharpoonright_\varphi} = h'$.

The stratification is strict and, once more, straightforward: $[\![\langle Q, D, F\rangle]\!] = Q$.

Finally, the satisfaction relations of NFA capture the acceptance of words by automata. An automaton $\langle Q, \Delta, F\rangle$ over some alphabet A satisfies (accepts) a sentence (word) $a_1 a_2 \cdots a_n \in A^*$ at a state $q \in Q$ when there exist $q_1, q_2, \ldots, q_{n+1} \in Q$ such that $q_1 = q$, $(q_i, q_{i+1}) \in \Delta_{a_i}$ for all $1 \leq i \leq n$, and $q_{n+1} \in F$.

Deterministic finite automata give rise to a sub-institution DFA of NFA. The signatures and sentences of DFA are the same as those of NFA, but the models are restricted to automata $\langle Q, \Delta, F\rangle$ with functional transition relations: for all symbols $a \in A$ and all states $q, s, t \in Q$, if $(q, s) \in \Delta_a$ and $(q, t) \in \Delta_a$, then $s = t$.

A useful variation of DFA is $\mathcal{P}$DFA, which has the same signatures, models, and stratification as the institution DFA, and sentences defined as sets of words, or languages. Hence $\mathrm{Sen}^{\mathcal{P}\mathrm{DFA}} = \mathrm{Sen}^{\mathrm{DFA}} \mathbin{;} \mathcal{P}$, the latter being the covariant power-set functor $\mathcal{P}\colon \mathbb{S}\mathrm{et} \to \mathbb{S}\mathrm{et}$. The satisfaction of $\mathcal{P}$DFA-sentences is defined by the expected conjunctive interpretation of sets of sentences: for every automaton $\langle Q, \Delta, F\rangle$ over some alphabet A, every state $q \in Q$, and every language $L \subseteq A^*$, we have $\langle Q, \Delta, F\rangle \models_A^{\mathcal{P}\mathrm{DFA},q} L$ if and only if $\langle Q, \Delta, F\rangle$ accepts all the words in L starting from the state q – which is the same as the abbreviation $\langle Q, \Delta, F\rangle \models_A^{\mathrm{DFA},q} L$.

Example 2.6 (Open formulae)**.** The *internal stratification* of ordinary institutions developed in [AD07] gives us further examples of stratified institutions where the stratification generalizes valuations of free variables. For instance, let OFOL be an institution obtained from the internal stratification of first-order logic. Its signatures are pairs $\langle \Sigma, X\rangle$ consisting of a first-order signature Σ and a set X of Σ-variables – see [GB92; Dia08] for details on how to formalize first-order logic as an institution. The signature morphisms $\langle \Sigma, X\rangle \to \langle \Sigma', X'\rangle$ require X to be a subset of X' and, when that holds, are simply first-order signature morphisms $\varphi\colon \Sigma \to \Sigma'$.

Both sentences and models for an OFOL-signature $\langle \Sigma, X\rangle$ are defined as in first-order logic: for sentences, we consider open first-order Σ-formulae with free variables in X; and for models, we consider all Σ-models. Their translations and reductions along signature morphisms are also defined as in first-order logic.

For every signature $\langle \Sigma, X \rangle$, the stratification $[\![\_]\!]_{\langle \Sigma, X \rangle}$ maps every first-order Σ-model M to the set $|M|^X$ of M-valuations of X – i.e., functions from X into the carrier set of M. And for every signature morphism $\varphi \colon \langle \Sigma, X \rangle \to \langle \Sigma', X' \rangle$ and $\langle \Sigma', X' \rangle$-model M', the state-reduction function $[\![M']\!]_\varphi \colon |M'|^{X'} \to |M'|^X$ maps every M'-valuation $w' \colon X' \to |M'|$ to $(X \subseteq X') \mathbin{;} w'$, which is the restriction of w' to X. Therefore, the stratification of OFOL is a proper non-strong lax natural transformation. Its 2-morphism components $[\![M']\!]_\varphi$ are surjective functions for all models M' that have non-empty carriers – a property which is guaranteed, e.g., whenever the signature Σ' of M' is rich enough so as to allow the formation of terms.

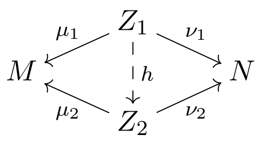
To define the satisfaction relations, notice that any first-order signature Σ and any set X of Σ-variables determine an extended first-order signature $\Sigma(X)$ obtained by adding the elements in X to Σ as new constant-operation symbols. Based on this extension, for every $\langle \Sigma, X \rangle$-model M, every valuation $w \colon X \to |M|$, and every open Σ-formula ρ with variables in X, we let $M \vDash^w \rho$ if and only if ρ holds – according to the definition of the satisfaction relations of first-order logic – in the unique $\Sigma(X)$-expansion of M that interprets every symbol $x \in X$ as $w(x)$.

3 Abstract bisimulations

The general approach to bisimilarity that we consider in this paper builds on the idea that bisimulations are derived relations between models that arise from combinations of certain structure-preserving maps. Those maps are known as *bounded morphisms* in the model theory of conventional modal logics [see Gol89; GO07], and they are frequently presented as particular bisimulations induced by functions – even though their role is more fundamental. In what follows, we introduce them in an axiomatic manner, through a functorial selection of abstract homomorphisms.

Definition 3.1 (Bounded homomorphism)**.** We say that a stratified institution $\mathcal{S}$ *has bounded homomorphisms* when it is equipped with a subfunctor BH of Mod that defines, for every signature Σ, a broad subcategory $\mathrm{BH}(\Sigma) \subseteq \mathrm{Mod}(\Sigma)$. When that happens, we call the arrows in $\mathrm{BH}(\Sigma)$ *bounded Σ-homomorphisms*.

To formalize bisimulations in an arbitrary but fixed stratified institution, we resort to categorical relations, which are represented by spans $\langle \mu_i, Z_i, \nu_i \rangle$ as in the diagram below. We also make use of the following preorder relation on spans: for any two objects M and N, in any category, $\langle \mu_1, Z_1, \nu_1 \rangle \leq \langle \mu_2, Z_2, \nu_2 \rangle$ if and only if there exists an arrow $h\colon Z_1 \to Z_2$ such that $\mu_1 = h \mathbin{;} \mu_2$ and $\nu_1 = h \mathbin{;} \nu_2$.



We say that the class of spans between two arbitrary but fixed objects M and N is *κ-directed* for some cardinal number κ, usually infinite, when every subclass of it of cardinality less than κ has an upper bound.

Definition 3.2 (Bisimulation)**.** Given a stratified institution $\mathcal{S}$ with bounded homomorphisms, a *bisimulation* (*representation*) between two Σ-models M and N consists in a span $\langle \mu, Z, \nu \rangle$ of bounded homomorphisms. We say that the stratified models M and N are *bisimilar* if there exists a bisimulation between them.

In addition, we say that a bisimulation $\langle \mu, Z, \nu \rangle$ witnesses the bisimilarity of two states $u \in [\![M]\!]$ and $v \in [\![N]\!]$, or that the pointed models $\langle M, u \rangle$ and $\langle N, v \rangle$ are *bisimilar*, if there exists a state $z \in [\![Z]\!]$ such that $[\![\mu]\!]z = u$ and $[\![\nu]\!]z = v$.

For the rest of the paper, whenever we refer to bisimulations or to bisimilar models or states, we implicitly assume that the stratified institution within which we work is equipped with a functorial selection BH of bounded homomorphisms.

Definition 3.3 (Hennessy-Milner property)**.** A stratified institution $\mathcal{S}$ has the *Hennessy-Milner* (*HM*) *property for a signature* Σ when the bisimilarity and the elementary-equivalence relation for Σ are indistinguishable. That is, for any two Σ-models M and N, and for any states $u \in [\![M]\!]$ and $v \in [\![N]\!]$, we have:

[*bisimilarity*] the pointed models $\langle M, u \rangle$ and $\langle N, v \rangle$ are bisimilar

if and only if

[*elementary equivalence*] $M \models^u \rho$ if and only if $N \models^v \rho$ for all Σ-sentences ρ.

We say that the institution $\mathcal{S}$ has the *HM property*, or that it is an *HM institution*, if the above condition holds for all signatures Σ of $\mathcal{S}$.

When the stratified models M and N are fixed, the HM property can also be understood as the equality of two relations $\sim_{M,N}, \equiv_{M,N} \subseteq [\![M]\!] \times [\![N]\!]$ between the states of M and N, where $u \sim_{M,N} v$ stands for the bisimilarity of $\langle M, u \rangle$ and $\langle N, v \rangle$, and $u \equiv_{M,N} v$ stands for their elementary equivalence.

The forward implication of Definition 3.3 is related to the following notion of elementary homomorphism, which originates from [AD07].

Definition 3.4 (Elementary homomorphism)**.** For any signature Σ in a stratified institution and any set Γ of Σ-sentences, a Σ-homomorphism $h\colon M \to N$ is *Γ-elementary* when, for every state $w \in [\![M]\!]$ and every Σ-sentence $\rho \in \Gamma$, $M \models^w \rho$ if and only if $N \models^{[\![h]\!]w} \rho$. We say that h is *elementary* when it is $\mathrm{Sen}(\Sigma)$-elementary.

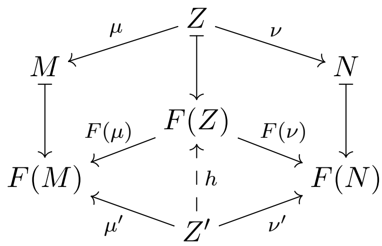
Proposition 3.5. *Bisimilarity implies elementary equivalence if and only if bounded homomorphisms are elementary.*

Proof. For the 'if' part, suppose $\langle \mu, Z, \nu \rangle$ is a bisimulation between stratified models M and N with distinguished states $u \in [\![M]\!]$, $v \in [\![N]\!]$, and $z \in [\![Z]\!]$ such that $[\![\mu]\!]z = u$ and $[\![\nu]\!]z = v$, and let ρ be a Σ-sentence. Since the homomorphism $\mu \colon Z \to M$ is bounded, and thus elementary, it follows that $M \vDash^u \rho$ if and only if $Z \vDash^z \rho$, which then holds if and only if $N \vDash^v \rho$ because $\nu \colon Z \to N$ is bounded.

For the 'only if' part, let $h \colon M \to N$ be a bounded homomorphism, w a state of M, and ρ a Σ-sentence. It follows that $\langle id_M, M, h \rangle$ is a bisimulation representation that witnesses the bisimilarity of the states w and $[\![h]\!]w$. Therefore, since bisimilarity implies elementary equivalence, we get $M \vDash^w \rho$ if and only if $N \vDash^{[\![h]\!]w} \rho$. □

In order to characterize as well the backward implication of Definition 3.3, we make use of a category-theoretic concept of cover of a span along a functor.

Definition 3.6 (Cover)**.** Let $F \colon \mathbb{C} \to \mathbb{C}'$ be a functor. We say that a span $\langle \mu, Z, \nu \rangle$ in $\mathbb{C}$ between objects M and N is an *F-cover* of another span $\langle \mu', Z', \nu' \rangle$ in $\mathbb{C}'$ between $F(M)$ and $F(N)$, or that F *covers* $\langle \mu', Z', \nu' \rangle$, when $F(\langle \mu, Z, \nu \rangle) \geq \langle \mu', Z', \nu' \rangle$.



An F-cover $\langle \mu, Z, \nu \rangle$ is *strong* when the inequality $F(\langle \mu, Z, \nu \rangle) \geq \langle \mu', Z', \nu' \rangle$ is evidenced by an isomorphism $h \colon Z' \to F(Z)$ as in the commutative diagram above.

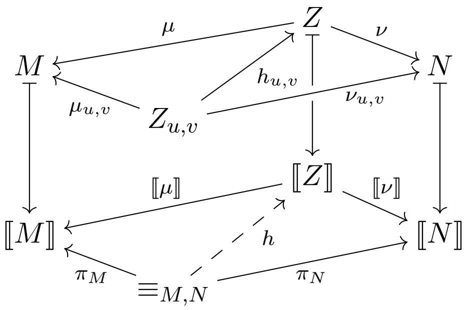
In particular, when F is $\mathbb{S}$et-valued and the span $\langle \mu', Z', \nu' \rangle$ that it covers is the tabulation of a relation $Z' \subseteq F(M) \times F(N)$, we may also say that F covers Z'.

Proposition 3.7. *Elementary equivalence implies bisimilarity when, for all Σ-models M and N, the composite functor $(\mathrm{BH}(\Sigma) \subseteq \mathrm{Mod}(\Sigma))\, ⨾\, [\![\_]\!]$ covers the relation $\equiv_{M,N}$. Conversely, if the class of bisimulation representations between M and N is κ-directed for some cardinal number $\kappa > \mathrm{card}(\equiv_{M,N})$, then the bisimilarity of elementary-equivalent states implies that $(\mathrm{BH}(\Sigma) \subseteq \mathrm{Mod}(\Sigma))\, ⨾\, [\![\_]\!]$ covers the relation $\equiv_{M,N}$.*

Proof. The first part of the statement is straightforward since, for any pair of stratified models M and N, every cover of $\equiv_{M,N}$ along $(\mathrm{BH}(\Sigma) \subseteq \mathrm{Mod}(\Sigma))\, ⨾\, [\![\_]\!]$ gives rise to a bisimulation $\langle \mu, Z, \nu \rangle$ between M and N. Moreover, for any such bisimulation, there exists, by definition, a function $h \colon \equiv_{M,N} \to [\![Z]\!]$ such that $[\![\mu]\!]h(u, v) = u$ and $[\![\nu]\!]h(u, v) = v$ for all states $u \in [\![M]\!]$ and $v \in [\![N]\!]$ such that $u \equiv_{M,N} v$. Therefore, $\langle \mu, Z, \nu \rangle$ witnesses the bisimilarity of any two elementary-equivalent states.

For the second part of the statement, let $\langle \mu_{u,v}, Z_{u,v}, \nu_{u,v} \rangle$ be a bounded span that witnesses the bisimilarity of any two given elementary-equivalent states $u \in [\![M]\!]$ and $v \in [\![N]\!]$, and take $\langle \mu, Z, \nu \rangle$ as an upper bound of $\{\langle \mu_{u,v}, Z_{u,v}, \nu_{u,v} \rangle \mid u \equiv_{M,N} v\}$.

It is easy to check that $\langle \mu, Z, \nu \rangle$ is a cover of $\equiv_{M,N}$ along $(\mathrm{BH}(\Sigma) \subseteq \mathrm{Mod}(\Sigma)) \fatsemi [\![\_]\!]$: For every pair of elementary-equivalent states u and v as above, there exists, on the one hand, a state $z_{u,v} \in [\![Z_{u,v}]\!]$ such that $[\![\mu_{u,v}]\!] z_{u,v} = u$ and $[\![\nu_{u,v}]\!] z_{u,v} = v$; and on the other hand, a homomorphism $h_{u,v} \colon Z_{u,v} \to Z$, depicted in the upper part of the commutative diagram below, by which $\langle \mu_{u,v}, Z_{u,v}, \nu_{u,v} \rangle \leq \langle \mu, Z, \nu \rangle$.



The mapping $(u, v) \mapsto [\![h_{u,v}]\!] z_{u,v}$ gives us a function $h \colon \equiv_{M,N} \to [\![Z]\!]$ that satisfies $[\![\mu]\!] h(u, v) = [\![\mu]\!][\![h_{u,v}]\!] z_{u,v} = [\![\mu \fatsemi h_{u,v}]\!] z_{u,v} = [\![\mu_{u,v}]\!] z_{u,v} = u$ and, similarly, $[\![\nu]\!] h(u, v) = v$, for all $u \equiv_{M,N} v$. Therefore, $[\![\langle \mu, Z, \nu \rangle]\!] \geq \langle \pi_M, \equiv_{M,N}, \pi_N \rangle$. □

The premise in the second part of Proposition 3.7 poses no challenges in practice because concrete bisimulations are typically closed under arbitrary unions, so there is always a largest bisimulation between two fixed models, also known as the *bisimilarity* relation. That is generally the case with modal logics as well as more abstract frameworks as in coalgebraic approaches to bisimilarity [e.g., Gol06].

We conclude this section by putting together Propositions 3.5 and 3.7 and a well-known result in cardinal arithmetic: if $[\![M]\!]$ and $[\![N]\!]$ have cardinalities bounded by some infinite cardinal κ, then $\equiv_{M,N} \subseteq [\![M]\!] \times [\![N]\!]$ has this property too.

Corollary 3.8. *A stratified institution has the HM property for a signature* Σ *if:*

1. *all bounded* Σ*-homomorphism are elementary; and*
2. $(\mathrm{BH}(\Sigma) \subseteq \mathrm{Mod}(\Sigma)) \fatsemi [\![\_]\!]$ *covers* $\equiv_{M,N}$ *for all* Σ*-models* M *and* N*.*

When the class of bisimulations between any two Σ*-models is* κ*-directed for some infinite cardinal number* κ *that is larger than the cardinal of the state space of any* Σ*-model, the two conditions listed above are not only sufficient, but also necessary.* □

HM by translation

The HM property for a signature can sometimes be inherited from another signature, provided that the two signatures are linked by a suitable morphism. In the next two propositions we examine properties of signature morphisms that enable us to transfer HM results from one signature to another.

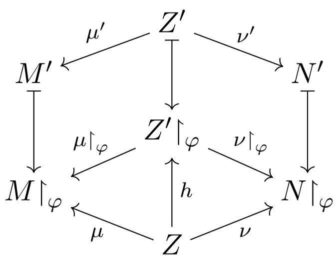
Proposition 3.9. *Let $\varphi\colon \Sigma \to \Sigma'$ be a signature morphism such that every bounded φ-expansion of an elementary Σ-homomorphism is elementary. If the HM property holds for Σ, then bisimilarity implies elementary equivalence for Σ'.*

Proof. By Proposition 3.5, it suffices to show that all bounded Σ'-homomorphisms are elementary, so let h' be a homomorphism in $\mathrm{BH}(\Sigma')$. It follows that $h'{\restriction_\varphi}$ is a bounded Σ-homomorphism, and thus elementary – by Corollary 3.8 – because the HM property holds for Σ. Therefore, given that h' is a bounded φ-expansion of an elementary Σ-homomorphism, we get that h' is elementary as well. □

Proposition 3.10. *Let $\varphi\colon \Sigma \to \Sigma'$ be a signature morphism in a strong stratified institution such that* $\mathrm{BH}(\varphi)$ *covers spans. If the HM property holds for Σ, then elementary equivalence implies bisimilarity for Σ'.*

Proof. Let u' and v' be states in Σ'-models M' and N', respectively, such that $M' \models^{u'} \rho'$ if and only if $N' \models^{v'} \rho'$ for all Σ'-sentences ρ', and let $u = [\![M]\!]_\varphi u'$ and $v = [\![N']\!]_\varphi v'$. By the satisfaction condition for φ, it follows that $M'{\restriction_\varphi} \models^{u} \rho$ if and only if $N'{\restriction_\varphi} \models^{v} \rho$ for all Σ-sentences ρ. Therefore, since the HM property holds for Σ, there exists a bisimulation representation $\langle \mu, Z, \nu \rangle$ between $M'{\restriction_\varphi}$ and $N'{\restriction_\varphi}$ and a state $z \in [\![Z]\!]$ such that $[\![\mu]\!]z = u$ and $[\![\nu]\!]z = v$.

Since $\mathrm{BH}(\varphi)$ covers spans, we obtain a bisimulation $\langle \mu', Z', \nu' \rangle$ between M' and N' and a Σ-homomorphism $h\colon Z \to Z'{\restriction_\varphi}$ such that the following diagram commutes:



Given that the stratification is strong, it follows that $[\![M']\!]_\varphi$, $[\![Z']\!]_\varphi$, and $[\![N']\!]_\varphi$ are all bijections. In particular, the surjectivity of $[\![Z']\!]_\varphi$ entails that there exists a state $z' \in [\![Z']\!]$ such that $[\![Z']\!]_\varphi z' = [\![h]\!]z$. Moreover, based on the injectivity of $[\![M']\!]_\varphi$ and on the sequence of equalities below, we infer that $[\![\mu']\!]z' = u'$.

$$\begin{aligned}
&[\![M']\!]_\varphi [\![\mu']\!]z' \\
&\quad = [\![\mu'{\restriction_\varphi}]\!][\![Z']\!]_\varphi z' && \text{by the naturality of } [\![_]\!]_\varphi \\
&\quad = [\![\mu'{\restriction_\varphi}]\!][\![h]\!]z && \text{since } [\![Z']\!]_\varphi z' = [\![h]\!]z \\
&\quad = [\![h \mathbin{;} \mu'{\restriction_\varphi}]\!]z && \text{because } [\![_]\!]_\Sigma \text{ is a functor} \\
&\quad = [\![\mu]\!]z && \text{by the commutativity of the diagram above} \\
&\quad = [\![M']\!]_\varphi u' && \text{because } [\![\mu]\!]z = u = [\![M']\!]_\varphi u'.
\end{aligned}$$

In a similar manner, we show that $[\![\nu']\!]z' = v'$, which enables us to conclude that the pointed models $\langle M', u'\rangle$ and $\langle N', v'\rangle$ are bisimilar. □

Corollary 3.11. *In strong stratified institutions, the HM property is preserved along all signature morphisms $\varphi\colon \Sigma \to \Sigma'$ such that:*

1. *bounded φ-expansions of elementary Σ-homomorphism are elementary; and*
2. *the model-reduct functor* $\mathrm{BH}(\varphi)$ *covers spans.* □

4 Frame-bounded homomorphisms

The results we have developed thus far hold for arbitrary stratified institutions where we have little to no knowledge about the structure of models. But in many concrete examples of stratified institutions, including those presented in Section 2, stratified models are structures built on top of some Kripke frame. In this section, we make explicit the underlying Kripke frames of models and we use the additional information to arrive at a refined form of Corollary 3.8. For this purpose, we first introduce categories of Kripke frames, which are parameterized by sets of modalities.

Frame extractions

For every set Λ, whose elements we regard as *modalities*, we let $\mathbb{K}(\Lambda)$ be the category of *Kripke frames* with accessibility relations indexed by Λ. Each of its objects W consists of a set $|W|$ (typically non-empty) of *possible worlds* together with a *binary accessibility relation* $W_\lambda \subseteq |W| \times |W|$ for every modality $\lambda \in \Lambda$. Its arrows $W_1 \to W_2$ are Λ-frame homomorphisms, meaning functions $h\colon |W_1| \to |W_2|$ that preserve the accessibility of worlds: $h(W_{1,\lambda}) \subseteq W_{2,\lambda}$ for all modalities $\lambda \in \Lambda$.

Remark 4.1. The map $\mathbb{K}$ extends to a contravariant functor $\mathbb{K}\colon \mathbb{S}\mathrm{et}^{\mathrm{op}} \to \mathbb{C}\mathrm{at}$, where, for every function $l\colon \Lambda \to \Lambda'$ and every frame W' in $\mathbb{K}(\Lambda')$, $\mathbb{K}(l)(W')$ is the frame W in $\mathbb{K}(\Lambda)$ given by $|W| = |W'|$ and $W_\lambda = W'_{l(\lambda)}$ for all $\lambda \in \Lambda$. Building on the functoriality of $\mathbb{K}$, the notation used for the underlying set of possible worlds of a Kripke frame can be extended as well to a natural transformation $|\_|\colon \mathbb{K} \Rightarrow \mathrm{Set}$.

The following concept of frame extraction, based on which we can regard stratified models as abstract Kripke structures, originates from [Dia17]. For simplicity, we only consider binary frame extractions here; however, the definition and all the results that follow can be easily upgraded to modalities of arbitrary type [see Dia17; Dia22].

Definition 4.2 (Frame extraction)**.** A *binary frame extraction* for a stratified institution is a pair $\langle \mathrm{L}, \mathrm{Fr} \rangle$ consisting of a functor $\mathrm{L}\colon \mathbb{S}\mathrm{ig} \to \mathbb{S}\mathrm{et}$ and a lax natural transformation $\mathrm{Fr}\colon \mathrm{Mod} \Rightarrow \mathrm{L} \mathbin{;} \mathbb{K}$ such that $[\![\_]\!] = \mathrm{Fr} \mathbin{;} \mathrm{L}|\_|$.

Example 4.3. Both modal and hybrid logics have obvious frame extractions given by the underlying Kripke frames of their models. For instance, for MPL, we let $\mathrm{L}(P)$ be the singleton $\{\lambda\}$ for every set P of propositional symbols, and we let $\mathrm{Fr}(\langle W, M \rangle)$ be the Kripke frame W for every P-model $\langle W, M \rangle$. The same holds for HPL and for many other variants of modal and hybrid logic. In each of those cases, similarly to the stratification $[\![\_]\!]$, the lax natural transformation Fr is strict.

Example 4.4. For the stratified institutions of finite automata from Example 2.5, we let $\mathrm{L}(A) = A$ for every signature (alphabet) A, and $\mathrm{Fr}(\langle Q, \Delta, F \rangle) = \langle Q, \Delta \rangle$ for every automaton $\langle Q, \Delta, F \rangle$ – i.e., $|\mathrm{Fr}(\langle Q, \Delta, F \rangle)| = Q$ and $\mathrm{Fr}(\langle Q, \Delta, F \rangle)_a = \Delta_a$ for all $a \in A$. The lax natural transformations Fr are strict in this case too.

Example 4.5. The stratified institution OFOL also admits a non-trivial frame extraction where $\mathrm{L}(\langle \Sigma, X \rangle) = X$ for every signature $\langle \Sigma, X \rangle$. Therefore, in this case, variables may be regarded as modalities. Concerning accessibility relations, for every $\langle \Sigma, X \rangle$-model M and variable $x \in X$, we let $\mathrm{Fr}(M)_x$ be the set of all pairs of valuations $u, v\colon X \to |M|$ that agree on all variables save possibly x; or, in symbols: $(u, v) \in \mathrm{Fr}(M)_x$ if and only if $(X \setminus \{x\} \subseteq X) \mathbin{;} u = (X \setminus \{x\} \subseteq X) \mathbin{;} v$.

Frame extraction enables us to introduce an institutional notion of frame-bounded homomorphism, which captures concrete bounded homomorphisms, also known as *p-morphisms* [Seg71], from traditional modal and hybrid logics.

Definition 4.6 (Frame-bounded homomorphism)**.** Let $\mathcal{S}$ be a stratified institution equipped with a subfunctor $\mathrm{E} \subseteq \mathrm{Sen}$ and a binary frame extraction $\langle \mathrm{L}, \mathrm{Fr} \rangle$. A Σ-homomorphism $h\colon M \to N$ is *frame-bounded* when, for all $w \in [\![M]\!]$:

- $M \vDash^{w} \rho$ if and only if $N \vDash^{[\![h]\!]w} \rho$ for every sentence $\rho \in \mathrm{E}(\Sigma)$; and
- $\mathrm{Fr}(N)_\lambda([\![h]\!]w) \subseteq [\![h]\!](\mathrm{Fr}(M)_\lambda w)$ for every modality $\lambda \in \mathrm{L}(\Sigma)$,

where $\mathrm{Fr}(M)_\lambda w = \{s \in [\![M]\!] \mid (w, s) \in \mathrm{Fr}(M)_\lambda\}$ is the set of λ-successors of w in M and likewise $\mathrm{Fr}(N)_\lambda([\![h]\!]w) = \{t \in [\![N]\!] \mid ([\![h]\!]w, t) \in \mathrm{Fr}(N)_\lambda\}$.

In practice, the sentences in $\mathrm{E}(\Sigma)$ are typically atomic, and play a key role in defining bisimulations. For this reason, we call them *bisimulation-essential*.

Example 4.7. For MPL, we let $\mathrm{E}(P) = P$, the set of all propositional atoms. Similarly, for HPL, the bisimulation-essential sentences are given by all propositional and nominal atoms. For the stratified institutions of automata from Example 2.5, $\mathrm{E}(A)$ consists of only one sentence: ε, the empty word, for NFA and DFA; and $\{\varepsilon\}$ for $\mathcal{P}$DFA. And for the stratified institution of open formulae from Example 2.6, $\mathrm{E}(\langle\Sigma, X\rangle)$ is the set of all equational and relational Σ-atoms with variables in X.

Frame-boundedness combines two orthogonal properties of homomorphisms. One is the elementary property for bisimulation-essential sentences – cf. Definition 3.4 – while the other is the standard boundedness property of Kripke-frame homomorphisms: a frame homomorphism $h\colon W_1 \to W_2$ in $\mathbb{K}(\Lambda)$ is *bounded* [see Gol89; BB07] when, for every possible world $w \in |W_1|$, every modality $\lambda \in \Lambda$, and every transition $(h(w), s_2) \in W_{2,\lambda}$, there exists a transition $(w, s_1) \in W_{1,\lambda}$ such that $h(s_1) = s_2$. In other words, the function induced by h between the set of λ-successors of w in W_1 and the set of λ-successors of $h(w)$ in W_2 is surjective. Based on this observation, and in order to avoid confusion between frame-bounded and bounded frame homomorphisms, we henceforth call the latter frame homomorphisms *successor-surjective*.

Remark 4.8. A homomorphism h of stratified Σ-models is frame-bounded if and only if it is $\mathrm{E}(\Sigma)$-elementary and $\mathrm{Fr}(h)$ is a successor-surjective $\mathrm{L}(\Sigma)$-frame homomorphism.

This view eases the task of checking that frame-bounded homomorphisms fit into the general framework introduced in Section 3. To see why that is the case, we first consider two important properties of successor-surjective frame homomorphisms. The proofs of the following lemmas are straightforward.

Lemma 4.9. *For every set Λ of modalities, the successor-surjective Λ-frame homomorphisms form a broad subcategory $\mathbb{S}(\Lambda) \subseteq \mathbb{K}(\Lambda)$. Moreover, successor-surjectivity is preserved by frame-reduction functors, hence $\mathbb{S}$ extends to a subfunctor of $\mathbb{K}$.* □

Lemma 4.10. *For every pair of composable homomorphisms f and g in $\mathbb{K}(\Lambda)$, if f is surjective and $f \mathbin{;} g$ is successor-surjective, then g is successor-surjective as well.* □

Proposition 4.11. *Suppose $\varphi\colon \Sigma \to \Sigma'$ is a signature morphism in a stratified institution as in Definition 4.6 such that, for every Σ'-model M', $[\![M']\!]_\varphi$ is surjective. Then the φ-reduct of any $\mathrm{E}(\Sigma')$-elementary homomorphism is $\mathrm{E}(\Sigma)$-elementary.*

If, in addition, for every Σ'-model M', $\mathrm{Fr}_\varphi(M')$ is successor-surjective, then the φ-reduct of any frame-bounded Σ'-homomorphism is frame-bounded as well.

Proof. Let $h'\colon M' \to N'$ be an $\mathrm{E}(\Sigma')$-elementary Σ'-homomorphism. For the first part of the proposition, we need to check that the satisfaction of any sentence $\rho \in \mathrm{E}(\Sigma)$ is invariant along $h' {\restriction}_\varphi$. So let w be a state of $M' {\restriction}_\varphi$. Since $[\![M']\!]_\varphi$ is surjective, there exists $w' \in [\![M']\!]$ such that $w = [\![M']\!]_\varphi w'$. The equi-satisfaction of ρ at w' and $[\![h {\restriction}_\varphi]\!] w'$ follows from the next sequence of equivalent statements:

1 $\quad M' {\restriction}_\varphi \models^w \rho$

2 $M'{\restriction_\varphi} \models^{[\![M']\!]_\varphi w'} \rho$ — because $w = [\![M']\!]_\varphi w'$

3 $M' \models^{w'} \varphi(\rho)$ — by the satisfaction condition for φ

4 $N' \models^{[\![h']\!] w'} \varphi(\rho)$ — since h' is frame-bounded and $\varphi(\rho) \in \mathrm{E}(\Sigma')$

5 $N'{\restriction_\varphi} \models^{[\![N']\!]_\varphi [\![h']\!] w'} \rho$ — by the satisfaction condition for φ

6 $N'{\restriction_\varphi} \models^{[\![h'{\restriction_\varphi}]\!] [\![M']\!]_\varphi w'} \rho$ — by the naturality of $[\![\_]\!]_\varphi$

7 $N'{\restriction_\varphi} \models^{[\![h'{\restriction_\varphi}]\!] w} \rho$ — once more, because $w = [\![M']\!]_\varphi w'$.

For the second part, based on Remark 4.8, it suffices to show that $\mathrm{Fr}(h'{\restriction_\varphi})$ is successor-surjective. For that purpose, consider the next commutative diagram in $\mathbb{K}(\mathrm{L}(\Sigma))$, which arises from the naturality of $\mathrm{Fr}_\varphi \colon \mathrm{Fr}_{\Sigma'} \mathbin{⨾} \mathbb{K}(\mathrm{L}(\varphi)) \Rightarrow \mathrm{Mod}(\varphi) \mathbin{⨾} \mathrm{Fr}_\Sigma$.

$$\begin{array}{ccccc} M' & \quad & \mathbb{K}(\mathrm{L}(\varphi))(\mathrm{Fr}(M')) & \xrightarrow{\mathrm{Fr}_\varphi(M')} & \mathrm{Fr}(M'{\restriction_\varphi}) \\ {\scriptstyle h'}\big\downarrow & & {\scriptstyle \mathbb{K}(\mathrm{L}(\varphi))(\mathrm{Fr}(h'))}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Fr}(h'{\restriction_\varphi})} \\ N' & & \mathbb{K}(\mathrm{L}(\varphi))(\mathrm{Fr}(N')) & \xrightarrow[\mathrm{Fr}_\varphi(N')]{} & \mathrm{Fr}(N'{\restriction_\varphi}) \end{array}$$

Since the model homomorphism h' is frame-bounded, it follows that $\mathrm{Fr}(h')$ is successor-surjective. By Lemma 4.9, the frame homomorphism $\mathbb{K}(\mathrm{L}(\varphi))(\mathrm{Fr}(h'))$ is also successor-surjective, hence both the left-hand side and bottom homomorphisms in the diagram above are successor-surjective – the latter being so by assumption. Therefore, the composed morphism $\mathrm{Fr}_\varphi(M') \mathbin{⨾} \mathrm{Fr}(h'{\restriction_\varphi})$ is successor-surjective; and, in addition, the left factor of that composition is surjective on possible worlds because, by assumption, $|\mathrm{Fr}_\varphi(M')| = [\![M']\!]_\varphi$ is surjective. This enables us to conclude, based on Lemma 4.10, that the frame homomorphism $\mathrm{Fr}(h'{\restriction_\varphi})$ is successor-surjective. □

The conditions of Proposition 4.11 are normally inconsequential as all our examples of stratified institutions satisfy them. In fact, in many cases, including modal, hybrid logics, and variations thereof, both $[\![M']\!]_\varphi$ and $\mathrm{Fr}_\varphi(M')$ are identities, albeit in different categories. Besides being stable under reducts, it is also easy to see that any identity homomorphism is frame-bounded and that, for any signature Σ, both $\mathrm{E}(\Sigma)$-elementary and frame-bounded homomorphisms are closed under composition. This observation leads to the following result.

Corollary 4.12. *Let $\mathcal{S}$ be a stratified institution equipped with frame-bounded homomorphisms such that, for every signature morphism $\varphi \colon \Sigma \to \Sigma'$ and Σ'-model M', the state-reduction function $[\![M']\!]_\varphi$ is surjective. Then the functor $\mathrm{E} \subseteq \mathrm{Sen}$ generates a subfunctor $\mathrm{EH} \subseteq \mathrm{Mod}$ where, for every signature Σ, $\mathrm{EH}(\Sigma)$ is the broad subcategory of $\mathrm{Mod}(\Sigma)$ consisting of $\mathrm{E}(\Sigma)$-elementary homomorphisms.*

If, in addition, for every Σ'-model M', $\mathrm{Fr}_\varphi(M')$ is successor-surjective, then frame-bounded homomorphisms give rise to a subfunctor $\mathrm{FBH} \subseteq \mathrm{Mod}$ as in Definition 3.1.

Zig-zag relations

Frame-bounded homomorphisms are closely related to the following notion of 'zig-zag' relation. The terminology we use here is based on [Ben01].

Definition 4.13 (Zig-zag)**.** Let M and N be two Σ-models in a stratified institution equipped with a binary frame extraction $\langle \mathrm{L}, \mathrm{Fr} \rangle$. A relation $R \subseteq [\![M]\!] \times [\![N]\!]$ is *zig-zag* when, for every $(u, v) \in R$ and $\lambda \in \mathrm{L}(\Sigma)$, it simultaneously satisfies:

[*zig*] for every $s \in \mathrm{Fr}(M)_\lambda u$ there exists $t \in \mathrm{Fr}(N)_\lambda v$ such that $(s, t) \in R$;

[*zag*] for every $t \in \mathrm{Fr}(N)_\lambda v$ there exists $s \in \mathrm{Fr}(M)_\lambda u$ such that $(s, t) \in R$.

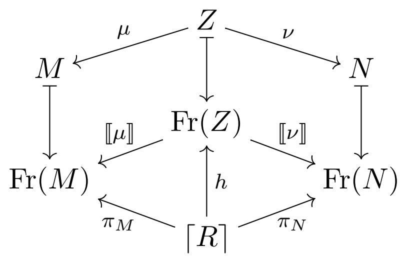
Remark 4.14. Every span $\langle \mu, Z, \nu \rangle$ of frame-bounded homomorphisms between models M and N determines a zig-zag relation given by $\{([\![\mu]\!] z, [\![\nu]\!] z) \in [\![M]\!] \times [\![N]\!] \mid z \in [\![Z]\!]\}$.

The converse of the property above does not hold universally, for arbitrary stratified models M and N, not even in the simple case of modal propositional logic, because bounded spans have a much richer theory than that of zig-zag relations. The remaining part of this section deals with conditions under which zig-zag relations can be covered by spans of frame-bounded homomorphisms.

Remark 4.15. Every relation R between state spaces of Σ-models M and N can be equipped with accessibility relations to form a Kripke frame $\lceil R \rceil$ given by $|\lceil R \rceil| = R$ and $\lceil R \rceil_\lambda = \pi_M^{-1}(\mathrm{Fr}(M)_\lambda) \cap \pi_N^{-1}(\mathrm{Fr}(N)_\lambda)$ for all modalities $\lambda \in \mathrm{L}(\Sigma)$. Together with the obvious projections $\pi_M \colon R \to [\![M]\!]$ and $\pi_N \colon R \to [\![N]\!]$, this construction yields a strong $|\_|$-cover $\langle \pi_M, \lceil R \rceil, \pi_N \rangle$ of R that is maximal among all such covers.

Proposition 4.16. *Let R be a zig-zag relation between the states spaces of two Σ-models M and N in a stratified institution with frame-bounded homomorphisms. Every strong cover of $\langle \pi_M, \lceil R \rceil, \pi_N \rangle$ along the functor* $(\mathrm{EH}(\Sigma) \subseteq \mathrm{Mod}(\Sigma)) \mathbin{;} \mathrm{Fr}$ *is also a cover of the relation R along* $(\mathrm{FBH}(\Sigma) \subseteq \mathrm{Mod}(\Sigma)) \mathbin{;} [\![\_]\!]$.

Proof. Let $\langle \mu, Z, \nu \rangle$ be a strong cover of $\langle \pi_M, \lceil R \rceil, \pi_N \rangle$ along $(\mathrm{EH}(\Sigma) \subseteq \mathrm{Mod}(\Sigma)) \mathbin{;} \mathrm{Fr}$. It follows that there exists an isomorphism of Kripke frames $h \colon \lceil R \rceil \to \mathrm{Fr}(Z)$ such that the next diagram commutes.



Since $[\![\_]\!] = \mathrm{Fr} \mathbin{;} \mathrm{L}|\_|$ by the very definition of frame extractions, it suffices to show that the homomorphisms μ and ν are frame-bounded. We already know that they are

$E(\Sigma)$-elementary, because $\langle \mu, Z, \nu \rangle$ is a cover along $(EH(\Sigma) \subseteq Mod(\Sigma)) \mathbin{;} Fr$. Hence all we need to check is that they satisfy the second property of Definition 4.6.

Suppose $([\![\mu]\!]z, s) \in Fr(M)_\lambda$ for some state $z \in [\![Z]\!]$ and modality $\lambda \in L(\Sigma)$. Since h has an inverse, it follows that $([\![\mu]\!]z, [\![\nu]\!]z) \in R$. So there exists a transition $([\![\nu]\!]z, t) \in Fr(N)_\lambda$ such that $(s, t) \in R$, because R is zig-zag. This gives us a state $h(s, t)$ of Z, which – by the definition of $\lceil R \rceil$, and because h preserves transitions – is a λ-successor of z satisfying $[\![\mu]\!]h(s, t) = s$. Consequently, μ is frame-bounded. The frame-boundedness of ν can be shown in a similar manner. □

Together with Corollary 3.8, Proposition 4.16 leads to the following main result:

Theorem 4.17. *Any stratified institution with frame-bounded homomorphisms has the HM property for any signature Σ such that:*

1. *all frame-bounded Σ-homomorphisms are elementary; and*
2. *for all Σ-models M and N, the elementary-equivalence relation $\equiv_{M,N}$ is zig-zag and the functor* $(EH(\Sigma) \subseteq Mod(\Sigma)) \mathbin{;} Fr$ *strongly covers* $\langle \pi_M, \lceil \equiv_{M,N} \rceil, \pi_N \rangle$. □

5 Saturated models

In many cases, the elementary-equivalence relation is not zig-zag. There are plenty examples both in the modal-logic literature [e.g., BRV01] and in automata theory [Par81] that illustrate this point. However, it is a zig-zag relation for certain classes of models. For instance, in its original formulation [HM85], the Hennessy-Milner theorem considers image-finite models. Their definition can be easily adapted to stratified institutions equipped with a frame extraction: a Σ-model M is *image-finite* when, for every state $w \in [\![M]\!]$ and every modality $\lambda \in L(\Sigma)$, the set $Fr(M)_\lambda w$ of λ-successors of w in M is finite. In the model theory of conventional modal logic [GO07], image-finiteness has a well-known generalization in the form of *modal saturation* for which the HM property can be shown as well [Gol95].

Definition 5.1 (Saturated model)**.** Let Σ be signature in a stratified institution equipped with a binary frame extraction $\langle L, Fr \rangle$. A set Γ of Σ-sentences is (infinitely) *satisfiable* in a set $S \subseteq [\![M]\!]$ of states of a Σ-model M when there exists $w \in S$ such that $M \models^w \Gamma$; and it is *κ-satisfiable* in S, for some cardinal number κ, when every subset of Γ of cardinality less than κ is satisfiable in S.

A Σ-model M is *modally κ-saturated* when, for every $\lambda \in L(\Sigma)$ and $w \in [\![M]\!]$, every set $\Gamma \subseteq Sen(\Sigma)$ that is κ-satisfiable in $Fr(M)_\lambda w$ is satisfiable in $Fr(M)_\lambda w$. We say that M is *finitely-saturated* when it is κ-saturated for every infinite cardinal κ.

Remark 5.2. Every image-finite model is finitely-saturated.

Like bounded homomorphisms, for every signature Σ, κ-saturated models also form a subcategory of $\mathrm{Mod}(\Sigma)$; but this time a full subcategory, which we denote by κ-satMod(Σ). The next result shows that the inclusion $\kappa\text{-satMod}(\Sigma) \subseteq \mathrm{Mod}(\Sigma)$ is natural in Σ, hence κ-satMod extends to another subfunctor of Mod.

Proposition 5.3. *Let $\varphi\colon \Sigma \to \Sigma'$ be a signature morphism in a stratified institution equipped with a binary frame extraction $\langle \mathrm{L}, \mathrm{Fr} \rangle$. For every κ-saturated Σ'-model M', the reduct $M'{\restriction_\varphi}$ is κ-saturated too, provided that the state-reduction function $[\![M']\!]_\varphi$ is surjective and the frame homomorphism $\mathrm{Fr}_\varphi(M')$ is successor-surjective.*

Proof. Suppose M' is a κ-saturated Σ'-model and let $\lambda \in \mathrm{L}(\Sigma)$, $w \in [\![M'{\restriction_\varphi}]\!]$, and $\Gamma \subseteq \mathrm{Sen}(\Sigma)$ such that Γ is κ-satisfiable in $\mathrm{Fr}(M'{\restriction_\varphi})_\lambda w$. Since the function $[\![M']\!]_\varphi$ is surjective, there exists $w' \in [\![M']\!]$ such that $w = [\![M']\!]_\varphi w'$.

We begin by showing that $\varphi(\Gamma)$ is κ-satisfiable in $\mathrm{Fr}(M')_{\lambda'} w'$ for $\lambda' = \mathrm{L}(\varphi)(\lambda)$. With that goal, take $\Delta' \subseteq \varphi(\Gamma)$ of cardinality less than κ. It follows that we can choose a equinumerous subset $\Delta \subseteq \Gamma \cap \varphi^{-1}(\Delta')$ such that $\varphi(\Delta) = \Delta'$. Since Γ is κ-satisfiable in $\mathrm{Fr}(M'{\restriction_\varphi})_\lambda w$, there exists a λ-successor s of w in $M'{\restriction_\varphi}$ such that $M'{\restriction_\varphi} \vDash^s \Delta$. Then $s = [\![M']\!]_\varphi s'$ for some λ'-successor s' of w' in M', because $\mathrm{Fr}(M')_\varphi$ is successor-surjective; so, by the satisfaction condition for φ, $M' \vDash^{s'} \varphi(\Delta)$.

Since the model M' is κ-saturated, it follows that the set $\varphi(\Gamma)$ is satisfiable at some state $u' \in \mathrm{Fr}(M')_{\lambda'} w'$. Therefore, by the satisfaction condition for φ, we have $M'{\restriction_\varphi} \vDash^{[\![M']\!]_\varphi u'} \gamma$ for all sentences $\gamma \in \Gamma$. Coupled with the fact that $[\![M']\!]_\varphi u'$ is a λ-successor of $w = [\![M']\!]_\varphi w'$ in $M'{\restriction_\varphi}$, which holds because $\mathrm{Fr}_\varphi(M')$ is a frame homomorphism, this entails that Γ is satisfiable in $\mathrm{Fr}(M'{\restriction_\varphi})_\lambda w$. ☐

Corollary 5.4. *Let $\mathcal{S}$ be a stratified institution with surjective reduction functions $[\![M']\!]_\varphi$ and successor-surjective homomorphisms $\mathrm{Fr}_\varphi(M')$ for all signature morphisms $\varphi\colon \Sigma \to \Sigma'$ and all Σ'-models M'. By restricting the stratified models of $\mathcal{S}$ to those that are κ-saturated we obtain a sub-institution of $\mathcal{S}$, which we denote by κ-sat$(\mathcal{S})$.* ☐

The κ-saturation property is not only preserved by model-reduct functors but also reflected by homomorphisms that are frame-bounded and elementary. The next proposition is used in the final part of the section to show that bisimulations between κ-saturated Σ-models are represented by spans in κ-satMod(Σ).

Lemma 5.5. *Let $h\colon M \to N$ be a frame-bounded and elementary Σ-homomorphism in an arbitrary stratified institution. If N is κ-saturated, then so is M.*

Proof. Take a state $w \in [\![M]\!]$ and let Γ be a set of Σ-sentences that is κ-satisfiable in $\mathrm{Fr}(M)_\lambda w$ for some modality $\lambda \in \mathrm{L}(\Sigma)$. This means that for any subset $\Delta \subseteq \Gamma$ of cardinality less than κ, there exists a λ-successor s of w in M such that $M \vDash^s \Delta$. Since h is elementary, it follows that $N \vDash^{[\![h]\!]s} \Delta$. Moreover, the state $[\![h]\!]s$ must be a λ-successor of $[\![h]\!]w$ in N. So Γ is κ-satisfiable in $\mathrm{Fr}(N)_\lambda([\![h]\!]w)$.

Since N is κ-saturated, the whole set Γ is satisfiable in $\mathrm{Fr}(N)_\lambda(\llbracket h \rrbracket w)$. Therefore, Γ is also satisfiable in $\llbracket h \rrbracket(\mathrm{Fr}(M)_\lambda w)$ because h is frame-bounded. In other words, there exists a λ-successor x of w in M such that $N \vDash^{\llbracket h \rrbracket x} \Gamma$. The last property is equivalent to $M \vDash^x \Gamma$ because h is elementary, hence Γ is satisfiable in $\mathrm{Fr}(M)_\lambda w$. $\square$

It only remains to argue that, for saturated models, the familiar zig-zag property of concrete elementary-equivalence relations from modal logics is an instance of a more general phenomenon that can be described for arbitrary stratified institutions. For this purpose, we need two more ingredients: a semantic treatment of connectives, which we recall from [Dia17], and consistent sets of sentences [see, e.g., Tar86].

Definition 5.6 (Connectives)**.** For any signature Σ in a stratified institution:

- a sentence ρ is a *semantic conjunction* of $\Gamma \subseteq \mathrm{Sen}(\Sigma)$ when, for every Σ-model M and state $w \in \llbracket M \rrbracket$, $M \vDash^w \rho$ if and only if $M \vDash^w \gamma$ for all $\gamma \in \Gamma$;
- a sentence ρ is a *semantic negation* of another sentence γ when, for every Σ-model M and state $w \in \llbracket M \rrbracket$, $M \vDash^w \rho$ if and only if $M \nvDash^w \gamma$.

If, in addition, the stratified institution is equipped with a binary frame extraction $\langle \mathrm{L}, \mathrm{Fr} \rangle$, then for every modality $\lambda \in \mathrm{L}(\Sigma)$:

- a sentence ρ is a *semantic λ-possibility* of $\gamma \in \mathrm{Sen}(\Sigma)$ when, for every Σ-model M and state $w \in \llbracket M \rrbracket$, $M \vDash^w \rho$ if and only if $M \vDash^s \gamma$ for some $s \in \mathrm{Fr}(M)_\lambda w$.

We say that a stratified institution $\mathcal{S}$ *has semantic κ-conjunctions* for some cardinal number κ when, for every signature Σ, every subset $\Gamma \subseteq \mathrm{Sen}(\Sigma)$ of cardinality less than κ admits a semantic conjunction. We also way that $\mathcal{S}$ *has semantics negations* when each of its sentences admits a semantic negation; and that it has *semantic possibilities* when, for every modality $\lambda \in \mathrm{L}(\Sigma)$, every Σ-sentence admits a λ-possibility.

When semantic conjunctions, negations, or possibilities exist, we typically denote them by $\bigwedge \Gamma$, $\neg\gamma$, or $\langle \lambda \rangle \gamma$, respectively. This notation does not uniquely identify them as sentences, nor does it imply the existence of corresponding syntactic connectives.

Example 5.7. It is easy to see that, for non-empty signatures, both MPL and HPL have finite semantic conjunctions, negations, and possibilities. The 'non-empty' requirement is important here because, otherwise, the empty set of sentences does not admit a semantic conjunction – which follows from the observation that, for MPL and HPL, the empty signature determines an empty set of sentences. For every non-empty signature, say containing a propositional symbol $\pi \in P$, the sentence $\neg(\pi \wedge \neg\pi)$, which is equivalent to *true*, is obviously a semantic conjunction of $\emptyset$.

The institutions NFA and DFA do not have semantic conjunctions or negations, but they have possibilities: given an alphabet A, the composite word aw is a semantic a-possibility of w for every input symbol $a \in A$ and every sentence $w \in A^*$. In the power-set extension $\mathcal{P}$DFA, we recover semantic conjunctions, which are given by set-theoretic unions of languages, but negations are still missing.

Like MPL and HPL, the stratified institution OFOL admits finite semantic conjunctions and negations for all signatures that have a non-empty set of atomic sentences. Moreover, given a signature $\langle \Sigma, X \rangle$, for every modality $x \in \mathrm{L}(\langle \Sigma, X \rangle)$ – i.e., following Example 4.5, for every variable $x \in X$ – the existentially quantified sentence $\exists x \cdot \rho$ is a semantic x-possibility of ρ, for all $\langle \Sigma, X \rangle$-sentences ρ.

Definition 5.8 (Consistency)**.** A set Γ of Σ-sentences in an arbitrary stratified institution is *consistent* when there exist a Σ-model M and a state $w \in [\![M]\!]$ such that $M \vDash^w \Gamma$. And it is *maximally consistent* when it is consistent and maximal with this property – i.e., it has no proper consistent superset.

We say that $\mathcal{S}$ has *maximally consistent state theories* when, for every Σ-model M and $w \in [\![M]\!]$, the set $\langle M, w \rangle^* = \{\rho \in \mathrm{Sen}(\Sigma) \mid M \vDash^w \rho\}$ is maximally consistent.

Remark 5.9. If a stratified institution has semantic negations, then it also has maximally consistent state theories. For this reason, except for the logics that have automata as models, all other examples of stratified institutions from Section 2 have maximally consistent state theories. The stratified institution NFA and its variants have neither negations nor maximally consistent state theories. That is because, for any alphabet (signature) A in NFA or DFA, the state theories of A-automata correspond to regular languages with symbols from A, of which only A^*, the language consisting of all words, is maximally consistent. A similar observation applies to $\mathcal{P}$DFA, whose state theories are power-sets of regular languages.

Proposition 5.10. *Consider a stratified institution $\mathcal{S}$ with semantic κ-conjunctions and possibilities, and let u and v be states of Σ-models M and N, respectively, such that $\langle M, u \rangle^* \subseteq \langle N, v \rangle^*$. If N is κ-saturated, then for every modality $\lambda \in \mathrm{L}(\Sigma)$ and state $s \in \mathrm{Fr}(M)_\lambda u$ there exists a state $t \in \mathrm{Fr}(N)_\lambda v$ such that $\langle M, s \rangle^* \subseteq \langle N, t \rangle^*$.*

Proof. Suppose s is a λ-successor of u in M. Since $\mathcal{S}$ has semantic κ-conjunctions and possibilities, it follows that, for every subset $\Delta \subseteq \langle M, s \rangle^*$ of cardinality less than κ, we have $M \vDash^u \langle \lambda \rangle \bigwedge \Delta$. By assumption, $N \vDash^v \langle \lambda \rangle \bigwedge \Delta$, hence Δ is satisfiable in $\mathrm{Fr}(N)_\lambda v$. The κ-saturation of N further implies that $\langle M, s \rangle^*$ is satisfiable in $\mathrm{Fr}(N)_\lambda v$, so there exists a λ-successor t of v in N at which all the sentences in $\langle M, s \rangle^*$ hold. □

Applying Proposition 5.10 twice, from left to right, and vice versa, for elementary-equivalent states – which satisfy $\langle M, u \rangle^* = \langle N, v \rangle^*$ – yields the following property.

Corollary 5.11. *In any stratified institution with maximally consistent state theories, semantic κ-conjunctions, and possibilities, the elementary-equivalence relation $\equiv_{M,N}$ is zig-zag for all κ-saturated models M and N.* □

This allows us to further refine Theorem 4.17 into a result that applies to κ-saturated models. To that end, notice that every selection of bounded homomorphisms for an institution S trivially determines a selection of bounded homomorphisms for κ-sat$(\mathcal{S})$. Moreover, since κ-saturation only restricts the models of $\mathcal{S}$, it follows that κ-sat$(\mathcal{S})$ has maximally consistent state theories, as well as semantic κ-conjunctions and possibilities whenever $\mathcal{S}$ has those properties. By Corollary 5.11, we can safely drop the zig-zag requirement on $\equiv_{M,N}$ from condition 2 of Theorem 4.17; and by Proposition 4.16 and Lemma 5.5 we know that, for κ-saturated Σ-models M and N, any cover of $\langle \pi_M, \lceil \equiv_{M,N} \rceil, \pi_N \rangle$ along $(\mathrm{EH}(\Sigma) \subseteq \mathrm{Mod}(\Sigma)) \mathbin{⨾} \mathrm{Fr}$ is a span in κ-satMod(Σ). All these observations lead to the following simplified formulation.

Theorem 5.12. *Let $\mathcal{S}$ be a stratified institution with frame-bounded homomorphisms that has maximally consistent state theories, semantic κ-conjunctions, and possibilities. Then κ-sat$(\mathcal{S})$ has the HM property for every signature Σ such that:*

1. *all frame-bounded Σ-homomorphisms are elementary; and*
2. *the functor $(\mathrm{EH}(\Sigma) \subseteq \mathrm{Mod}(\Sigma)) \mathbin{⨾} \mathrm{Fr}$ strongly covers $\langle \pi_M, \lceil \equiv_{M,N} \rceil, \pi_N \rangle$ for all κ-saturated Σ-models M and N.* □

Except for the stratified institution NFA, for which the HM property is known to fail [e.g., Par81], all other logics presented in Section 2 admit Hennessy-Milner theorems based either on Theorem 4.17 or on Theorem 5.12. The latter applies to stratified institutions of modal and hybrid logics and to OFOL, whereas the former applies to the stratified institution $\mathcal{P}$DFA and – indirectly – to DFA.

To be more precise, condition 1, which entails the 'soundness' of bisimilarity, generally amounts to proving that the elementary property of frame-bounded homomorphisms extends from atoms to more complex sentences. The proofs are routinely done by structural induction [e.g., BB07, Bisimulation Invariance Lemma].

The second condition, from which the 'completeness' of bisimilarity follows, is also easy to check in conventional settings. For instance, in MPL, for any pair of finitely-saturated models M and N over some signature P, the cover of $\equiv_{M,N}$ that we need can be obtained by building an adequate submodel Z of the product $M \times N$: its underlying Kripke frame is given by the construction outlined in Remark 4.15, hence $\mathrm{Fr}(Z) = \lceil \equiv_{M,N} \rceil$; and for any pair $(u, v) \in [\![M]\!] \times [\![N]\!]$ of elementary-equivalent worlds and any propositional symbol $\pi \in P$, we let $Z \models^{(u,v)} \pi$ if and only if $M \models^{u} \pi$ – or, equivalently, $N \models^{v} \pi$. Similar constructions apply to HPL and OFOL, and to many other stratified institutions of modal and hybrid logics.

For $\mathcal{P}$DFA, the second condition of Theorem 4.17 follows from an argument that is analogous to the one presented above for MPL, save that the zig-zag property of $\equiv_{M,N}$ is not entailed by the existence of maximally consistent state theories; instead, it follows from Proposition 5.10 and the functionality of the transition relations of deterministic automata. The HM property of DFA follows from that of $\mathcal{P}$DFA *by translation*, based on the observation that the two institutions have the same signatures, models, and stratification, and that two pointed models are elementary equivalent in DFA if and only if they are elementary equivalent in $\mathcal{P}$DFA.

6 Conclusions

In this paper, we have shown that stratified institutions can be easily enriched with a notion of bounded homomorphism to provide an institution-independent framework for studying bisimulations. Within that framework, we have examined the relationship between the concepts of bisimilarity and elementary equivalence at several model-theoretic levels of complexity: for unconstrained stratified institutions, for institutions equipped with a binary frame extraction, and in the context of modally saturated models. For each of those levels, we have presented additional constructions and sufficient conditions under which – similarly to Hennessy and Milner's famous characterization of process invariance using modal satisfaction [HM85] – the bisimilarity and the elementary equivalence of states are proved to coincide.

This work establishes guidelines for selecting bounded homomorphisms that support the subsequent development of Hennessy-Milner theorems for logical systems that are much more complex than those presented in Section 2. A vast family of such logics can be obtained, for instance, by iterating the modalization and hybridization processes outlined in Examples 2.3 and 2.4. The unconstrained version of those constructions yields stratified institutions to which we can apply with ease the results presented in Section 5. In that sense, the Hennessy-Milner theorem for hierarchical hybrid logic from [Mad+17] can be obtained as a corollary of Theorem 5.12. In contrast, for constrained models [e.g., Dia16; GŢ19; ŢCF21], the second condition of Theorem 5.12 is more challenging to check and open to further investigation.

The framework we have developed here also provides a basis upon which a richer theory of stratified bisimulations can be developed. For example, following Definition 5.1, the construction and study of ultrafilter extensions seems to be well within reach. In addition, following Corollary 3.11, other transfer properties can be shown for maps between stratified institutions along the lines of [CM97]. We have already seen such a technique in action, in a significantly simplified form, when we established that the HM property holds for the stratified institution DFA.

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