Transporting connectives along parchment addenda*

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Institutional parchments provide algebraic presentations of logics formalized as institutions by using initial-algebra semantics to define both the syntax of sentences and their evaluation in models. In this paper, we advance a notion of stratified parchment that is adapted to presenting institutions where the satisfaction relation between models and sentences is parameterized by model states. We show how stratified parchments can be used to capture logical connectives, and introduce parchment addenda, which allow connectives to be transported between logical systems by means of universal category-theoretic constructions.

Keywords: Stratified institution, Abstract connective, Parchment, Parchment addendum, Logic combination

1 Introduction

The model-theoretic approach to formalizing the intuitive notion of logical system has led to the development of several families of categorical frameworks that focus on different aspects of logics. In increasing order of complexity, we have: specification frames [EG94], which account for categories of models indexed by abstract specifications; institutions [GB83; GB92], which show how specifications can arise from signatures and sentences that are connected to models by means of abstract satisfaction relations; from the same authors, we also have charters [GB86], which

*This work was supported by a grant of the Romanian Ministry of Education and Research, CNCS – UEFISCDI, project number PN-III-P4-ID-PCE-2020-0446, within PNCDI III.

generate institutions out of adjunctions between categories of syntactic entities; and parchments [GB86], which present charters by means of initial, term algebras that capture the abstract syntax of sentences. Each of those frameworks include specialized categorical structures – and corresponding theories built around them – to cater for various aspects of logics or logical phenomena that are particular to some application domain. For example, general logics [Mes89] provide support for syntactic entailment [see also Mos+07], context institutions [Paw95] include explicit open formulae and valuations, substitution systems [ŢF17] focus on logical systems used in logic programming, and stratified institutions [AD07] capture logics where the satisfaction relations between models and sentences are parameterized by so-called *states* of models. Likewise, for parchments, we have λ-parchments [MTP97a], model-theoretic parchments [MTP97b], and **c**-parchments [Cal+01], all of which support theories that tackle the difficult problem of combining logics.

In this paper, we introduce stratified parchments as a form of model-theoretic parchments that are specifically designed to present stratified institutions. Compared with previous notions of parchment, which rely on many-sorted algebra, stratified parchments integrate initial-algebra semantics with power-set algebras, which can be regarded as algebras with power sets as carriers. This makes stratified parchments particularly well suited for dealing with logical connectives, whose semantics can naturally be presented in terms of operations on sets. To illustrate their use, consider a sentence of the form $p \wedge q$, say in basic modal logic, which can be viewed as a term over an algebraic signature where $\wedge$ is a binary operation symbol. Then, in any Kripke structure $\langle W, M \rangle$, the sentence $p \wedge q$ holds precisely at those possible worlds (or states, in terminology that is specific to stratified institutions) where both p and q hold. We capture this property by defining a power-set algebra for $\langle W, M \rangle$ that interprets the conjunction as intersection (applied to sets of states).

Another distinctive attribute of stratified parchments is that, for any signature Σ – meant to introduce non-logical symbols such as p and q in the example above – we impose no special constraints on the algebraic signature $\mathrm{L}(\Sigma)$ that defines the syntax of Σ-sentences, and also no constraints on the power-set algebras that correspond to Σ-models. This means that, unlike [GB86; and also MTP97b; Cal+01], we do not require $\mathrm{L}(\Sigma)$ to contain a distinguished sort $*$ of sentences, nor to interpret any of the sorts in $\mathrm{L}(\Sigma)$ as a set of truth values. Instead, in the stratified approach, all the sorts in $\mathrm{L}(\Sigma)$ may be regarded as sentence types, and their interpretations in power-set algebras correspond to (sets of) sets of states where certain properties hold. Admitting multiple sentence types turns out to be convenient for formalizing logics with quantifiers, because every set X of variables may be regarded as the type of open formulae with variables in X – in that case, the semantics of such a formula is given by the set of all valuations of X for which the formula holds.

Within this framework, we examine a simplified form of logic combination whereby a given base logic, formalized as a stratified parchment, can be extended by 'borrowing' logical connectives that are defined externally, in another stratified parchment. To that end, we introduce parchment addenda, which may be seen as recipes for borrowing connectives. We define concrete addenda for logical connectives used in modal and first-order logics, and study conditions that allow multiple addenda can be sequentially applied to a parchment. Then we show that most addenda can be equivalently presented as cospans in a category of morphisms of stratified parchments, and that extending a base parchment along given addenda is the same as taking limits of their corresponding cospans. The advantage of this connective and addenda-oriented construction, as opposed to using arbitrary limits, is that it yields results that require no further adjustments – which is known to be a recurring complication in combining parchments by means of universal constructions.

The paper is organized as follows. In Section 2, we review basic notions from many-sorted algebra and briefly present power-set algebras. In Section 3, we introduce the concept of stratified parchment, give a few examples, and establish the connection with stratified institutions. In Section 4, we discuss addenda and demonstrate how they can be used to build parchments of modal and first-order logics. Finally, in Section 5, we introduce morphisms of stratified parchments and show that parchment extensions along addenda can often be obtained through iterated pullbacks.

2 Algebraic preliminaries

The framework we propose in this paper makes ample use of many-sorted algebra. Hence, to fix notations and terminology, we begin by recalling a few elementary algebraic concepts and facts. For details, see, for instance, the monograph [ST11].

An algebraic signature is a pair $\langle S, F\rangle$, where S is a set whose elements we call *sorts*, and F is a family of sets $F_{w\to s}$ of *operation symbols* indexed by arities $w \in S^*$ and sorts $s \in S$. We often write $\sigma\colon w \to s$ to indicate that σ is an operation symbol of arity w and sort s. In addition, we denote the empty arity by ε and refer to the elements of $F_{\varepsilon\to s}$ as *constant-operation symbols* of sort s.

An $\langle S, F\rangle$-algebra A interprets each sort $s \in S$ as a set A_s, called the *carrier set* of s in A, and each operation symbol $\sigma\colon w \to s$ in F as a function $A_\sigma\colon A_w \to A_s$, where $A_w = A_{s_1} \times A_{s_2} \times \cdots \times A_{s_n}$ for $w = s_1 s_2 \cdots s_n$. An $\langle S, F\rangle$-homomorphism h between algebras A and B is an S-sorted function $h\colon A \to B$ such that, for every operation symbol $\sigma\colon w \to s$ in F, we have $h_s(A_\sigma(a)) = B_\sigma(h_w(a))$ for all $a \in A_w$, where $h_w\colon A_w \to B_w$ is the obvious extension of h to tuples of type w.

For every algebraic signature $\langle S, F\rangle$, there is a particular algebra, denoted $\mathrm{T}_{\langle S,F\rangle}$ or just T_F when S can be easily inferred, whose elements are terms built from the operation symbols in F. Its carriers form the least S-sorted set T_F such that "$\sigma(t)$" $\in \mathrm{T}_{F,s}$ for all operation symbols $\sigma\colon w \to s$ in F and all tuples $t \in \mathrm{T}_{F,w}$. In addition, the algebra T_F interprets each operation symbol $\sigma\colon w \to s$ in F as the function $\mathrm{T}_{F,\sigma}$ that maps every tuple $t \in \mathrm{T}_{F,w}$ to the term "$\sigma(t)$" $\in \mathrm{T}_{F,s}$. The characteristic property of T_F is that, for any $\langle S, F\rangle$-algebra A, there exists a unique 'evaluation' homomorphism $\mathrm{T}_F \to A$. We denote the image of a term t under that homomorphism – i.e., the evaluation of the term t in A – by A_t.

Algebraic signature morphisms $\varphi\colon \langle S, F\rangle \to \langle S', F'\rangle$ map sorts $s \in S$ to sorts $\varphi(s) \in S'$ and operation symbols $\sigma \in F_{w\to s}$ to symbols $\varphi(\sigma) \in F'_{\varphi(w)\to\varphi(s)}$. They determine contravariant reductions of algebras where, for every $\langle S', F'\rangle$-algebra A', $A'\!\upharpoonright_\varphi$ is the $\langle S, F\rangle$-algebra given by $(A'\!\upharpoonright_\varphi)_\varsigma = A'_{\varphi(\varsigma)}$ for all sorts or operation symbols ς in $\langle S, F\rangle$. The same holds for $\langle S', F'\rangle$-homomorphisms.

Power-set algebras are algebras whose carrier sets are power sets. To be more precise, a power-set $\langle S, F\rangle$-algebra A interprets each sort $s \in S$ as a set A_s – just like ordinary algebras do – and each operation symbol $\sigma\colon w \to s$ in F as a function $A_\sigma\colon \mathcal{P}(A)_w \to \mathcal{P}(A)_s$, where $\mathcal{P}(A)$ is the S-sorted set given by $\mathcal{P}(A)_s = \mathcal{P}(A_s)$ for all $s \in S$. We say that an S-sorted map $h\colon A \to B$ between power-set $\langle S, F\rangle$-algebras *preserves* an operation symbol $\sigma\colon w \to s$ in F when $h_s(A_\sigma(X)) \subseteq B_\sigma(h_w(X))$ for all $X \in \mathcal{P}(A)_w$, and that it *reflects* σ when the opposite inclusion holds. Therefore, power-set maps $h\colon A \to B$ that preserve and reflect all the operation symbols in F are ordinary $\langle S, F\rangle$-homomorphisms between the algebras A and B.

Besides many-sorted algebra, throughout the paper we assume familiarity with basic notions of category and institution theory; we refer the interested reader to [Dia08; ST11] for a detailed introduction to institutions. The category-theoretic terminology and notations that we use are primarily based on [Mac98], except for the composition of morphisms f and g, which we prefer to write in diagrammatic order as $f \mathbin{⨟} g$, and for natural transformations, which we write using a double arrow. We let $\mathbb{S}\mathrm{et}$ denote the category of sets and functions, $\mathbb{A}\mathrm{lg}\mathbb{S}\mathrm{ig}$ denote the category of algebraic signatures and their morphisms, and $\mathbb{C}\mathrm{at}$ denote the higher category of categories and functors. We also let $\mathcal{P}\mathrm{Alg}$ be the contravariant functor $\mathbb{A}\mathrm{lg}\mathbb{S}\mathrm{ig} \to \mathbb{C}\mathrm{at}$ that maps (a) every algebraic signature $\langle S, F\rangle$ to the category $\mathcal{P}\mathrm{Alg}(S, F)$ whose objects are power-set $\langle S, F\rangle$-algebras and whose arrows are S-sorted maps; and (b) every signature morphism $\varphi\colon \langle S, F\rangle \to \langle S', F'\rangle$ to the functor $\mathcal{P}\mathrm{Alg}(\varphi)\colon \mathcal{P}\mathrm{Alg}(S', F') \to \mathcal{P}\mathrm{Alg}(S, F)$ given by $\mathcal{P}\mathrm{Alg}(\varphi)(A') = A'\!\upharpoonright_\varphi$ for all power-set $\langle S', F'\rangle$-algebras A'.

3 Many-sorted strata

Stratified parchments arise from the combination of two main ideas, each of which underlies an important line of development in institution theory. On the one hand, there is the idea that the sentences of a logic can be introduced as terms over carefully designed algebraic signatures, which then allows for evaluating sentences in models using the initiality property of the term algebra. This is the approach taken in institutional parchments [e.g., GB86; MTP97b; Mos+14]. On the other hand, there is an idea that stems from work on stratified institutions [AD07; Dia17] – and is emblematic of conventional modal logics [BRV01] – according to which models can be equipped with states where sentences are evaluated; hence, the evaluation of a sentence in a model gives rise, in this setting, to a many-valued outcome consisting in the set of all states where that sentence holds. We bring these views together by adding a corresponding algebraic structure on sets of model states.

Definition 3.1. A *stratified parchment* is a tuple $\langle \mathbb{S}\mathrm{ig}, \mathrm{L}, \mathrm{Mod}, \mathrm{K} \rangle$ consisting of:

- a category $\mathbb{S}\mathrm{ig}$ of *signatures* and *signature morphisms*;
- a *language* functor $\mathrm{L}\colon \mathbb{S}\mathrm{ig} \to \mathbb{A}\mathrm{lg}\mathbb{S}\mathrm{ig}$ defining, for every signature Σ, a '*grammar*' $\mathrm{L}(\Sigma)$ of Σ-*sentences* – in the form of an algebraic signature – and, for every signature morphism $\varphi\colon \Sigma \to \Sigma'$, a *syntax-translation map* $\mathrm{L}(\varphi)\colon \mathrm{L}(\Sigma) \to \mathrm{L}(\Sigma')$;
- a *model* functor $\mathrm{Mod}\colon \mathbb{S}\mathrm{ig}^{\mathrm{op}} \to \mathbb{C}\mathrm{at}$ defining, for every signature Σ, a category $\mathrm{Mod}(\Sigma)$ of Σ-*models* and *homomorphisms* and, for every signature morphism $\varphi\colon \Sigma \to \Sigma'$, a *model-reduction functor* $\mathrm{Mod}(\varphi)\colon \mathrm{Mod}(\Sigma') \to \mathrm{Mod}(\Sigma)$; and
- a *stratification* lax natural transformation $\mathrm{K}\colon \mathrm{Mod} \Rightarrow \mathrm{L}^{\mathrm{op}} \mathbin{⨾} \mathcal{P}\mathrm{Alg}$ defining, for every signature Σ, a *state-space functor* $\mathrm{K}_\Sigma\colon \mathrm{Mod}(\Sigma) \to \mathcal{P}\mathrm{Alg}(\mathrm{L}(\Sigma))$ and, for every signature morphism $\varphi\colon \Sigma \to \Sigma'$, a *state-reduction* natural transformation $\mathrm{K}_\varphi\colon \mathrm{K}_{\Sigma'} \mathbin{⨾} \mathcal{P}\mathrm{Alg}(\mathrm{L}(\varphi)) \Rightarrow \mathrm{Mod}(\varphi) \mathbin{⨾} \mathrm{K}_\Sigma$

such that, for every signature morphism $\varphi\colon \Sigma \to \Sigma'$ and every Σ'-model M', the map $\mathrm{K}_\varphi(M')$ preserves and reflects the interpretation of all symbols in $\mathrm{L}(\Sigma)$.

Notations and terminology When dealing with multiple stratified parchments, we use names as superscripts to distinguish the components of various parchments. For example, we may write $\mathbb{S}\mathrm{ig}^{\mathcal{S}}$ to denote the category of signatures of a parchment $\mathcal{S}$, and $\mathrm{K}^{\mathcal{S}}_\Sigma$ to denote the stratification for a signature Σ in a parchment $\mathcal{S}$.

Given a signature Σ, we let $\mathrm{L}(\Sigma) = \langle \mathrm{ST}(\Sigma), \mathrm{LC}(\Sigma) \rangle$. We refer to the elements of $\mathrm{ST}(\Sigma)$ as *sentence/stratum types* of Σ, to the constant-operation symbols in $\mathrm{LC}(\Sigma)_{\varepsilon \to s}$ as Σ-*primitives* of type $s \in \mathrm{ST}(\Sigma)$, and to the operation symbols in $\mathrm{LC}(\Sigma)_{a \to s}$, for non-empty $a \in \mathrm{ST}(\Sigma)^+$ and $s \in \mathrm{ST}(\Sigma)$, as *logical connectives* of arity a and type s. We also refer to the $\mathrm{L}(\Sigma)$-terms of sort s as Σ-*sentences* of type s. So, a sentence is either a primitive (e.g., an atomic statement) or a compound expression obtained from primitives through repeated applications of logical connectives.

To anticipate the connection with stratified institutions, for every Σ-model M and every stratum type $s \in \mathrm{ST}(\Sigma)$, we also denote the carrier $\mathrm{K}_\Sigma(M)_s$ of s in $\mathrm{K}_\Sigma(M)$ by $[\![M]\!]_s$, and we refer to the elements of $[\![M]\!]_s$ as *states* of M of type s.

In order to simplify the notations used, we follow a common practice in institution theory and write $\varphi(s)$, $\varphi(\kappa)$, and $\varphi(\rho)$ for the translation of a stratum type s, logical connective κ, or sentence ρ, respectively, along the algebraic signature morphism $\mathrm{L}(\varphi)$. We also denote the reduct functor $\mathrm{Mod}(\varphi)$ by $\_{\restriction_\varphi}$, and we say that a model M is the φ-*reduct* of M', or that M' is a φ-*expansion* of M, whenever $M = M'{\restriction_\varphi}$. Under these notations, the map $\mathrm{K}_\varphi(M')$ that captures the reduction of the states of a Σ'-model M' along a signature morphism $\varphi\colon \Sigma \to \Sigma'$, yielding states of the reduced Σ-model $M'\restriction_\varphi$, may also be written as $\mathrm{K}_\varphi(M')\colon \mathrm{K}_{\Sigma'}(M')\restriction_{\mathrm{L}(\varphi)} \to \mathrm{K}_\Sigma(M'\restriction_\varphi)$. When the context allows it, we may further overload the reduct notation, and denote the φ-reduct $\mathrm{K}_\varphi(M')(w')$ of a state w' of M' by $w'\restriction_\varphi$.

Strict stratifications All examples and most of the results that we present herein concern *strict* stratified parchments, i.e., parchments whose stratification is a strict natural transformation. In other words, for every signature morphism $\varphi\colon \Sigma \to \Sigma'$ and Σ'-model M' in such a parchment, the state-reduction map $\mathrm{K}_\varphi(M')$ is an identity. This leads to a much simpler conceptual framework that is convenient for introducing parchment addenda and combinations of logical connectives in the later parts of the paper. However, in general, stratified parchments need not be strict. We use the original lax definition when discussing the connection between stratified parchments and stratified institutions, and the strict variant in most other places.

Example 3.2 (MPL)**.** Modal logic [BRV01] provides some of the simplest yet instructive examples of stratified institutions and parchments. For instance, let MPL be the strict parchment of modal propositional logic – which, by the subsequent Proposition 3.6, can be shown to correspond to the stratified institution of modal propositional logic [Dia17]. The signatures of MPL are sets of *propositional symbols*, while signature morphisms are ordinary functions; hence $\mathbb{S}\mathrm{ig}^{\mathrm{MPL}} = \mathbb{S}\mathrm{et}$. For every signature P, $\mathrm{L}^{\mathrm{MPL}}(P)$ is the algebraic signature given by:

sort Sen
ops "p": $\varepsilon \to Sen$ [for every $p \in P$]
$false\colon \varepsilon \to Sen$
$\Rightarrow\colon Sen\, Sen \to Sen$
$\Diamond\colon Sen \to Sen$

For every function $\varphi\colon P \to P'$, $\mathrm{L}^{\mathrm{MPL}}(\varphi)$ is the algebraic signature morphism that maps "p" to "$\varphi(p)$" for every propositional symbol $p \in P$ and leaves unchanged all other operation symbols – i.e., Boolean and modal connectives – in $\mathrm{L}^{\mathrm{MPL}}(P)$.

The models in $\mathrm{Mod}^{\mathrm{MPL}}(P)$ are Kripke *structures* $\langle W, M\rangle$ where:

- W is a Kripke *frame* consisting of a set $|W|$ whose elements are called *possible worlds* and a binary *accessibility relation* W_λ on $|W|$, and
- M is a function assigning a subset $M(w) \subseteq P$ to every possible world $w \in |W|$.

Homomorphisms $\langle W_1, M_1\rangle \to \langle W_2, M_2\rangle$ are functions $h\colon |W_1| \to |W_2|$ such that $h(W_{1,\lambda}) \subseteq W_{2,\lambda}$, meaning that h is a frame homomorphism, and $M_1(w) \subseteq M_2(h(w))$ for every possible world w in W_1. The reduction of a P'-model $\langle W', M'\rangle$ along a function $\varphi\colon P \to P'$ is the Kripke structure $\langle W', M\rangle$ with the same possible worlds and accessibility relation as the original model and with $M(w) = \varphi^{-1}(M'(w))$ for all $w \in |W'|$. Similarly, for P'-homomorphisms h', we have $h'\restriction_{\varphi} = h'$.

Concerning the stratification of MPL, for every Kripke structure $\langle W, M\rangle$ of signature P, $\mathrm{K}^{\mathrm{MPL}}_P(W, M)$ is the power-set algebra with $\mathrm{K}^{\mathrm{MPL}}_P(W, M)_{Sen} = |W|$ and

- $\mathrm{K}^{\mathrm{MPL}}_P(W, M)_{\text{“}p\text{”}} = \{w \in |W| \mid p \in M(w)\}$,
- $\mathrm{K}^{\mathrm{MPL}}_P(W, M)_{false} = \emptyset$,
- $\mathrm{K}^{\mathrm{MPL}}_P(W, M)_{\Rightarrow}(U, V) = |W| \setminus (U \setminus V)$,
- $\mathrm{K}^{\mathrm{MPL}}_P(W, M)_{\Diamond}(U) = W_{\lambda}^{-1}(U)$.

Kripke-structure homomorphisms $h\colon \langle W_1, M_1\rangle \to \langle W_2, M_2\rangle$ are, once more, left unchanged, in the sense that $\mathrm{K}^{\mathrm{MPL}}_P(h)_{Sen} = h$.

The basic modal logic presented above knows many variations in the modal-logic literature. Many of them have already been formalized as stratified institutions [see Dia22b], and all those formalizations can be easily adapted as examples of modal stratified parchments. The variations we mentioned typically go in two directions: extensions of the base logic, essentially replacing propositional logic with richer logics that may support, for instance, quantifiers or other complex features – see [FM98] and [DS07] for an institutional account; or changes of the modal ‘layer’ of the logic. The latter may be restricted, which means imposing constraints on the accessibility relations of Kripke structures; for example, they may be required to be reflexive, preorders, or equivalences, leading to the T, S4, or S5 variants, respectively, of modal propositional logic. Or it can be extended by considering additional accessibility relations – and new corresponding modal operators – or by allowing the arities of the accessibility relations to be arbitrarily large instead of being binary. Going further, we may also consider different modal operators such as the local-satisfaction operators used in hybrid logics [Bla00]. To simplify the presentation, and because most of the features of those logics are either Boolean, modal, or first-order, in the sense that they involve first-order quantifiers, in this paper we refer only to modal proportional logic and to first-order logic, which we discuss next.

Example 3.3 (FOL)**.** When formalized as a stratified institution [e.g. Dia17; Dia22a; Ţuţ23], first-order logic typically presents proper lax stratifications because the model states are structurally tied to the signatures over which they are defined. Here, we give an alternative formalization as a strict stratified parchment denoted FOL.

Many-sorted first-order signatures extend the algebraic signatures presented in Section 2 with distinguished sets of predicate symbols, while signature morphisms are structured families of functions that map sorts to sorts, operation symbols to operation symbols, and predicate symbols to predicate symbols – see, e.g., [GB92; Dia08] for details on how to formalize first-order logic as an institution, as most of those constructions apply nearly verbatim to stratified parchments.

To define the language functor of FOL, let a first fix a set *Var* of *variable names*. For every first-order signature Σ, with S as its underlying set of sorts, a Σ-variable is a pair (v, s), usually denoted $v\colon s$, where $v \in \mathit{Var}$ is its name and $s \in S$ is its sort. A *block* of Σ-variables is just a set of variables such that different variables have different names. Blocks of variables may also be regarded as S-sorted sets X of variable names such that $X_s \cap X_t = \emptyset$ for any pair of distinct sorts $s, t \in S$. We often switch between these views depending on which one is more convenient.

For every first-order signature Σ, $\mathrm{L}^{\mathrm{FOL}}(\Sigma)$ is the algebraic signature given by:

sorts $\mathit{Fm}(X)$ [for every block of Σ-variables X]
ops "ρ": $\varepsilon \to \mathit{Fm}(X)$ [for every atomic Σ-formula ρ with free variables in X]
$\quad \mathit{false}\colon \varepsilon \to \mathit{Fm}(X)$
$\quad \Rightarrow\colon \mathit{Fm}(X)\,\mathit{Fm}(X) \to \mathit{Fm}(X)$
$\quad \exists\{x\colon s\}\colon \mathit{Fm}(X) \to \mathit{Fm}(X \setminus \{x\colon s\})$ [for every variable $x\colon s$ in X]

Signature morphisms $\varphi\colon \Sigma \to \Sigma'$ translate blocks of Σ-variables X to blocks of Σ'-variables $\varphi(X)$ given by $\varphi(X)_{s'} = \bigcup\{X_s \mid s \in \varphi^{-1}(s')\}$. Consequently, $\mathrm{L}^{\mathrm{FOL}}(\varphi)$ maps types $\mathit{Fm}(X)$ to $\mathit{Fm}(\varphi(X))$, atomic formulae ρ to $\varphi(\rho)$ – consult, e.g., [GB92] for more details – and quantification operators $\exists\{x\colon s\}$ to $\exists\{x\colon \varphi(s)\}$.

Models, model homomorphisms, and reducts are defined as usual in institutional formalizations of first-order logic: for instance, Σ-models M interpret sorts s in Σ as sets M_s, operation symbols $\sigma\colon w \to s$ as functions $M_\sigma\colon M_w \to M_s$ (similarly to algebras), and predicate symbols $\pi\colon w$ as relations $M_\pi \subseteq M_w$.

For every block of Σ-variables X, and every Σ-model M, a state of type $\mathit{Fm}(X)$ of M is simply an M-valuation of X, by which we mean a function

$$v\colon \bigcup\{X_s \mid s \in S\} \to \bigcup\{M_s \mid s \in S \text{ and } X_s \neq \emptyset\},$$

where S is the underlying set of sorts of Σ, such that $v(x) \in M_s$ for all $s \in S$ and $x \in X_s$. We write $[X \to M]$ for the set of all M-valuations of X; $v \setminus \{x\colon s\}$ to denote the restriction of a valuation $v \in [X \to M]$ to $X \setminus \{x\colon s\}$, for $x \in X_s$; $u \mathbin{+\!\!+} \{x\colon s\}$ for the set of all valuations $v \in [X \to M]$ such that $v \setminus \{x\colon s\} = u$; and $M, v \vDash \rho$ to indicate that an atomic formula ρ is holds in a model M for a valuation v.

Then $\mathrm{K}^{\mathrm{FOL}}_\Sigma(M)$ is the power-set algebra with $\mathrm{K}^{\mathrm{FOL}}_\Sigma(M)_{\mathit{Fm}(X)} = [X \to M]$ and

- $\mathrm{K}^{\mathrm{FOL}}_\Sigma(M)_{\text{“}\rho\text{”}} = \{v \in [X \to M] \mid M, v \vDash \rho\}$,
- $\mathrm{K}^{\mathrm{FOL}}_\Sigma(M)_{\mathit{false}} = \emptyset$,
- $\mathrm{K}^{\mathrm{FOL}}_\Sigma(M)_{\Rightarrow}(U, V) = [X \to M] \setminus (U \setminus V)$,
- $\mathrm{K}^{\mathrm{FOL}}_\Sigma(M)_{\exists\{x\colon s\}}(V) = \{v \setminus \{x\colon s\} \mid v \in V\}$.

Given a first-order Σ-homomorphism $h\colon M_1 \to M_2$ and a block X of Σ-variables, $\mathrm{K}^{\mathrm{FOL}}_\Sigma(h)_{\mathit{Fm}(X)}$ is the function $[X \to M_1] \to [X \to M_2]$ that maps every M_1-valuation v_1 of X to the M_2-valuation v_2 given by $v_2(x\colon s) = h_s(v_1(x\colon s))$.

For every signature morphism $\varphi\colon \Sigma \to \Sigma'$, every Σ'-model M', and every block of Σ-variables X, notice that $[X \to M'\restriction_\varphi] = [\varphi(X) \to M']$. This allows us to define the state reduction $\mathrm{K}^{\mathrm{FOL}}_\varphi(M')_{Fm(X)}$ as $id_{[\varphi(X)\to M']}$. It ought to be noted, however, that different formalizations of valuations – e.g., as many-sorted functions – may lead to stratification natural transformations that are not strict [see Dia17].

Example 3.4 (Connectives)**.** The logical connectives used in Examples 3.2 and 3.3 can also be considered in isolation and formalized so as stratified parchments. For example, the Boolean implication may be captured by a strict parchment $\mathcal{C}_\Rightarrow$ over $\mathbb{S}\mathrm{et}$ (viewed as a category of signatures) where:

- The functor $\mathrm{L}^{\mathcal{C}_\Rightarrow}$ maps every set S to the algebraic signature $\mathrm{L}^{\mathcal{C}_\Rightarrow}(S)$ given by:

  sorts s [for all $s \in S$]
  ops $\Rightarrow\colon s\,s \to s$

- The model functor $\mathrm{Mod}^{\mathcal{C}_\Rightarrow}$ maps every set S to the category of S-sorted sets.

- For every S-sorted set A, $\mathrm{K}^{\mathcal{C}_\Rightarrow}_S(A)$ is the power-set algebra that interprets every sort $s \in S$ as A_s and the binary operation symbol $\Rightarrow\colon s\,s \to s$, for every sort $s \in S$, as the function mapping subsets $U, V \subseteq A_s$ to $A_s \setminus (U \setminus V)$.

Similar parchments may be defined for other Boolean operators. We denote them by: $\mathcal{C}_{false}$ for falsity, interpreted as the empty set; $\mathcal{C}_{true}$ for truth, interpreted as the total set; $\mathcal{C}_\neg$ for negation, interpreted as complementation; $\mathcal{C}_\wedge$ for conjunction, interpreted as intersection; $\mathcal{C}_\vee$ for disjunction, interpreted as union; etc.

To capture modal operators and quantifiers in the same way, we need parchments whose models have a richer structure. Take, for example, the possibility operator. We formalize it by a strict parchment $\mathcal{C}_\Diamond$ over $\mathbb{S}\mathrm{et}$ as follows:

- The functor $\mathrm{L}^{\mathcal{C}_\Diamond}$ maps every set S to the algebraic signature $\mathrm{L}^{\mathcal{C}_\Diamond}(S)$ given by:

  sorts s [for all $s \in S$]
  ops $\Diamond\colon s \to s$

- The model functor $\mathrm{Mod}^{\mathcal{C}_\Diamond}$ maps every set S to the category of S-sorted Kripke frames, by which we mean S-sorted sets W equipped with a binary accessibility relation $W_{\lambda\colon s}$ on W_s for each sort $s \in S$. When $S = \{s\}$ is a singleton, we obtain the more familiar notion of Kripke frame, where $|W| = W_s$.

- For every S-sorted frame W, $\mathrm{K}^{\mathcal{C}_\Diamond}_S(W)$ is the power-set algebra with the same carriers as W that interprets the symbol $\Diamond\colon s \to s$, for every sort $s \in S$, as the function mapping subsets $U \subseteq W_s$ to $W^{-1}_{\lambda\colon s}(U)$.

The necessity operator can be formalized similarly, by defining $\mathrm{K}_S^{\mathcal{C}_\Box}(W)$ as the power-set algebra with the same carriers as W that interprets the symbol $\Box\colon s \to s$ as the function mapping subsets $U \subseteq W_s$ to $\{w \in W_s \mid W_{\lambda\colon s}(w) \subseteq U\}$.

For quantifiers, following the parchment formalization of first-order logic in Example 3.3, we begin with a fixed set Var (typically countable) of variable names. Then $\mathcal{C}_\exists$ is the strict stratified parchment over $\mathbb{S}\mathrm{et}$ such that:

- The functor $\mathrm{L}^{\mathcal{C}_\exists}$ maps every set S to the algebraic signature $\mathrm{L}^{\mathcal{C}_\exists}(S)$ given by:

  sorts $Fm(X)$ [for every S-indexed block of variables X]
  ops $\exists\{x\colon s\}\colon Fm(X) \to Fm(X \setminus \{x\colon s\})$ [for every variable $x\colon s$ in X]

- The model functor $\mathrm{Mod}^{\mathcal{C}_\exists}$ maps every set S to the category of S-sorted sets.

- For every S-sorted set A, $\mathrm{K}_S^{\mathcal{C}_\exists}(A)$ is the power-set algebra that interprets every type $Fm(X)$ in $\mathrm{L}^{\mathcal{C}_\exists}(S)$ as the set $[X \to A]$ of A-valuations of X and every operation symbol $\exists\{x\colon s\}\colon Fm(X) \to Fm(X \setminus \{x\colon s\})$ as the function mapping sets V of A-valuations of X to $\{v \setminus \{x\colon s\} \mid v \in V\}$.

Universal quantifiers can be defined analogously by letting $\mathrm{K}_S^{\mathcal{C}_\forall}(A)$ be the power-set algebra that interprets operation symbols $\forall\{x\colon s\}\colon Fm(X) \to Fm(X \setminus \{x\colon s\})$ as functions given by $V \mapsto \{u \in [X \setminus \{x\colon s\} \to A] \mid u +\!\!+ \{x\colon s\} \subseteq V\}$.

Satisfaction relations The stratification functors K_Σ enable us to define local satisfaction relations between models and sentences using ordinary set membership. For every type $s \in \mathrm{ST}(\Sigma)$, Σ-model M, state $w \in [\![M]\!]_s$, and Σ-sentence ρ of type s, we write $M \vDash_\Sigma^w \rho$, and read M *satisfies* ρ *at* w, if and only if $w \in \mathrm{K}_\Sigma(M)_\rho$.

Remark 3.5. When, for every signature Σ, the grammar $\mathrm{L}(\Sigma)$ is single-sorted and consists only of constant symbols – disregarding the syntactic structure of sentences – we recover the usual notion of stratified institution [Dia17]. In that case, it is fitting to replace the language functor L with a *sentence functor* $\mathrm{Sen}\colon \mathbb{S}\mathrm{ig} \to \mathbb{S}\mathrm{et}$ and the stratification K with a lax natural transformation $[\![\_]\!]\colon \mathrm{Mod} \Rightarrow \mathrm{Set}$ (where by Set we mean the constant functor $\mathbb{S}\mathrm{ig}^{\mathrm{op}} \to \mathbb{C}\mathrm{at}$ that maps every signature Σ to $\mathbb{S}\mathrm{et}$ and every signature morphism to $id_{\mathbb{S}\mathrm{et}}$) together with the signature-indexed family of satisfaction relations $\vDash_\Sigma$ between Σ-models and Σ-sentences introduced above. Hence stratified institutions are tuples of the form $\langle \mathbb{S}\mathrm{ig}, \mathrm{Sen}, \mathrm{Mod}, [\![\_]\!], \vDash \rangle$.

The preservation and reflection properties of the state-reduction maps $\mathrm{K}_\varphi(M')$ from Definition 3.1 then correspond to the institution-theoretic *satisfaction condition* for φ: for every Σ'-model M', state $w' \in [\![M']\!]$, and Σ-sentence ρ, we have

$$M' \vDash_{\Sigma'}^{w'} \varphi(\rho) \quad \text{if and only if} \quad M'{\restriction_\varphi} \vDash_\Sigma^w \rho, \text{ where } w = w'{\restriction_\varphi}.$$

More precisely, the preservation property of $\mathrm{K}_\varphi(M')$ ensures that the ‘only if’ part of the equivalence holds, while the reflection property entails the ‘if’ part.

As with the original concept of parchment, it is also possible to move in the other direction, from stratified parchments to stratified institutions. The following preliminary result can be checked in much the same way as 'writing' charters for institutions [GB86]. To avoid repetition, we present it without a proof.

Proposition 3.6. *Every stratified parchment* $\langle \mathbb{S}\mathrm{ig}, \mathrm{L}, \mathrm{Mod}, \mathrm{K} \rangle$ *can be 'flattened' to a stratified institution* $\langle \mathbb{S}\mathrm{ig}^\flat, \mathrm{Sen}^\flat, \mathrm{Mod}^\flat, [\![\_]\!]^\flat, \models \rangle$ *where:*

- *The signatures in* $\mathbb{S}\mathrm{ig}^\flat$ *are pairs* $\langle \Sigma, s \rangle$ *consisting of a signature* $\Sigma \in |\mathbb{S}\mathrm{ig}|$ *and a type* $s \in \mathrm{ST}(\Sigma)$*, while the signature morphisms* $\varphi \colon \langle \Sigma, s \rangle \to \langle \Sigma', s' \rangle$ *in* $\mathbb{S}\mathrm{ig}^\flat$ *are morphisms* $\varphi \colon \Sigma \to \Sigma'$ *in* $\mathbb{S}\mathrm{ig}$ *such that* $\varphi(s) = s'$.
- *The sentence functor is defined on objects by* $\mathrm{Sen}^\flat(\Sigma, s) = \mathrm{T}_{\mathrm{L}(\Sigma),s}$. *And for every morphism* $\varphi \colon \langle \Sigma, s \rangle \to \langle \Sigma', s' \rangle$*, the function* $\mathrm{Sen}^\flat(\varphi)$ *is given by the* s*-component of the unique* $\mathrm{L}(\Sigma)$*-homomorphism* $\mathrm{T}_{\mathrm{L}(\Sigma)} \to \mathrm{T}_{\mathrm{L}(\Sigma')}\restriction_{\mathrm{L}(\varphi)}$.
- *The model functor is defined by* $\mathrm{Mod}^\flat(\Sigma, s) = \mathrm{Mod}(\Sigma)$*, discarding the type.*
- *For every signature* $\langle \Sigma, s \rangle$ *in* $\mathbb{S}\mathrm{ig}^\flat$ *and every* Σ*-model* M*,* $[\![M]\!]^\flat_\Sigma = \mathrm{K}_\Sigma(M)_s$.
- *The satisfaction relations coincide with those of the parchment.* □

4 Borrowing connectives

As Examples 3.2 to 3.4 indicate, most logical connectives can be developed on top of some base parchment by transporting them from another parchment where they are defined. The following notion of addendum makes precise the basic ingredients needed in order to extend parchments with new logical connectives.

Definition 4.1 (Addendum)**.** Let $\mathcal{B}$, called *base*, and $\mathcal{F}$, called *feature*, be two strict stratified parchments. An $\mathcal{F}$*-addendum* to $\mathcal{B}$ is a functor $\Psi \colon \mathbb{S}\mathrm{ig}^{\mathcal{B}} \to \mathbb{S}\mathrm{ig}^{\mathcal{F}}$ such that, for any base signature morphism $\varphi \colon \Sigma \to \Sigma'$, the algebraic signature morphisms $\mathrm{L}^{\mathcal{B}}(\varphi)$ and $\mathrm{L}^{\mathcal{F}}(\Psi(\varphi))$ are compatible, meaning that the diagram below commutes.

$$\begin{array}{ccccc}
\mathrm{L}^{\mathcal{B}}(\Sigma) & \xleftarrow{\supseteq} & \mathrm{L}^{\mathcal{B}}(\Sigma) \cap \mathrm{L}^{\mathcal{F}}(\Psi(\Sigma)) & \xrightarrow{\subseteq} & \mathrm{L}^{\mathcal{F}}(\Psi(\Sigma)) \\
{\scriptstyle \mathrm{L}^{\mathcal{B}}(\varphi)}\big\downarrow & & & & \big\downarrow{\scriptstyle \mathrm{L}^{\mathcal{F}}(\Psi(\varphi))} \\
\mathrm{L}^{\mathcal{B}}(\Sigma') & \xrightarrow[\subseteq]{} & \mathrm{L}^{\mathcal{B}}(\Sigma') \cup \mathrm{L}^{\mathcal{F}}(\Psi(\Sigma')) & \xleftarrow[\supseteq]{} & \mathrm{L}^{\mathcal{F}}(\Psi(\Sigma'))
\end{array}$$

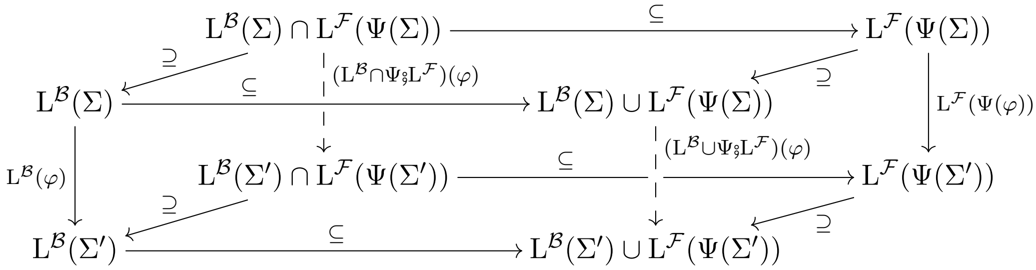
When working with multiple base or feature parchments, in order to distinguish different addenda with the same underlying functor, we may also denote them as triples $\langle \mathcal{B}, \mathcal{F}, \Psi \rangle$, or as pairs $\langle \mathcal{F}, \Psi \rangle$ if the base can be easily inferred.

Example 4.2. For every base parchment $\mathcal{B}$, notice that the mapping $\Sigma \mapsto \mathrm{ST}^{\mathcal{B}}(\Sigma)$ extends to a functor $\mathrm{ST}^{\mathcal{B}}\colon \mathbb{S}\mathrm{ig}^{\mathcal{B}} \to \mathbb{S}\mathrm{et}$ where, for every signature morphism $\varphi\colon \Sigma \to \Sigma'$ and every sentence type $s \in \mathrm{ST}^{\mathcal{B}}(\Sigma)$, $\mathrm{ST}^{\mathcal{B}}(\varphi)(s) = \varphi(s)$. This gives us several addenda to $\mathcal{B}$ of the form $\langle \mathcal{F}, \mathrm{ST}^{\mathcal{B}} \rangle$, where $\mathcal{F}$ is any of the Boolean or modal parchments presented in Example 3.4. When the base $\mathcal{B}$ is clear, then we may also denote the addendum $\langle \mathcal{C}_{false}, \mathrm{ST}^{\mathcal{B}} \rangle$ by $[false]$, $\langle \mathcal{C}_{\Rightarrow}, \mathrm{ST}^{\mathcal{B}} \rangle$ by $[\Rightarrow]$, and so on.

These addenda are polymorphic, in the sense that they can be defined for any base parchment. In contrast, for quantifiers we need bases of suitable form. Take, for instance, the 'atomic' fragment of FOL, whose parchment we denote by FOL_0. Its signatures and models are the same as those of FOL, while its sentence grammars, and their corresponding interpretations in models, are reduced to atomic sentences. Then the forgetful functor $sorts\colon \mathbb{S}\mathrm{ig}^{\mathrm{FOL}} \to \mathbb{S}\mathrm{et}$ that maps every signature Σ to its underlying set of sorts defines two addenda to FOL_0: $\langle \mathcal{C}_{\exists}, sorts \rangle$, which we abbreviate as $[\exists]$, for existential quantifiers; and $\langle \mathcal{C}_{\forall}, sorts \rangle$, or $[\forall]$, for universal quantifiers.

The next property of parchment addenda is an immediate consequence of the fact that every intersection-union square of algebraic signatures is both a pullback and a pushout square in the category $\mathbb{A}\mathrm{lg}\mathbb{S}\mathrm{ig}$ – see [DŢ11; Ţuţ14] for details.

Remark 4.3. For any $\mathcal{F}$-addendum Ψ to a stratified parchment $\mathcal{B}$ and any base signature morphism $\varphi\colon \Sigma \to \Sigma'$ there exist two unique algebraic signature morphisms $(\mathrm{L}^{\mathcal{B}} \cap \Psi \mathbin{⨾} \mathrm{L}^{\mathcal{F}})(\varphi)$ and $(\mathrm{L}^{\mathcal{B}} \cup \Psi \mathbin{⨾} \mathrm{L}^{\mathcal{F}})(\varphi)$ such that the following diagram commutes.



Both signature morphisms emerge from universal properties of limits and colimits in $\mathbb{A}\mathrm{lg}\mathbb{S}\mathrm{ig}$. For the first one, consider the bottom intersection-union square of the diagram above, which describes a pullback; and for the second one, consider the top intersection-union square, viewed in this case as a pushout square.

The same universal properties entail that the mappings $\varphi \mapsto (\mathrm{L}^{\mathcal{B}} \cap \Psi \mathbin{⨾} \mathrm{L}^{\mathcal{F}})(\varphi)$ and $\varphi \mapsto (\mathrm{L}^{\mathcal{B}} \cup \Psi \mathbin{⨾} \mathrm{L}^{\mathcal{F}})(\varphi)$ preserve both identities and the composition of algebraic signature morphisms, leading to functors $\mathrm{L}^{\mathcal{B}} \cap \Psi \mathbin{⨾} \mathrm{L}^{\mathcal{F}}, \mathrm{L}^{\mathcal{B}} \cup \Psi \mathbin{⨾} \mathrm{L}^{\mathcal{F}}\colon \mathbb{S}\mathrm{ig}^{\mathcal{B}} \to \mathbb{A}\mathrm{lg}\mathbb{S}\mathrm{ig}$ with natural inclusions $\mathrm{L}^{\mathcal{B}} \supseteq \mathrm{L}^{\mathcal{B}} \cap \Psi \mathbin{⨾} \mathrm{L}^{\mathcal{F}} \subseteq \Psi \mathbin{⨾} \mathrm{L}^{\mathcal{F}}$ and $\mathrm{L}^{\mathcal{B}} \subseteq \mathrm{L}^{\mathcal{B}} \cup \Psi \mathbin{⨾} \mathrm{L}^{\mathcal{F}} \supseteq \Psi \mathbin{⨾} \mathrm{L}^{\mathcal{F}}$. To simplify notations in the rest of the section, we denote the natural transformations corresponding to these inclusions, in order, by $\eta^{\mathcal{B}}$, $\eta^{\mathcal{F}}$, $\theta^{\mathcal{B}}$, and $\theta^{\mathcal{F}}$.

Addenda allow us to extend parchments by use of amalgamation, which is an elementary but very important property of logical systems that forms the basis of a wide range of developments in institution theory; e.g., [Tar85; ST88; DGS93; Bor02; AD07; ŢF17], to mention only a few. Intuitively, amalgamation refers to the process of combining models or homomorphisms of different but related signatures, provided that they have a common reduct to some shared signature. More precisely, a square of signature morphisms (sometimes assumed to be commutative) as depicted below

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \varphi'_1} \\ \Sigma_2 & \xrightarrow[\varphi'_2]{} & \Sigma' \end{array}$$

is said to *admit amalgamation* when for every two models (or model homomorphisms) M_1 of Σ_1 and M_2 of Σ_2 such that $M_1\upharpoonright_{\varphi_1} = M_2\upharpoonright_{\varphi_2}$ there exists a unique Σ'-model M', called the *amalgamation* of M_1 and M_2, satisfying $M'\upharpoonright_{\varphi'_1} = M_1$ and $M'\upharpoonright_{\varphi'_2} = M_2$. In category-theoretic terms, this means that the diagram above determines a pullback square of reduct functors in the higher-level category $\mathbb{C}\mathrm{at}$ of categories.

Example 4.4. To illustrate amalgamation, suppose Ω_1 and Ω_2 are algebraic signatures and let $\Omega = \Omega_1 \cap \Omega_2$ and $\Omega' = \Omega_1 \cup \Omega_2$. Then every two power-set algebras A_1 of Ω_1 and A_2 of Ω_2 that interpret the symbols in Ω in the same way admit a Ω'-amalgamation A' given by $A'_\varsigma = A_{i,\varsigma}$ for all sorts and operation symbols ς in Ω_i.

The following result corresponds to an institution-theoretic property known as *semi-exactness* [Dia08, Section 4.3; ST11, Section 4.4], which was originally developed for many-sorted algebras, but holds for power-set algebras just as well.

Lemma 4.5. *The category* $\mathbb{A}\mathrm{lg}\mathbb{S}\mathrm{ig}$ *has pushouts and the contravariant functor* $\mathcal{P}\mathrm{Alg}\colon \mathbb{A}\mathrm{lg}\mathbb{S}\mathrm{ig} \to \mathbb{C}\mathrm{at}$ *preserves them, meaning that it maps pushouts of algebraic signature morphisms to pullbacks of categories of power-set algebras.* □

Based on this lemma, we can now introduce a general method for developing new features on top of existing base parchments according to some given addenda.

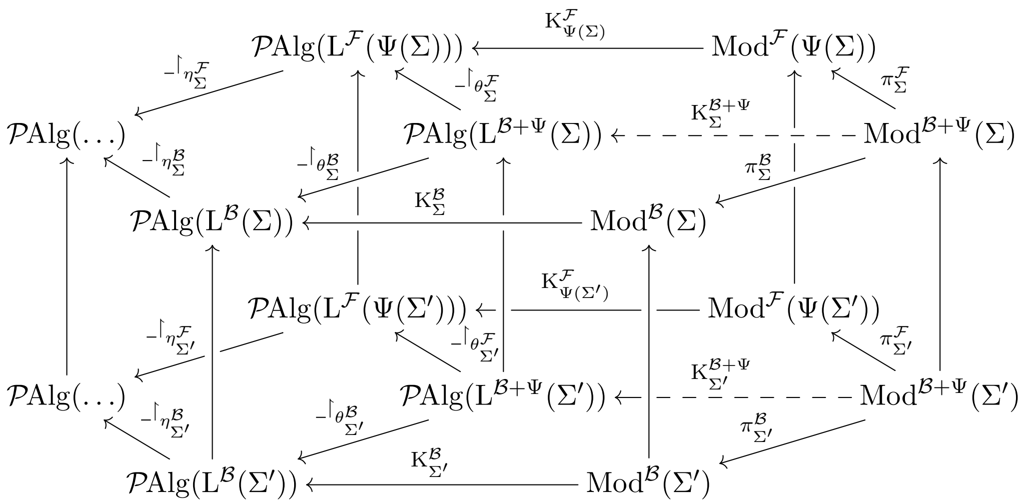
Proposition 4.6. *Any* $\mathcal{F}$*-addendum* Ψ *to a parchment* $\mathcal{B}$ *gives rise to a strict 'extended' stratified parchment* $\mathcal{B} + \Psi = \langle \mathbb{S}\mathrm{ig}^{\mathcal{B}}, \mathrm{L}^{\mathcal{B}} \cup \Psi \mathbin{;} \mathrm{L}^{\mathcal{F}}, \mathrm{Mod}^{\mathcal{B}+\Psi}, \mathrm{K}^{\mathcal{B}+\Psi} \rangle$ *where:*

- *For every base signature* Σ*, the objects in* $\mathrm{Mod}^{\mathcal{B}+\Psi}(\Sigma)$ *are pairs* (M, F) *consisting of a base* Σ*-model* M *and a feature* $\Psi(\Sigma)$*-model* F *in such a way that* $\mathrm{K}^{\mathcal{B}}_{\Sigma}(M)\upharpoonright_{\eta^{\mathcal{B}}_{\Sigma}} = \mathrm{K}^{\mathcal{F}}_{\Psi(\Sigma)}(F)\upharpoonright_{\eta^{\mathcal{F}}_{\Sigma}}$*. The arrows in* $\mathrm{Mod}^{\mathcal{B}+\Psi}(\Sigma)$ *are defined in a similar manner and they compose componentwise.*

- *For every base signature* Σ *and every extended* Σ*-model* (M, F)*,* $\mathrm{K}^{\mathcal{B}+\Psi}_{\Sigma}(M, F)$ *is the unique power-set algebra arising from the amalgamation of* $\mathrm{K}^{\mathcal{B}}_{\Sigma}(M)$ *and* $\mathrm{K}^{\mathcal{F}}_{\Psi(\Sigma)}(F)$*; the same applies to* Σ*-homomorphisms.*

Proof. We need to show that $\mathrm{Mod}^{\mathcal{B}+\Psi}$ is a contravariant functor $\mathbb{S}\mathrm{ig}^{\mathcal{B}} \to \mathbb{C}\mathrm{at}$ and that $\mathrm{K}^{\mathcal{B}+\Psi}$ is a natural transformation $\mathrm{Mod}^{\mathcal{B}+\Psi} \Rightarrow (\mathrm{L}^{\mathcal{B}} \cup \Psi \fatsemi \mathrm{L}^{\mathcal{F}})^{\mathrm{op}} \fatsemi \mathcal{P}\mathrm{Alg}$. The first part of the proof is straightforward and can be established by mere calculations. Notice that, for every base signature Σ, the category $\mathrm{Mod}^{\mathcal{B}+\Psi}(\Sigma)$ together with the obvious projection functors $\pi^{\mathcal{B}}_{\Sigma}$ and $\pi^{\mathcal{F}}_{\Sigma}$ into $\mathrm{Mod}^{\mathcal{B}}(\Sigma)$ and $\mathrm{Mod}^{\mathcal{F}}(\Psi(\Sigma))$, respectively, forms a pullback of $\mathrm{K}^{\mathcal{B}}_{\Sigma} \fatsemi \mathcal{P}\mathrm{Alg}(\eta^{\mathcal{B}}_{\Sigma})$ and $\mathrm{K}^{\mathcal{F}}_{\Psi(\Sigma)} \fatsemi \mathcal{P}\mathrm{Alg}(\eta^{\mathcal{F}}_{\Sigma})$. The action of $\mathrm{Mod}^{\mathcal{B}+\Psi}$ on signature morphisms, its functoriality, and the fact that the projections $\pi^{\mathcal{B}}_{\Sigma}$ and $\pi^{\mathcal{F}}_{\Sigma}$ are natural in Σ, all follow from the universal property of pullbacks.

For the second part of the proof, by Lemma 4.5, we know that the functors $\mathcal{P}\mathrm{Alg}(\theta^{\mathcal{B}}_{\Sigma})$ and $\mathcal{P}\mathrm{Alg}(\theta^{\mathcal{F}}_{\Sigma})$ form a pullback of $\mathcal{P}\mathrm{Alg}(\eta^{\mathcal{B}}_{\Sigma})$ and $\mathcal{P}\mathrm{Alg}(\eta^{\mathcal{F}}_{\Sigma})$. Based on this observation, and considering that $\pi^{\mathcal{B}} \fatsemi \mathrm{K}^{\mathcal{B}} \fatsemi \eta^{\mathcal{B}}\mathcal{P}\mathrm{Alg} = \pi^{\mathcal{F}} \fatsemi \Psi\mathrm{K}^{\mathcal{F}} \fatsemi \eta^{\mathcal{F}}\mathcal{P}\mathrm{Alg}$, it is easy to see that $\mathrm{K}^{\mathcal{B}+\Psi}_{\Sigma}$ is the unique functor $\mathrm{Mod}^{\mathcal{B}+\Psi}(\Sigma) \to \mathcal{P}\mathrm{Alg}\big(\mathrm{L}^{\mathcal{B}}(\Sigma) \cup \mathrm{L}^{\mathcal{F}}(\Psi(\Sigma))\big)$ satisfying $\mathrm{K}^{\mathcal{B}+\Psi}_{\Sigma} \fatsemi \mathcal{P}\mathrm{Alg}(\theta^{\mathcal{B}}_{\Sigma}) = \pi^{\mathcal{B}}_{\Sigma} \fatsemi \mathrm{K}^{\mathcal{B}}_{\Sigma}$ and $\mathrm{K}^{\mathcal{B}+\Psi}_{\Sigma} \fatsemi \mathcal{P}\mathrm{Alg}(\theta^{\mathcal{F}}_{\Sigma}) = \pi^{\mathcal{F}}_{\Sigma} \fatsemi \mathrm{K}^{\mathcal{F}}_{\Phi(\Sigma)}$. The top part of the next diagram may help us visualize these equalities. To avoid cluttering the diagram, we write $\mathcal{P}\mathrm{Alg}(\ldots)$ for categories produced by the functor $(\mathrm{L}^{\mathcal{B}} \cap \Psi \fatsemi \mathrm{L}^{\mathcal{F}}) \fatsemi \mathcal{P}\mathrm{Alg}$, and $\mathrm{L}^{\mathcal{B}+\Psi}$ instead of $\mathrm{L}^{\mathcal{B}} \cup \Psi \fatsemi \mathrm{L}^{\mathcal{F}}$. We also leave unlabelled all vertical arrows, which correspond to reduction functors along a signature morphism $\varphi\colon \Sigma \to \Sigma'$ or along other signature morphisms derived from φ: $\Psi(\varphi)$, $\mathrm{L}^{\mathcal{B}}(\varphi)$, $\mathrm{L}^{\mathcal{F}}(\Psi(\varphi))$, etc.



To show that $\mathrm{K}^{\mathcal{B}+\Psi}$ is a natural transformation, we use the fact that, for every signature Σ, the functors $\mathcal{P}\mathrm{Alg}(\theta^{\mathcal{B}}_{\Sigma})$ and $\mathcal{P}\mathrm{Alg}(\theta^{\mathcal{F}}_{\Sigma})$ form a monomorphic family. Therefore, it suffices to prove, for every signature morphism $\varphi\colon \Sigma \to \Sigma'$, that the following property holds for both $* \in \{\mathcal{B}, \mathcal{F}\}$:

$$\mathrm{Mod}^{\mathcal{B}+\Psi}(\varphi) \fatsemi \mathrm{K}^{\mathcal{B}+\Psi}_{\Sigma} \fatsemi \mathcal{P}\mathrm{Alg}(\theta^{*}_{\Sigma}) = \mathrm{K}^{\mathcal{B}+\Psi}_{\Sigma'} \fatsemi \mathcal{P}\mathrm{Alg}(\mathrm{L}^{\mathcal{B}+\Psi}(\varphi)) \fatsemi \mathcal{P}\mathrm{Alg}(\theta^{*}_{\Sigma}).$$

Both equalities can be checked with ease by diagram chasing, based on the definition of $\mathrm{K}^{\mathcal{B}+\Psi}$ and the naturality of $\mathrm{K}^{\mathcal{B}}$, $\Psi\mathrm{K}^{\mathcal{F}}$, θ^{*}, and π^{*}. $\square$

Example 4.7. Let MPL_0 be the 'atomic' fragment of MPL. Its signatures are the same as those of MPL, and both its language and model functor are derived from those of MPL by dropping all Boolean and modal operators, their interpretations in models, and the accessibility relations. Therefore, an MPL_0-model over some signature P is a pair $\langle W, M\rangle$ consisting of a set W and a function $M\colon W \to \mathcal{P}(P)$.

Then MPL can be obtained from MPL_0 by adding the logical connectives back using the construction presented in Proposition 4.6. To be more precise, the parchment $\mathrm{MPL}_0 + [\mathit{false}] + [\Rightarrow] + [\Diamond]$ has the same signatures and language functor as MPL, and a model functor and stratification that are isomorphic to those of MPL.

In a similar fashion, first-order logic can be obtained from the base parchment FOL_0 by considering the extension $\mathrm{FOL}_0 + [\mathit{false}] + [\Rightarrow] + [\exists]$.

Using this construction, it would also be possible to obtain variants of modal first-order logic, but not from MPL_0 or FOL_0. Instead, we would need to select a base, say MFOL_0, whose models are pairs $\langle W, M\rangle$, where W is a set of possible worlds and M is a W-indexed family of first-order models – required to satisfy, perhaps, certain sharing constraints [see Dia17]. This exemplifies some of the limitations of using addenda, which are essentially syntactic devices for extending parchments.

To apply multiple addenda as in the example above we need to check that they do not interfere with one another in a way that prevents further extensions.

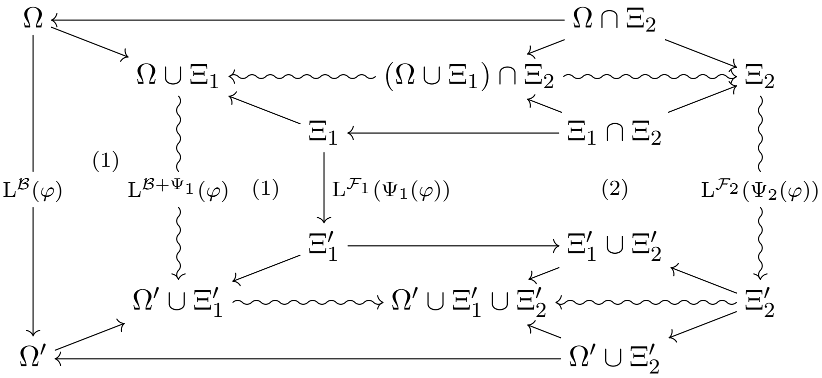
Definition 4.8 (Independent addenda)**.** Let Ψ_i be an $\mathcal{F}_i$-addendum to a parchment $\mathcal{B}$, for $i \in \{1, 2\}$. We say that Ψ_1 and Ψ_2 are *independent* when $\mathrm{L}^{\mathcal{F}_1}(\Psi_1(\varphi))$ and $\mathrm{L}^{\mathcal{F}_2}(\Psi_2(\varphi))$ are compatible for all base signature morphisms φ.

The following observation introduces a much simpler criteria, met by all examples we consider in this paper, for establishing the independence of addenda.

Remark 4.9. If Ψ_1 and Ψ_2 are addenda such that $\mathrm{L}^{\mathcal{F}_1}(\Psi_1(\Sigma)) \cap \mathrm{L}^{\mathcal{F}_2}(\Psi_2(\Sigma)) \subseteq \mathrm{L}^{\mathcal{B}}(\Sigma)$ for all base signatures Σ, then Ψ_1 and Ψ_2 are independent. Whenever this additional condition holds, we say that Ψ_1 and Ψ_2 are *strongly independent.*

Proposition 4.10. *If Ψ_1 and Ψ_2 are independent addenda to a parchment $\mathcal{B}$, then Ψ_2 is also an addendum to the extended parchment $\mathcal{B} + \Psi_1$.*

Proof. Suppose $\mathcal{F}_1$ and $\mathcal{F}_2$ are the feature parchments corresponding to Ψ_1 and Ψ_2, respectively. We need to show that, for every signature morphism $\varphi\colon \Sigma \to \Sigma'$, $\mathrm{L}^{\mathcal{B}+\Psi_1}(\varphi)$ and $\mathrm{L}^{\mathcal{F}_2}(\Psi_2(\varphi))$ are compatible. To that end, consider the following diagram where, in order to simplify notations, we write $\Omega^{(\prime)}$ in place of $\mathrm{L}^{\mathcal{B}}(\Sigma^{(\prime)})$ and $\Xi_i^{(\prime)}$ in place of $\mathrm{L}^{\mathcal{F}_i}(\Psi_i(\Sigma^{(\prime)}))$, for $i \in \{1, 2\}$. All unlabelled arrows denote inclusions.



The compatibility of $L^{\mathcal{B}+\Psi_1}(\varphi)$ and $L^{\mathcal{F}_2}(\Psi_2(\varphi))$ amounts to checking that the mid, wavy rectangle is commutative. We know that the outer polygon is commutative because Ψ_2 is an addendum to $\mathcal{B}$. The inner-left squares labelled (1) are commutative because Ψ_1 is an addendum to $\mathcal{B}$. And (2) is commutative, by definition, because the addenda Ψ_1 and Ψ_2 are independent. All other inner squares and triangles are evidently commutative because they consist only of inclusions.

Since the inclusions $\Omega \cap \Xi_2 \leftarrow (\Omega \cup \Xi_1) \cap \Xi_2 \rightarrow \Xi_1 \cap \Xi_2$ are jointly epimorphic, it suffices to check that the following two equalities hold:

$$(\Omega \cap \Xi_2 \to (\Omega \cup \Xi_1) \cap \Xi_2 \to \Omega \cup \Xi_1) \fatsemi L^{\mathcal{B}+\Psi_1}(\varphi) \fatsemi (\Omega' \cup \Xi_1' \to \Omega' \cup \Xi_1' \cup \Xi_2')$$
$$= (\Omega \cap \Xi_2 \to \Xi_2) \fatsemi L^{\mathcal{F}_2}(\Psi_2(\varphi)) \fatsemi (\Xi_2' \to \Omega' \cup \Xi_1' \cup \Xi_2')$$

and

$$(\Xi_1 \cap \Xi_2 \to (\Omega \cup \Xi_1) \cap \Xi_2 \to \Omega \cup \Xi_1) \fatsemi L^{\mathcal{B}+\Psi_1}(\varphi) \fatsemi (\Omega' \cup \Xi_1' \to \Omega' \cup \Xi_1' \cup \Xi_2')$$
$$= (\Xi_1 \cap \Xi_2 \to \Xi_2) \fatsemi L^{\mathcal{F}_2}(\Psi_2(\varphi)) \fatsemi (\Xi_2' \to \Omega' \cup \Xi_1' \cup \Xi_2').$$

Both can be established by diagrammatic reasoning. For example, for the first one:

$$\begin{aligned}
&(\Omega \cap \Xi_2 \to \Omega \cup \Xi_1) \fatsemi L^{\mathcal{B}+\Psi_1}(\varphi) \fatsemi (\Omega' \cup \Xi_1' \to \Omega' \cup \Xi_1' \cup \Xi_2') \\
&\quad = (\Omega \cap \Xi_2 \to \Omega) \fatsemi L^{\mathcal{B}}(\varphi) \fatsemi (\Omega' \to \Omega' \cup \Xi_1' \cup \Xi_2') && \text{(since } \Psi_1 \text{ is an addendum)} \\
&\quad = (\Omega \cap \Xi_2 \to \Xi_2) \fatsemi L^{\mathcal{F}_2}(\Psi_2(\varphi)) \fatsemi (\Xi_2' \to \Omega' \cup \Xi_1' \cup \Xi_2') && \text{(since } \Psi_2 \text{ is an addendum)}
\end{aligned}$$

The second equality follows in essentially the same way, based on the fact that Ψ_1 is an addendum and on the independence of Ψ_1 and Ψ_2. □

Notice that, unlike the defining property of addenda, their independence does not involve the language functor of the base parchment. Therefore, the independence property of two addenda is trivially preserved along parchment extensions. A similar result holds for strong independence by noticing that $L^{\mathcal{B}} \subseteq L^{\mathcal{B}+\Psi_1}$.

Remark 4.11. If Ψ_1, Ψ_2, and Ψ_3 are pairwise (strongly) independent addenda to a parchment $\mathcal{B}$, then Ψ_2 and Ψ_3 are also (strongly) independent addenda to $\mathcal{B} + \Psi_1$.

All in all, this shows that, in order to sequentially extend a parchment $\mathcal{B}$ with addenda Ψ_1, Ψ_2, …, Ψ_n, it suffices to check that Ψ_1, Ψ_2, …, Ψ_n are pairwise (and perhaps strongly) independent with respect to the original parchment $\mathcal{B}$. The stronger variant of this property holds for all addenda in Example 4.2.

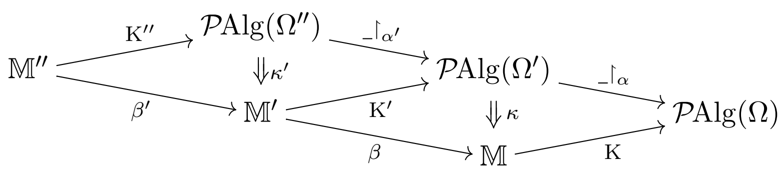
5 Addendum extensions as universal constructions

Much like institutions [GB92] and ordinary parchments [Mos95], stratified parchments can be described as functors into a special category of rooms.

Definition 5.1 (Rooms and corridors)**.** A *stratified-parchment room* is a triple $\langle \Omega, \mathbb{M}, \mathrm{K} \rangle$ where Ω is an algebraic signature of *sentences*, $\mathbb{M}$ is a category of *models* and *homomorphisms*, and $\mathrm{K}\colon \mathbb{M} \to \mathcal{P}\mathrm{Alg}(\Omega)$ is a *stratification* functor.

A *corridor* $\langle \Omega, \mathbb{M}, \mathrm{K} \rangle \to \langle \Omega', \mathbb{M}', \mathrm{K}' \rangle$ is a triple $\langle \alpha, \beta, \kappa \rangle$ consisting of an algebraic signature morphism $\alpha\colon \Omega \to \Omega'$, a functor $\beta\colon \mathbb{M}' \to \mathbb{M}$, and a natural transformation $\kappa\colon \mathrm{K}' \mathbin{⨾} \mathcal{P}\mathrm{Alg}(\alpha) \Rightarrow \beta \mathbin{⨾} \mathrm{K}$ such that, for every model $M' \in |\mathbb{M}'|$, the map $\kappa_{M'}\colon \mathrm{K}'(M')\restriction_\alpha \to \mathrm{K}(\beta(M'))$ is an Ω-algebra homomorphism. We say that a corridor $\langle \alpha, \beta, \kappa \rangle$ is *strict* when the natural transformation κ is an identity.

Both identities and the composition of corridors are defined componentwise: e.g., $\langle \alpha, \beta, \kappa \rangle \mathbin{⨾} \langle \alpha', \beta', \kappa' \rangle = \langle \alpha \mathbin{⨾} \alpha', \beta' \mathbin{⨾} \beta, \kappa' \mathcal{P}\mathrm{Alg}(\alpha) \mathbin{⨾} \beta' \kappa \rangle$, where α, β, κ and α', β', κ' correspond to corridors depicted in the next diagram; the outcome is well formed because both $\mathcal{P}\mathrm{Alg}(\alpha)$ and the composition of arrows in $\mathcal{P}\mathrm{Alg}(\Omega)$ preserve homomorphisms.



Remark 5.2. Thanks to the interchange law for the vertical and horizontal composition of natural transformations, the composition of corridors can be shown to be associative, leading to a category $\mathbb{SPR}\mathrm{oom}$ of stratified-parchment rooms and corridors. Moreover, strict stratified-parchment corridors define a subcategory $\mathrm{s}\mathbb{SPR}\mathrm{oom} \subseteq \mathbb{SPR}\mathrm{oom}$.

Then stratified parchments can also be understood as functors into $\mathbb{SPR}\mathrm{oom}$.

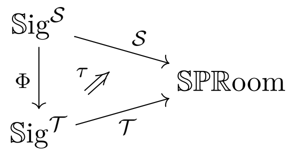
Remark 5.3. There is a one-to-one correspondence between stratified parchments $\langle \mathbb{S}\mathrm{ig}^{\mathcal{S}}, \mathrm{L}^{\mathcal{S}}, \mathrm{Mod}^{\mathcal{S}}, \mathrm{K}^{\mathcal{S}} \rangle$ and (homonymous) functors $\mathcal{S}\colon \mathbb{S}\mathrm{ig}^{\mathcal{S}} \to \mathbb{SPR}\mathrm{oom}$ given by:

- $\mathcal{S}(\Sigma) = \langle \mathrm{L}^{\mathcal{S}}(\Sigma), \mathrm{Mod}^{\mathcal{S}}(\Sigma), \mathrm{K}^{\mathcal{S}}_{\Sigma} \rangle$ for every signature Σ; and
- $\mathcal{S}(\varphi) = \langle \mathrm{L}^{\mathcal{S}}(\varphi), \mathrm{Mod}^{\mathcal{S}}(\varphi), \mathrm{K}^{\mathcal{S}}_{\varphi} \rangle$ for every signature morphism φ.

If $\mathcal{S}$ is strict, then it can also be presented as a functor $\mathcal{S}\colon \mathbb{S}\mathrm{ig}^{\mathcal{S}} \to \mathrm{s}\mathbb{SPR}\mathrm{oom}$.

This observation allows us to introduce morphisms of stratified parchments as arrows in the Grothendieck category [e.g., JY21, Chapter 10] $\int \mathrm{Parch}$ determined by the functor $\mathrm{Parch}\colon \mathbb{C}\mathrm{at}^{\mathrm{op}} \to \mathbb{C}\mathrm{at}$ where (*a*) for every category $\mathbb{S}\mathrm{ig}$ (intuitively, of signatures), $\mathrm{Parch}(\mathbb{S}\mathrm{ig})$ is the opposite category $[\mathbb{S}\mathrm{ig} \to \mathbb{SPR}\mathrm{oom}]^{\mathrm{op}}$ of the category of functors $\mathbb{S}\mathrm{ig} \to \mathbb{SPR}\mathrm{oom}$, and (*b*) for every functor $\Phi\colon \mathbb{S}\mathrm{ig} \to \mathbb{S}\mathrm{ig}'$, $\mathrm{Parch}(\Phi)$ is the pre-composition functor $(\Phi\_)^{\mathrm{op}}$ mapping $\mathbb{S}\mathrm{ig}'$-parchments $\mathcal{S}'\colon \mathbb{S}\mathrm{ig}' \to \mathbb{SPR}\mathrm{oom}$ to $\Phi \fatsemi \mathcal{S}'$, and natural transformations τ' in $\mathrm{Parch}(\mathbb{S}\mathrm{ig}')$ to $\Phi\tau'$. By spelling out the details of the arrows in $\int \mathrm{Parch}$, we obtain the following notion.

Definition 5.4. A morphism between stratified parchments $\mathcal{S}\colon \mathbb{S}\mathrm{ig}^{\mathcal{S}} \to \mathbb{SPR}\mathrm{oom}$ and $\mathcal{T}\colon \mathbb{S}\mathrm{ig}^{\mathcal{T}} \to \mathbb{SPR}\mathrm{oom}$ consists of a functor $\Phi\colon \mathbb{S}\mathrm{ig}^{\mathcal{S}} \to \mathbb{S}\mathrm{ig}^{\mathcal{T}}$ and a natural transformation $\tau\colon \Phi \fatsemi \mathcal{T} \Rightarrow \mathcal{S}$ as in the diagram below.



More concretely, by changing perspective from the higher abstraction level of stratified-parchment corridors to the level of their underlying language, model, and stratification components, the natural transformation τ is equivalent to defining:

- a *language-translation* natural transformation $\alpha\colon \Phi \fatsemi \mathrm{L}^{\mathcal{T}} \Rightarrow \mathrm{L}^{\mathcal{S}}$;
- a *model-reduction* natural transformation $\beta\colon \mathrm{Mod}^{\mathcal{S}} \Rightarrow \Phi^{\mathrm{op}} \fatsemi \mathrm{Mod}^{\mathcal{T}}$; and
- a modification $\kappa\colon \mathrm{K}^{\mathcal{S}} \fatsemi \alpha\mathcal{P}\mathrm{Alg} \Rightarrow \beta \fatsemi \Phi^{\mathrm{op}}\mathrm{K}^{\mathcal{T}}$ with algebra homomorphisms as components; i.e., for every $\mathcal{S}$-signature Σ and every Σ-model M, the many-sorted map $\kappa_\Sigma(M)\colon \mathrm{K}^{\mathcal{S}}_{\Sigma}(M){\restriction}_{\alpha_\Sigma} \to \mathrm{K}^{\mathcal{T}}_{\Phi(\Sigma)}(\beta_\Sigma(M))$ preserves and reflects the interpretation of all symbols in the grammar $\mathrm{L}^{\mathcal{T}}(\Phi(\Sigma))$.

The 'strict' attribute of stratified parchments and corridors also applies to morphisms: a stratified-parchment morphism $\langle \alpha, \beta, \kappa \rangle\colon \mathcal{S} \to \mathcal{T}$ is *strict* when the modification κ is an identity; in other words, when $\mathrm{K}^{\mathcal{S}} \fatsemi \alpha\mathcal{P}\mathrm{Alg} = \beta \fatsemi \Phi^{\mathrm{op}}\mathrm{K}^{\mathcal{T}}$.

Remark 5.5. Strict stratified parchments, together with strict morphisms between them, form a subcategory $\int \mathrm{sParch} \subseteq \int \mathrm{Parch}$, where $\mathrm{sParch}\colon \mathbb{C}\mathrm{at}^{\mathrm{op}} \to \mathbb{C}\mathrm{at}$ is defined similarly to Parch except that, for every category $\mathbb{S}\mathrm{ig}$ of signatures, the objects of $\mathrm{sParch}(\mathbb{S}\mathrm{ig})$ are functors into $\mathrm{s}\mathbb{SPR}\mathrm{oom}$ instead of $\mathbb{SPR}\mathrm{oom}$.

We say that a strict parchment $\mathcal{T}$ is *trivial* when its stratification, $\mathrm{K}^{\mathcal{T}}$, is an identity. Hence, the model functor of a trivial parchment $\mathcal{T}$ can be derived from its language functor: $\mathrm{Mod}^{\mathcal{T}} = (\mathrm{L}^{\mathcal{T}})^{\mathrm{op}} \fatsemi \mathcal{P}\mathrm{Alg}$. Conversely, every functor $F\colon \mathbb{C} \to \mathbb{A}\mathrm{lg}\mathbb{S}\mathrm{ig}$ generates a trivial parchment $\mathrm{Triv}(F)$ given by $\mathbb{S}\mathrm{ig}^{\mathrm{Triv}(F)} = \mathbb{C}$ and $\mathrm{L}^{\mathrm{Triv}(F)} = F$.

Remark 5.6. Any strict morphism $\langle \Phi, \alpha, \beta \rangle \colon \mathcal{S} \to \mathcal{T}$ into a trivial parchment is fully determined by its signature functor Φ and the language translation α: $\beta = \mathrm{K}^{\mathcal{S}} ⨾ \alpha \mathcal{P}\mathrm{Alg}$.

Definition 5.7 (Split)**.** An $\mathcal{F}$-addendum Ψ to a parchment $\mathcal{B}$ *splits* when there exists a subfunctor $S \subseteq \mathrm{L}^{\mathcal{F}}$, called a *split* of Ψ, such that $\Psi ⨾ S = \mathrm{L}^{\mathcal{B}} \cap \Psi ⨾ \mathrm{L}^{\mathcal{F}}$.

Example 5.8. All the concrete addenda considered in Section 4 split. For every feature parchment $\mathcal{F}$ used in those examples, define $S \colon \mathbb{S}\mathrm{ig}^{\mathcal{F}} \to \mathbb{A}\mathrm{lg}\mathbb{S}\mathrm{ig}$ as the functor that maps every feature signature Σ to $\langle \mathrm{ST}^{\mathcal{F}}(\Sigma), \emptyset \rangle$ – i.e., to the algebraic signature with the same sentence types as Σ and with no logical connectives.

However, not every addendum splits, as the following example illustrates.

Example 5.9. Suppose $\mathcal{B}$ and $\mathcal{F}$ are parchments such that:

- $\mathbb{S}\mathrm{ig}^{\mathcal{B}}$ is a terminal category; hence it consists of only one object, say X.
- $\mathrm{L}^{\mathcal{B}}(X)$ is the algebraic signature $\langle \{s\}, \emptyset \rangle$, where s is an arbitrary stratum type.
- $\mathbb{S}\mathrm{ig}^{\mathcal{F}}$ is the category with one object, X, generated by an arrow $f \colon X \to X$.
- $\mathrm{L}^{\mathcal{F}}(X) = \langle \{s, t\}, \emptyset \rangle$ and $\mathrm{L}^{\mathcal{F}}(f)$ maps both s and t to t.

Since $\mathbb{S}\mathrm{ig}^{\mathcal{B}}$ is a discrete category, the inclusion functor $\Psi \colon \mathbb{S}\mathrm{ig}^{\mathcal{B}} \to \mathbb{S}\mathrm{ig}^{\mathcal{F}}$ defines an $\mathcal{F}$-addendum to $\mathcal{B}$. Then it is easy to see that $\mathrm{L}^{\mathcal{F}}$ is the only subfunctor $S \subseteq \mathrm{L}^{\mathcal{F}}$ such that $\mathrm{L}^{\mathcal{B}}(X) \cap \mathrm{L}^{\mathcal{F}}(X) = \langle \{s\}, \emptyset \rangle \subseteq S(X)$ – because $S(f)$ should agree with $\mathrm{L}^{\mathcal{F}}(f)$ on the type s. Therefore, no subfunctor $S \subseteq \mathrm{L}^{\mathcal{F}}$ satisfies the equality $\Psi ⨾ S = \mathrm{L}^{\mathcal{B}} \cap \Psi ⨾ \mathrm{L}^{\mathcal{F}}$.

Remark 5.10. Every split S of an $\mathcal{F}$-addendum Ψ to a parchment $\mathcal{B}$ determines a cospan in the category $\int \mathrm{sParch}$ of strict stratified parchments

$$\mathcal{B} \xrightarrow{\langle \Psi, \alpha^{\mathcal{B}}, \beta^{\mathcal{B}} \rangle} \mathrm{Triv}(S) \xleftarrow{\langle id, \alpha^{\mathcal{F}}, \beta^{\mathcal{F}} \rangle} \mathcal{F}$$

where the language translations $\alpha^{\mathcal{B}}$ and $\alpha^{\mathcal{F}}$ correspond to the natural inclusions $\Psi ⨾ S \subseteq \mathrm{L}^{\mathcal{B}}$ and $S \subseteq \mathrm{L}^{\mathcal{F}}$, respectively – the former of which is also denoted by $\eta^{\mathcal{B}}$ in Proposition 4.6 – and the model reductions $\beta^{\mathcal{B}}$ and $\beta^{\mathcal{F}}$ are given by the composite transformations $\mathrm{K}^{\mathcal{B}} ⨾ \alpha^{\mathcal{B}} \mathcal{P}\mathrm{Alg}$ and $\mathrm{K}^{\mathcal{F}} ⨾ \alpha^{\mathcal{F}} \mathcal{P}\mathrm{Alg}$ as per Remark 5.6.

Proposition 5.11. *Suppose Ψ is an $\mathcal{F}$-addendum to a parchment $\mathcal{B}$ and let S be one of its splits. Then the extended stratified parchment $\mathcal{B} + \Psi$ is the vertex of a pullback in $\int \mathrm{sParch}$ of $\langle \Psi, \alpha^{\mathcal{B}}, \beta^{\mathcal{B}} \rangle$ and $\langle id, \alpha^{\mathcal{F}}, \beta^{\mathcal{F}} \rangle$*

$$\mathcal{B} \xleftarrow{\langle id, \theta^{\mathcal{B}}, \pi^{\mathcal{B}} \rangle} \mathcal{B} + \Psi \xrightarrow{\langle \Psi, \theta^{\mathcal{F}}, \pi^{\mathcal{F}} \rangle} \mathcal{F}$$

where $\theta^{\mathcal{B}}$ and $\theta^{\mathcal{F}}$ are the natural inclusions $\mathrm{L}^{\mathcal{B}} \subseteq \mathrm{L}^{\mathcal{B}} \cup \Psi ⨾ \mathrm{L}^{\mathcal{F}} \supseteq \Psi ⨾ \mathrm{L}^{\mathcal{F}}$ – see Remark 4.3 – and $\pi^{\mathcal{B}}$ and $\pi^{\mathcal{F}}$ are the obvious projections $\mathrm{Mod}^{\mathcal{B}} \Leftarrow \mathrm{Mod}^{\mathcal{B}+\Psi} \Rightarrow \Psi ⨾ \mathrm{Mod}^{\mathcal{F}}$.

Proof. To start, notice that, according to the construction introduced in Proposition 4.6, the parchment morphisms $\langle id, \theta^{\mathcal{B}}, \pi^{\mathcal{B}} \rangle$ and $\langle \Psi, \theta^{\mathcal{F}}, \pi^{\mathcal{F}} \rangle$ do form a cone over the cospan representation of Ψ. Hence we only need to check its universal property. Let $\langle \Phi^{\mathcal{B}}, \sigma^{\mathcal{B}}, \tau^{\mathcal{B}} \rangle \colon \mathcal{G} \to \mathcal{B}$ and $\langle \Phi^{\mathcal{F}}, \sigma^{\mathcal{F}}, \tau^{\mathcal{F}} \rangle \colon \mathcal{G} \to \mathcal{F}$ be two other strict morphisms such that $\langle \Phi^{\mathcal{B}}, \sigma^{\mathcal{B}}, \tau^{\mathcal{B}} \rangle \mathbin{;} \langle \Psi, \alpha^{\mathcal{B}}, \beta^{\mathcal{B}} \rangle = \langle \Phi^{\mathcal{F}}, \sigma^{\mathcal{F}}, \tau^{\mathcal{F}} \rangle \mathbin{;} \langle id, \alpha^{\mathcal{F}}, \beta^{\mathcal{F}} \rangle$. By evaluating these compositions we obtain the following equalities:

(*a*) $\Phi^{\mathcal{B}} \mathbin{;} \Psi = \Phi^{\mathcal{F}}$,

(*b*) $\Phi^{\mathcal{B}} \alpha^{\mathcal{B}} \mathbin{;} \sigma^{\mathcal{B}} = \Phi^{\mathcal{F}} \alpha^{\mathcal{F}} \mathbin{;} \sigma^{\mathcal{F}}$, and

(*c*) $\tau^{\mathcal{B}} \mathbin{;} (\Phi^{\mathcal{B}})^{\mathrm{op}} \beta^{\mathcal{B}} = \tau^{\mathcal{F}} \mathbin{;} (\Phi^{\mathcal{F}})^{\mathrm{op}} \beta^{\mathcal{F}}$.

Consider a signature Σ of the parchment $\mathcal{G}$. Combining (*a*) and (*b*), it follows that $\alpha^{\mathcal{B}}_{\Phi^{\mathcal{B}}(\Sigma)} \mathbin{;} \sigma^{\mathcal{B}}_{\Sigma} = \alpha^{\mathcal{F}}_{\Psi(\Phi^{\mathcal{B}}(\Sigma))} \mathbin{;} \sigma^{\mathcal{F}}_{\Sigma}$. Since $\theta^{\mathcal{B}}_{\Phi^{\mathcal{B}}(\Sigma)}$ and $\theta^{\mathcal{F}}_{\Phi^{\mathcal{B}}(\Sigma)}$ define a pushout in $\mathbb{A}\mathrm{lg}\mathbb{S}\mathrm{ig}$ of $\alpha^{\mathcal{B}}_{\Phi^{\mathcal{B}}(\Sigma)} = \eta^{\mathcal{B}}_{\Phi^{\mathcal{B}}(\Sigma)}$ and $\alpha^{\mathcal{F}}_{\Psi(\Phi^{\mathcal{B}}(\Sigma))} = \eta^{\mathcal{F}}_{\Phi^{\mathcal{B}}(\Sigma)}$, we infer that the exists a unique algebraic signature morphism $\delta_{\Sigma} \colon \mathrm{L}^{\mathcal{B}+\Psi}(\Phi^{\mathcal{B}}(\Sigma)) \to \mathrm{L}^{\mathcal{G}}(\Sigma)$ such that $\theta^{\mathcal{B}}_{\Phi^{\mathcal{B}}(\Sigma)} \mathbin{;} \delta_{\Sigma} = \sigma^{\mathcal{B}}_{\Sigma}$ and $\theta^{\mathcal{F}}_{\Phi^{\mathcal{B}}(\Sigma)} \mathbin{;} \delta_{\Sigma} = \sigma^{\mathcal{F}}_{\Sigma}$. And by putting together these signature morphisms, for all $\mathcal{G}$-signatures Σ, we obtain a natural transformation $\delta \colon \Phi^{\mathcal{B}} \mathbin{;} \mathrm{L}^{\mathcal{B}+\Psi} \Rightarrow \mathrm{L}^{\mathcal{G}}$ that satisfies the equalities $\Phi^{\mathcal{B}} \theta^{\mathcal{B}} \mathbin{;} \delta = \sigma^{\mathcal{B}}$ and $\Phi^{\mathcal{B}} \theta^{\mathcal{F}} \mathbin{;} \delta = \sigma^{\mathcal{F}}$.

In a similar manner, combining (*a*) and (*c*), and based on the observation that the components of $\pi^{\mathcal{B}}$ and $\pi^{\mathcal{F}}$ correspond to pullbacks of functors given by $\beta^{\mathcal{B}}$ and $\beta^{\mathcal{F}}$, we obtain a natural transformation $\zeta \colon \mathrm{Mod}^{\mathcal{G}} \Rightarrow (\Phi^{\mathcal{B}})^{\mathrm{op}} \mathbin{;} \mathrm{Mod}^{\mathcal{B}+\Psi}$ such that $\zeta \mathbin{;} (\Phi^{\mathcal{B}})^{\mathrm{op}} \pi^{\mathcal{B}} = \tau^{\mathcal{B}}$ and $\zeta \mathbin{;} (\Phi^{\mathcal{B}})^{\mathrm{op}} \pi^{\mathcal{F}} = \tau^{\mathcal{F}}$. It is easy to see now that the triple $\langle \Phi^{\mathcal{B}}, \delta, \zeta \rangle$ forms a morphism of stratified parchments $\mathcal{G} \to \mathcal{B} + \Psi$ such that $\langle \Phi^{\mathcal{B}}, \delta, \zeta \rangle \mathbin{;} \langle id, \theta^{\mathcal{B}}, \pi^{\mathcal{B}} \rangle = \langle \Phi^{\mathcal{B}}, \sigma^{\mathcal{B}}, \tau^{\mathcal{B}} \rangle$ and $\langle \Phi^{\mathcal{B}}, \delta, \zeta \rangle \mathbin{;} \langle \Psi, \theta^{\mathcal{F}}, \pi^{\mathcal{F}} \rangle = \langle \Phi^{\mathcal{F}}, \sigma^{\mathcal{F}}, \tau^{\mathcal{F}} \rangle$ – and, moreover, that it is uniquely determined by these equalities. □

The 'split' property of addenda is often preserved when extending parchments.

Remark 5.12. Let Ψ_1 and Ψ_2 be two strongly independent addenda to a stratified parchment $\mathcal{B}$. Then every split of Ψ_2 with respect to $\mathcal{B}$ is also a split of Ψ_2 with respect to the extended parchment $\mathcal{B} + \Psi_1$, and vice versa.

However, if the addenda Ψ_1 and Ψ_2 are independent but not strongly so, then the splits of Ψ_2 might not be preserved under the extension $\mathcal{B} + \Psi_1$.

Example 5.13. Consider the parchments $\mathcal{B}$ and $\mathcal{F}$ from Example 5.9, and let $\mathcal{G}$ be a parchment with the same category of signatures as $\mathcal{B}$ and with $\mathrm{L}^{\mathcal{G}}(X) = \emptyset$, i.e., the empty algebraic signature. Both the identity $id_{\mathbb{S}\mathrm{ig}^{\mathcal{B}}}$ and the inclusion functor $\Psi \colon \mathbb{S}\mathrm{ig}^{\mathcal{G}} \to \mathbb{S}\mathrm{ig}^{\mathcal{F}}$ define addenda to $\mathcal{G}$ – one is a $\mathcal{B}$-addendum, while the other is an $\mathcal{F}$-addendum. They are trivially independent (because $\mathbb{S}\mathrm{ig}^{\mathcal{G}}$ is discrete), but not strongly independent; and both admit splits with respect to $\mathcal{G}$ that map X to $\emptyset$. Yet the extended parchment $\mathcal{G} + id_{\mathbb{S}\mathrm{ig}^{\mathcal{B}}}$ fits within the premisses of Example 5.9, hence the addendum Ψ does not split with respect to $\mathcal{G} + id_{\mathbb{S}\mathrm{ig}^{\mathcal{B}}}$.

Lemma 5.14. *Let Ψ_1 and Ψ_2 be two split and strongly independent addenda to a parchment $\mathcal{B}$, with corresponding feature parchments $\mathcal{F}_1$ and $\mathcal{F}_2$, and with splits S_1 and S_2. And let us also denote the morphisms involved in computing $\mathcal{B} + \Psi_1$ by:*

$$\begin{array}{ccc}
\mathcal{B} + \Psi_1 & \xrightarrow{\mu_1^{\mathcal{F}}=\langle \Psi_1,\theta_1^{\mathcal{F}},\pi_1^{\mathcal{F}}\rangle} & \mathcal{F}_1 \\
{\scriptstyle \mu_1^{\mathcal{B}}=\langle id,\theta_1^{\mathcal{B}},\pi_1^{\mathcal{B}}\rangle}\downarrow & & \downarrow{\scriptstyle \nu_1^{\mathcal{F}}=\langle id,\alpha_1^{\mathcal{F}},\beta_1^{\mathcal{F}}\rangle} \\
\mathcal{B} & \xrightarrow[\nu_1^{\mathcal{B}}=\langle \Psi_1,\alpha_1^{\mathcal{B}},\beta_1^{\mathcal{B}}\rangle]{} & \mathrm{Triv}(S_1)
\end{array}$$

For every cospan representation $(\iota_2^{\mathcal{B}}, \iota_2^{\mathcal{F}})$ as depicted below of the $\mathcal{F}_2$-addendum Ψ_2 to $\mathcal{B}$, the pair $(\mu_1^{\mathcal{B}} \mathbin{⨟} \iota_2^{\mathcal{B}}, \iota_2^{\mathcal{F}})$ is a cospan representation of the addendum Ψ_2 to $\mathcal{B} + \Psi_1$.

$$\mathcal{B} \xrightarrow{\iota_2^{\mathcal{B}}=\langle \Psi_2,\alpha_2^{\mathcal{B}},\beta_2^{\mathcal{B}}\rangle} \mathrm{Triv}(S_2) \xleftarrow{\iota_2^{\mathcal{F}}=\langle id,\alpha_2^{\mathcal{F}},\beta_2^{\mathcal{F}}\rangle} \mathcal{F}_2$$

Proof. First, notice that, by Remark 5.12, the functor S_2 is a split of Ψ_2 not only with respect to $\mathcal{B}$, but also with respect to $\mathcal{B} + \Psi_1$. By evaluating the composition $\mu_1^{\mathcal{B}} \mathbin{⨟} \iota_2^{\mathcal{B}}$ we obtain the parchment morphism $\langle \Psi_2, \alpha_2^{\mathcal{B}} \mathbin{⨟} \theta_1^{\mathcal{B}}, \pi_1^{\mathcal{B}} \mathbin{⨟} \beta_2^{\mathcal{B}}\rangle$. So, according to Remark 5.10, it suffices to check that the composed language translation $\alpha_2^{\mathcal{B}} \mathbin{⨟} \theta_1^{\mathcal{B}}$ is the natural inclusion $\Psi_2 \mathbin{⨟} S_2 \subseteq \mathrm{L}^{\mathcal{B}+\Psi_1}$. But this is straightforward since, by Remark 5.10 (applied to Ψ_1) and Proposition 5.11, $\alpha_2^{\mathcal{B}}$ and $\theta_1^{\mathcal{B}}$ correspond to the natural inclusions $\Psi_2 \mathbin{⨟} S_2 \subseteq \mathrm{L}^{\mathcal{B}}$ and $\mathrm{L}^{\mathcal{B}} \subseteq \mathrm{L}^{\mathcal{B}+\Psi_1}$, respectively. □

Proposition 5.15. *Suppose $(\Psi_i \mid 1 \leq i \leq n)$ is a sequence of split and strongly independent addenda to a parchment $\mathcal{B}$. Then the extended parchment $\mathcal{B}+\Psi_1+\cdots+\Psi_n$ is the vertex of a limit in $\int \mathrm{sParch}$ of their cospan representations.*

Proof. For each index i, let $(\iota_i^{\mathcal{B}}, \iota_i^{\mathcal{F}})$ be a cospan representation of Ψ_i. Let us also define the sequence of parchments $(\mathcal{B}_i \mid 0 \leq 1 \leq n)$ by $\mathcal{B}_0 = \mathcal{B}$ and $\mathcal{B}_{i+1} = \mathcal{B}_i + \Psi_{i+1}$ for all $0 \leq i < n$, and denote the morphisms involved in computing $\mathcal{B}_i + \Psi_{i+1}$ by:

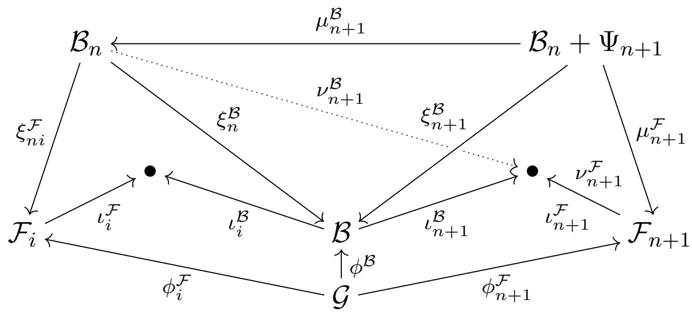
$$\begin{array}{ccc}
\mathcal{B}_i + \Psi_{i+1} & \xrightarrow{\mu_{i+1}^{\mathcal{F}}} & \mathcal{F}_{i+1} \\
{\scriptstyle \mu_{i+1}^{\mathcal{B}}}\downarrow & & \downarrow{\scriptstyle \nu_{i+1}^{\mathcal{F}}} \\
\mathcal{B}_i & \xrightarrow[\nu_{i+1}^{\mathcal{B}}]{} & \mathrm{Triv}(S_{i+1})
\end{array}$$

In addition, let $(\xi_i^{\mathcal{B}} \colon \mathcal{B}_i \to \mathcal{B} \mid 0 \leq i \leq n)$ and $(\xi_{ij}^{\mathcal{F}} \colon \mathcal{B}_i \to \mathcal{F}_j \mid 1 \leq j \leq i \leq n)$ be the families of parchment morphisms defined by:

- $\xi_0^{\mathcal{B}} = id_{\mathcal{B}}$ and $\xi_{i+1}^{\mathcal{B}} = \mu_{i+1}^{\mathcal{B}} \mathbin{⨟} \xi_i^{\mathcal{B}}$ for all $0 \leq i < n$;
- $\xi_{ii}^{\mathcal{F}} = \mu_i^{\mathcal{F}}$ for all $1 \leq i \leq n$, and $\xi_{(i+1)j}^{\mathcal{F}} = \mu_{i+1}^{\mathcal{B}} \mathbin{⨟} \xi_{ij}^{\mathcal{F}}$ for all $1 \leq j \leq i < n$.

Then we claim that the morphism $\xi_n^{\mathcal{B}}$ together with the 'projections' $\xi_{ni}^{\mathcal{F}}$, for all indices $1 \leq i \leq n$, define a limiting cone of the addenda representations $\left(\iota_i^{\mathcal{B}}, \iota_i^{\mathcal{F}}\right)_{1 \leq i \leq n}$.

We prove the statement by induction on the number n of addenda. The base case is addressed in Proposition 5.11. For the induction step, consider the top part of the following diagram, which summarizes the construction of $\mathcal{B}_n + \Psi_{n+1}$:



First, we need to check that the morphism $\xi_{n+1}^{\mathcal{B}}$ together with the projections $\xi_{(n+1)i}^{\mathcal{F}}$ is indeed a cone of $\left(\iota_i^{\mathcal{B}}, \iota_i^{\mathcal{F}}\right)_{1 \leq i \leq n+1}$. To that end, notice that, by (repeated applications of) Lemma 5.14, $\nu_{i+1}^{\mathcal{B}} = \xi_i^{\mathcal{B}} \fatsemi \iota_{i+1}^{\mathcal{B}}$ and $\nu_{i+1}^{\mathcal{F}} = \iota_{i+1}^{\mathcal{F}}$ for all indices $0 \leq i \leq n$. Based on this observation, for $i = n+1$, the cone property we are interested in follows from the next sequence of equalities:

$$
\begin{aligned}
&\xi_{n+1}^{\mathcal{B}} \fatsemi \iota_{n+1}^{\mathcal{B}} \\
&\quad = \mu_{n+1}^{\mathcal{B}} \fatsemi \xi_n^{\mathcal{B}} \fatsemi \iota_{n+1}^{\mathcal{B}} && \text{since } \xi_{n+1}^{\mathcal{B}} = \mu_{n+1}^{\mathcal{B}} \fatsemi \xi_n^{\mathcal{B}} \\
&\quad = \mu_{n+1}^{\mathcal{B}} \fatsemi \nu_{n+1}^{\mathcal{B}} && \text{since, by Lemma 5.14, } \xi_n^{\mathcal{B}} \fatsemi \iota_{n+1}^{\mathcal{B}} = \nu_{n+1}^{\mathcal{B}} \\
&\quad = \mu_{n+1}^{\mathcal{F}} \fatsemi \nu_{n+1}^{\mathcal{F}} && \text{by the construction of } \mathcal{B}_n + \Psi_{n+1} \\
&\quad = \mu_{n+1}^{\mathcal{F}} \fatsemi \iota_{n+1}^{\mathcal{F}} && \text{since, by Lemma 5.14, } \nu_{i+1}^{\mathcal{F}} = \iota_{i+1}^{\mathcal{F}}.
\end{aligned}
$$

For $i < n+1$, we have:

$$
\begin{aligned}
&\xi_{n+1}^{\mathcal{B}} \fatsemi \iota_i^{\mathcal{B}} \\
&\quad = \mu_{n+1}^{\mathcal{B}} \fatsemi \xi_n^{\mathcal{B}} \fatsemi \iota_i^{\mathcal{B}} && \text{since } \xi_{n+1}^{\mathcal{B}} = \mu_{n+1}^{\mathcal{B}} \fatsemi \xi_n^{\mathcal{B}} \\
&\quad = \mu_{n+1}^{\mathcal{B}} \fatsemi \xi_{ni}^{\mathcal{F}} \fatsemi \iota_i^{\mathcal{F}} && \text{by the induction hypothesis} \\
&\quad = \xi_{(n+1)i}^{\mathcal{F}} \fatsemi \iota_i^{\mathcal{F}} && \text{since } \mu_{n+1}^{\mathcal{B}} \fatsemi \xi_{ni}^{\mathcal{F}} = \xi_{(n+1)i}^{\mathcal{F}}.
\end{aligned}
$$

For the universal property of this cone, suppose $\mathcal{G}$ is another stratified parchment together with morphisms $\phi^{\mathcal{B}} \colon \mathcal{G} \to \mathcal{B}$ and $\phi_i^{\mathcal{F}} \colon \mathcal{G} \to \mathcal{F}_i$, for $1 \leq i \leq n+1$, such that the diagram above commutes. Since, by the induction hypothesis, the morphisms $\xi_n^{\mathcal{B}}$ and $\xi_{ni}^{\mathcal{F}}$, with $1 \leq i \leq n$, form a limiting cone, it follows that there exists a unique parchment morphism $\phi_n^{\mathcal{B}} \colon \mathcal{G} \to \mathcal{B}_n$ such that $\phi_n^{\mathcal{B}} \fatsemi \xi_n^{\mathcal{B}} = \phi^{\mathcal{B}}$ and $\phi_n^{\mathcal{B}} \fatsemi \xi_{ni}^{\mathcal{F}} = \phi_i^{\mathcal{F}}$ for all $1 \leq i \leq n$. And because $\xi_n^{\mathcal{B}} \fatsemi \iota_{n+1}^{\mathcal{B}} = \nu_{n+1}^{\mathcal{B}}$, we deduce that $\phi_n^{\mathcal{B}}$ and $\phi_{n+1}^{\mathcal{F}}$ form a cone of $\nu_{n+1}^{\mathcal{B}}$ and $\nu_{n+1}^{\mathcal{F}}$. From the universal property of $\mathcal{B}_n + \Psi_{n+1}$, it follows that there exists a unique morphism $\phi^{\mathcal{B}}_{n+1} : \mathcal{G} \to \mathcal{B}_n + \Psi_{n+1}$ such that $\phi^{\mathcal{B}}_{n+1} ⨟ \mu^{\mathcal{B}}_{n+1} = \phi^{\mathcal{B}}_n$ and $\phi^{\mathcal{B}}_{n+1} ⨟ \mu^{\mathcal{F}}_{n+1} = \phi^{\mathcal{F}}_{n+1}$. To conclude, notice that:

$$
\begin{aligned}
&\phi^{\mathcal{B}}_{n+1} ⨟ \xi^{\mathcal{B}}_{n+1} \\
&\quad = \phi^{\mathcal{B}}_{n+1} ⨟ \mu^{\mathcal{B}}_{n+1} ⨟ \xi^{\mathcal{B}}_n && \text{since } \xi^{\mathcal{B}}_{n+1} = \mu^{\mathcal{B}}_{n+1} ⨟ \xi^{\mathcal{B}}_n \\
&\quad = \phi^{\mathcal{B}}_n ⨟ \xi^{\mathcal{B}}_n && \text{since } \phi^{\mathcal{B}}_{n+1} ⨟ \mu^{\mathcal{B}}_{n+1} = \phi^{\mathcal{B}}_n \\
&\quad = \phi^{\mathcal{B}} && \text{from the universal property of } \mathcal{B}_n
\end{aligned}
$$

and, in a similar manner, $\phi^{\mathcal{B}}_{n+1} ⨟ \xi^{\mathcal{F}}_{(n+1)i} = \phi^{\mathcal{F}}_i$ for all $1 \leq i \leq n+1$. □

Corollary 5.16. *If Ψ_1, Ψ_2, …, Ψ_n are split and strongly independent addenda to a parchment $\mathcal{B}$, then for any permutation γ of their indices, the extended parchments $\mathcal{B} + \Psi_1 + \cdots + \Psi_n$ and $\mathcal{B} + \Psi_{\gamma(1)} + \cdots + \Psi_{\gamma(n)}$ are isomorphic.* □

6 Conclusions

The framework presented in this paper has arisen from the serendipitous observation that (for logics with only one sentence type) selections of connectives for arbitrary but fixed stratified institutions – as defined in [AD07] – may also be loosely regarded as parchments of those institutions. To make that connection precise, we have introduced a new concept of stratified parchment that refines previous model-theoretic notions of parchment by taking into account models with states and by defining the semantics of logical connectives using operations on sets of model states.

Besides presenting stratified institutions, stratified parchments are also useful in showing how connectives can be gradually developed on top of base logical systems. For this purpose, we have presented addenda, which allow for simple combinations between a base parchment (of a logic that we aim to extend) and a feature parchment (that captures a connective). We have also examined conditions that allow addenda to be applied sequentially, and we have proved that the results obtained in this way are vertices of limits in a category of strict morphisms of parchments.

We have refrained from using limits of arbitrary diagrams for combining stratified parchments in order to simplify the constructions and make the presentation accessible to a wider audience. However, as with other kinds of institutional parchments, this is of course possible. In fact, it is not difficult to see, based on general results on indexed and comma categories [TBG91] that $\int$ sParch is complete: this follows by noticing that the category s$\mathbb{SPR}$oom of strict stratified corridors can be presented as the opposite of the comma category $\mathbb{C}$at / $\mathcal{P}$Alg, which is complete because $\mathbb{A}$lg$\mathbb{S}$ig is complete and $\mathcal{P}$Alg preserves limits (generalizing Lemma 4.5). Limits in $\int$ Parch can be obtained similarly by noticing that the category $\mathbb{SPR}$oom of lax stratified corridors arises from a secondary Grothendieck construction. We aim to explore these properties and their potential applications to non-conventional logics that combine modal and first-order features as in [ŢCF21] in future work.

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