Scientific Report for Phase III of the Project

1 Scientific results

The following lines give a short description of the research results obtained by the members of the team during the third phase of the project.

• P. Baird and J. C. Wood have solved completely a problem of classification of harmonic morphisms with one-dimensional fibres from open domains in $\mathbb{R}^3$, posed by Jacobi. This problem is to describe (locally) foliations in lines on open domains in $\mathbb{R}^3$ which verify a geometric extra-condition, horizontal weak conformality (HWC). In the paper *Holomorphic vector bundles on Kahler manifolds and totally geodesic foliations on Euclidian domains* (by *Monica Aprodu* and Marian Aprodu, paper **submitted for publication**), there were extended the results of Baird and Wood to foliations of higher codimension. It was established a relation between holomorphic maps to an orthogonal Grassmannian, together with a section in the universal bundle, and certain foliations with totally geodesic leaves on Euclidean open domains. It was found a geometric condition for a totally geodesic foliation to originate in a holomorphic vector bundle. For codimension-two foliations, this description recovers the results of Baird and Wood. The universal objects that play a key role are the orthogonal Grassmannians.

• In the paper *On cotangent manifolds, complex structures and generalized geometry* (by *Liana David*, paper **submitted for publication**), we studied the generalized complex geometry and its relations to special complex geometry. Generalized complex geometry was discovered by N. Hitchin as a unification of complex and symplectic geometry. A (skew-symmetric) generalized complex structure on a (smooth) manifold M is a field of endomorphisms $\mathcal{J}$ of the generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$, which satisfies $\mathcal{J}^2 = -\mathrm{Id}$ and which is skew-symmetric relative to the standard natural metric g_{can} of $\mathbb{T}M$ of neutral signature. There is a notion of integrability of generalized complex structures, which generalizes the usual notion of integrability for almost complex and for almost symplectic structures. In the paper mentioned above, we studied generalized complex structures (i.e. endomorphisms $\mathcal{J}$ of $\mathbb{T}M$ with $\mathcal{J}^2 = -\mathrm{Id}$), which are *symmetric* with respect to the metric g_{can}. Various properties of such structures (in relation to their holomorphic space and B-field transformations) were developed. As a main application, a large class of invariant (integrable) complex structures on the cotangent manifold T^*G of a semisimple Lie group G were constructed. Simpler proofs of various results from *Special complex manifolds* (by D. Alekseevsky, V. Cortes and Devchand, paper published in *Journal of Geometry and Physics* (2002)), were also developed.

• In the book *Parabolic Geometry I* (by A. Cap and J. Slovak, Mathematical Surveys and Monographs (2009)), there were studied in a systematic way the Cartan connections of semisimple type. A Cartan connection $\omega \in \Omega^1(P, \mathfrak{h})$ on a Q-fiber bundle $\pi : P \to M$ is of semisimple type, if $\mathfrak{h}$ is a semisimple (graded) Lie algebra and $\mathrm{Lie}(Q) = \mathfrak{h}_{\geq 0}$ is its non-negative part. A. Cap and J. Slovak proved that any (regular) Cartan connection of semisimple type induces a Tanaka structure (which is a generalization of the notion of G-structure) on the base space M and conversely, any Tanaka structure on M arises from a (not necessarily unique) such a Cartan connection. Using the fact that the Killing form of $\mathfrak{h}$ is non-degenerate, they constructed a natural, positive definite, adapted to the gradation metric on $\mathfrak{h}$ which was used to define the notion of normal Cartan connection (as a Cartan connection with coclosed curvature). The central result of the book by A. Cap and J. Slovak states that there is a bijective correspondence between Tanaka structures and normal Cartan connections. In the paper *Tanaka structures (non-holonomic G-structures) and Cartan connections* (by Dmitri Alekseevsky and *Liana David*; paper **under evaluation**), we extended this theory to the more general case when the Lie algebra $\mathfrak{h}$ is graded (but not necessarily semisimple). We proved that most of the results of A. Cap and J. Slovak remain true in this more general setting

• In the paper *On the twistor space of a (co-)CR quaternionic manifold* (by *Radu Pantilie*; paper published in **New-York J. Math.**), we characterised, in the setting of the Kodaira–Spencer deformation theory, the twistor spaces of (co-)CR quaternionic manifolds. As an application, we proved that, locally, the leaf space of any nowhere zero quaternionic vector field on a quaternionic manifold is endowed with a natural co-CR quaternionic structure. Also, for any positive integers k and l , with kl even, we obtained the geometric objects whose twistorial counterparts are complex manifolds endowed with a conjugation without fixed points and which preserves an embedded Riemann sphere with normal bundle $l\mathcal{O}(k)$. We applied these results to prove the existence of natural classes of co-CR quaternionic manifolds.

• Let M be a quaternionic manifold of dimension $4k$, whose twistor space is a Fano manifold. In the paper *On the quaternionic manifolds whose twistor spaces are Fano manifolds* (by *Radu Pantilie*; paper published in **Tohoku J. Math.**), we prove the following: (a) M admits a reduction to $\mathrm{Sp}(1) \times \mathrm{GL}(k, \mathbb{H})$ if and only if $M = \mathbb{H}P^k$; (b) either $b_2(M) = 0$ or $M = \mathrm{Gr}_2(k+2, \mathbb{C})$. This generalizes results of S. Salamon and C. R. LeBrun, respectively, who obtained the same conclusions under the assumption that M is a complete quaternionic-Kähler manifold with positive scalar curvature.

2 Presentations as "Invited Speakers"

• **Gabriel Baditoiu** gave the prezentation *Classification of pseudo-Riemannian submersions with totally geodesic fibers from pseudo-hyperbolic spaces*, at "Real and Complex Differential Geometry", Universitaty of Bucharest, 8-13 September 2014;

• **Liana David** gave the prezentation *On cotangent manifolds, complex structures and generalized geometry*, at "Real and Complex Differential Geometry", University of Bucharest, 8-13 September 2014;

• **Radu Pantilie** gave the prezentation *The Penrose transform in quaternionic geometry*, at 'The 10th Differential Geometry Day and The Differential Geometry Workshop', The Centre for Mathematical Sciences, Lund University, Sweeden, 14–16 Mai 2014.

3 Visits abroad

Radu Pantilie visited Lund University (Sweeden), where he gave the prezentation mentioned in the previous section. The visit was financially supported by the budget of the project.

4 Foreign visitors

Prof. Dmitri Alekseevsky (Institute of Information and Transmission Problems, Moscova - Russia, and Masaryk University, Brno - Cehia) visited, as an *Invitated Professor*, the host institution of the project, IMAR. The visit is intended also for collaboration with the members of the team, on topics from the project. The *travel expenses* were supported by the budget of the grant.

5 Participation at seminars

The members of the team attended on a regular basis the differential geometry seminar from IMAR (the host institution of the project), where they presented the research results obtained on the topics of the project.

6 Expected results for 2015

The Research Objectives for 2015, as they appear in AAD-2014 (Additional Document 2014), are the followings: 1) Totally geodesic foliations on open subsets in Euclidian space; 2) Riemannian submersions; 3) Cartan connections in parabolic geometry (continuation from 2014).

We mention that Objectives 1) and 3) above are achieved during 2014 (see the first and third paper from the section Scientific Results, from this document). Instead of these objectives, we intend to work on: 1) F-manifolds and their relations to Frobenius manifolds and meromorphic connections (a continuation of the objective from 2012); Harmonic morphisms; 3) Geometry of stratifolds. We will also continue our research on Riemanian submersions (Objective 2 from the above paragraph).