Scientific report for the period October 2011 - October 2013

1 Publications

1.1 Published papers

P1 U. Kohlenbach, L. Leuştean, Effective metastability of Halpern iterates in CAT(0) spaces, Advances in Mathematics, Vol. 231 (2012), 2526-2556.

P2 U. Kohlenbach, L. Leuştean, On the computational content of convergence proofs via Banach limits, Philosophical Transactions of the Royal Society A, Vol. 370 (2012), 3449-3463. Theme Issue *The foundations of computation, physics and mentality: the Turing legacy.*

P3 A. Fernandez-Leon, A. Nicolae, Averaged alternating reflections in geodesic spaces, Journal of Mathematical Analysis and Applications, Vol. 402 (2013), 558-566.

P4 A. Nicolae, Asymptotic behavior of averaged and firmly nonexpansive mappings in geodesic spaces, Nonlinear Analysis - Theory, Methods & Applications, Vol. 87 (2013), 102-115.

P5 L. Leuştean, A. Nicolae, Effective results on compositions of nonexpansive mappings, Journal of Mathematical Analysis and Applications, Vol. 410 (2014), 902-907.

1.2 Papers accepted for publication

P6 D. Ariza-Ruiz, L. Leuştean, G. Lopez-Acedo, Firmly nonexpansive mappings in classes of geodesic spaces, arXiv:1203.1432v2 [math.FA]; accepted for publication in Transactions of the American Mathematical Society.

P7 M. Buliga, Graphic lambda calculus, arXiv:1305.5786v2[cs.LO]; accepted for publication in Complex Systems.

P8 U. Kohlenbach, L. Leuştean, Addendum to 'Effective metastability of Halpern iterates in CAT(0) spaces'(Adv.Math. 231 (2012) 2526-2556); accepted for publication in Advances in Mathematics.

P9 A. Fernandez Leon, A. Nicolae, Best proximity pair results for relatively nonexpansive mappings in geodesic spaces, arXiv:1306.0358v2 [math.FA], 2013; accepted for publication (with minor revision) in Numerical Functional Analysis and Optimization.

1.3 Submitted papers

P10 M. Buliga, Sub-riemannian geometry from intrinsic viewpoint, arXiv:1206.3093v1 [math.MG], 2012.

P11 L. Leuştean, A. Nicolae, Effective results on nonlinear ergodic averages in CAT(k) spaces, arXiv:1305.5916 [math.FA], 2013.

2 Directions of research

Proof mining is a paradigm of research developed by Ulrich Kohlenbach in the 90's, concerned with the extraction of hidden finitary and combinatorial content from proofs that make use of highly infinitary principles. The new information can be both of quantitative nature, such as algorithms and effective bounds, as well as of qualitative nature, such as uniformities in the bounds or weakening the premises. This line of research has its roots in Kreisel's program of *unwinding of proofs*, initiated in the 50's.

Recently, Terence Tao [34] arrived at a proposal of so-called *hard analysis*, based on finitary arguments, instead of the infinitary arguments from *soft analysis*, inspired by the methods used by him and Green [14] in their proof that there are arithmetic progressions of arbitrary length in the prime numbers. Proof mining allows us to obtain results in hard analysis, as Tao remarks [34]: *"There are rigorous results from proof theory, such as Herbrand's theorem, which can allow one to automatically convert certain types of qualitative arguments into quantitative ones."*

Applications of proof mining consist in preprocessing the original mathematical proof in such a way that the statement of the theorem and the main concepts have a suitable logical form, followed by the identification of the key steps in the proof that require a computational interpretation. As a result, we get direct proofs for the explicit quantitative versions of the original results, that is proofs that no longer rely on any logical tools.

In the following we present the results obtained.

2.1 Effective results in nonlinear ergodic theory

Let us recall the Hilbert space formulation of the celebrated von Neumann mean ergodic theorem.

Teorema 2.1. *Let H be a Hilbert space and $U : H \to H$ be a unitary operator. Then for all $x \in H$, the Cesàro mean $x_n = \frac{1}{n}\sum_{i=0}^{n-1} U^i x$ converges strongly to $P_{Fix(U)}x$, the projection of x on the set of fixed points of U.*

If $\mathcal{X} = (X, \mathcal{B}, \mu, T)$ is a probability measure-preserving system, $H = L^2(\mathcal{X})$ and $U = U_T : L^2(\mathcal{X}) \to L^2(\mathcal{X})$, $f \mapsto f \circ T$ is the induced operator, the Cesàro mean starting with $f \in L^2(\mathcal{X})$ becomes the ergodic average $A_n f = \frac{1}{n}\sum_{i=0}^{n-1} f \circ T^i$.

Avigad, Gerhardy and Towsner [1] showed that we can not obtain in general computable rates of convergence. In this situation, one can consider the following equivalent reformulation of the Cauchy property of (x_n):

$$\forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists N \forall i, j \in [N, N + g(N)] \; \left(\|x_i - x_j\| < 2^{-k}\right). \tag{1}$$

This is known in logic as the *no-counterexample* interpretation [21, 22] of the Cauchy property and it was popularized in the last years under the name of *metastability* by Tao [34, 35]. In [35], Tao generalized the mean ergodic theorem to multiple commuting measure-preserving transformations, by deducing it from a finitary norm convergence result, expressed in terms of metastability. Recently, Walsh [36] used again metastability to show the L^2-convergence of multiple polynomial ergodic averages arising from nilpotent groups of measure-preserving transformations. Logical metatheorems [17] show that, from wide classes of mathematical proofs one can extract upper bounds $\Phi(\varepsilon, g)$ on $\exists N$ in (1). Thus, taking $\varepsilon > 0$ instead of 2^{-k}, we define a *rate of metastability* as a functional $\Phi : (0, \infty) \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ satisfying

$$\forall \varepsilon > 0 \, \forall g : \mathbb{N} \to \mathbb{N} \, \exists N \leq \Phi(\varepsilon, g) \, \forall i, j \in [N, N + g(N)] \; (\|x_i - x_j\| < \varepsilon) . \tag{2}$$

In [19] we obtained a quantitative version of the mean ergodic theorem for uniformly convex Banach spaces [3], computing an effective uniform rate of metastability for the Cesàro mean. Immediate consequences are the results obtained by Avigad, Gerhardy and Towsner [1] for Hilbert spaces and by Tao [35] for a particular dynamical system.

An important generalization of the von Neumann mean ergodic theorem was obtained by Wittmann [37] in 1992.

Teorema 2.2. *[37] Let C be a bounded closed convex subset of a Hilbert space X, $T : C \to C$ a nonexpansive mapping and $(\lambda_n)_{n \geq 1}$ be a sequence in $[0, 1]$. Assume that (λ_n) satisfies*

$$\lim_{n \to \infty} \lambda_n = 0, \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda_n = \infty \tag{3}$$

For any $x, u \in C$, define

$$x_0 := u, \quad x_{n+1} := \lambda_{n+1} u + (1 - \lambda_{n+1}) T x_n. \tag{4}$$

Then (x_n) converges to $P_{Fix(T)} u$.

One can easily see that (x_n) coincides with the Cesàro mean when T is linear and $\lambda_n = \dfrac{1}{n+1}$. The iteration (x_n) is known as the Halpern iteration, as it was introduced by Halpern [15] for the special case $u = 0$. The Halpern iteration can be defined similarly in more general spaces, like the geodesic ones.

2.1.1 Effective uniform rates of asymptotic regularity

The first step towards proving the weak or strong convergence of an iteration consists in obtaining the so-called *asymptotic regularity* and this can be done in a very general setting. Asymptotic regularity is a very important concept, introduced by Browder and Petryshyn [5] in the 60's for the Picard iteration, but it can be defined in general for any iteration (x_n) associated with a mapping T on a metric space (X, d): (x_n) is asymptotically regular if $\lim\limits_{n \to \infty} d(x_n, T x_n) = 0$ for all $x \in C$. A rate of convergence of the sequence $(d(x_n, T x_n))$ towards 0 will be called a *rate of asymptotic regularity.*

In the paper **P1** we obtained quantitative results on the asymptotic regularity of the Halpern iteration in CAT(0) spaces for general (λ_n), by considering as a hypothesis both $\sum_{n=1}^{\infty} \lambda_{n+1} = \infty$ and the equivalent condition $\prod_{n=1}^{\infty} (1 - \lambda_{n+1}) = 0$. As an immediate corollary, one obtains a quadratic rate of asymptotic regularity, generalizing in this way the result proved for Hilbert spaces by Kohlenbach [18].

Teorema 2.3. *Assume that* $\lambda_n = \dfrac{1}{n+1}, n \geq 1$. *Then for every* $\varepsilon \in (0,1)$,

$$\forall n \geq \Psi(\varepsilon, M) \ \ (d(x_n, Tx_n) \leq \varepsilon)\,,$$

where $\Psi(\varepsilon, M) = \left\lceil \dfrac{4M}{\varepsilon} + \dfrac{16M^2}{\varepsilon^2} \right\rceil - 1$, *with* $M \in \mathbb{Z}_+$ *such that* $M \geq d_C$.

The method used in **P1** for CAT(0) spaces can not be applied in the case of CAT(κ) spaces (with $\kappa > 0$). For these spaces, we computed an exponential rate of asymptotic regularity in **P11**.

In the paper **P5** we extended this result to finite families of nonexpansive mappings and to (r, δ)-convex spaces, introduced by us as a generalization of CAT(κ) spaces and of the metric spaces with a convex geodesic bicombing (examples of such spaces are the normed ones, Busemann spaces, hyperconvex spaces or W-hyperbolic spaces in the sense of [16]). Immediate consequences of our extension are the results in [24, 25].

2.1.2 Effective uniform rates of metastability

In the papers **P1**, **P2**, **P8** şi **P11** we obtained finitary versions, with effective and highly uniform rates of metastability for generalizations of Theorem 2.2 proved in [33, 32, 29].

These results constitute a significant extension of the actual context of proof mining, as the proofs in [32, 33] make use of Banach limits, inspired by Lorentz' seminal paper [27], in which almost convergence was introduced. Reich [30] initiated the use of almost convergence in nonlinear ergodic theory, while Bruck and Reich [7] applied for the first time Banach limits to the subject of Halpern iteration. The existence of Banach limits is either proved by applying the Hahn-Banach theorem to l^∞ or via ultralimits, in both cases the axiom of choice being needed.

In **P1** we develop a method to convert such proofs into more elementary proofs which no longer rely on Banach limits and can be analyzed by the existing logical machinery. The way Banach limits are used in these proofs seems to be rather typical for other proofs in nonlinear ergodic theory. Therefore, our method can be used to obtain similar results in those cases too.

The following theorem, proved by Saejung [32] using Banach limits, generalizes Wittmann's theorem to CAT(0) spaces.

Teorema 2.4. *Let* C *be a bounded closed convex subset of a complete* CAT(0) *space* X *and* $T : C \to C$ *a nonexpansive mapping. Assume that* (λ_n) *satisfies* (3). *Then for any* $u, x \in C$, *the iteration* (x_n) *converges to a fixed point of* T.

While we can not expect to obtain effective rates of convergence for (x_n), the existence of an effective and highly uniform rate of metastability is guaranteed, after the elimination of Banach limits, by [16, Teorema 3.7.3].

Teorema 2.5. *In the hypotheses of Theorem 2.4, let* α *be a rate of convergence of* (λ_n), β *be a Cauchy modulus of* $s_n := \sum_{i=1}^{n} |\lambda_{i+1} - \lambda_i|$ *and* θ *be a rate of divergence of* $\sum_{n=1}^{\infty} \lambda_{n+1}$.

Then for all $\varepsilon \in (0,2)$ *and* $g : \mathbb{N} \to \mathbb{N}$,

$$\exists N \leq \Sigma(\varepsilon, g, M, \theta, \alpha, \beta)\ \forall m, n \in [N, N + g(N)]\ (d(x_n, x_m) \leq \varepsilon),$$

where $M \in \mathbb{Z}_+$ *is an upper bound on the diameter of* C.

The rate of metastability Σ, extracted in Theorem 4.2 from **P1** does not depend on T, the starting point $x \in C$ and depends weakly on C, via its diameter. We remark that in practical cases, such as $\lambda_n = \frac{1}{n+1}$, the rates α, β, θ are easy to compute. In **P8** we remark that the quantitative analysis of Saejung's proof has as a result the complete elimination of any contribution of the use of Banach limits, which results in simpler bounds.

In **P2** we apply the same method of eliminating Banach limits from the proof given by Shioji and Takahashi [33] to a generalization of Wittmann's theorem to Banach spaces with a uniformly Gâteaux differentiable norm. Furthermore, we prove a logical metatheorem for a class of Banach spaces introduced in **P2** under the name of *spaces with a uniformly continuous duality selection map.*

The paper **P11** is dedicated to the extraction of a uniform rate of metastability for the generalization of Wittmann's theorem to CAT(κ) spaces (with $\kappa > 0$). We use the same general method as in **P1**, but the proofs from this paper ar much more involved than the ones from **P1**. In the case $\lambda_n = \dfrac{1}{n+1}$, we obtain a rate having a very simple logical form, similar with the one described in [20]:

$$\Sigma(\varepsilon, g, \kappa, M) = A_{\varepsilon,\kappa,M}\left(\widetilde{f^*}^{B_{\varepsilon,\kappa,M}}(0) + \left\lceil \frac{1}{\varepsilon_0} \right\rceil\right),$$

computed in Corollary 3.5 from **P11**. Thus, the function g appears only in the definition of $\widetilde{f^*}$, the mappings $A_{\varepsilon,\kappa,M}, B_{\varepsilon,\kappa,M}$ do not depend at all on g.

2.2 Asymptotic behaviour of classes of nonlinear mappinga

In the papers **P3**, **P4**, **P6** şi **P9** important classes of mappings (firmly nonexpansive, averaged, reflections and relatively nonexpansive) are studied in different classes of geodesic spaces and proof mining methods are applied to obtain effective results on the asymptotic behaviour of the associated Picard or Krasnoselskii iterations.

Firmly nonexpansive mappings, introduced by Browder [4] in Hilbert spaces and by Bruck [6] in Banach spaces, play a very important role in nonlinear analysis and convex optimization, due to their correspondence with maximal monotone operators proved by Minty [28]. Bruck's definition extends immediately to W-hyperbolic spaces (in the sense of Kohlenbach [16]). A mapping $T : C \subseteq X \to C$ is called firmly nonexpansive if for all $x, y \in C$ and for all $\lambda \in (0,1)$,

$$d(Tx, Ty) \leq d((1-\lambda)x + \lambda Tx, (1-\lambda)y + \lambda Ty). \tag{5}$$

A first main result of the paper **P6** is a fixed point theorem for firmly nonexpansive mappings defined on unions of closed convex subsets of a complete UCW-hyperbolic space. These spaces [23, 26] are a class of uniformly convex spaces generalizing both CAT(0) spaces and uniformly convex Banach spaces. A second main result of the paper **P6** generalizes results obtained by Reich and Shafrir [31] in Banach spaces or in the Hilbert ball.

Teorema 2.6. *Let C be a subset of a W-hyperbolic space X and $T : C \to C$ be a firmly nonexpansive mapping. Then for all $x \in X$ and $k \in \mathbb{Z}_+$,*

$$\lim_{n\to\infty} d(T^{n+1}x, T^n x) = \frac{1}{k}\lim_{n\to\infty} d(T^{n+k}x, T^n x) = \lim_{n\to\infty}\frac{d(T^n x, x)}{n} = r_C(T),$$

where $r_C(T) := \inf\{d(x,Tx) \mid x \in C\}$.

In **P4** we prove a quantitative version of Theorem 2.6, having as an immediate consequence an exponential rate of asymptotic regularity for the Picard iteration, the only one known even for Banach space. Using different methods, rates of asymptotic regularity for UCW-hyperbolic spaces are computed in **P6**, which turn out to be quadratic for CAT(0) spaces or polynomial for L_p spaces, $1 < p < \infty$.

sive retractions [13].

In the papers **P3** şi **P4** different algorithms for convex feasibility problem in geodesic spaces are studied. Thus, effective rates of asymptotic regularity are obtained in **P4** for the well-known alternating projections method introduced by von Neumann, as well as for a method defined in terms of weighted averages of nonexpansive retractions [13]. The paper **P3** studies the convergence, in spaces of constant curvature, of the algorithm AAR (*Averaged Alternating Reflection*), introduced by Bauschke, Combettes and Luke [2].

In **P9** there are obtained existence results of best proximity points for cyclic and noncyclic relatively nonexpansive mappings in the context of Busemann convex reflexive metric spaces. Moreover, polynomial bounds on the existence of approximate fixed points for such mappings in UCW-hyperbolic spaces are computed.

2.3 Dilation structures

2.3.1 Dilation structures in sub-riemannian geometry

The paper **P10** presents a description of sub-riemannian geometry with the help of dilation structures and it is based mainly on [8], with numerous improvements, as Theorem 8.10 giving an intrinsic characterization of riemannian metric spaces, Section 2.5 Curvdimension and curvature or the extended proof of Theorem 8.8 on the Γ-convergence of the length functionals for tempered dilation structures.

2.3.2 Graphic lambda calculus: dilation structures and logic

The paper **P7** introduces and studies graphic lambda calculus, which consists in a class of graphs endowed with moves between them. Graphic lambda calculus can be used for representing terms and reductions from untyped lambda calculus, its main move being called graphic beta move for its relation to the beta reduction in lambda calculus. This formalism can also be used for computations in emergent algebras [9] or for tangle diagrams. The paper **P7** is a massive revision of the descriptions from [10, 11, 12] and Section 5 of the paper is based on [9].

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