Scientific report for the period December 2013 - November 2014

1 Publications

1.1 Published papers

- D. Ariza-Ruiz, L. Leuştean, G. Lopez-Acedo, Firmly nonexpansive mappings in classes of geodesic spaces, Transactions of the American Mathematical Society 366 (2014), 4299-4322.

- A. Fernandez Leon, A. Nicolae, Best proximity pair results for relatively nonexpansive mappings in geodesic spaces, Numerical Functional Analysis and Optimization 35 (2014), 1399-1418.

- U. Kohlenbach, L. Leuştean, Addendum to "Effective metastability of Halpern iterates in CAT(0) spaces" [Adv.Math. 231 (5) (2012) 2526-2556], Advances in Mathematics 250 (2014) 650-651.

- L. Leuştean, A. Nicolae, Effective results on compositions of nonexpansive mappings, Journal of Mathematical Analysis and Applications 410 (2014), 902-907.

1.2 Submitted papers

- D. Ivan, L. Leuştean, A rate of asymptotic regularity for the Mann iteration of κ-strict pseudo-contractions, arXiv:1411.3389 [math.FA], 2014.

- L. Leuştean, A. Nicolae, A note on an ergodic theorem in weakly uniformly convex geodesic spaces, arXiv:1411.1095 [math.DS], 2014.

- L. Leuştean, A. Nicolae, Effective results on nonlinear ergodic averages in CAT(k) spaces, arXiv:1305.5916 [math.FA], 2014.

1.3 Work in progress

- U. Kohlenbach, L. Leuştean, A. Nicolae, Rates of metastability for generalized Féjer monotone sequences.

2 Conference and Seminar Talks

- L. Leuştean, Proof theoretic methods in nonlinear analysis III. Quantitative results on Fejer monotone sequences, Oberwolfach Workshop 1147: Mathematical Logic: Proof Theory, Type Theory and Constructive Mathematics, Mathematisches Forschungsinstitut Oberwolfach, Germany, 16 - 22.11.2014.

- L. Leuştean, An invitation to proof mining, Topics in Geometric Group Theory, IMAR, Bucharest, 29.09 - 05.10 2014.

- L. Leuştean, Effective results on the asymptotic behavior of nonexpansive iterations, Logic Colloquium 2014, Vienna, 14.07 - 19.07.2014.

- L. Leuştean, An invitation to proof mining II (Effective results on the mean ergodic theorem), Logic in Computer Science Seminar, University of Bucharest, 22.05.2014.

- L. Leuştean, An invitation to proof mining I, Logic in Computer Science Seminar, University of Bucharest, 16.05.2014.

- L. Leuştean, Effective methods in geodesic spaces, IRTG 1529 Research Seminar "Proof Mining and Nonlinear Analysis",Technische Universität Darmstadt, 26.02 - 28.02.2014.

3 Scientific seminars organized

- Logic Seminar (2012 - present), http://imar.ro/~leustean/seminar-logic.html.

4 Short description of the results

In the following we give a brief presentations of the results obtained this year.

- D. Ivan, L. Leuştean, A rate of asymptotic regularity for the Mann iteration of κ-strict pseudo-contractions, arXiv:1411.3389 [math.FA], 2014.

In this paper we apply methods of proof mining to obtain a uniform effective rate of asymptotic regularity for the Mann iteration associated to κ-strict pseudo-contractions on convex subsets of Hilbert spaces. This class of nonlinear mappings was introduced by Browder and Petryshyn [2].
The main result of the paper is the following:

Teorema 1. *Let H be a Hilbert space, $C \subseteq H$ a nonempty convex subset and $T : C \to C$ be a κ-strict pseudo-contraction, where $0 \leq \kappa < 1$. Assume that*

- *(λ_n) is a sequence in $(\kappa, 1)$ satisfying $\sum_{n=0}^{\infty}(\lambda_n - \kappa)(1 - \lambda_n) = \infty$ with rate of divergence $\theta : \mathbb{N} \to \mathbb{N}$.*
- *$x \in C, b > 0$ are such that $\|x - Tx\| \leq b$ and T has approximate fixed points in a b-neighborhood of x.*

Let (x_n) be the Mann iteration [7, 8, 4] starting with $x \in C$, defined by

$$x_0 := x, \quad x_{n+1} := (1 - \lambda_n)x_n + \lambda_n T x_n.$$

Then $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and

$$\forall \varepsilon > 0 \forall n \geq \Phi(\varepsilon, b, \theta)\big(\|x_n - Tx_n\| < \varepsilon\big), \quad \textit{where} \quad \Phi(\varepsilon, b, \theta) = \theta\left(\left\lceil \frac{b^2}{\varepsilon^2} \right\rceil\right).$$

As a consequence of our main result, we obtain for bounded C and $\lambda_n = \lambda \in (\kappa, 1)$ a quadratic in $1/\varepsilon$ rate of asymptotic regularity for the Krasnoselskii iteration (x_n):

$$\Phi(\varepsilon, b, \lambda, \kappa) = \left\lceil \frac{1}{(\lambda - \kappa)(1 - \lambda)} \right\rceil \left\lceil \frac{b^2}{\varepsilon^2} \right\rceil,$$

where b is an upper bound for the diameter d_C of C.

- L. Leuştean, A. Nicolae, A note on an ergodic theorem in weakly uniformly convex geodesic spaces, arXiv:1411.1095 [math.DS], 2014.

Karlsson and Margulis [5] proved in the setting of uniformly convex geodesic spaces, which additionally satisfy a nonpositive curvature condition, an ergodic theorem that focuses on the asymptotic behavior of integrable cocycles of nonexpansive mappings over an ergodic measure-preserving transformation.
In this paper we show that this result holds true when assuming a weaker notion of uniform convexity on the space. More precisely, we prove that we can relax the uniform convexity assumption on Y used in [5] as follows: a geodesic space Y is said to be *weakly uniformly convex* if for any $a \in Y$, $r > 0$ and $\varepsilon \in (0, 2]$,

$$\delta(a, r, \varepsilon) = \inf \left\{ 1 - \frac{1}{r} d\left(a, \frac{1}{2}x + \frac{1}{2}y\right) : d(a, x), d(a, y) \leq r, d(x, y) \geq \varepsilon r \right\} > 0.$$

The mapping δ is called *the modulus of convexity* of Y. This notion was used by Reich and Shafrir [9] in the setting of hyperbolic spaces. In addition, we assume that for every $a \in Y$ and $\varepsilon > 0$ there exists $s > 0$ such that

$$\inf_{r \geq s} \delta(a, r, \varepsilon) > 0. \tag{1}$$

Assume that Y is a geodesic space, $D \subseteq Y$, S is a semigroup of nonexpansive self-mappings defined on D endowed with the Borel σ-algebra induced by the compact-open topology on S, (X, μ) is a probability measure space, $T : X \to X$ is an ergodic measure-preserving transformation and $w : X \to S$ is a measurable map. Define the cocycles

$$a_n(x) = w(x)w(Tx) \cdots w(T^{n-1}x).$$

Fix $y \in D$ and suppose that

$$\int_X d(y, a_1(x)y) d\mu(x) < \infty.$$

As an immediate application of Kingman's subadditive ergodic theorem [6] one gets that

$$\lim_{n \to \infty} \frac{1}{n} d(y, a_n(x)y) = A \quad \text{for almost every } x \in X.$$

The main result of the paper is the following:

Teorema 2. *Assume that Y is a complete Busemann convex geodesic space that is weakly uniformly convex and satisfies (1). If $A > 0$, then for almost every $x \in X$, there exists a unique geodesic ray γ in Y that issues at y and depends on x such that*

$$\lim_{n\to\infty} \frac{1}{n} d(\gamma(An), a_n(x)y) = 0.$$

- L. Leuştean, A. Nicolae, Effective results on nonlinear ergodic averages in CAT(k) spaces, arXiv:1305.5916 [math.FA], 2014.

In this paper we apply proof mining techniques to compute, in the setting of CAT(κ) spaces (with $\kappa > 0$), effective and highly uniform rates of asymptotic regularity and metastability for a nonlinear generalization of the ergodic averages, known as the Halpern iteration. In this way, we obtain a uniform quantitative version of a nonlinear extension of the classical von Neumann mean ergodic theorem.
The 2014 version of the paper is an enhanced version of the one from 2013.

- U. Kohlenbach, L. Leuştean, A. Nicolae, Rates of metastability for generalized Féjer monotone sequences.

In this work in progress, we provide in a unified way quantitative forms of strong convergence results for iterative procedures which satisfy a general type of Fejér monotonicity where the convergence uses the compactness of the underlying set. Fejér monotonicity is a key notion in convex optimization and fixed point theory (see, e.g., [1, 3]).

References

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