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Energy Minimizing Brittle Crack Propagation

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Abstract. We propose a minimizing movement model for quasi-static brittle crack evolution. Cracks (fissures) appear and/or grow without any prescription of their shape or location when time-dependent displacements are imposed on the exterior boundary of the body. We use an energetic approach based on Mumford–Shah type functionals. By the discretization of the time variable we obtain a sequence of free discontinuity problems.

We find exact solutions and estimations which lead us to the conclusion that in this model crack appearance is allowed but the constant of Griffith G and the critical stress which causes the fracture in an uni-dimensional traction experiment cannot be both constants of material.

A weak formulation of the model is given in the frame of special functions with bounded deformation. We prove the existence of weak constrained incremental solutions of the model. A partial existence result for the minimizing movement model is obtained under the assumption of uniformly bounded (in time) power communicated to the body by the rest of the universe.

The model is of applicative interest. A numerical approach and examples, using an Ambrosio–Tortorelli variational approximation of the energy functional, are given in the last section.

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1. Introduction

This paper concerns the study of quasi-static brittle crack evolution. We work under the following assumptions: a linear elastic body, with or without initial cracks inside, evolves in a quasi-static manner under an imposed path of boundary displacements. During its evolution cracks with unprescribed geometry may appear and/or grow.

The difficulty of brittle crack propagation problems consists in the nature of the main unknown: the crack itself, at various moments in time. The research in this field concerns mainly the constitutive behavior of a brittle material, like the basic paper of Griffith [27]. Amongst the basic references we can quote: Eshelby [24], Irwin [30], Gurtin [28], [29], Rice [38].

In almost all the studies the geometry of the crack is prescribed. There are few exceptions, as the papers of Ohtsuka [34–37] or Stumpf and Le [39]. The geometry of the crack can be prescribed in a strong form, like in the case of a plane rectangu-

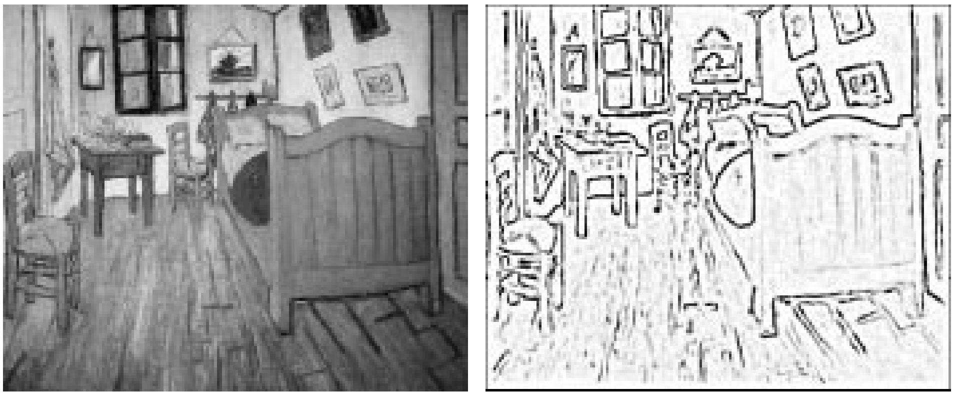


Figure 1. Example of image segmentation with the Mumford–Shah functional. The left figure is a black-and-white copy of a Van Gogh's painting; in the right figure we see the set of edges.

lar or elliptic crack which is supposed to remain plane rectangular or elliptic during its growth. We find a weak prescription upon the evolution of the crack in the case of a body with two-dimensional configuration, when the crack is supposed to have only an edge, which is a point. Therefore, in this case, the evolution of the crack is conveniently reduced to the movement of a point. Under these assumptions the geometrical nature of the main unknown is obscured.

A new direction of research in brittle fracture mechanics begins with the article of Mumford and Shah [33] regarding the problem of image segmentation. This problem, which consists in finding the set of edges of an image and constructing a smoothed (away from the edges) version of that image, turns out to be intimately related to the problem of brittle crack evolution.

In the article mentioned above Mumford and Shah propose the following variational approach to the problem of image segmentation: let $g: \Omega \subset R^2 \to [0, 1]$ be the original picture, given as a distribution of grey levels (1 is white and 0 is black). Let $u: \Omega \to R$ be the output picture and let K be the set of edges of the objects in the picture. K is (contained in) the set where u has jumps, i.e. $u \in C^1(\Omega \setminus K, R)$. The pair formed by the smoothed picture u and the set of edges K minimizes then the functional

$$I(u, K) = \int_\Omega \alpha |\nabla u|^2 \, \mathrm{d}x + \int_\Omega \beta |u - g|^2 \, \mathrm{d}x + \gamma \mathcal{H}^1(K). \tag{1}$$

The parameter α controls the smoothness away from the edges of the new picture u, β controls the L^2 distance between the smoothed picture and the original one and γ controls the total length of the edges given by this variational method. The authors remark that for $\beta = 0$ the functional I might be useful for an energetic treatment of fracture mechanics. In the followings is presented a model of brittle crack appearance in the case of imposed boundary displacements.

The state of a brittle body with reference configuration Ω is described by a pair displacement-crack. $(\mathbf{u}, K)$ is such a pair if:

(1) K is a crack in the body, seen as a surface,
(2) $\mathbf{u}$ is a displacement of the body with the crack $K \subset \overline{\Omega}$, compatible with the imposed boundary displacement $\mathbf{u}_0$, i.e. $\mathbf{u} \in C^1(\Omega \setminus K)$ and $\mathbf{u} = \mathbf{u}_0$ on $\partial\Omega$.

The total energy of the body in the state $(\mathbf{u}, K)$ is a Mumford–Shah functional of the form

$$E(\mathbf{u}, K; \mathbf{u}_0) = \int_\Omega w(\nabla \mathbf{u})\,\mathrm{d}x + F(\mathbf{u}_0, K).$$

The first term of the functional E represents the elastic energy of the body with the displacement $\mathbf{u}$. The second term represents the energy consumed to produce the crack K in the body, with the boundary displacement $\mathbf{u}_0$ as parameter.

In this model the brittle crack appearance is seen as an equilibrium problem. When the displacement $\mathbf{u}_0$ is imposed on the (exterior) boundary $\partial\Omega$ the state $(\mathbf{v}, S)$ of the brittle body is a minimizer of the total energy $E(\cdot, \cdot, ; \mathbf{u}_0)$. The crack predicted by the model is S. Notice that S may be the empty-set; in this case the model predicts that no crack appears when $\mathbf{u}_0$ is imposed.

Brittle crack appearance and image segmentation are free discontinuity problems. The unknowns, the crack or the collection of edges, are discontinuity surfaces for the displacement field or for the smoothed image; their location is entirely unprescribed.

We shall use an energetic approach to quasi-static brittle crack evolution. Therefore, we proceed to a time discretization which transforms the problem of crack evolution into a sequence of energy minimization problems. Francfort and Marigo [26] proceed in the same way in the case of brittle brutal damage evolution. However, it is only a belief that when the time step goes to zero, the discretized evolution converges to an almost continuous (in time) evolution. We have found in the frame of generalized minimizing movements, introduced by De Giorgi [20], stronger mathematical reasons to support this belief. That is why we introduce in Section 2 the notion of energy minimizing movement as a particular case of a generalized minimizing movement.

In Section 3, after the preliminaries concerning the statics of a brittle body, the Griffith criterion of brittle crack propagation is presented in Subsection 3.3, as a selection criterion amongst all possible crack evolutions. At the end of this section we formulate the problem of quasi-static brittle crack evolution in the form (14).

In Subsection 4.1 we give an energy minimizing movement formulation to this problem using a Mumford–Shah energy functional (Definition 4.1). In this model we have only one material constant connected to fracture, namely the constant of Griffith G. Some features of the model are explored in Subsection 4.2 in the anti-plane and uni-dimensional cases. We prove that crack appearance is allowed (we refer to [18] for more information, especially concerning fiber-matrix debonding in composites). The relation (23) contains the expression of σ_c, the critical stress which lead to fracture in an uni-dimensional traction experiment. We infer from this relation that σ_c and G cannot be both constants of material in this model.

Section 5 concerns the weak formulation of the incremental (that is discretized in time) model of crack evolution introduced in Definition 4.1. Subsection 5.1 deals with special functions with bounded variation or deformation. The existence of weak constrained incremental solutions of the model (Definition 5.1, Theorem 5.3) is a consequence of more general results due to De Giorgi and Ambrosio [21], Ambrosio [1–2], Bellettini, Coscia and Dal Maso, [15], Ambrosio, Coscia and Dal Maso [6]. The anti-plane case is discussed in Subsection 5.3. We compare the notions of weak (according to Definition 5.1) and strong (Definition 4.1) solution in Subsection 5.4.

In Section 6 a comparison is made with the model of Ambrosio and Braides [4], also based on generalized minimizing movements. In this model viscosity forces are introduced and crack propagation under imposed constant boundary displacement is allowed; on the contrary, crack appearance can not occur in a physically acceptable way.

In Section 7 we prove a partial existence result of the energy minimizing movement described in the model, under the assumption of uniformly bounded power communicated by the rest of the universe to the body during its evolution.

Section 8 is devoted to the numerical approach to the model. We use here functional convergence results of Ambrosio and Tortorelli [11–12] and the numerical method of Richardson and Mitter [32].

This paper continues a part of the work [17].

2. General Energy Minimizing Movements

An energy minimizing movement is a particular case of a generalized minimizing movement. The latter notion has been introduced by De Giorgi in [20], inspired by the paper [13] of Almgren, Taylor and Wang. The definition of a generalized minimizing movement (according to Ambrosio [3]) is presented below

DEFINITION 2.1. Let S be a topological space and

$$F\colon (1, +\infty) \times N \times S \times S \to R \cup \{+\infty\}$$

be a function. For any $u_0 \in S$, a function $u\colon [0, +\infty) \to S$ is a generalized minimizing movement associated to F with initial datum u_0, and we write $u \in GMM(F, u_0)$, if there exists a diverging sequence $(s_i)_{i\in N}$, $s_i > 1$, and there are functions $u_i\colon N \to S$ such that:

(i) $u_i(0) = u_0$;
(ii) for any $k \in N$ and any i, $u_i(k+1)$ minimizes the functional

$$v \mapsto F(s_i, k, v, u_i(k))$$

over S;

(iii) for any $t \geqslant 0$, $u_i([s_i t]) \to u(t)$ in S as $i \to +\infty$.

As the name tells, the notion of a generalized minimizing movement extends the notion of minimizing movement. With S, F and $u_0 \in S$ as in Definition 2.1, $u\colon [0,+\infty) \to S$ is a minimizing movement associated to F with initial datum u_0, and we write $u \in MM(F, u_0)$, if there are functions $u_s(k)$, for any $s > 1$ and $k \in N$, such that:

(i) $u_s(0) = u_0$;
(ii) for any $k \in N$ and any $s \in (0, +\infty)$, $u_s(k+1)$ minimizes the functional

$$v \mapsto F(s, k, v, u_s(k))$$

over S;
(iii) for any $t \geqslant 0$, $u_s([st]) \to u(t)$ in S as $s \to +\infty$.

The canonical example of (generalized) minimizing movement is given by the choice: $S = R^n$, $f\colon R^n \to R$ Lipschitz continuous and C^2 and

$$F(s, k, u, v) = f(u) + \frac{s}{2}|u - v|^2.$$

In this case, for any $u_0 \in R^n$ there is only one minimizing movement, namely the unique solution of the Cauchy problem

$$u'(t) = -\nabla f(u(t)), \qquad u(0) = u_0.$$

Notice that the minimizing movement associated to F and u_0 might not be unique, mainly because the functional $v \mapsto F(s, k, u_s(k), v)$ may have more than one minimizer. The nonuniqueness of a generalized minimizing movement is of higher order, because there might be different generalized minimizing movements depending on the choice of the diverging sequence s_i. For examples and techniques of investigation of the sets $MM(F, u_0)$ and $GMM(F, u_0)$ we refer to Ambrosio [3].

An energy minimizing movement is a generalized minimizing movement associated to a particular function F. It is designed to be a 'weak stable' solution of an evolution problem of the following type

$$\begin{cases} \mathbf{A}(u(t), \alpha(t), t) = 0, & \forall t \geqslant 0 \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\alpha(t) \leqslant \mathbf{L}(\alpha(t), u(t)), & \forall t \geqslant 0 \\ u(0) = u_0, \quad \alpha(0) = \alpha_0. \end{cases} \tag{2}$$

There are two unknowns in this problem: u and α. The evolution of the unknown u is quasi-static. Suppose that we don't have a proper law of evolution of α, or that the law of evolution that we have gives too many solutions. We may assume that we have the expression of the total energy $f(u, \alpha)$ of the system in the state (u, α) and a set of constraints, not in a differential form, upon the evolution of α. We make then a time discretization with time step δ and recursively find $(u^\delta_{k+1}, \alpha^\delta_{k+1})$ from $(u^\delta_k, \alpha^\delta_k)$, by a minimization process of the total energy f under some constraints. A weak stable solution of the previous problem is a limit of sequences $(u^\delta_k, \alpha^\delta_k)_k$ when the time step δ converges to 0.

In the next definition S may be seen as the space of all pairs $x = (u, \alpha)$, endowed with a topology.

DEFINITION 2.2. Let S be a topological space and

$$F: (1, +\infty) \times N \times S \times S \to R \cup \{+\infty\},$$

$$F(s, k, x, y) = f(s, x, y) + \psi(k/s, y)$$

be a function, with $f: N \times S \times S \to R$ and $\psi: [0, \infty) \times S \to \{0, +\infty\}$. For any $x_0 \in S$, an energy minimizing movement associated to the energy f with the constraints ψ and initial datum x_0 is any generalized minimizing movement $x: [0, +\infty) \to S$, $x \in GMM(F, x_0)$.

Let us denote by $S(\lambda)$ the following set

$$S(\lambda) = \{y \in S: \psi(\lambda, y) = 0\}.$$

From Definition 2.2 we notice that $x: [0, +\infty) \to S$ is an energy minimizing evolution associated to f, with the constraints ψ and initial datum x_0 if there exists a diverging sequence $(s_i)_{i \in N}$, $s_i > 1$, and there are functions $x_i: N \to S$ such that:

(i) $x_i(0) = x_0$;
(ii) for any $k \in N$ and any $i \in N$, $x_i(k + 1)$ minimizes the functional f over the set $S(k/s_i)$ (in particular $x_i(k + 1)$ belongs to $S(k/s_i)$);
(iii) for any $t > 0$, $x_i([s_i t]) \to x(t)$ in S as $i \to +\infty$.

3. Notations and Preliminaries

3.1. Notations and constitutive assumptions

The open bounded set $\Omega \subset R^3$ represents the reference configuration of an elastic body and $\mathbf{u}: \Omega \to R^3$ is the displacement field of the body. We shall always suppose, without mentioning further, that the open set Ω and its closure have the same topological boundary.

The expression of the elastic (or free) energy of the body is

$$\int_\Omega w(\nabla \mathbf{u})\, \mathrm{d}x.$$

The first Piola-Kirchhoff stress tensor $\boldsymbol{S}$ is

$$\boldsymbol{S}(\mathbf{u}) = \frac{\mathrm{d}w}{\mathrm{d}\nabla}(\nabla \mathbf{u})$$

and the equilibrium equation of the body in the absence of volumic forces is

$$\operatorname{div} \boldsymbol{S}(\mathbf{u}) = 0 \quad \text{in } \Omega.$$

In this paper we suppose that the body is linear elastic and homogeneous, i.e. the function $w(\nabla\mathbf{u})$ has the form

$$w(\nabla\mathbf{u}) = \tfrac{1}{2}\boldsymbol{C}\nabla\mathbf{u} : \nabla\mathbf{u},$$

with the elasticity 4-tensor $\boldsymbol{C}$ having the symmetries

$$\boldsymbol{C}_{ijkl} = \boldsymbol{C}_{jikl} = \boldsymbol{C}_{klij}.$$

Under these assumptions the stress tensor $\boldsymbol{S}$ becomes the Cauchy stress tensor

$$\sigma = \sigma(\mathbf{u}) = \boldsymbol{C}\nabla\mathbf{u} = \boldsymbol{C}\varepsilon(\mathbf{u}),$$

where $\varepsilon(\mathbf{u})$ is the symmetric part of $\nabla\mathbf{u}$, i.e.

$$\varepsilon(\mathbf{u}) = \tfrac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T).$$

We suppose moreover that w satisfies the growth conditions

$$\forall\ \mathbf{F} \in R^9, \qquad \mathbf{F} = \mathbf{F}^T, \qquad c|\mathbf{F}|^2 \leqslant w(\mathbf{F}) \leqslant C|\mathbf{F}|^2,$$

where c and C belong to $(0, +\infty)$.

In the cases of plane or anti-plane displacements the domain $\Omega \subset R^2$ represents a section in the cylindrical reference configuration of the body $\Omega \times R$ and the body is supposed to be isotropic.

If $\mathbf{u}: \Omega \to R^2$ is a plane displacement then the displacement relative to the three-dimensional configuration of the body has the following expression

$$(x_1, x_2, x_3) \in \Omega \times R \mapsto (u_1(x_1, x_2), u_2(x_1, x_2), 0) \in R^3.$$

The anti-plane displacement is a function $u: \Omega \to R$. The three-dimensional displacement has the following form

$$(x_1, x_2, x_3) \in \Omega \times R \mapsto (0, 0, u(x_1, x_2)) \in R^3.$$

In this case the elastic energy takes the form

$$\int_\Omega \mu|\nabla u|^2\, \mathrm{d}x,$$

where μ is one of the two Lamé's constants.

3.2. STATICS OF A FRACTURED ELASTIC BODY

For any measurable set $B \subset R^n$, $|B| = \mathcal{L}^n(B)$ denotes the Lebesgue measure of B and $\mathcal{H}^k(B)$ the k-dimensional Hausdorff measure of B.

By a crack set in the body Ω we mean (according to Ball [14]) a topologically closed countably rectifiable set, generically denoted by K. We shall always suppose that K is a subset of $\overline{\Omega}$.

Given the function f, a point $x \in \Omega \subset R^n$ and an unit vector (or direction) $\mathbf{n} \in R^n$, the approximate limit of f in x associated to the direction $\mathbf{n}$ is denoted by $\tilde{f}(x, \mathbf{n})$ and it is defined by the following expression

$$\lim_{\rho \to 0_+} \frac{\int_{B_\rho(x) \cap \{y : (y-x)\cdot \mathbf{n} \geqslant 0\}} |f(y) - \tilde{f}(x, \mathbf{n})| \, \mathrm{d}y}{|B_\rho(x) \cap \{y : (y-x) \cdot \mathbf{n} \geqslant 0\}|} = 0. \tag{3}$$

Given a field of unit vectors $x \in K \mapsto \mathbf{n}(x)$ normal to K, the lateral limits f^+ and f^- of any function $f : \Omega \setminus K \to R^n$ are $f^+ : K \to R$ and $f^- : K \to R$, defined by

$$f^+(x) = \tilde{f}(x, \mathbf{n}(x)), \qquad f^-(x) = \tilde{f}(x, -\mathbf{n}(x)).$$

This means that f^+ and f^- satisfy the equalities

$$\forall x \in K, \quad \lim_{\rho \to 0_+} \frac{\int_{B_\rho(x) \cap \{y : (y-x)\cdot \mathbf{n} \geqslant 0\}} |f(y) - f^+(x)| \, \mathrm{d}y}{|B_\rho(x) \cap \{y : (y-x) \cdot \mathbf{n} \geqslant 0\}|} = 0,$$

$$\forall x \in K, \quad \lim_{\rho \to 0_+} \frac{\int_{B_\rho(x) \cap \{y : (y-x)\cdot \mathbf{n} \leqslant 0\}} |f(y) - f^-(x)| \, \mathrm{d}y}{|B_\rho(x) \cap \{y : (y-x) \cdot \mathbf{n} \leqslant 0\}|} = 0.$$

Note that for any $x \in K$ the triplet $(f^+(x), f^-(x), \mathbf{n}(x))$ is unique up to a change of sign of $\mathbf{n}$ and a permutation of f^+, f^-, i.e.

$$(f^+(x), f^-(x), \mathbf{n}(x)) \sim (f^-(x), f^+(x), -\mathbf{n}(x)).$$

We denote by $[f] = f^+ - f^-$ the jump of f over K. Notice that the tensor field over K defined by $[f] \otimes \mathbf{n}$ is uniquely determined by f and K. If f takes values in R^n then the same is true for the symmetric part of the tensor field defined above, namely

$$\{[f] \odot \mathbf{n}\}_{ij} = \tfrac{1}{2}([f]_i \mathbf{n}_j + [f]_j \mathbf{n}_i).$$

The jump of f over the crack set K can be described by the following measure

$$\mathrm{j}(f, K) = [f] \odot \mathbf{n} \, \mathrm{d}\mathcal{H}^{n-1}_K, \qquad \mathrm{j}(f, K)(B) = \int_{B \cap K} [f] \odot \mathbf{n} \, \mathrm{d}\mathcal{H}^{n-1}. \tag{4}$$

Consider a crack set $K \subset \Omega$ formed by a finite collection of smooth surfaces. By a displacement compatible with K we mean a function $\mathbf{u} : \overline{\Omega} \setminus K \to R^k$ (where

k might be 1, 2 or 3) which is C^1 and has continuous lateral limits on K. In this section we shall consider the space $W^{1,2}(\Omega \setminus K)$ as the set of weak displacements compatible with the crack set K.

Let n be the dimension of the reference configuration Ω. For any $u_0 \in H^{1/2}(\partial\Omega, R^n)$ and for any crack set K, such that $\mathcal{H}^{n-1}(\partial\Omega \setminus K) > 0$, a solution (if any) of the following problem

$$\begin{cases} \operatorname{div} \sigma(\mathbf{u}) = 0, & \text{in } \Omega \setminus K \\ \sigma^+(\mathbf{u})\mathbf{n} = \sigma^-(\mathbf{u})\mathbf{n} = 0, & \text{on } K \\ \mathbf{u} = \mathbf{u}_0, & \text{on } \partial\Omega \setminus K \end{cases} \tag{5}$$

will be denoted by $\mathbf{u} = \mathbf{u}(\mathbf{u}_0, K)$. The solution is unique up to rigid displacements of $\Omega \setminus K$ equal to 0 on $\partial\Omega$. If K and $\partial\Omega$ are such that a Korn inequality holds on the space $W^{1,2}(\Omega \setminus K)$, then the problem (5) has a solution. For this paper the fact that $\mathbf{u}(\mathbf{u}_0, K)$ is unique up to a class of rigid displacements is irrelevant, therefore $\mathbf{u}(\mathbf{u}_0, K)$ will be called 'the solution' of the problem (5).

We use the same notation – $u = u(u_0, K)$ – in the anti-plane case, when $n = 2$, $k = 1$ and the problem (5) becomes

$$\begin{cases} \mu \operatorname{div} \nabla u = 0, & \text{in } \Omega \setminus K, \\ (\nabla u)^+\mathbf{n} = (\nabla u)^-\mathbf{n} = 0, & \text{on } K, \\ u = u_0, & \text{on } \partial\Omega \setminus K. \end{cases} \tag{6}$$

The solution $\mathbf{u}(\mathbf{u}_0, K)$ of the problem (5) minimizes the functional

$$E(\mathbf{v}) = \int_\Omega w(\nabla \mathbf{v})\, \mathrm{d}x$$

over the following set of weak displacements compatible with the crack set K and the boundary displacement $\mathbf{u}_0$

$$\{\mathbf{v} \in W^{1,2}(\Omega \setminus K, R^n) : \mathbf{v} = \mathbf{u}_0 \quad \text{on } \partial\Omega \setminus K\}.$$

By standard arguments the functional

$$\mathbf{v} \in W^{1,2}(\Omega, R^n) \mapsto \int_\Omega \sigma(\mathbf{u}(\mathbf{u}_0, K)) : \nabla \mathbf{v}\, \mathrm{d}x$$

depends only on the trace of $\mathbf{v}$ on $\partial\Omega$, hence it gives raise to the linear continuous function

$$\mathbf{T}(K) \colon H^{1/2}(\partial\Omega, R^n) \to H^{-(1/2)}(\partial\Omega, R^n),$$

$$\langle \mathbf{T}(K)\mathbf{u}_0, \mathbf{v}\rangle = \int_\Omega \sigma(\mathbf{u}(\mathbf{u}_0, K)) : \nabla \mathbf{v}' \, \mathrm{d}x \quad \text{for any } \mathbf{v}' = \mathbf{v} \text{ on } \partial\Omega. \tag{7}$$

In the latter definition $\langle\cdot, \cdot\rangle$ is the duality product of the pair of spaces $H^{1/2}(\partial\Omega, R^n)$ and $H^{-(1/2)}(\partial\Omega, R^n)$.

The function $\mathbf{T}(K)$ is called the Dirichlet-to-Neumann map of the elastic body Ω with the crack set K. Under the assumptions concerning the symmetries of the elasticity tensor $\boldsymbol{C}$, the function $\mathbf{T}(K)$ is self-adjoint, that is for any $\mathbf{u}$, $\mathbf{v}$ we have

$$\langle \mathbf{T}(K)\mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{T}(K)\mathbf{v}, \mathbf{u}\rangle. \tag{8}$$

In the same way the Dirichlet-to Neumann map associated to the problem (6) is defined.

Finally, we remark that the elastic energy of the body can be expressed using the Dirichlet-to-Neumann map. Indeed, we have

$$\int_\Omega w(\nabla \mathbf{u}(\mathbf{u}_0, K)) \, \mathrm{d}x = \tfrac{1}{2}\langle \mathbf{T}(K)\mathbf{u}_0, \mathbf{u}_0\rangle. \tag{9}$$

3.3. The Griffith Criterion of Brittle Crack Propagation

Let us consider in the elastic body Ω an initial crack set K_0 which evolves and becomes at the moment t the crack set K_t. We assume that the crack set always increases in time, i.e.,

$$\forall 0 < t < t', \qquad K_t \subset K_{t'}. \tag{10}$$

We suppose that the evolution of the body is quasi-static. At the moment t the state of the body is characterized by the displacement-crack pair $(\mathbf{u}(t), K_t)$, where $\mathbf{u}(t)$ is the displacement of the body, compatible with the crack set K_t. Let us denote by $\mathbf{u}_0(t)$ the trace of $\mathbf{u}(t)$ on $\partial\Omega$. We have then the equality $\mathbf{u}(t) = \mathbf{u}(\mathbf{u}_0(t), K_t)$. We make the assumption that the function $t \mapsto \mathbf{u}_0(t)$ is sufficiently regular in time.

The power given to the body by the rest of the universe at the moment t has the following expression

$$P(t) = \int_{\partial\Omega} \boldsymbol{S}(\mathbf{u}(t))\mathbf{n} \cdot \dot{\mathbf{u}}_0(t) \, \mathrm{d}x = \langle \mathbf{T}(K_t)\mathbf{u}_0(t), \dot{\mathbf{u}}_0(t)\rangle.$$

Let us consider a given curve $t \mapsto (\mathbf{u}(t), K_t)$, such that for any t we have $\mathbf{u}(t) = \mathbf{u}(\mathbf{u}_0(t), K_t)$. For a given t we introduce the following curve of displacements

$$\forall \tau \geqslant 0, \qquad \tilde{\mathbf{u}}(\tau) = \mathbf{u}(\mathbf{u}_0(t+\tau), K_t).$$

$\tilde{\mathbf{u}}(\tau)$ represents the displacement of the body at the moment $t+\tau$ in the presence of the crack K_t. An easy calculation leads us to the equality

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_\Omega w(\nabla\tilde{\mathbf{u}}(\tau))\,\mathrm{d}x_{|_{\tau=0}} = P(t). \tag{11}$$

Therefore $P(t)$ represents the power consumed at the moment t by the body in order to modify its displacement, constrained to follow the path of imposed boundary displacements $t \mapsto \mathbf{u}_0(t)$, without any modification of the actual crack set K_t.

The Griffith criterion of brittle crack propagation asserts that during the propagation of the crack K_t the following inequality is true at any moment t

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_\Omega w(\nabla\mathbf{u}(t))\,\mathrm{d}x + G\mathcal{H}^{n-1}(K_t)\right\} \leqslant P(t). \tag{12}$$

Here G is the constant of Griffith, supposed to be a material constant.

The relation (12) can be written in a different form using the map $\mathbf{T}(K_t)$. Let us assume that the crack evolution is smooth in the sense that the function $t \mapsto \mathbf{T}(K_t)$ is differentiable, i.e., the Dirichlet-to-Neumann map varies smoothly in time. The Griffith criterion takes the following form

$$\begin{aligned}
&\frac{1}{2}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{T}(K_t)]\mathbf{u}_0(t), \mathbf{u}_0(t)\right\rangle + \tfrac{1}{2}\langle \mathbf{T}(K_t)\dot{\mathbf{u}}_0(t), \mathbf{u}_0(t)\rangle \\
&\quad + \tfrac{1}{2}\langle \mathbf{T}(K_t)\mathbf{u}_0(t), \dot{\mathbf{u}}_0(t)\rangle + G\frac{\mathrm{d}}{\mathrm{d}t}\{\mathcal{H}^{n-1}(K_t)\} \\
&\qquad \leqslant \langle \mathbf{T}(K_t)\mathbf{u}_0(t), \dot{\mathbf{u}}_0(t)\rangle.
\end{aligned}$$

The function $\mathbf{T}(K_t)$ is self-adjoint, therefore we obtain the following expression of the Griffith criterion

$$\frac{1}{2}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{T}(K_t)]\mathbf{u}_0(t), \mathbf{u}_0(t)\right\rangle + G\frac{\mathrm{d}}{\mathrm{d}t}\{\mathcal{H}^{n-1}(K_t)\} \leqslant 0. \tag{13}$$

Notice that we have the following equality

$$P(t) - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w(\nabla\mathbf{u}(t))\,\mathrm{d}x = -\frac{1}{2}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{T}(K_t)]\mathbf{u}_0(t), \mathbf{u}_0(t)\right\rangle.$$

The left-hand member of the previous equality is usually called the energy release rate due only to the crack propagation.

$\mathbf{u}_0(t)$ plays the role of a time-dependent parameter, since in the last inequality $\dot{\mathbf{u}}_0(t)$ does not appear. As we have seen, this is a consequence of relations (8), (9) and (12).

The problem of quasi-static brittle propagation of an initial crack in an elastic body under a time-dependent imposed displacement $\mathbf{u}_0(t)$ is of the type (2). If we put apart the constraint (10), we have the following formulation

$$\begin{cases} \mathbf{u}(t) - \mathbf{u}(\mathbf{u}_0(t), K_t) = 0, & \forall t \geqslant 0, \\ \frac{1}{2}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{T}(K_t)]\mathbf{u}_0(t), \mathbf{u}_0(t)\right\rangle + G\frac{\mathrm{d}}{\mathrm{d}t}\{\mathcal{H}^{n-1}(K_t)\} \leqslant 0, & \forall t \geqslant 0, \\ \mathbf{u}(0) = \mathbf{u}_0, \quad K_0 = K. \end{cases} \tag{14}$$

4. The Model

In the left term of the Griffith criterion (12) there appears the time-derivative of an energetic functional. Let us consider the set M of all admissible displacement-crack pairs $(\mathbf{u}, K)$ with the following properties

(1) $K \subset \overline{\Omega}$ is a crack set;
(2) $\mathbf{u} \in C^1(\overline{\Omega} \setminus K, R^n)$;
(3) for $\mathcal{H}^{n-1}$-almost any $x \in K$ there exist the normal $\mathbf{n}(x)$ at K in x and $\mathbf{u}^+(x)$, $\mathbf{u}^-(x)$.

Notice that the field $\mathbf{n}$ of normals induces an orientation in the neighborhood of K. The item (3) in the definition of M can be replaced by imposing the existence of traces $\mathbf{u}^+$ and $\mathbf{u}^-$ of $\mathbf{u}$ on K with respect to this orientation.

The Mumford–Shah energy functional over M has the following expression

$$I: M \to R \cup \{+\infty\}, \qquad I(u, K) = \int_\Omega w(\nabla \mathbf{u})\,\mathrm{d}x + G\mathcal{H}^{n-1}(K). \tag{15}$$

4.1. Introduction of the Model

According to Definition 2.2 and the constraint (10) we give an energy minimizing movement formulation to the problem (14) using the functional defined in (15).

DEFINITION 4.1. Let us define the functions

$$J: M \times M \to R,$$

$$J((\mathbf{u}, K), (\mathbf{v}, L)) = \int_\Omega w(\nabla \mathbf{v})\,\mathrm{d}x + G\mathcal{H}^{n-1}(L \setminus K),$$

$$\Psi: [0, \infty) \times M \to \{0, +\infty\},$$

$$\Psi(\lambda, (\mathbf{v}, K)) = \begin{cases} 0, & \text{if } \mathbf{v} = \mathbf{u}_0(\lambda) \text{ on } \partial\Omega \setminus K \\ +\infty, & \text{otherwise.} \end{cases}$$

We consider the initial data $(\mathbf{u}_0, K) \in M$ such that $\mathbf{u}_0 = \mathbf{u}(\mathbf{u}_0(0), K)$. For any $s \geqslant 1$ we define the sequences

$$k \in N \mapsto \mathbf{u}^s(k), \qquad L^s(k), \qquad K^s(k),$$

$(\mathbf{u}^s(k), L^s(k)) \in M$ and $(\mathbf{u}^s(k), K^s(k)) \in M$, recursively:

(i) $(\mathbf{u}^s, K^s)(0) = (\mathbf{u}_0, K)$, $L^s(0) = K$,

(ii) for any $k \in N$ $(\mathbf{u}^s, L^s)(k+1) \in M$ minimizes the functional

$$(\mathbf{v}, L) \in M \mapsto J((\mathbf{u}^s, K^s)(k), (\mathbf{v}, L)) + \Psi((k+1)/s, (\mathbf{v}, L))$$

over M. In order to verify the constraint (10), $K^s(k+1)$ is defined by the formula:

$$K^s(k+1) = K^s(k) \cup L^s(k+1).$$

$(\mathbf{u}, L)\colon [0, +\infty) \to M$ is an energy minimizing movement associated to J with the constraints (10), Ψ and initial data $(\mathbf{u}_0, K)$, and we write $(\mathbf{u}, L) \in GMM(\mathbf{u}_0, K, \Psi)$, if there is a diverging sequence (s_i) such that for any $t > 0$ we have

$$\begin{cases} \mathbf{u}^{s_i}([s_i t]) \to \mathbf{u}(t) \quad \text{in } L^2(\Omega, R^n), \\ j(\mathbf{u}^{s_i}, L^{s_i})([s_i t]) \to j(\mathbf{u}, L)(t) \quad \text{weakly as Radon measures} \end{cases} \tag{16}$$

as $i \to \infty$ and

$$\mathcal{H}^{n-1}(L(t)) \leqslant \liminf_{i\to\infty} \mathcal{H}(L^{s_i}([s_i t])). \tag{17}$$

In the previous definition $1/s$ is the step of the discretization of the time variable. The approximate displacement of the body at the moment k/s is $\mathbf{u}^s(k)$. The *active crack* at the same moment is $L^s(k)$ and the *total crack* is $K^s(k)$. The state of the brittle body is $(\mathbf{u}^s(k), L^s(k))$ while $K^s(k)$ takes account of the history of fissuration. Any sequence $k \mapsto (\mathbf{u}^s, L^s, K^s)(k)$ constructed using the rules (i) and (ii) from the Definition 4.1 is called an incremental solution. We use the same name for a sequence of displacement-crack pairs $k \mapsto (\mathbf{u}^s, L^s)(k)$. Notice that in rule (ii) the triplet $(\mathbf{u}^s, L^s, K^s)(k)$ appears in the expression of the functional j only through $K^s(k)$.

The time step goes to 0 as i converges to ∞ and the incremental solution $(\mathbf{u}^{s_i}, L^{s_i})([s_i t])$ converges to $(\mathbf{u}, L)(t)$, for any $t > 0$. $L(t)$ is called the *active crack* at the moment t and

$$K(t) = \cup_{s\in[0,t]} L(s)$$

is called the *damaged region* of the body at the same moment. Notice that the damaged region $K(t)$ might not be a crack set, because it is *a priori* a noncountable union of surfaces.

The convergence of the incremental solution to the energy minimizing movement deserves a discussion. The measure $\mathrm{j}(\mathbf{u}, L)$ associated to a displacement-crack pair contains information about the placement and the opening of the crack L under the displacement $\mathbf{u}$. The weak convergence of $\mathrm{j}(\mathbf{u}^{s_i}, L^{s_i})([s_i t])$ to $\mathrm{j}(\mathbf{u}, L)(t)$ as Radon measures means that for any $\phi \in C_0(\Omega, M^{n\times n})$ we have

$$\lim_{i\to\infty} \int_{L^{s_i}([s_i t])} ([\mathbf{u}^{s_i}([s_i t])] \odot \mathbf{n}) : \phi \,\mathrm{d}\mathcal{H}^{n-1}$$
$$= \int_{L(t)} ([\mathbf{u}(t)] \odot \mathbf{n}) : \phi \,\mathrm{d}\mathcal{H}^{n-1}.$$

Therefore (16) asserts that the incremental displacement converges to the displacement at the moment t and (in a weak sense) the placement and the opening of the incremental crack set converges to the placement and the opening of the active crack set at the same moment. Generally $\mathrm{j}(\mathbf{u}, L)$ is not null on the part of L where the jump of $\mathbf{u}$ is not null, therefore this measure gives information only about the opened crack. The role of the condition (17) is to control the area of the crack $L(t)$, in order to eliminate the parts of the active crack which are not opened.

4.2. Features of the Model

We investigate further the behavior of the model proposed in Definition 4.1 in the particular case of anti-plane displacements. There are some obvious adjustments to be made. Ω is now a bounded domain in R^2 and the displacement is a scalar function u. The functional J will take the form

$$J((u, K), (v, L)) = \int_{\Omega} \mu |\nabla v|^2 \,\mathrm{d}x + G\mathcal{H}^1(L \setminus K). \tag{18}$$

For a displacement-crack pair (u, L) we introduce the notation

$$\mathrm{j}(u, L) = [u]\,\mathrm{d}\mathcal{H}^1_{|L}.$$

Let us consider a particular type of imposed displacement on $\partial\Omega$. We split the boundary of the body into three parts

$$\partial\Omega = \overline{\Gamma^1_u} \cup \overline{\Gamma^2_u} \cup \overline{\Gamma_f},$$

$$\Gamma^i_u \cap \Gamma_f = \emptyset, \qquad \overline{\Gamma^1_u} \cap \overline{\Gamma^2_u} = \emptyset, \qquad \mathcal{H}^1(\Gamma^1_u)\cdot\mathcal{H}^1(\Gamma^2_u)\cdot\mathcal{H}^1(\Gamma_f) > 0.$$

At any moment $t \geqslant 0$, Γ_f is force free, i.e. the displacement is not prescribed on this part of the boundary. On Γ^1_u and Γ^2_u the imposed displacement is defined by

$$u_0(t)(x) = \begin{cases} 0 & \text{on } \Gamma^1_u \\ t\delta & \text{on } \Gamma^2_u \end{cases},$$

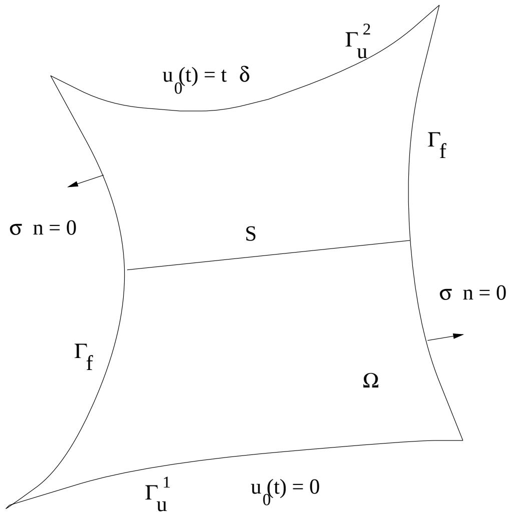


Figure 2. The geometry of the body and imposed displacement.

where δ is a positive constant with dimension of speed. This displacement is homogeneous in the time variable

$$\forall t > 0, \quad u_0(t) = tu_0(1).$$

We suppose that at the moment $t = 0$ there are no cracks in the body. This assumption takes the form $K = \emptyset$. At $t = 0$ we have $u_0(0) = 0$, hence the initial data are $(u_0 = 0, K = \emptyset)$.

Let us consider a time discretization given by the parameter $1/s$ and the incremental solution $k \in N \mapsto (u^s, L^s)(k)$ introduced in Definition 4.1 for the initial data and the imposed boundary conditions described above. In order to shorten the notations we shall omit for the moment the superscript s.

The incremental solution $(u, L): N \to M$ is recursively defined by the following two rules:

(i) $u(0) = 0$ and $K(0) = \emptyset$;

(ii) for any $k \in N$ we seek to determine the crack set $L(k+1)$ and the displacement $u(k+1)$ such that $(u(k+1), L(k+1)) \in M$, $u(k+1) = (k+1)/su_0(1)$ on $(\Gamma_u^1 \cup \Gamma_u^2) \setminus L(k+1)$ and $(u(k+1), L(k+1))$ is a minimizer of the functional

$$(v, L) \mapsto J((u(k), K(k)), (v, L)),$$

where $(v, L) \in M$, $v = (k+1)/su_0(1)$ on $(\Gamma_u^1 \cup \Gamma_u^2) \setminus L$. The set $K(k+1)$ is given by the formula

$$K(k+1) = K(k) \cup L(k+1).$$

Let $u_\emptyset$ denote the displacement of the body Ω, without cracks, under the prescribed displacement on the boundary $u_0(1)$. With the use of a notation made earlier, $u_\emptyset$ is defined by $u_\emptyset = u(u_0(1), \emptyset)$. For any $k \in N$ we have $(k/su_\emptyset, \emptyset) \in M$ and $k/su_\emptyset = k/su_0(1)$ on $\Gamma_u^1 \cup \Gamma_u^2$. Therefore, with the notation

$$J_k = J((u(k), K(k)), (u(k+1), L(k+1)))$$

for any $k \in N$ we have

$$J_k \leqslant J((u(k), K(k)), ((k+1)/su_\emptyset, \emptyset)).$$

The last inequality may be written as

$$J_k \leqslant \left(\frac{k}{s}\right)^2 \int_\Omega \mu |\nabla u_\emptyset|^2 \, \mathrm{d}x, \tag{19}$$

$$J_k = \int_\Omega \mu |\nabla u(k+1)|^2 \, \mathrm{d}x + G\mathcal{H}^1(L(k+1) \setminus K(k)). \tag{20}$$

We can always find a curve in $\overline{\Omega}$ which is a length minimizer in the family of all curves in $\overline{\Omega}$ separating Γ_u^1 from Γ_u^2. Let us denote such a curve by S (which exists but it might not be unique). The curve S splits the domain $\overline{\Omega}$

$$\overline{\Omega} = \Omega^1 \cup \Omega^2, \qquad \Gamma_u^1 \subset \Omega^1, \qquad \Gamma_u^2 \subset \Omega^2,$$

$$\Omega^1 \cap \Omega^2 = \emptyset, \qquad \overline{\Omega^1} \cap \overline{\Omega^2} = S.$$

We define the following displacement

$$u_S(x) = \begin{cases} 0 & x \in \Omega^1 \\ \delta & x \in \Omega^2. \end{cases}$$

It is easy to see that for any $k \in N$ the pair $(k/su_S, S)$ belongs to M and $k/su_S = k/su_0(1)$ on $(\Gamma_u^1 \cup \Gamma_u^2) \setminus S$. We have therefore the inequality

$$J_k \leqslant G\mathcal{H}^1(S \setminus K(k)), \tag{21}$$

with J_k given by (20). From (21) we derive the following conclusion: *for large time k/s the crack set $K(k)$ is not void.* Indeed, suppose that the function $k \in N \mapsto (k/su_\emptyset, \emptyset)$ is an incremental solution constructed by the rules (i) and (ii) above. Then for any $k \in N$ the inequality (21) becomes an equality and the inequality (21) takes the form

$$\left(\frac{k}{s}\right)^2 \int_\Omega \mu |\nabla u_\emptyset|^2 \, \mathrm{d}x \leqslant G\mathcal{H}^1(S), \tag{22}$$

which lead to a contradiction. Therefore this model can predict crack appearance.

We get more information about the behavior of the model if we use it in the case of an uni-axial traction experiment. The body with modulus of elasticity E has the configuration $\Omega = (0, L) \subset R$ and any crack set is a finite collection of points in the interval Ω, so the body is either undamaged or totally broken. The imposed displacement at the time t is

$$u_0(t) = tu_0(1),$$

where $u_0(1) = 0$ at $x = 0$ and $u_0(1) = D$ at $x = L$. The function $J((u, K), (v, L))$ takes the expression

$$J((u, K), (v, L)) = \int_0^L \tfrac{1}{2}E(v'(x))^2\,\mathrm{d}x + G\#(L \setminus K),$$

where $\#(M)$ is the number of elements of the set M.

At the time k/s we have only two kinds of displacement-crack pairs which compete. These are:

(1) $(k/su_{\emptyset}, \emptyset)$, where $u_{\emptyset}(x) = xD/L$;
(2) $(k/su_S, S)$, where $S = \{x_1, \ldots, x_N\}$ is a crack set and u_S is a piecewise constant function on $[0, L] \setminus S$ such that $u_S(0) = 0$ and $u_S(1) = D$.

For any displacement-crack pair (u, K) we have

$$J((u, K), (k/su_S, S)) = \begin{cases} (k/s)^2 \int_0^L \tfrac{1}{2}E(u_{\emptyset}')^2\,\mathrm{d}x & \text{if } S = \emptyset, \\ G\#(S \setminus K) & \text{if } S \neq \emptyset, \end{cases}$$

therefore among all pairs $(k/su_S, S)$ it is sufficient to consider only the pairs with $\#(S) = 1$ or $S = \emptyset$.

For small time k/s the body remains uncracked and for large time k/s a crack appears in the body. Precisely, for small k/s we have

$$(u(k), K(k)) = (k/su_{\emptyset}, \emptyset)$$

and for large k/s we have

$$(u(k), K(k)) = (k/su_S, S),$$

with $\#(S) = 1$. An inequality similar to (22) leads us to an equation for the critical time t_c when the crack appears

$$t_c^2 \int_0^1 \tfrac{1}{2}E(u_{\emptyset}')^2\,\mathrm{d}x = G.$$

We obtain the following expression of the uni-axial stress $\sigma_c = t_c E u'_{\emptyset}$, existing in the uncracked body when the model predicts its fracture:

$$\sigma_c = \left(\frac{2EG}{L}\right)^{1/2}. \tag{23}$$

We see that the stress σ_c and the quantity G cannot be both constants of material in this model.

5. Existence of weak incremental solutions

5.1. The spaces SBV and SBD

This section is dedicated to a brief voyage through the spaces **SBV** and **SBD**.

We use the notation $\mu \ll \lambda$ if the measure μ is absolutely continuous with respect to the measure λ. For any measure μ we denote by $|\mu|(B)$ the variation of μ over the Borel set $B \subset \Omega$, defined by the relation

$$|\mu|(B) = \sup\left\{\sum_{i=1}^{\infty} |\mu(A_i)| : \cup_{i=1}^{\infty} A_i \subset B, A_i \cap A_j = \emptyset \ \forall i \neq j\right\}.$$

The measure μ has finite total variation (over Ω) if $|\mu|(\Omega) < +\infty$.

$\mathbf{BV}(\Omega, R^n)$ is the space of functions $\mathbf{u} \in L^1(\Omega, R^n)$ with the distributional derivative $D\mathbf{u}$ representable as a vector measure with finite total variation. We refer to the book of Evans and Gariepy [25] for the main properties of such functions. The approximate limit of $\mathbf{u}$ at the point $x \in \Omega$ is that $\tilde{\mathbf{u}}(x)$ defined by the equality

$$\lim_{\rho \to 0_+} \frac{\int_{B_\rho(x)} |\mathbf{u}(y) - \tilde{\mathbf{u}}(x)| \,\mathrm{d}y}{|B_\rho(x)|} = 0.$$

The Lebesgue set of $\mathbf{u}$ is the set of points where $\mathbf{u}$ has an approximate limit. The complementary set is a $\mathcal{L}^n$ negligible set denoted by $\mathbf{S_u}$. De Giorgi proved in [23] that for any $\mathbf{u} \in \mathbf{BV}(\Omega, R^n)$ the set $\mathbf{S_u}$ is countably rectifiable. Moreover, for $\mathcal{H}^{n-1}$ almost every $x \in \mathbf{S_u}$ there is a triplet $(\mathbf{u}^+(x), \mathbf{u}^-(x), \mathbf{n}(x))$ such that

(1) $\mathbf{n}(x)$ is a unit vector normal to $\mathbf{S_u}$ at x;
(2) $(\mathbf{u}^+(x), \mathbf{u}^-(x))$ are the approximate limits of $\mathbf{u}$ in x associated with the direction $\mathbf{n}(x)$ (for the definition see (3)).

This triplet is uniquely determined up to a change of sign of $\mathbf{n}$ and an interchange of $\mathbf{u}^+$, $\mathbf{u}^-$. The jump of $\mathbf{u}$ across $\mathbf{S_u}$ is $[\mathbf{u}] = \mathbf{u}^+ - \mathbf{u}^-$; notice that the tensor field $[\mathbf{u}] \otimes \mathbf{n}$ over $\mathbf{S_u}$ is independent of the choice of the field of normals $\mathbf{n}$.

For any $\mathbf{u} \in \mathbf{BV}(\Omega, R^n)$ the measure $D\mathbf{u}$ admits the decomposition into absolute continuous and singular parts with respect to the Lebesgue measure dx: $D\mathbf{u} =$

$D^a\mathbf{u} + D^s\mathbf{u}$. Calderon and Zygmund [19] theorem gives the following decomposition of the measure $D\mathbf{u}$ into three mutually singular parts

$$D\mathbf{u} = \nabla\mathbf{u}(x)\,\mathrm{d}x + [\mathbf{u}] \otimes \mathbf{n}\,\mathrm{d}\mathcal{H}^{n-1}_{|\mathbf{S_u}} + C(\mathbf{u}).$$

$\nabla\mathbf{u}$ is the approximate gradient of $\mathbf{u}$ defined for almost every $x \in \Omega$ by the equality

$$\lim_{\rho\to 0_+} \frac{\int_{B_\rho(x)} |\mathbf{u}(y) - \mathbf{u}(x) - \nabla\mathbf{u}(x)\cdot(y-x)|\,\mathrm{d}y}{|B_\rho(x)||y-x|} = 0.$$

The jump part of $D\mathbf{u}$ is

$$D^j\mathbf{u} = [\mathbf{u}] \otimes \mathbf{n}\,\mathrm{d}\mathcal{H}^{n-1}_{|\mathbf{S_u}}.$$

$C(\mathbf{u})$ is called the Cantor part of $D\mathbf{u}$; for any Borel set $B \subset \Omega$ the quantity $C(\mathbf{u})(B)$ is defined by $C(\mathbf{u})(B) = D^s\mathbf{u}(B \setminus \mathbf{S_u})$. We have therefore

$$D^a\mathbf{u} = \nabla\mathbf{u}\,\mathrm{d}x, \qquad D^s\mathbf{u} = [\mathbf{u}] \otimes \mathbf{n}\,\mathrm{d}\mathcal{H}_{|K} + C(\mathbf{u}).$$

The space $\mathbf{SBV}(\Omega, R^n)$ of special functions with bounded variation was introduced by De Giorgi and Ambrosio in the study of a class of free discontinuity problems ([21], [1], [2]). A general reference to **SBV** and free-discontinuity problems is Ambrosio, Fusco and Pallara [10]. This space is defined as follows:

$$\mathbf{SBV}(\Omega, R^n) = \{\mathbf{u} \in \mathbf{BV}(\Omega, R^n) : |D^s\mathbf{u}|(\Omega \setminus \mathbf{S_u}) = 0\}.$$

For any $\mathbf{u} \in \mathbf{BV}(\Omega, R^n)$, $\mathbf{u}$ is a special function with bounded variation if and only if the Cantor part of $D\mathbf{u}$ is null.

For several versions of the compactness theorem in **SBV** we refer to the aforementioned papers of De Giorgi and Ambrosio. We shall use this theorem in the following form:

THEOREM 5.1. *Let* $(\mathbf{u}_h)_h$ *be a sequence in* $\mathbf{SBV}(\Omega, R^k)$ *and* C *be a constant such that for any* h

$$\int_\Omega |\nabla\mathbf{u}_h|^2\,\mathrm{d}x + \mathcal{H}(\mathbf{S}_{\mathbf{u}_h}) + \|\mathbf{u}_h\|_{L^\infty} \leqslant C.$$

Then there exist $\mathbf{u} \in \mathbf{SBV}(\Omega, R^k)$ *and a subsequence, still denoted by* $(\mathbf{u}_h)_h$*, such that*

$$\begin{cases} \mathbf{u}_h \to \mathbf{u} \quad \textit{in } L^2(\Omega, R^k), \\ \nabla\mathbf{u}_h \to \nabla\mathbf{u} \quad \textit{weakly in } L^2(\Omega, M^{n\times k}), \\ D^j\mathbf{u}_h \to D^j\mathbf{u} \quad \textit{weakly as Radon measures}, \end{cases}$$

and

$$\mathcal{H}^{n-1}(\mathbf{S_u}) \leqslant \liminf_{h\to\infty} \mathcal{H}^{n-1}(\mathbf{S}_{\mathbf{u}_h}).$$

A description of the space of special functions with bounded deformation $\mathbf{SBD}(\Omega)$, can be found in Ambrosio, Coscia and Dal Maso [6]. Any function $\mathbf{u} \in L^1(\Omega, R^n)$ belongs to $\mathbf{BD}(\Omega)$ if $E\mathbf{u}$, the symmetric part of the distributional derivative of $\mathbf{u}$, is representable as a vector measure with finite total variation.

For any $\mathbf{u} \in \mathbf{BD}(\Omega)$ the measure $E\mathbf{u}$ decomposes with respect to the Lebesgue measure into absolute continuous and singular parts

$$E\mathbf{u} = E^a\mathbf{u} + E^s\mathbf{u}.$$

We denote by $|E\mathbf{u}|$ the variation of the measure $E\mathbf{u}$. Kohn introduced in [31] the set $\Theta_\mathbf{u}$

$$\Theta_\mathbf{u} = \left\{ x \in \Omega\colon \limsup_{\rho\to 0^+} \frac{|E\mathbf{u}|(B_\rho(x))}{\rho^{n-1}} > 0 \right\}$$

and proved that it is countably rectifiable. Let $\mathbf{J_u}$ be the subset of Ω of all points $x \in \Omega$ such that there is a unit vector $\nu(x)$ with the property that $\mathbf{u}$ has different approximate limits $\mathbf{u}^+(x) = \tilde{\mathbf{u}}(x, \nu(x))$, $\mathbf{u}^-(x) = \tilde{\mathbf{u}}(x, -\nu(x))$ defined by the relation (3). It is straightforward that $\mathbf{J_u} \subset \mathbf{S_u}$. However, $\mathbf{S_u}$ may not be countably rectifiable. In [6] it is proved that $\Theta_\mathbf{u}$ coincides with $\mathbf{J_u}$ up to a $\mathcal{H}^{n-1}$ negligible set, therefore $\mathbf{J_u}$ is countably rectifiable. The triplet $(\mathbf{u}^+(x), \mathbf{u}^-(x), \mathbf{n}(x))$ exists for $\mathcal{H}^{n-1}$ almost every $x \in J_\mathbf{u}$, where $\mathbf{n}(x)$ is the normal unit vector to $\Theta_\mathbf{u}$ at x; as previously the tensor field over $\mathbf{J_u}$ defined by $[\mathbf{u}] \otimes \mathbf{n}$ is uniquely determined. We denote by $[\mathbf{u}] \odot \mathbf{n}$ its symmetric part.

The difference between $\mathbf{S_u}$ and $\mathbf{J_u}$ is subtle. Let us quote only the fact that for a function $\mathbf{u} \in \mathbf{SBV}(\Omega, R^n)$ these sets coincide up to a $\mathcal{H}$-negligible set.

The following decomposition theorem is due to Ambrosio, Coscia and Dal Maso [6] and asserts that

$$E\mathbf{u} = \varepsilon(\mathbf{u})(x)\,\mathrm{d}x + [\mathbf{u}] \odot \mathbf{n}\,\mathrm{d}\mathcal{H}^{n-1}_{|\mathbf{J_u}} + E^c(\mathbf{u}).$$

Here $\varepsilon(\mathbf{u})$ is the approximate symmetric gradient, defined for almost every $x \in \Omega$ by

$$\lim_{\rho\to 0^+} \frac{1}{\rho^n} \int_{B_\rho(x)} \frac{(\mathbf{u}(y) - \mathbf{u}(x) - \varepsilon(\mathbf{u})(x)(y-x)) \cdot (y-x)}{|y-x|^2}\,\mathrm{d}y = 0.$$

The jump part of $E\mathbf{u}$ is

$$E^j\mathbf{u} = [\mathbf{u}] \odot \mathbf{n}\,\mathrm{d}\mathcal{H}^{n-1}_{|\mathbf{J_u}}.$$

$E^c\mathbf{u}$ is the Cantor part of $E\mathbf{u}$, that is the part of $E^s\mathbf{u}$ not concentrated on $\mathbf{J_u}$. Therefore we have

$$E^a\mathbf{u} = \varepsilon(\mathbf{u})\,\mathrm{d}x, \qquad E^s\mathbf{u} = E^j\mathbf{u} + E^c\mathbf{u}.$$

The definition of $\mathbf{SBD}(\Omega)$ is the following:

$$\mathbf{SBD}(\Omega, R^n) = \{\mathbf{u} \in \mathbf{BD}(\Omega) : |E^s\mathbf{u}|(\Omega \setminus \mathbf{J_u}) = 0\}.$$

We have the inclusion

$$\mathbf{SBV}(\Omega, R^n) \subset \mathbf{SBD}(\Omega).$$

For the compactness theorem in **SBD** we refer to Bellettini, Coscia and Dal Maso [15]. We shall use this theorem in the following form:

THEOREM 5.2. *Let us consider the function*

$$\mathbf{F} \in M^{n\times n}_{\mathrm{sym}} \mapsto w(\mathbf{F}) = (1/2)\mathbf{CF} : \mathbf{F},$$

with $\boldsymbol{C}$ *a positive definite symmetric 4-order tensor. Let* $(\mathbf{u}_h)_h$ *be a sequence in* $\mathbf{SBD}(\Omega)$ *and* C *a constant such that for any* h

$$\int_\Omega w(\varepsilon(\mathbf{u}_h))\,\mathrm{d}x + \mathcal{H}^{n-1}(\mathbf{J}_{\mathbf{u}_h}) + \|\mathbf{u}_h\|_{L^\infty} \leqslant C.$$

Then there exist $\mathbf{u} \in \mathbf{SBV}(\Omega, R^k)$ *and a subsequence, still denoted by* $(\mathbf{u}_h)_h$*, such that*

$$\begin{cases} \mathbf{u}_h \to \mathbf{u} \quad \textit{in} \;\; L^2(\Omega, R^k), \\ \varepsilon(\mathbf{u}_h) \to \varepsilon(\mathbf{u}) \quad \textit{weakly in} \;\; L^2(\Omega, M^{n\times n}_{\mathrm{sym}}), \\ E^j\mathbf{u}_h \to E^j\mathbf{u} \quad \textit{weakly as Radon measures}, \end{cases}$$

and

$$\mathcal{H}^{n-1}(\mathbf{J_u}) \leqslant \liminf_{h\to\infty} \mathcal{H}^{n-1}(\mathbf{J}_{\mathbf{u}_h}).$$

5.2. Existence of weak constrained incremental solutions

In order to give a weak formulation of the model described in Definition 4.1 let us weaken first the space M of displacement-crack pairs. We introduce the new set of weak displacement-crack pairs $\mathcal{M}$

$$\begin{aligned} \mathcal{M} \;=\; &\{(\mathbf{u}, K) : K \text{ is } \sigma\text{-rectifiable}, \mathbf{u} \in \mathbf{SBD}(\Omega) \text{ and} \\ &\quad |E^s\mathbf{u}|(\Omega \setminus K) = 0\}. \end{aligned} \qquad (24)$$

Given $(\mathbf{u}, K) \in \mathcal{M}$, the set K is countably rectifiable but it is not necessarily closed; we have also weaker conditions on the regularity of the displacement $\mathbf{u}$. A direct consequence of (29) is that any (strong) displacement-crack pair $(\mathbf{u}, K)$ such that $\mathbf{u} \in L^{\infty}(\Omega, R^n)$ belongs to the set $\mathcal{M}$.

Let us define the functional $\mathcal{J}$, the weak version of the functional J introduced at Definition 4.1: $\mathcal{J}: \mathcal{M} \times \mathcal{M} \to R$,

$$\mathcal{J}((\mathbf{u}, K), (\mathbf{v}, L)) = \int_{\Omega} w(\varepsilon(\mathbf{v}))\, \mathrm{d}x + G\mathcal{H}^{n-1}(L \setminus K). \tag{25}$$

Before the introduction of the weak form of the function Ψ from the same definition, let us explain what we mean by $\mathbf{u} = \mathbf{u}_0$ on the boundary of Ω. We consider, for technical reasons, that there is an open bounded set Λ with piecewise Lipschitz boundary such that $\overline{\Omega} \subset \Lambda$. The imposed boundary displacement is $\mathbf{u}_0 \in \mathbf{SBD}(\Lambda)$ such that $\mathbf{J}_{\mathbf{u}_0} \cap \overline{\Omega} = \emptyset$. Then, for any $\mathbf{u} \in \mathbf{SBD}(\lambda)$, $\mathbf{u} = \mathbf{u}_0$ on $\partial\Omega$ means that $\mathbf{u} = \mathbf{u}_0$ in $\Lambda \setminus \Omega$. We denote the set of all such functions $\mathbf{u}$ by $\mathbf{SBD}(\Omega, \mathbf{u}_0)$. The reason for this choice of defining boundary conditions is that the space $\mathbf{SBD}(\Omega, \mathbf{u}_0)$ is closed in $\mathbf{SBD}(\Lambda)$ in the L^2 convergence. Note that $\mathbf{SBD}(\Omega, \mathbf{u}_0)$ can be identified with a subspace of $\mathbf{SBD}(\Omega)$ by the inclusion map $\mathbf{u} \mapsto \mathbf{u}_{|\Omega}$.

Let us consider a curve of imposed displacements $\lambda \mapsto \mathbf{u}_0(\lambda)$ such that $\|\mathbf{u}_0(\lambda)\|_{L^{\infty}(\Lambda)} < +\infty$. We impose a supplementary condition for a displacement field $\mathbf{u}$ to be admissible at the time λ, namely

$$\|\mathbf{u}\|_{L^{\infty}(\Lambda)} \leqslant \|\mathbf{u}_0(\lambda)\|_{L^{\infty}(\Lambda)}. \tag{26}$$

The space of all $\mathbf{u} \in \mathbf{SBD}(\Omega, \mathbf{u}_0(\lambda))$ such that the constraint (26) holds will be denoted by $\mathbf{SBD}^{\infty}(\Omega, \mathbf{u}_0(\lambda))$.

The function $\tilde{\Psi}$, introduced instead of Ψ, is defined as follows

$$\tilde{\Psi}: [0, +\infty) \times \mathcal{M} \to \{0, +\infty\},$$

$$\tilde{\Psi}(\lambda, (\mathbf{u}, K)) = \begin{cases} 0 & \text{if } \mathbf{u} \in \mathbf{SBD}^{\infty}(\Omega, \mathbf{u}_0(\lambda)) \text{ and} \\ & \mathcal{H}^{n-1}(K \setminus \mathbf{J}_{\mathbf{u}}) = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

DEFINITION 5.1 (weak version of Definition 4.1). Let us consider the space $\mathcal{M}$ endowed with the topology given by the convergence

$$(\mathbf{u}_h, K_h) \to (\mathbf{u}, K) \quad \text{if } \mathbf{u}_h L^2 \to \mathbf{u}. \tag{27}$$

Let us consider also the function $\mathcal{J}$, the curve of imposed displacements $t \mapsto \mathbf{u}_0(t)$ with the associated function $\tilde{\Psi}$ and the initial data $(\mathbf{u}_0, K) \in M$ such that $\mathbf{u}_0 = \mathbf{u}(\mathbf{u}_0(0), K)$.

For any $s \geqslant 1$ we recursively define $(\mathbf{u}^s, K^s): N \to \mathcal{M}$ as:

(i) $(\mathbf{u}^s, K^s)(0) = (\mathbf{u}_0, K)$;

(ii) for any $k \in N$ $(\mathbf{u}^s, \mathbf{J}_{\mathbf{u}^s})(k+1) \in \mathcal{M}$ minimizes the functional

$$(\mathbf{v}, L) \in \mathcal{M} \mapsto \mathcal{J}((\mathbf{u}^s, K^s)(k), (\mathbf{v}, L)) + \tilde{\Psi}((k+1)/s, (\mathbf{v}, L))$$

over $\mathcal{M}$. In order to verify the constraint (10), $K^s(k+1)$ is defined by the formula

$$K^s(k+1) = K^s(k) \cup \mathbf{J}_{\mathbf{u}^s(k+1)}. \tag{28}$$

An energy minimizing movement associated to $\mathcal{J}$ with the constraints (10), $\tilde{\Psi}$ and initial data $(\mathbf{u}_0, K)$ is any $(\mathbf{u}, \mathbf{J}_{\mathbf{u}})\colon [0, +\infty) \to \mathcal{M}$ having the property: there is a diverging sequence (s_i) such that for any $t > 0$ $\mathbf{u}^{s_i}([s_i t]) \to \mathbf{u}(t) \in \mathbf{SBD}^\infty(\Omega, \mathbf{u}_0(t))$ in $L^2(\Omega, R^n)$ as $i \to \infty$. The active crack at the time t is $\mathbf{J}_{\mathbf{u}(t)}$ and the damaged region at the same instant is

$$K(t) = \cup_{s \in [0,t]} \mathbf{J}_{\mathbf{u}(s)}.$$

Let us remark that the disappearance of the set $L^s(k+1)$ from the definition of the incremental solution (28) is only apparent, because if $(\mathbf{u}^s, L^s)(k+1)$ minimizes the functional

$$(\mathbf{v}, L) \in \mathcal{M} \mapsto \mathcal{J}((\mathbf{u}^s, K^s)(k), (\mathbf{v}, L)) + \tilde{\Psi}((k+1)/s, (\mathbf{v}, L))$$

then $\tilde{\Psi}((k+1)/s, (\mathbf{u}^s, L^s)(k+1)) = 0$, hence

$$\mathcal{H}^{n-1}(K \setminus \mathbf{J}_{\mathbf{u}}) = 0.$$

From Theorem 5.2 we notice that functionals like $\mathcal{J}$ are L^2 sequential lower semi-continuous and coercive on closed sets $\mathbf{V} \subset \mathbf{SBD}(\Omega)$ of functions equally bounded in L^∞ norm. If we consider in particular the functional

$$\mathbf{v} \in \mathbf{V} \mapsto \mathcal{J}((\mathbf{u}^s, K^s)(k), (\mathbf{v}, \mathbf{J}_{\mathbf{v}}))$$

the following theorem is true by a trivial induction:

THEOREM 5.3 (existence of weak incremental constrained solutions). *Let $\Omega, \Lambda \subset R^n$ be bounded open sets with piecewise smooth boundary such that $\overline{\Omega} \subset \Lambda$. Let*

$$\mathbf{u}_0\colon N \to \mathbf{SBD}(\Lambda) \cap L^\infty(\Lambda)$$

be a given sequence of imposed displacements such that $\mathbf{J}_{\mathbf{u}_0(\lambda)} \cap \overline{\Omega} = \emptyset$ and let $(\mathbf{u}_0, K)$ be a given admissible displacement-crack pair in Ω such that $\mathbf{u}_0 = \mathbf{u}(\mathbf{u}_0(0), K)$ on $\partial\Omega$.

Then there exists a sequence $(\mathbf{u}, K)\colon N \to \mathcal{M}$ such that:

(i) $\mathbf{u}(0) = \mathbf{u}_0$ *and* $K(0) = K$;

(ii) *for any* $k \in N$ *there is* $(\mathbf{u}(k+1), \mathbf{J}_{\mathbf{u}(k+1)}) \in \mathcal{M}$*, such that* $\mathbf{u}(k+1) = \mathbf{u}_0(k+1)$ *on* $\partial\Omega$ *and* $(\mathbf{u}(k+1), \mathbf{J}_{\mathbf{u}(k+1)})$ *is a minimizer of the functional*

$$(\mathbf{v}, L) \in \mathcal{M}, \mathbf{v} = \mathbf{u}_0(k+1) \quad on \ \partial\Omega \mapsto \mathcal{J}((\mathbf{u}(k), K(k)), \mathbf{v}, L)).$$

The set $K(k+1)$ *is given by the formula*

$$K(k+1) = K(k) \cup \mathbf{J}_{\mathbf{u}(k+1)}.$$

5.3. THE ANTI-PLANE CASE

In the anti-plane case we have to replace $\mathbf{SBD}(\Omega)$ by $\mathbf{SBV}(\Omega, R)$. Let us consider a larger domain $\overline{\Omega} \subset \Lambda \subset R^2$, a boundary condition $\mathbf{u}_0 \in \mathbf{SBV}(\Lambda, R) \cap L^\infty(\Lambda)$ and $u \in \mathbf{SBV}(\Lambda, R)$ such that $\mathbf{u} = \mathbf{u}_0$ in $\Lambda \setminus \overline{\Omega}$. We don't need the constraint (26) because in this case we have a maximum principle. Indeed, with the notations

$$I(\mathbf{u}) = \int_\Omega \mu |\nabla \mathbf{u}|^2 \,\mathrm{d}x + G\mathcal{H}^1(\mathbf{S_u}),$$

$$\overline{\mathbf{u}}(x) = \begin{cases} \mathbf{u}(x) & \text{if } |\mathbf{u}(x)| \leqslant \|\mathbf{u}_0\|_{L^\infty(\Lambda)}, \\ \|\mathbf{u}_0\|_{L^\infty(\Lambda)} & \text{otherwise}, \end{cases}$$

we have the inequality $I(\overline{\mathbf{u}}) \leqslant I(\mathbf{u})$ and we notice that $\overline{\mathbf{u}} = \mathbf{u}_0$ on $\Lambda \setminus \overline{\Omega}$.

The set of **SBV** displacements compatible with the boundary displacement $\mathbf{u}_0$ is denoted by $\mathbf{SBV}(\Omega, \mathbf{u}_0)$.

The set of weak displacement-crack pairs will be

$$\mathcal{N} = \{(\mathbf{u}, K) : K \text{ is } \sigma\text{-rectifiable}, \mathbf{u} \in \mathbf{SBV}(\Omega) \text{ and } |D^s\mathbf{u}|(\Omega \setminus K) = 0\}.$$

For a given path of imposed boundary displacements $\lambda \mapsto \mathbf{u}_0(\lambda) \in \mathbf{SBV}(\Lambda, R) \cap L^\infty(\Lambda, R)$ we define

$$\tilde{\Phi} : [0, +\infty) \times \mathcal{N} \to \{0, +\infty\},$$

$$\tilde{\Phi}(\lambda, (\mathbf{u}, K)) = \begin{cases} 0 & \text{if } \mathbf{u} \in \mathbf{SBV}(\Omega, \mathbf{u}_0(\lambda)) \quad \text{and} \\ & \mathcal{H}^{n-1}(K \setminus \mathbf{S_u}) = 0, \\ +\infty & \text{otherwise}. \end{cases}$$

With this setting we obtain the notion of a weak incremental solution in the case of anti-plane displacements as in Definition 5.1. All we have to do is to replace the space $\mathcal{M}$ by $\mathcal{N}$, the function $\tilde{\Psi}$ by $\tilde{\Phi}$ and $\mathbf{J_u}$ by $\mathbf{S_u}$. The existence of weak incremental solutions is a consequence of Theorem 5.1.

The partial regularity results of De Giorgi, Carriero and Leaci [22] tell us that weak incremental solutions give raise to strong incremental solutions. The existence of incremental solutions is therefore true in the anti-plane case. For $\mathcal{H}^{n-1}$-smoothness of $\mathbf{S_u}$, where $\mathbf{u}$ is a minimizer of the Mumford–Shah functional, we refer to Ambrosio and Pallara [7], Ambrosio, Fusco and Pallara [8], [9].

5.4. JUSTIFICATION OF THE WEAK FORMULATION

Let us compare the Definitions 4.1 and 5.1, where strong, respectively weak (constrained) energy minimizing movements were introduced.

We consider the Sobolev space associated to the crack set K (see [5])

$$W^{1,2}_K = \Big\{\mathbf{u} \in \mathbf{SBV}(\Omega, R^n)\colon \int_\Omega |\nabla \mathbf{u}|^2 \,\mathrm{d}x + \int_K [\mathbf{u}]^2 \,\mathrm{d}\mathcal{H}^{n-1} < +\infty, \\ |D^s\mathbf{u}| \ll \mathcal{H}^{n-1}_{|K}\Big\}.$$

The following equality has been proved in [22]

$$W^{1,2}(\Omega \setminus K, R^n) \cap L^\infty(\Omega, R^n) = W^{1,2}_K(\Omega, R^n) \cap L^\infty(\Omega, R^n). \tag{29}$$

Therefore if $\mathbf{u} = \mathbf{u}(\mathbf{u}_0, K)$ and $\mathbf{u} \in L^\infty(\Omega, R^n)$ then $\mathbf{u}$ is a special function with bounded variation. Also, if $(\mathbf{u}, K) \in M$ is a displacement-crack pair and $\mathbf{u}$ is essentially bounded, then $\mathbf{u} \in \mathbf{SBV}(\Omega, R^n)$ and $\overline{\mathbf{S_u}} \subset K$. These inclusions may lead to the introduction of the following space of weak displacement-crack pairs

$$\mathcal{M}' = \{(\mathbf{u}, K)\colon K \text{ is } \sigma\text{-rectifiable}, \mathbf{u} \in \mathbf{SBV}(\Omega, R^n) \text{ and} \\ |D^s\mathbf{u}|(\Omega \setminus K) = 0\}.$$

However the bulk part of the functional J (in weak form $\mathcal{J}$) controls only the symmetric part of the gradient of the displacement. This is the reason of considering the larger space $\mathcal{M}$ defined at (24). In conclusion, the pair $(\mathbf{u}, L)$ is replaced by the pair $(\mathbf{u}, \mathbf{J_u})$ (or, in the anti-plane case, by $(u, \mathbf{S}_u)$). The weak version of the measure $\mathrm{j}(\mathbf{u}, L)$ is then $E^j\mathbf{u}$ (or $D^j u$ in the anti-plane case).

The following proposition is a direct consequence of Theorem 5.2.

PROPOSITION 5.1. *Let $\mathbf{u}_h$ be a sequence in $\mathbf{SBD}(\Omega)$ which converges in $L^2(\Omega, R^n)$ to $\mathbf{u} \in \mathbf{SBD}(\Omega)$ such that*

$$\int_\Omega w(\varepsilon(\mathbf{u}_h))\,\mathrm{d}x + \mathcal{H}^{n-1}(\mathbf{J}_{\mathbf{u}_h}) + \|\mathbf{u}_h\|_{L^\infty} \leqslant C \tag{30}$$

for some constant C independent of h. Then there exists a subsequence, still denoted by $\mathbf{u}_h$, such that

$$\begin{cases} \varepsilon(\mathbf{u}_h) \to \varepsilon(\mathbf{u}) & \text{weakly in } L^2(\Omega, M^{n\times n}_{\text{sym}}), \\ E^j\mathbf{u}_h \to E^j\mathbf{u} & \text{as Radon measures} \end{cases}$$

and

$$\mathcal{H}^{n-1}(\mathbf{J}_{\mathbf{u}}) \leqslant \liminf_{h\to\infty} \mathcal{H}^{n-1}(\mathbf{J}_{\mathbf{u}_h}).$$

From this proposition we infer the following corollary

COROLLARY 5.1. *Let us consider* $(\mathbf{u}, \mathbf{J}_{\mathbf{u}})\colon [0, +\infty) \to \mathcal{M}$ *an energy minimizing movement,* (s_i) *a diverging sequence and* $(\mathbf{u}^s(k), \mathbf{J}_{\mathbf{u}^s(k)})$ *an incremental solution as in Definition* 5.1*, such that*

$$\mathbf{u}^{s_i}([s_i t]) \to \mathbf{u}(t) \tag{31}$$

in $L^2(\Omega, R^n)$ *as* $i \to \infty$*, for any* $t > 0$*. We have then*

$$\begin{cases} E^j \mathbf{u}^{s_i}([s_i t]) \to E^j \mathbf{u}(t) \quad \text{as Radon measures} \\ \mathcal{H}^{n-1}(\mathbf{J}_{\mathbf{u}(t)}) \leqslant \liminf_{i\to\infty} \mathcal{H}^{n-1}(\mathbf{J}_{\mathbf{u}^{s_i}([s_i t])}). \end{cases} \tag{32}$$

Therefore the relations (31), (32) are the weak version of (16). Moreover, we notice that (32) is a consequence of (31). However, this is a priori true only in the case of weak *constrained* energy minimizing movements.

6. Introduction of Small Viscosity

In the paper [4] Ambrosio and Braides introduce a generalized minimizing movement based model for the propagation of a crack in the presence of viscous forces in the body. They give as initial datum at $t = 0$ the anti-plane displacement $u_0 \in \mathbf{SBV}(\Omega, R) \cap L^\infty(\Omega, R)$. For a given s they recursively define a sequence $(u_k^s)_k$ in $\mathbf{SBV}(\Omega, R)$ and an increasing sequence of closed rectifiable sets $(K_k^s)_k$ as follows: $u_0^s = u_0$, $K_0^s = \emptyset$ and $u_{k+1}^s = w$, $K_{k+1}^s = \overline{\mathbf{S}_w} \cup K_k^s$, where w is a minimizer of the functional

$$v \mapsto \int_\Omega |\nabla v|^2 \, \mathrm{d}x + \mathcal{H}^1(\mathbf{S}_v \setminus K_k^s) + s \int_\Omega |v - u_k^s|^2 \, \mathrm{d}x \tag{33}$$

over the set of all v such that

$$v \in \mathbf{SBV}(\Omega, R), \qquad \|v\|_\infty \leqslant \|u_0\|_\infty.$$

The generalized minimizing movements obtained as limits of such incremental solutions, when s diverges, correspond to the following situation: a body evolves from the initial state u_0, with the initial crack $\mathbf{S}_{u_0}$, under a constant imposed boundary displacement. The equation of evolution for the displacement is

$$\operatorname{div} \nabla u(t) + \dot{u}(t) = 0.$$

The authors obtain an existence result for the generalized minimizing movement introduced by them. After the introduction of the piecewise constant function

$$u^s(t) = u^s_{[st]},$$

they find the following uniform estimate

$$\|u^s(t') - u^s(t)\|_{L^2} \leqslant M\sqrt{t' - t + \frac{1}{s}} \quad \text{if } t' \geqslant t. \tag{34}$$

Therefore there exists a diverging sequence $(s_i)_i$ such that u^{s_i} converges to u uniformly in $L^\infty([0, T], L^2(\Omega, R))$, for all $T > 0$ and

$$u \in C^{0,1/2}([0, +\infty); L^2(\Omega, R)). \tag{35}$$

This result is obtained under the assumption of constant imposed boundary displacement, equal to the trace on the boundary of the initial datum u_0.

It is natural to introduce the Lamé constant μ and the viscosity λ in the expression of the functional (33) and modify it as follows

$$v \mapsto \int_\Omega \mu|\nabla v|^2 \,\mathrm{d}x + \mathcal{H}^{n-1}(\mathbf{S}_v \setminus K^s_k) + \lambda s \int_\Omega |v - u^s_k|^2 \,\mathrm{d}x.$$

We obtain the more physical case of an anti-plane displacement satisfying at any moment t the equation

$$\operatorname{div} \mu \nabla u(t) + \lambda \dot{u}(t) = 0.$$

The estimate (34) becomes

$$\|u^s(t') - u^s(t)\|_{L^2} \leqslant M\sqrt{t' - t + \frac{1}{\lambda s}} \quad \text{if } t' \geqslant t.$$

We expect to obtain our model, in the case of anti-plane displacements, when the viscosity λ converges to 0. It is easy to see that if λ converges to 0 then the uniform estimate from above is lost.

We notice that the crack appearance can not occur in this model in a physically acceptable way.

Indeed suppose that for any $t > t' > 0$ we have

$$\mathbf{S}_{u(t')} \subset \mathbf{S}_{u(t)}.$$

This hypothesis means that the damaged region

$$K(t) = \cup_{s \in [0,t]} \mathbf{S}_{u(s)}$$

is the active crack $\mathbf{S}_{u(t)}$. We suppose moreover that the energy

$$E(t) = \int_\Omega |\nabla u(t)|^2 \,\mathrm{d}x + \mathcal{H}^1(K(t))$$

is a decreasing function.

From the above suppositions, the compactness theorem in **SBV** and (35) we infer that:

(a) the function $t \mapsto E(t)$ is decreasing lower semicontinuous,
(b) the function $t \mapsto \mathcal{H}^1(K(t))$ is increasing lower semicontinuous,
(c) the elastic energy

$$t \mapsto \int_\Omega |\nabla u(t)|^2 \,\mathrm{d}x$$

is a decreasing function (from (a) and (b)) and it is lower semicontinuous.

A straightforward consequence of items (a), (b), (c) is that for any t the lateral limits of the function $s \mapsto \mathcal{H}^1(K(s))$ at the moment t are both equal to the value $\mathcal{H}^1(K(t))$, that is the length of the crack grows continuously with time.

We mention however that we don't know if for any minimizing movement $u(t)$ the energy $E(t)$ decreases with time. Again from the compactness theorem in **SBV** all we can prove is that for any $t < t'$ we have the inequalities

$$\liminf_{i\to\infty} E(u^{s_i}(t)) \geqslant \liminf_{i\to\infty} E(u^{s_i}(t')),$$

$$E(u(t)) \leqslant \liminf_{i\to\infty} E(u^{s_i}(t)), \qquad E(u(t')) \leqslant \liminf_{i\to\infty} E(u^{s_i}(t')),$$

where

$$E(u) = \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \mathcal{H}^1(\mathbf{S}_u).$$

7. A Partial Existence Result

The main open theoretical problem is the general existence of an energy minimizing movement according to our definitions. Below is described a partial existence result based on a sound physical assumption (36). Nevertheless, we do not know if (36) can be proved from the basic assumptions of the model.

THEOREM 7.1. *Let us consider for any given s an incremental solution $k \mapsto (\mathbf{u}^s(k), K^s(k)) \in M$, according to Definition* 4.1, *such that $\mathbf{u}^s(k)$ are equally bounded in L^∞. Let us suppose that the power communicated by the rest of the universe to the body is uniformly bounded at any time t. The incremental form of* *this assumption consists in the existence of a constant P such that for any k and s we have*

$$\langle \mathbf{T}^s_k \tfrac{1}{2}(\mathbf{u}_0((k+1)/s) + \mathbf{u}_0(k/s)), \Delta\mathbf{u}_0(k,s)\rangle \leqslant P/s, \tag{36}$$

where $\mathbf{T}^s_k = \mathbf{T}(K^s(k))$ and $\Delta\mathbf{u}_0(k,s) = \mathbf{u}_0((k+1)/s) - \mathbf{u}_0(k/s)$. Then for any $t > 0$ there exist diverging sequences $(s_i)_i$ and $(k_i)_i$ such that k_i/s_i converges to t,

$$\begin{cases} \mathbf{u}^{s_i}(k_i) \to \mathbf{u}(t) \quad \textit{in } L^2(\Omega, R^n), \\ j(\mathbf{u}^{s_i}, L^{s_i})(k_i) \to j(\mathbf{u}, L)(t) \quad \textit{weakly as Radon measures} \end{cases} \tag{37}$$

as $i \to \infty$ and

$$\mathcal{H}^{n-1}(L(t)) \leqslant \liminf_{i\to\infty} \mathcal{H}(L^{s_i}(k_i)). \tag{38}$$

Proof. For any $k \in N$ we introduce the displacement

$$\mathbf{v}^s(k+1) = \mathbf{u}(\mathbf{u}_0((k+1)/s), K^s(k)).$$

From the minimality assumption on the incremental solution we have for any $k \in N$ the inequality

$$J((\mathbf{u}^s(k), K^s(k)), (\mathbf{v}^s(k+1), K^s(k))) \geqslant J(k,s),$$

$$J(k,s) = J((\mathbf{u}^s(k), K^s(k)), (\mathbf{u}^s(k+1), K^s(k+1))).$$

Also, because $\mathbf{u}^s(k)$ is uniformly (with respect to k and s) essentially bounded, from the relation (29) and the minimality of the incremental solution we have $\mathcal{H}^{n-1}(L^s(k) \setminus \mathbf{J}_{\mathbf{u}^s(k)}) = 0$ for any s, k. The latter inequality means that

$$\int_\Omega w(\nabla\mathbf{v}^s(k+1))\,\mathrm{d}x \;\geqslant\; \int_\Omega w(\nabla\mathbf{u}^s(k+1))\,\mathrm{d}x + G\mathcal{H}^{n-1}(K^s(k+1) \setminus K^s(k)).$$

The crack growth condition $K^s(k) \subset K^s(k+1)$ implies that the latter relation can be written as

$$\left(\int_\Omega w(\nabla\mathbf{v}^s(k+1))\,\mathrm{d}x - \int_\Omega w(\nabla\mathbf{u}^s(k))\,\mathrm{d}x\right) + \int_\Omega w(\nabla\mathbf{u}^s(k))\,\mathrm{d}x + G\mathcal{H}^{n-1}(K^s(k)) \geqslant \int_\Omega w(\nabla\mathbf{u}^s(k+1))\,\mathrm{d}x + G\mathcal{H}^{n-1}(K^s(k+1)).$$

This is the incremental form of the Griffith criterion of crack propagation (12). Indeed, we have

$$\int_\Omega w(\nabla \mathbf{v}^s(k+1))\,\mathrm{d}x = \tfrac{1}{2}\langle \mathbf{T}(K^s(k))\mathbf{u}_0((k+1)/s), \mathbf{u}_0((k+1)/s)\rangle,$$

$$\int_\Omega w(\nabla \mathbf{u}^s(k))\,\mathrm{d}x = \tfrac{1}{2}\langle \mathbf{T}(K^s(k))\mathbf{u}_0(k/s), \mathbf{u}_0(k/s)\rangle,$$

therefore

$$\begin{aligned}\int_\Omega w(\nabla \mathbf{v}^s(k+1))\,\mathrm{d}x - \int_\Omega w(\nabla \mathbf{u}^s(k))\,\mathrm{d}x \\ = \langle \mathbf{T}(K^s(k))\tfrac{1}{2}(\mathbf{u}_0((k+1)/s) + \mathbf{u}_0(k/s)), \\ \mathbf{u}_0((k+1)/s) - \mathbf{u}_0(k/s)\rangle.\end{aligned}$$

$\mathbf{v}^s(k+1)$ represents the displacement of the body with the boundary displacement $\mathbf{u}_0(k/s+1/s)$ in the presence of the crack $K^s(k)$. $\mathbf{u}^s(k)$ represents the displacement of the body with the boundary displacement $\mathbf{u}_0(k/s)$ in the presence of the same crack $K^s(k)$. According to (11), the quantity

$$\left(\int_\Omega w(\nabla \mathbf{v}^s(k+1))\,\mathrm{d}x - \int_\Omega w(\nabla \mathbf{u}^s(k))\,\mathrm{d}x\right) \Big/ \left(\frac{1}{s}\right)$$

is the discretized expression of the power communicated by the rest of the universe to the body at the time k/s, when a time discretization with step $1/s$ is considered.

We deduce from the inequality (39) that

$$\begin{aligned}P/s + \int_\Omega w(\nabla \mathbf{u}^s(k))\,\mathrm{d}x + G\mathcal{H}^{n-1}(K^s(k)) \\ \geqslant \int_\Omega w(\nabla \mathbf{u}^s(k+1))\,\mathrm{d}x + G\mathcal{H}^{n-1}(K^s(k+1)).\end{aligned}$$

We have therefore

$$Pk/s \geqslant \int_\Omega w(\nabla \mathbf{u}^s(k+1))\,\mathrm{d}x + G\mathcal{H}^{n-1}(K^s(k+1)).$$

From $L^s(k+1) \subset K^s(k+1)$ we infer that

$$Pk/s \geqslant \int_\Omega w(\nabla \mathbf{u}^s(k+1))\,\mathrm{d}x + G\mathcal{H}^{n-1}(L^s(k+1)).$$

The latter inequality and the equally boundedness of $\mathbf{u}^s(k)$ allow us to apply the compactness Theorem for **SBD** 5.2. We deduce that for any $t > 0$ there exist diverging sequences $(s_i)_i$ and $(k_i)_i$ such that k_i/s_i converges to t and $(\mathbf{u}^{s_i}, L^{s_i})(k_i)$ converges to an element of $M\ (\mathbf{u}, L)(t)$ in the sense of the relations (37), (38).

8. Numerical Approach to the Model

The models presented in this paper are of applicative interest. In order to use them we have to know how to minimize a Mumford–Shah functional. This can be done by approximating, in the sense of variational convergence, the original functional by a volume integral. There are several ways to approximate the Mumford–Shah functional by volume integrals (for a general reference we quote Braides [16]). One idea is to replace the displacement-crack pair $(\mathbf{u}, K)$ with the pair $(\mathbf{u}, f)$, where f is a smoothed version of the characteristic function of the crack set K, taking values in the interval [0, 1]. The original functional may be replaced by an Ambrosio–Tortorelli approximation, introduced in [11], [12].

Let us consider, for given $g\colon \Omega \to R$ and $c > 0$, functionals of the form

$$I_c(u, f) = \int_\Omega \left\{ \alpha\phi(f)|\nabla u|^2 + \beta(u-g)^2 + \right. \\ \left. +\gamma \left[c\psi(f)|\nabla f|^2 + \frac{f^2}{4c} \right] \right\} \mathrm{d}x. \tag{39}$$

We suppose that the functions ϕ, ψ have the following properties:

(a) $\psi(x) > 0$ for any $x \in (0, 1]$;
(b) $\int_0^1 2x\psi^{1/2}(x)\,\mathrm{d}x = 1$;
(c) $\phi(0) = 1$, $\phi(1) = 0$ and $\phi(x) \in (0, 1)$ for any $x \in (0, 1)$.

Under these assumptions it is known that when c converges to 0 then I_c converges in the variational sense (or Γ-convergence) to the Mumford–Shah functional

$$I(u) = \alpha \int_\Omega |\nabla u|^2\,\mathrm{d}x + \beta \int_\Omega |u-g|^2\,\mathrm{d}x + \gamma\mathcal{H}^1(\mathbf{S}_u). \tag{40}$$

This result, due to Ambrosio and Tortorelli, tells us that for any $u \in \mathbf{SBV}(\Omega, R)$ the followings are true:

(i) for any sequence (u_h, f_h, c_h) such that $u_h \to u$ and $f_h \to 0$ in L^2, $c_h \to 0$, we have

$$\liminf_{h\to\infty} I_{c_h}(u_h, f_h) \geqslant I(u);$$

(ii) there is a sequence (u_h, f_h, c_h) such that $u_h \to u$ and $f_h \to 0$ in L^2, $c_h \to 0$, and

$$\limsup_{h\to\infty} I_{c_h}(u_h, f_h) \leqslant I(u).$$

A consequence of this result is that if:

(i) (u_h, f_h) is a minimizer of the functional I_{c_h} and $c_h \to 0$; and
(ii) there is a function $u \in \mathbf{SBV}(\Omega, R)$ such that $u_h \to u$ and $f_h \to 0$ in L^2,

then u is a minimizer of the Mumford–Shah functional I.

The numerical approach to the problem of minimizing the Mumford–Shah functional consists in the replacement of this functional with an approximate functional I_c. After a numerical minimization of I_c over a conveniently chosen set we obtain a minimizing pair (u^c, f^c). The function f^c represents an approximation of the characteristic function of the set $\mathbf{S}_u$, where u is a minimizer of I.

We shall use this idea for the model presented here, in the anti-plane case. Instead of a sequence of incremental solutions (u_h, K_h) we shall consider a sequence of pairs (u_h^c, f_h^c). The crack-growth condition $K_h \subset K_{h+1}$ will be replaced by: $f_h^c(x) \leqslant f_{h+1}^c(x)$ for any $x \in \Omega$. Notice that f_h^c is an approximation of the characteristic function of the damaged region.

We shall not be concerned further with the regularity of the functions that we are dealing with. We set M to be the space of all pairs of smooth enough functions $u: \Omega \subset R^2 \to R$, $f: \Omega \to [0, 1]$. The number c and functions ϕ, ψ are given, as well as a sequence of imposed boundary displacements $u_0^n: \Gamma_u \subset \partial\Omega \to R$. As for the material constants, we set $\gamma = G/\mu$, which has the dimension of a length.

DEFINITION 8.1. Let us define the functions

$$J_c: M \times M \to R,$$

$$F(g) = \int_\Omega \left\{ \Phi(g)|\nabla v|^2 + \gamma \left[c\psi(g)|\nabla g|^2 + \frac{g^2}{4c} \right] \right\} \mathrm{d}x,$$

$$J_c((u, f), (v, g)) = \begin{cases} F(g) & \text{if } g \geqslant f, \\ +\infty & \text{otherwise,} \end{cases}$$

$$\Psi: N \times M \to \{0, +\infty\},$$

$$\Psi(n, (v, g)) = \begin{cases} 0 & \text{if } (1-g)(v - u_0^n) = 0 \text{ on } \Gamma_u, \\ +\infty & \text{otherwise.} \end{cases}$$

We consider the initial data (u_0, f_0) such that $u_0 = u(u_0^0, K)$ and f_0 satisfies

$$\sup\{|f(x) - \chi_K(x)|: x \in \Omega\} \leqslant c,$$

where χ_K is the characteristic function of the set K.

We recursively define the sequence (u_h^c, f_h^c) as follows:

(i) $(u_0^c, f_0^c) = (u_0, f_0)$;

(ii) for any $k \in N$ the pair (u^c_{k+1}, f^c_{k+1}) minimizes over M the functional

$$(v, g) \mapsto J_c((u^c_k, f^c_k), (v, g)) + \Psi(k+1, (v, g)).$$

For the approximate model described in Definition 8.1 we shall use the gradient descent method described in Richardson and Mitter [32]. The domain Ω is discretized in pixels and the various partial derivatives of functions u^c and f^c are replaced by finite differences. With the notation

$$J^k_c(u, f) = J_c((u^c_k, f^c_k), (u, f))$$

the gradient descent of the functional J^k_c has the form

$$\dot{u} = -C_u \partial_u J^k_c(u, f),$$
$$\dot{f} = -C_f \partial_f J^k_c(u, f)$$

with variable controls C_u and C_f. In order to respect the constraint Ψ, after each step of the descent a projection of f on the convex set

$$\{g\colon \Omega \to [0, 1]\colon g(x) \geqslant f^c_k(x) \ \ \forall x \in \Omega\}$$

is performed. The boundary condition for the displacement u is satisfied in the usual way by setting the value of u on the pixels of $\partial\Omega$ equal to the value of u^{k+1}_0.

The simplest choice for the functions ϕ and ψ is

$$\phi(x) = (1 - x)^2, \qquad \psi(x) = 1.$$

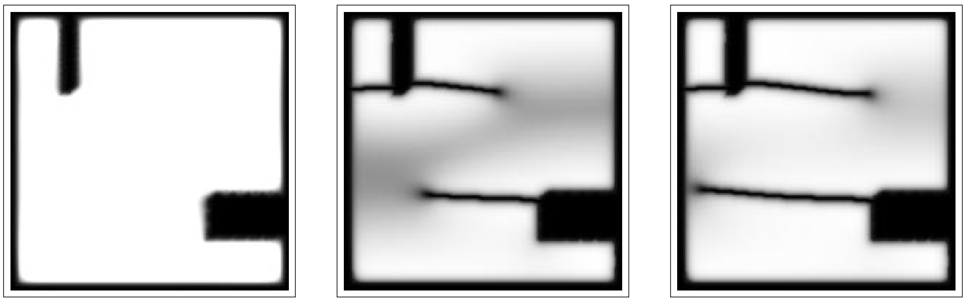
Richardson and Mitter remark in [32] that the parameter β (see (1)), which is equal to 0 in Definitions 4.1 and 8.1, has a strong influence on the speed of the gradient descent method they propose: small β causes low speed of the gradient descent. In our problem β is null and this causes a very slow rate of convergence. There is an empirical reason for which the Mumford–Shah functional behaves badly when β is zero, in the problem of crack evolution: unlike the case of image segmentation, where the information is scattered all over Ω, in the problem of crack evolution the displacement that causes the growth of the crack is a datum concentrated on the boundary of Ω. The viscous force induced by β should serve to transport this information inside Ω.

For numerical reasons we shall mix our model with an Ambrosio and Braides model with small, but not zero, viscosity. We replace the functional J_c by

$$J^*_c((u, f), (v, g)) = J_c((u, f), (v, g)) + \beta s \int_\Omega |v - u|^2 \, dx.$$

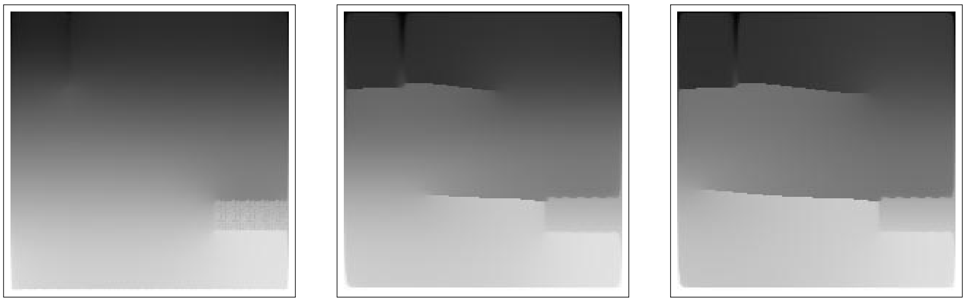
The sequence of imposed boundary displacements (u^n_0) is the discretized in time version of a path of displacements $u_0(t)$. For a fixed step of discretization $1/s$ we have

$$u^n_0 = u_0(n/s).$$



(a) (b) (c)

Figure 3. (a) The initial geometry of the body; (b) and (c) the aspect of the evolving cracks.



(a) (b) (c)

Figure 4. (a) The initial displacement of the body; (b) and (c) represent the displacement of the body fractured as in (b) and (c) previous pictures.

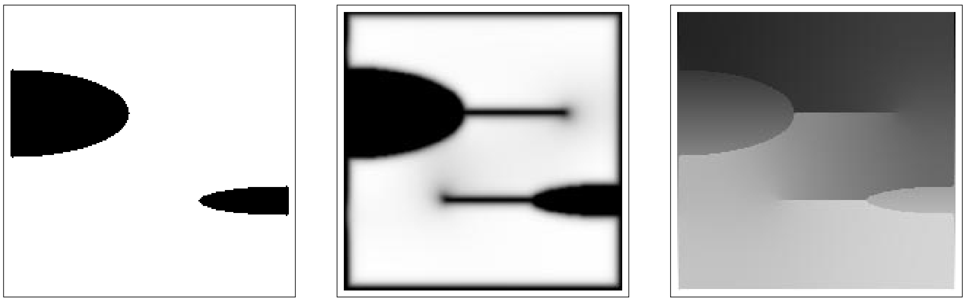
In order to eliminate the effects of the viscosity we replace also the sequence (u_0^n) with the following one, for a given natural P

$$\forall n \in N, \quad k \in \{0, 1, \dots P - 1\} U_0^{nP+k} = u_0^n.$$

Therefore at any time n/s, the boundary displacement becomes $u_0(n/s)$ and after that it remains constant in the interval $[n/s, (n+P)/s]$, in order to let the influence of the viscosity to become negligible.

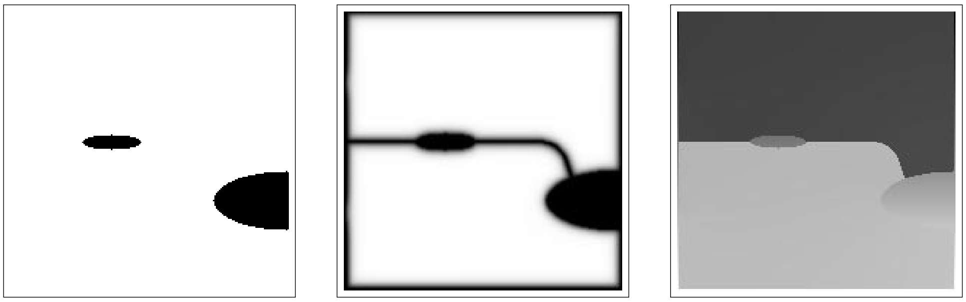
In Figure 1 we see how the Richardson and Mitter method works for the image segmentation problem. Recall that the Mumford–Shah functional (1) is used. The parameters α, β and γ have been left to our choice, in order to get a good result.

The results of the numerical method for a cylinder with a rectangular cross-section of $0.1\,\mathrm{m} \times 0.1\,\mathrm{m}$ are shown in the next four figures. We remove from this cross-section small rectangles (Figures 3 and 4) or parts of ellipsis (Figures 5 and 6) and study what happened with the body obtained in this way during an imposed path of boundary displacements. The material (carbon steel) has the constant $\gamma = G/\mu = 0.0000025\,\mathrm{m}$ and it has a pure elastic behavior. The boundary conditions are described further. The rectangular section is a square $[0, 0.1] \times [0, 0.1]$. The



(a) (b) (c)

Figure 5. (a) The initial geometry; (b) final aspect of the cracks; (c) final displacement.



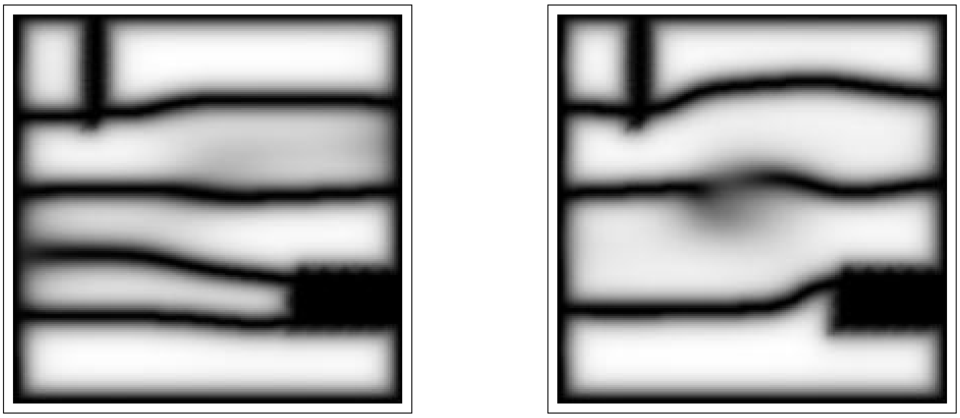
(a) (b) (c)

Figure 6. (a) The initial geometry of the body; (b) final aspect of the cracks (c) final displacement.

displacement u_0^n is imposed on the faces $[0, 0.1] \times \{0\}$, where u_0^n is constant and equal to 0, and $[0, 0.1] \times \{0.1\}$, where the displacement u_0^n is constant and grows slowly with n, from the value 0 m to the value 0.0041 m. The other two faces are force free.

The approximate characteristic function of the crack, i.e., the function $f : \Omega \to [0, 1]$ is represented with the following convention: there are 256 grey levels, numbered from 0 (black) to 255 (white); the number 0 (no crack there) corresponds to the level 255 and the number 1 (certainly a crack there) corresponds to the level 0. We have a linear correspondence between the numbers from $(0, 1)$ and the intermediary grey level. In this way we obtain a kind of picture of the shape of the crack in the cross-section of the body. Therefore a pixel is black either if there was no material there from the start, or if it belongs to the actual crack. Irrelevant black pixels appear on the boundary of the picture, maybe as an effect of error accumulation during the minimization process.

The displacement function u is represented in the complete square cross-section, but is irrelevant in the portions removed from the section. The representation was made with the following convention: the 255 level (white) correspond to the max-



(a) (b)

Figure 7. Examples of local minima.

imum value of u and the 0 level (black) correspond to the minimum value of u; all the intermediary values of u are represented as grey levels, with a linear law of correspondence.

9. Final Remarks

This energetic approach to quasi-static brittle fracture propagation has the quality that it does not contain any prescription of the shape or location of the cracks. We have seen that the model provides a way of working with cracks which suddenly appear in the body. We have partially investigated this feature of the model and we have concluded that the model is not compatible with a critical stress based model of damage of an elastic body.

In this paper we did not study the bifurcation of an existing crack. A crack bifurcates when its shape suffers a change of topology. The most common example is a crack in a two-dimensional configuration, initially with only one edge in the body, which develops in time new branches. During this phenomenon the number of edges of the crack increases.

The numerical results presented in the last section have the following feature: during the evolution of the crack new concentrations of the elastic energy density do not appear *in the interior of the body*. It may seem that we have an example of crack bifurcation in Figures 3(b) and (c), but the two branches from the top of the Figure 3(b) do not grow simultaneously. We have noticed that a first crack grows to the left until its edge reaches the boundary of the rectangle and, after that, a second crack grows to the right.

There is no method to find the global minimum of a functional like the Ambrosio–Tortorelli approximation. We have experimented with our programs for a large variety of data. We have obtained from time to time solutions which were obviously local but not global minima. We have found that some of these local minima loose old edges (Figure 7(a)), eventually developing instead new ones (Figure 7(b)).

Our numerical results indicate that there is a sort of conservation law of ‘edges’ (i.e. maxima or singularities of the elastic energy density) of the solutions of the model, asserting that during the evolution of the crack the number of these ‘edges’ can only decrease. If such a conservation law is true, it may be a consequence of the fact that in the Mumford–Shah functional there is no term which controls the creation of a new ‘edge’.

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