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A Characterization of Sub-Riemannian Spaces as Length Dilation Structures Constructed Via Coherent Projections

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Abstract

We introduce length dilation structures on metric spaces, tempered dilation structures and coherent projections and explore the relations between these objects and the Radon-Nikodym property and Gamma-convergence of length functionals. Then we show that the main properties of sub-riemannian spaces can be obtained from pairs of length dilation structures, the first being a tempered one and the second obtained via a coherent projection. Thus we get an intrinsic, synthetic, axiomatic description of sub-riemannian geometry, which transforms the classical construction of a Carnot-Carathéodory distance on a regular sub-riemannian manifold into a model for this abstract sub-riemannian geometry.

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1 Introduction

Motivation: sub-riemannian spaces as approximately self-similar metric spaces. Sub-riemannian geometry is the study of non-holonomic spaces (introduced by Vrănceanu [28], [29] in 1926) endowed with a Carnot-Carathéodory distance. Such spaces appear in applications to thermodynamics (the name ”Carnot-Carathéodory distance” is inspired by the work of Carathéodory [10] (1909) concerning a mathematical approach to Carnot work in thermodynamics), in non-holonomic dynamics (see the survey Vershik and Gershkovich [26]), in the study of hypo-elliptic operators Hörmander [16], in harmonic analysis on homogeneous cones Folland, Stein [14], and as boundaries of CR-manifolds.

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In several foundational papers on sub-Riemannian geometry, among them Mitchell [19], Bellaïche [3], the paper of Gromov asking for an intrinsic point of view for sub-riemannian geometry [15], Margulis, Mostow [17], [18], dedicated to Rademacher theorem for sub-riemannian manifolds and to the construction of a tangent bundle of such manifolds, Vodopyanov [23] [24], Vodopyanov and Karmanova [25], fundamental results concerning the intrinsic properties of sub-riemannian spaces endowed with the Carnot-Carathéodory distance were proved using differential geometry tools.

Among the fundamental results in sub-riemannian geometry, or more general non-holonomic geometry, a particular position occupies the result concerning the nilpotent group structure of the tangent space to a point of a regular non-holonomic manifold (for an evolution of this subject see [27], [2]). According to the introduction of the Agrachev and Marigo paper [2], non-holonomic tangent functors appear appear via constructions involving nilpotent or graded approximations of the geometrical objects to be studied. We cite from [2] page 112, 3rd paragraph: "A weak point of these constructions is their heavy dependence on the choice of coordinates. Because of that, the approximation looks like an auxiliary technical tool rather than a fundamental functorial operation; the geometric insight and the application of geometric machinery are highly impeded." Agrachev and Marigo propose therefore an intrinsic construction of the tangent bundle of a non-holonomical manifold. Their notion of "intrinsic" means "coordinate free", in the frame of differential geometry of manifolds.

The point of view of Gromov in [15] is that the only intrinsic object on a sub-riemannian manifold is the Carnot-Carathéodory distance. The underlying differential structure of the manifold is then clearly not intrinsic. Nevertheless, in all proofs in the before mentioned papers on the fundamentals of sub-riemannian geometry this differential structure is used in order to prove intrinsic statements.

In conclusion, for sub-riemannian geometry there seems to be two meanings of the word "intrinsic":

(a) the Carnot-Carathéodory distance as the only intrinsic object permits to formulate several important results, but the proof of these results is based on the constructions and approximations of the type mentioned by Agrachev and Marigo,

(b) if we see sub-riemannian geometry as a subspecies of non-holonomic geometry, then "intrinsic" is only the non-holonomic distribution and differential geometry is accepted as an "intrinsic" tool.

One interpretation of this situation is that the Carnot-Carathéodory distance is necessary, but not sufficient for an axiomatization of sub-riemannian geometry. In few words, the Carnot-Carathéodory distance is not enough and the whole formalism of differential geometry is too much for describing sub-riemannian geometry.

Together with the Carnot-Carathéodory distance, there is another object which persistently appears in all studies of sub-riemannian geometry: the (anisotropic) dilations.

To my knowledge, topological spaces with dilations were proposed for the first time in [4]. In the last section of of the paper by Bellaïche [3] it is explained how one can obtain the operation of addition in the the tangent space at a point by using dilations (dilatations, homotheties, re-scaling, and so on) and a reasoning based on uniform convergence. These arguments based on uniform convergence made me return to the subject of spaces with dilations. After a period of experimenting with different constructions related to sub-riemannian geometry on Lie groups, I proposed the notion of dilation structure [5]. A dilation structure encodes the approximate self-similarity of a metric space. A dilation structure gives to the metric space a tangent bundle with conical group operations in each fiber (tangent space to a point). There is also a notion of derivative associated to a dilation structure. Conical groups generalize Carnot groups. The affine geometry of conical groups was then studied in [6]. In [7] it is shown that regular sub-riemannian manifolds admit dilation structures constructed via normal frames. In that paper I tried to minimize the contribution of classical differential calculus in the proof of the basic results in sub-riemannian geometry, by showing that in fact the differential calculus on the underlying differential manifold of the sub-riemannian space is needed only for proving that normal frames exist, which implies the existence of dilation structures associated to the Carnot-Carathéodory distance.

The point of view of this paper is that sub-riemannian geometry may be described by a set of axioms concerning dilation structures. It is true that this viewpoint is less general than Gromov's (there are more intrinsic objects than the Carnot-Carathéodory distance). Nevertheless, in this approach we renounce at the differential structure (of the manifold) and we replace it with something which is much weaker, a dilation structure. We could then see sub-riemannian geometry (as well as riemannian geometry) as the geometry of dilation structures on length metric spaces.

In [8] I showed that there are many dilation structures on ultrametric spaces. The distance on these metric spaces is not a length distance, therefore such dilation structures are different from the ones appearing in sub-riemannian geometry.

With this motivation I propose here the notion of a length dilation structure (section 4) with the Radon-Nikodym property (RNP) (section 7). A dilation structure has the RNP property if any Lipschitz curve is almost everywhere derivable in the sense induced by the dilation structure. The dilation structure of a regular sub-riemannian manifold is a length dilation structure with RNP. We can equally endow a riemannian manifold with a dilation structure with RNP. Are there any other axioms which we have to add in order to obtain a class of dilation structures which describes the true (non-riemannian) sub-riemannian geometry?

The answer (theorem 10.10) is that regular sub-riemannian manifolds can be seen as length dilation structures (definition 4.3) which are constructed with the help of coherent projections (definition 9.1) and tempered dilation structures (definition 8.1).

Tempered dilation structures are a generalization of riemannian manifolds. They have the property that for any point x of the space, the dilation based at x is a bi-lipschitz transformation, in a uniform manner with respect to the magnification ε and the base point x. This property describes riemannian spaces, but not general sub-riemannian spaces.

In order to obtain sub-riemannian spaces we also need coherent projections, which are objects generalizing non-holonomic distributions. Indeed, consider M a real smooth n-dimensional manifold. Instead of a distribution D, which is a map associating to any point $x \in M$ a subspace $D_x \subset T_x M$, we could use a field of projections

$$Q^x : T_x M \to T_x M, \quad Q^x T_x M = D_x, \quad Q^x Q^x = Q^x.$$

Let us denote by $\bar{\delta}^x_\varepsilon u = \varepsilon u$ the multiplication by positive scalars in the tangent space of M

at x. Suppose that the distribution D is spanned by a family of vector fields which induces by the Chow condition a normal frame $\{X_i : i = 1, ..., n\}$, definition 6.5, and a non-isotropic dilation

$$\delta_\varepsilon^x \left(\sum_{i=1}^n a_i X_i(x) \right) = \left(\sum_{i=1}^n a_i \varepsilon^{deg X_i} X_i(x) \right),$$

as in theorem 6.6. Then the field of projections $x \mapsto Q^x$ is an uniform limit of "approximate projections":

$$Q^x u = \lim_{\varepsilon \to 0} Q_\varepsilon^x u, \quad Q_\varepsilon^x = \bar{\delta}_{\varepsilon^{-1}}^x \, \delta_\varepsilon^x u.$$

Under closer scrutiny, it appears that the existence of the limit Q^x (as a uniform limit, as well as having some other algebraic properties) is the basis which can be used for establishing sub-riemannian geometry.

Outline of the paper. After the introductory section 2 dedicated to basic notions concerning length in metric spaces, in section 3 we describe the notion of a dilation structure, introduced in [5]. A dilation structure on a metric space directly provides a notion of derivative, thus endowing the space with its own differential calculus. The class of metric spaces admitting dilation structures seems rather large, containing riemannian, sub-riemannian as well as some ultrametric spaces, as explained in [6], [7], [8]. The idea of dilation structures is that dilations (or dilations, or homotheties, or even contractions as considered in the case of contractible groups) are central objects for a differential calculus. The field δ of dilations on a metric space (X, d) obeys 5 axioms, see definition 3.1, stating algebraic and analytical properties of δ, as well as the compatibility between δ and the distance d.

In section 4 we propose an alternative notion, length dilation structures, which will be central in further considerations. In a length dilation structure, definition 4.3, the accent is put on the length functional induced by the distance d. We may imagine the field of dilations

$$(x, \varepsilon) \in X \times (0, 1] \mapsto \delta_\varepsilon^x : U(x) \subset X \to X$$

as a field of microscopes with magnification power ε, associating to any $x \in X$ a chart $U(x)$ of a ε-neighbourhood of x, as measured with the distance d. Imagine a curve in X as a road and the various charts provided by dilations as roadmaps. In a length dilation structure the lengths of the images of the true road, as seen in different roadmaps, have to agree. Also, these roadmaps have to be compatible in a clearly stated manner. Finally, the compatibility of the dilation field with the length functional induced by the distance d is further stated as a Gamma-convergence condition of induced length functionals, as $\varepsilon \to 0$.

In section 5 is explained the structure of the tangent bundle which comes with a strong dilation structure or a length dilation structure. The characterization of the tangent bundle for length dilation structures is new. A key notion which appears is the one of a conical group, studied in [6], which generalizes Carnot groups and contractible groups as well.

In order to facilitate the understanding of the abstract theory of tempered dilation structures and coherent projections (sections 8, 9 and 10), we explain in section 6 the case of dilation structures on sub-riemannian manifolds, following [7].

In section 7 we begin to study dilation structures satisfying the Radon-Nikodym property for metric spaces (or rectifiability property, or RNP), definition 7.3. This property says that Lipschitz curves are derivable almost everywhere in the sense provided by the dilation structure. We give examples, then we easily obtain a description of the length functional as if we were in a kind of a generalized Finsler manifold, theorem 7.4.

Tempered dilation structures, section 8, seem to be the habitat where generalizations of results of Buttazzo, De Pascale and Fragala [9] and Venturini [22] naturally live. A dilation structure is tempered, definition 8.1, if the charts provided by dilations are bi-lipschitz with the real distance, in a uniform manner with respect to the magnification ε and the base point x. This is locally the case for any C^1 riemannian manifold, but it is not true for sub-riemannian manifolds, for example. From corollary 8.4 to theorem 8.3 we find out that a tempered dilation structure with RNP is also a length dilation structure.

In section 9 coherent projections are introduced and studied. Coherent projections are generalizations of distributions. With the help of a coherent projection Q and a tempered dilation structure $(X,\bar{d},\bar{\delta})$ we get a new field of dilations δ and a new distance d, quite similar to a Carnot-Carathéodory distance. Notice however that in the case of sub-riemannian manifold we use as a tempered dilation structure the one coming from a riemannian manifold, which according to our language has two very special properties: it is locally linear (see the paper [6] for the affine geometry of a linear dilation structure) and it is commutative in the sense that the tangent spaces are commutative conical groups, that is they are vector spaces. In the general formalism of coherent projections and tempered dilation structures nothing like this is used.

The main problem that we solve, section 10, is if (X,d,δ) is a length dilation structure. This problem is solved for coherent projections which satisfy a generalized Chow condition. This condition is inspired by the classical Chow condition, but for the reader which becomes familiar with dilation structures is rather clear that Chow condition is only one among an infinity of other conditions with equivalent effect. Indeed, even if we shall not touch this in the present paper, the Chow condition seems to be only a convenient way to indicate an algorithm for going from point A to point B, in terms of vector field brackets. We explained in [5] that to dilation structures in general is associated a formalism of binary decorated planar trees. At the level of this formalism the algorithm from Chow condition, as formulated in this paper, appears as working on a very particular class of such binary trees.

In subsection 10.3 we finally get that coherent projections which satisfy condition (Cgen) and tempered dilation structures which satisfy some supplementary conditions (A) and (B) indeed induce length dilation structures. At the classical level, this implies the new result that on regular sub-riemannian manifolds the rescaled (with the magnification factor ε) lengths Gamma-converge to the length in the metric tangent space, for any point.

The paper ends with the conclusion section 11, where we state that Gromov' viewpoint, that the CC distance is the only intrinsic object in sub-riemannian geometry, should be supplemented with Siebert' result, that a homogeneous Lie group is just a locally compact group endowed with a contracting and continuous one parameter group of automorphisms. This is what we do in this paper, by replacing the classical differential structures with the more general dilation structures.

2 Length in metric spaces

For a detailed introduction into the subject see for example [1], chapter 1.

Definition 2.1. The **(upper) dilation of a map** $f : X \to Y$ between metric spaces, in a point $u \in Y$ is

$$Lip(f)(u) = \limsup_{\varepsilon \to 0} \sup \left\{ \frac{d_Y(f(v), f(w))}{d_X(v,w)} \ : \ v \neq w \ , \ v,w \in B(u,\varepsilon) \right\}.$$

In the particular case of a derivable function $f : \mathbb{R} \to \mathbb{R}^n$ the upper dilation is $Lip(f)(t) = \|\dot{f}(t)\|$. A function $f : (X,d) \to (Y,d')$ is Lipschitz if there is $C > 0$ such that for any $x, y \in X$ we have $d'(f(x), f(y)) \leq C\, d(x,y)$. The number $Lip(f)$ is the smallest such positive constant. For any $x \in X$ we have the obvious relation $Lip(f)(x) \ \leq \ Lip(f)$.

Definition 2.2. Let $c : [a,b] \to X$ be a curve a metric space (X,d).

(a) If c has L^1 upper dilation then the **length of the curve** c is $L(f) = \displaystyle\int_a^b Lip(c)(t)\, \mathrm{d}t$,

(b) The **variation of the curve** c is the quantity

$$Var(c) = \sup \left\{ \sum_{i=0}^{n} d(c(t_i), c(t_{i+1})) \ : \ a = t_0 < t_1 < \ldots < t_n < t_{n+1} = b \right\},$$

(c) The **length of the path** $A = c([a,b])$ is the one-dimensional Hausdorff measure of the path:

$$l(A) \ = \ \mathcal{H}^1(A) \ = \ (\lim_{\delta \to 0} \inf \left\{ \sum_{i \in I} diam\ E_i : diam\ E_i < \delta \ , \ A \subset \bigcup_{i \in I} E_i \right\}.$$

The definitions are not equivalent. For any Lipschitz curve $c : [a,b] \to X$, we have $L(c) \ = \ Var(c) \ \geq \ \mathcal{H}^1(c([a,b]))$. If c is moreover injective then $\mathcal{H}^1(c([a,b])) \ = \ Var(f)$. For further use we state the following reparametrisation theorem.

Theorem 2.3. *Any Lipschitz curve c admits a reparametrisation c' such that $Lip(c')(t) = 1$ for almost any $t \in [a,b]$.*

Definition 2.4. We shall denote by l_d the **length functional induced by the distance** d, defined only on the family of Lipschitz curves. If the metric space (X,d) is connected by Lipschitz curves, then the length induces a new distance d_l, given by:

$$d_l(x,y) \ = \ \inf \ \{l_d(c([a,b])) : c : [a,b] \to X \ \text{ Lipschitz} \ , \quad c(a) = x \ , \ c(b) = y\}.$$

A **length metric space** is a metric space (X,d) connected by Lipschitz curves, with $d = d_l$.

Definition 2.5. A curve $c : (a,b) \to (X,d)$ in a metric space is **absolutely continuous** if there exists $m \in L^1((a,b))$ (called an **upper gradient** of c) such that for any $a < s \leq t < b$ we have

$$d(c(s), c(t)) \leq \int_s^t m(r)\, \mathrm{d}r.$$

If c is a Lipschitz curve in a complete length metric space then $Lip(c)$ is an upper gradient, therefore Lipschitz curves in complete length metric spaces are absolutely continuous.

Definition 2.6. A curve $c : (a,b) \to X$ is **metrically derivable** in $t \in (a,b)$ if the limit

$$md(c)(t) = \lim_{s \to t} \frac{d(c(s),c(t))}{\mid s-t \mid}$$

exists and it is finite. In this case $md(c)(t)$ is called the **metric derivative** of c in t.

For the proof of the following theorem see [1], theorem 1.1.2, chapter 1.

Theorem 2.7. *Let $c : (a,b) \to (X,d)$ be an absolutely continuous curve in the complete metric space (X,d). Then c is metrically derivable for $\mathcal{L}^1$-a.e. $t \in (a,b)$, the metric derivative $md(c)$ belongs to $L^1((a,b))$ and for any upper gradient m of c we have $md(c)(t) \leq m(t)$ for $\mathcal{L}^1$-a.e. $t \in (a,b)$.*

3 Dilation structures

We shall use here a slightly particular version of dilation structures. For the general definition of a dilation structure see [5] (the general definition applies for dilation structures over ultrametric spaces as well).

Definition 3.1. Let (X,d) be a complete metric space such that for any $x \in X$ the closed ball $\bar{B}(x,3)$ is compact. A **dilation structure** (X,d,δ) over (X,d) is the assignment to any $x \in X$ and $\varepsilon \in (0,+\infty)$ of a homeomorphism, defined as: if $\varepsilon \in (0,1]$ then $\delta^x_\varepsilon : U(x) \to V_\varepsilon(x)$, else $\delta^x_\varepsilon : W_\varepsilon(x) \to U(x)$, with the following properties.

A0. For any $x \in X$ the sets $U(x), V_\varepsilon(x), W_\varepsilon(x)$ are open neighbourhoods of x. There are $1 < A < B$ such that for any $x \in X$ and any $\varepsilon \in (0,1)$ we have: $B_d(x,\varepsilon) \subset \delta^x_\varepsilon B_d(x,A) \subset V_\varepsilon(x) \subset W_{\varepsilon^{-1}}(x) \subset \delta^x_\varepsilon B_d(x,B)$. Moreover for any compact set $K \subset X$ there are $R = R(K) > 0$ and $\varepsilon_0 = \varepsilon(K) \in (0,1)$ such that for all $u,v \in \bar{B}_d(x,R)$ and all $\varepsilon \in (0,\varepsilon_0)$, we have $\delta^x_\varepsilon v \in W_{\varepsilon^{-1}}(\delta^x_\varepsilon u)$.

A1. For any $x \in X$ $\delta^x_\varepsilon x = x$ and $\delta^x_1 = id$. Consider the closure $Cl(dom\,\delta)$ of the set

$$dom\,\delta = \{(\varepsilon,x,y) \in (0,+\infty) \times X \times X : \text{if } \varepsilon \leq 1 \text{ then } y \in U(x)\text{ , else } y \in W_\varepsilon(x)\},$$

seen in $[0,+\infty) \times X \times X$ endowed with the product topology. . The function $\delta : dom\,\delta \to X$, $\delta(\varepsilon,x,y) = \delta^x_\varepsilon y$ is continuous, admits a continuous extension over $Cl(dom\,\delta)$ and we have $\lim_{\varepsilon \to 0} \delta^x_\varepsilon y = x$.

A2. For any $x, \in X$, $\varepsilon,\mu \in (0,+\infty)$ and $u \in U(x)$, whenever one of the sides are well defined we have the equality $\delta^x_\varepsilon \delta^x_\mu u = \delta^x_{\varepsilon\mu} u$.

A3. For any x there is a distance function $(u,v) \mapsto d^x(u,v)$, defined for any u,v in the closed ball (in distance d) $\bar{B}(x,A)$, such that uniformly with respect to x in compact set we have the limit:

$$\lim_{\varepsilon \to 0} \quad \sup\left\{ \mid \frac{1}{\varepsilon} d(\delta^x_\varepsilon u, \delta^x_\varepsilon v) \; - \; d^x(u,v) \mid \; : u,v \in \bar{B}_d(x,A) \right\} \; = \; 0.$$

The **dilation structure is strong** if it satisfies the following supplementary condition:

A4. Let us define $\Delta_{\varepsilon}^{x}(u,v) = \delta_{\varepsilon^{-1}}^{\delta_{\varepsilon}^{x}u}\delta_{\varepsilon}^{x}v$. Then we have the limit, uniformly with respect to x,u,v in compact set,

$$\lim_{\varepsilon\to 0}\Delta_{\varepsilon}^{x}(u,v) = \Delta^{x}(u,v).$$

Definition 3.2. Let (X,d,δ) be a strong dilation structure. A property $\mathcal{P}(x_1,x_2,x_3,...)$ is true for $x_1,x_2,x_3,... \in X$ **sufficiently close** if for any compact, non empty set $K \subset X$, there is a positive constant $C(K) > 0$ such that $\mathcal{P}(x_1,x_2,x_3,...)$ is true for any $x_1,x_2,x_3,... \in K$ with $d(x_i,x_j) \leq C(K)$.

4 Length dilation structures

Consider (X,d) a complete, locally compact metric space, and a triple (X,d,δ) which satisfies A0, A1, A2. Denote by $Lip([0,1],X,d)$ the space of d-Lipschitz curves $c:[0,1]\to X$. Let also l_d denote the length functional associated to the distance d.

Gamma-convergence of length functionals

Definition 4.1. For any $\varepsilon \in (0,1)$ we define the **length functional** $l_{\varepsilon}(x,c) = l_{\varepsilon}^{x}(c) = \frac{1}{\varepsilon}\, l_d(\delta_{\varepsilon}^{x}c)$, defined over the space of curves:

$$\mathcal{L}_{\varepsilon}(X,d,\delta) \;=\; \{(x,c) \in X \times C([0,1],X) : c:[0,1] \in U(x)\,,$$

$$\delta_{\varepsilon}^{x}\circ c \text{ is } d-Lip \text{ and } Lip(\delta_{\varepsilon}^{x}c) \;\leq\; 2\, l_d(\delta_{\varepsilon}^{x}c)\}\,.$$

For clarity we denoted by "$\delta_{\varepsilon}^{x}\circ c$" the composition of two functions, but in the following we shall use a simpler notation, like "$\delta_{\varepsilon}^{x}c$".

The last condition from the definition of $\mathcal{L}_{\varepsilon}(X,d,\delta)$ is a selection of parameterization of the path $c([0,1])$. Indeed, by the reparameterization theorem, if $\delta_{\varepsilon}^{x}c:[0,1]\to(X,d)$ is a d-Lipschitz curve of length $L = l_d(\delta_{\varepsilon}^{x}c)$ then $\delta_{\varepsilon}^{x}c([0,1])$ can be reparameterized by length, that is there exists a increasing function $\phi:[0,L]\to[0,1]$ such that $c' = \delta_{\varepsilon}^{x}c\circ\phi$ is a d-Lipschitz curve with $Lip(c') \leq 1$. But we can use a second affine reparameterization which sends $[0,L]$ back to $[0,1]$ and we get a Lipschitz curve $c"$ with $c"([0,1]) = c'([0,1])$ and $Lip(c") \leq 2 l_d(c)$.

We shall use the following definition of Gamma-convergence (see the book [12] for the notion of Gamma-convergence).

Definition 4.2. Let Z be a metric space with distance function D and $(l_{\varepsilon})_{\varepsilon>0}$ be a family of functionals $l_{\varepsilon}: Z_{\varepsilon}\subset Z\to[0,+\infty]$. Then l_{ε} **Gamma-converges** to the functional $l: Z_0 \subset Z \to [0,+\infty]$ if:

(a) (**liminf inequality**) for any function $\varepsilon\in(0,\infty)\mapsto x_{\varepsilon}\in Z_{\varepsilon}$ such that $\lim_{\varepsilon\to 0} x_{\varepsilon} = x_0 \in Z_0$ we have $l(x_0) \leq \liminf_{\varepsilon\to 0} l_{\varepsilon}(x_{\varepsilon})$.

(b) (**existence of a recovery sequence**) For any $x_0 \in Z_0$ and for any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \varepsilon_n = 0$ there is a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in Z_{\varepsilon_n}$ for any $n \in \mathbb{N}$, such that $l(x_0) = \lim_{n\to\infty} l_{\varepsilon_n}(x_n)$.

We shall take as the metric space Z the space $X \times C([0,1], X)$ with the distance

$$D((x,c),(x',c')) = \max\{d(x,x'), \sup\{d(c(t),c'(t)) : t \in [0,1]\}\}.$$

Let $\mathcal{L}(X,d,\delta)$be the class of all $(x,c) \in X \times C([0,1],X)$ which appear as limits $(x_n,c_n) \to (x,c)$, with $(x_n,c_n) \in \mathcal{L}_{\varepsilon_n}(X,d,\delta)$, the family $(c_n)_n$ is d-equicontinuous and $\varepsilon_n \to 0$ as $n \to \infty$.

Definition 4.3. A triple (X,d,δ) is a **length dilation structure** if (X,d) is a complete, locally compact metric space such that A0, A1, A2, are satisfied, together with the following axioms:

A3L. there is a functional $l : \mathcal{L}(X,d,\delta) \to [0,+\infty]$ such that for any $\varepsilon_n \to 0$ as $n \to \infty$ the sequence of functionals l_{ε_n} Gamma-converges to the functional l.

A4+ Let us define $\Delta^x_\varepsilon(u,v) = \delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \delta^x_\varepsilon v$ and $\Sigma^x_\varepsilon(u,v) = \delta^x_{\varepsilon^{-1}} \delta^{\delta^x_\varepsilon u}_\varepsilon v$. Then we have the limits

$$\lim_{\varepsilon\to 0} \Delta^x_\varepsilon(u,v) = \Delta^x(u,v), \quad \lim_{\varepsilon\to 0} \Sigma^x_\varepsilon(u,v) = \Sigma^x(u,v),$$

uniformly with respect to x,u,v in compact set.

Remark 4.4. For strong dilation structures the axioms A0 - A4 imply A4+, cf. corollary 9 [5]. The transformations $\Sigma^x_\varepsilon(u,\cdot)$ have the interpretation of approximate left translations in the tangent space of (X,d) at x.

For any $\varepsilon \in (0,1)$ and any $x \in X$ the length functional l^x_ε induces the distance $\mathring{d}^x_\varepsilon$ on $U(x)$:

$$\mathring{d}^x_\varepsilon(u,v) = \inf\{l^x_\varepsilon(c) : (x,c) \in \mathcal{L}_\varepsilon(X,d,\delta), c(0) = u, c(1) = v\}.$$

In the same way the length functional l from A3L induces a the distance $\mathring{d}^x$ on $U(x)$. We then have

$$\mathring{d}^x(u,v) \geq \limsup_{\varepsilon\to 0} \mathring{d}^x_\varepsilon(u,v). \tag{4.1}$$

Remark 4.5. Without supplementary hypotheses we cannot prove A3 from A3L, that is in principle length dilation structures are not strong dilation structures.

5 Properties of (length) dilation structures

For a dilation structure the metric tangent spaces have a group structure which is compatible with dilations.

We shall work further with local groups. Such objects are spaces endowed with a locally defined operation, satisfying the conditions of a uniform group. See section 3.3 [5] for details about the definition of local groups.

5.1 Normed conical groups

These have been introduced in section 8.2 [5] and studied further in section 4 [6]. In the following general definition appear a topological commutative group Γ endowed with a continuous morphism $\nu : \Gamma \to (0, +\infty)$ from Γ to the group $(0, +\infty)$ with multiplication. The morphism ν induces an invariant topological filter on Γ (other names for such an invariant filter are "absolute" or "end"). The convergence of a variable $\varepsilon \in \Gamma$ to this filter is denoted by $\varepsilon \to 0$ and it means simply $\nu(\varepsilon) \to 0$ in $\mathbb{R}$.

Particular, interesting examples of pairs (Γ, ν) are: $(0, +\infty)$ with identity, which is the case interesting for this paper, $\mathbb{C}^*$ with the modulus of complex numbers, or $\mathbb{N}$ (with addition) with the exponential, which is relevant for the case of normed contractible groups, section 4.3 [6].

Definition 5.1. A **normed group with dilations** $(G, \delta, \|\cdot\|)$ is a local group G with a local action of Γ (denoted by δ), on G such that

H0. the limit $\lim_{\varepsilon \to 0} \delta_\varepsilon x = e$ exists and is uniform with respect to x in a compact neighbourhood of the identity e,

H1. the limit $\beta(x, y) = \lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} ((\delta_\varepsilon x)(\delta_\varepsilon y))$ is well defined in a compact neighbourhood of e and the limit is uniform with respect to x, y,

H2. the following relation holds: $\lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \left((\delta_\varepsilon x)^{-1} \right) = x^{-1}$, where the limit from the left hand side exists in a neighbourhood $U \subset G$ of e and is uniform with respect to $x \in U$.

Moreover the group is endowed with a continuous norm function $\|\cdot\| : G \to \mathbb{R}$ which satisfies (locally, in a neighbourhood of the neutral element e) the properties:

(a) for any x we have $\|x\| \geq 0$; if $\|x\| = 0$ then $x = e$,

(b) for any x, y we have $\|xy\| \leq \|x\| + \|y\|$,

(c) for any x we have $\|x^{-1}\| = \|x\|$,

(d) the limit $\lim_{\varepsilon \to 0} \frac{1}{\nu(\varepsilon)} \|\delta_\varepsilon x\| = \|x\|^N$ exists, is uniform with respect to x in compact set,

(e) if $\|x\|^N = 0$ then $x = e$.

Theorem 5.2. *(Thm. 15 [5]) Let $(G, \delta, \|\cdot\|)$ be a locally compact normed local group with dilations. Then (G, d, δ) is a strong dilation structure, where the dilations δ and the distance d are defined by: $\delta^x_\varepsilon u = x \delta_\varepsilon (x^{-1} u)$, $\quad d(x, y) = \|x^{-1} y\|$.*

Definition 5.3. A **normed conical group** N is a normed group with dilations such that for any $\varepsilon \in \Gamma$ the dilation δ_ε is a group morphism and such that for any $\varepsilon > 0$ $\|\delta_\varepsilon x\| = \nu(\varepsilon) \|x\|$.

A normed conical group is the infinitesimal version of a normed group with dilations ([5] proposition 2).

Proposition 5.4. *Let $(G, \delta, \|\cdot\|)$ be a locally compact normed local group with dilations. Then $(G, \beta, \delta, \|\cdot\|^N)$ is a locally compact, local normed conical group, with operation β, dilations δ and homogeneous norm $\|\cdot\|^N$.*

5.2 Tangent bundle of a dilation structure

The following two theorems describe the most important metric and algebraic properties of a dilation structure. As presented here these are condensed statements, available in full length as theorems 7, 8, 10 in [5]. The first theorem does not need a proof (see theorem 7 [5]).

Theorem 5.5. *Let (X,d,δ) be a strong dilation structure. Then the metric space (X,d) admits a metric tangent space at x, for any point $x \in X$. More precisely we have the following limit:*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sup \{ \mid d(u,v) - d^x(u,v) \mid \ : \ d(x,u) \leq \varepsilon \ , \ d(x,v) \leq \varepsilon \} \ = \ 0 .$$

Length dilation structures were introduced in this paper. Straightforward modifications in the proof of the before mentioned theorems allow us to extend some results to length dilation structures.

Theorem 5.6. *(a) If (X,d,δ) is a strong dilation structure then for any $x \in X$ the triple (U^x, δ^x, d^x) is a locally compact normed conical group, with operation $\Sigma^x(\cdot,\cdot)$, neutral element x and inverse $inv^x(y) = \Delta^x(y,x)$.*

(b) If (X,d,δ) is a length dilation structure then for any $x \in X$ the triple $(U^x, \Sigma^x, \delta^x)$ is a conical local group, that is $(U^x, \Sigma^x, x, inv^x)$ is a group with operation $\Sigma^x(\cdot,\cdot)$, neutral element x and inverse $inv^x(y) = \Delta^x(y,x)$Moreover, the length functional $l^x = l(x,\cdot)$ is invariant with respect to left translations $\Sigma^x(y,\cdot)$, $y \in U(x)$, and for any $\mu \in (0,1]$ we have the equality

$$l(x, \delta^x_\mu c) \ = \ \mu \, l(x,c)$$

for any curve $c \in \mathcal{L}(X,d,\delta)$ with image in $U(x)$.

Proof. We shall only prove the statements concerning length dilation structures. For proving that the triple $(U^x, \Sigma^x, \delta^x)$ is a conical local group we need only A4+ and algebraic relations from theorem 11 [5] which are true only from A0, A1, A2. Indeed, we notice that the proof of theorem 10 [5] has two parts: in the first part is proved A4+ from the axioms od strong dilation structures, then we pass to the limit in the algebraic relations from theorem 11 [5]. For length dilation structures A4+ is true, so the second part of the mentioned proof can be repeated verbatim in order to obtain a proof of the first part of the statement (b).

For proving that $l^x = l(x,\cdot)$ is invariant with respect to left translations $\Sigma^x(y,\cdot)$, $y \in U(x)$, consider a curve c such that $(\delta^x_\varepsilon y, c) \in \mathcal{L}_\varepsilon(X,d,\delta)$ for any ε sufficiently small. Then $(x, \Sigma^x_\varepsilon(y,\cdot)c) \in \mathcal{L}_\varepsilon(X,d,\delta)$ and moreover $l_\varepsilon(\delta^x_\varepsilon y, c) = l_\varepsilon(x, \Sigma^x_\varepsilon(y,\cdot)c)$. Indeed, this is true because of the equality: $\delta^{\delta^x_\varepsilon y} c = \delta^x_\varepsilon \Sigma^x_\varepsilon(y,\cdot)c$. By passing to the limit with $\varepsilon \to 0$ and using A3L and A4+ we get $l(x,c) = l(x, \Sigma^x(y,\cdot)c)$.

For the last part of the statement (b) remark that for any $\varepsilon, \mu > 0$ (and sufficiently small) $(x,c) \in \mathcal{L}_{\varepsilon\mu}(X,d,\delta)$ is equivalent with $(x, \delta^x_\mu c) \in \mathcal{L}_\varepsilon(X,d,\delta)$ and moreover: $l_\varepsilon(x, \delta^x_\mu c) = \frac{1}{\varepsilon} l_d(\delta^x_{\varepsilon\mu} c) = \mu \, l_{\varepsilon\mu}(x,c)$. We pass to the limit with $\varepsilon \to 0$ and we get the desired equality. □

Definition 5.7. The conical group $(U(x), \Sigma^x, \delta^x)$ can be seen as the tangent space of (X,d,δ) at x. We shall denote it by $T_x(X,d,\delta) = (U(x), \Sigma^x, \delta^x)$, or by $T_x X$ if (d,δ) are clear from the context.

The following proposition is corollary 6.3 from [6], which gives a more precise description of the conical group $(U(x),\Sigma^x,\delta^x)$.

Proposition 5.8. *Let (X,d,δ) be a strong dilation structure. Then for any $x \in X$ the local group $(U(x),\Sigma^x)$ is locally a simply connected Lie group whose Lie algebra admits a positive graduation (a homogeneous group), given by the eigenspaces of δ^x_ε for an arbitrary $\varepsilon \in (0,1)$.*

Remark 5.9. S. Vodopyanov (private communication) made the observation that in the proof of corollary 6.3 [6] it is used Siebert' proposition 5.4 [21], which is true for conical groups (in our language), while I am using it for *local* conical groups. This is true and constitutes a gap in the proof of the corollary 6.3. Fortunately the recent paper [13] provides the needed result for local groups. Indeed, theorem 1.1 [13] states that a locally compact, locally connected, contractible (with Siebert' wording) group is locally isomorphic to a contractive Lie group.

5.3 Differentiability with respect to dilation structures

For any strong dilation structure or length dilation structure there is an associated notion of differentiability (section 7.2 [5]). First we need the definition of a morphism of conical groups.

Definition 5.10. Let (N,δ) and $(M,\bar{\delta})$ be two conical groups. A function $f : N \to M$ is a conical group morphism if f is a group morphism and for any $\varepsilon > 0$ and $u \in N$ we have $f(\delta_\varepsilon u) = \bar{\delta}_\varepsilon f(u)$.

The definition of the derivative, or differential, with respect to dilations structures follows. In the case of a pair of Carnot groups this is just the definition of the Pansu derivative introduced in [20].

Definition 5.11. Let (X,d,δ) and $(Y,\overline{d},\bar{\delta})$ be two strong dilation structures or length and $f : X \to Y$ be a continuous function. The function f is differentiable in x if there exists a conical group morphism $Df(x) : T_xX \to T_{f(x)}Y$, defined on a neighbourhood of x with values in a neighbourhood of $f(x)$ such that

$$\lim_{\varepsilon\to 0} \sup \left\{ \frac{1}{\varepsilon} \overline{d}\left(f(\delta^x_\varepsilon u), \bar{\delta}^{f(x)}_\varepsilon Df(x)(u)\right) : d(x,u) \leq \varepsilon \right\} = 0. \tag{5.1}$$

The morphism $Df(x)$ is called the derivative, or differential, of f at x.

The definition also makes sense if the function f is defined on a open subset of (X,d).

6 Dilation structures on sub-riemannian manifolds

In [7] we proved that we can associate dilation structures to regular sub-Riemannian manifolds. This result, explained further, is the source of inspiration of the notion of a coherent projection (section 9).

Let M be a connected n dimensional real manifold. A distribution is a smooth subbundle D of M. To any point $x \in M$ there is associated the vector space $D_x \subset T_x M$. The dimension of the distribution D at point $x \in M$ is $m(x) = dim\, D_x$, locally constant. We suppose further that the dimension of the distribution is globally constant and we denote it by m. Clearly $m \leq n$; we are interested in the case $m < n$.

A horizontal curve $c : [a,b] \to M$ is a curve which is almost everywhere derivable and for almost any $t \in [a,b]$ we have $\dot{c}(t) \in D_{c(t)}$. The class of horizontal curves will be denoted by $Hor(M,D)$.

Further we shall use the following notion of non-integrability of the distribution D. The Chow condition (C) [11] gives a sufficient condition for the distribution D to be completely non-integrable.

Definition 6.1. The distribution D is **completely non-integrable** if M is locally connected by horizontal curves curves $c \in Hor(M,D)$.

Theorem 6.2. *(Chow) Let D be a distribution of dimension m in the manifold M. Suppose there is a positive integer number k (called the rank of the distribution D) such that for any $x \in X$ there is a topological open ball $U(x) \subset M$ with $x \in U(x)$ such that there are smooth vector fields $X_1, ..., X_m$ in $U(x)$ with the property:*

(C) the vector fields $X_1, ..., X_m$ span D_y and these vector fields together with their iterated brackets of order at most k span the tangent space $T_y M$ at every point $y \in U(x)$.

Then the distribution D is completely non-integrable in the sense of definition 6.1.

Definition 6.3. A **sub-riemannian manifold** or SR manifold is a triple (M,D,g), where M is a connected manifold, D is a completely non-integrable distribution on M, and g is a metric (Euclidean inner-product) on the distribution (or horizontal bundle) D.

Given a distribution D which satisfies the hypothesis of Chow theorem 6.2, let us consider a point $x \in M$, a neighbourhood $U(x)$ of x and the vector fields $X_1, ..., X_m$ satisfying the condition (C). One can define on $U(x)$ a filtration of bundles as follows. Define first the class of horizontal vector fields on $U(x)$:

$$\mathcal{X}^1(U(x),D) \ = \ \left\{ X \in \mathcal{X}^\infty(U) : \forall y \in U(x) \ , \ X(y) \in D_y \right\} .$$

Next, define inductively for all positive integers j:

$$\mathcal{X}^{j+1}(U(x),D) \ = \ \mathcal{X}^j(U(x),D) + [\mathcal{X}^1(U(x),D), \mathcal{X}^j(U(x),D)] .$$

Here $[\cdot,\cdot]$ denotes the bracket of vector fields. We obtain therefore a filtration $\mathcal{X}^j(U(x),D) \subset \mathcal{X}^{j+1}(U(x),D)$. We evaluate now this filtration at $y \in U(x)$:

$$V^j(y,U(x),D) \ = \ \left\{ X(y) : X \in \mathcal{X}^j(U(x),D) \right\} .$$

According to the hypothesis of Chow theorem, there is a positive integer k such that for all $y \in U(x)$ we have

$$D_y = V^1(y,U(x),D) \subset V^2(y,U(x),D) \subset ... \subset V^k(y,U(x),D) = T_y M .$$

Consequently, to the sub-riemannian manifold is associated the string of numbers:

$$\nu_1(y) = \dim V^1(y, U(x), D) < \nu_2(y) = \dim V^2(y, U(x), D) < ... < n = \dim M.$$

Generally k, $\nu_j(y)$ may vary from a point to another. The number k is called the step of the distribution at y. The distribution D is regular if $\nu_j(y)$ are constant on the manifold M. The sub-riemannian manifold (M, D, g) is regular if D is regular and for any $x \in M$ there is a topological ball $U(x) \subset M$ with $x \in U(M)$ and an orthonormal (with respect to the metric g) family of smooth vector fields $\{X_1, ..., X_m\}$ in $U(x)$ which satisfy the condition (C).

The lenght of a horizontal curve is

$$l(c) \;=\; \int_a^b \left(g_{c(t)}(\dot{c}(t), \dot{c}(t))\right)^{\frac{1}{2}} \,\mathrm{d}t.$$

Definition 6.4. The Carnot-Carathéodory distance (or CC distance) associated to the sub-riemannian manifold is the distance induced by the length l of horizontal curves:

$$d(x,y) \;=\; \inf\{l(c) : c \in Hor(M, D)\ ,\ c(a) = x\ ,\ c(b) = y\}.$$

The Chow theorem ensures the existence of a horizontal path linking any two sufficiently close points, therefore the CC distance is locally finite. The distance depends only on the distribution D and metric g, and not on the choice of vector fields $X_1, ..., X_m$ satisfying the condition (C). The space (M, d) is locally compact and complete, and the topology induced by the distance d is the same as the topology of the manifold M. (These important details may be recovered from reading carefully the constructive proofs of Chow theorem given by Bellaïche [3] or Gromov [15].)

6.1 Normal frames

Chow condition (C) is used to construct an adapted frame starting from a family of vector fields which generate the distribution D. A fundamental result in sub-riemannian geometry is the existence of normal frames. This existence result is based on the accumulation of various results by Bellaïche [3], first to speak about normal frames, providing rigorous proofs for this existence in a flow of results between theorem 4.15 and ending in the first half of section 7.3 (page 62), Gromov [15] in his approximation theorem p. 135 (conclusion of the point (a) below), as well in his convergence results concerning the nilpotentization of vector fields (related to point (b) below), Vodopyanov and others [23] [24] [25] concerning the proof of basic results in sub-riemannian geometry under very weak regularity assumptions (for a discussion of this see [7]).

In the following we shall work further only with regular sub-riemannian manifolds. For such a manifold M we stay in some small open neighbourhood of an arbitrary, but fixed point x_0 of the manifold M. We shall no longer mention the dependence of various objects on x_0, on the neighbourhood $U(x_0)$, or the distribution D.

Adapted frames. An adapted frame $\{X_1, ..., X_n\}$ is a collection of smooth vector fields which is obtained by the following construction. We start with a collection $X_1, ..., X_m$ of vector fields which satisfy the condition (C). In particular for any point x the vectors

$X_1(x),...,X_m(x)$ form a basis for D_x. We further associate to any word $a_1....a_q$ with letters in the alphabet $1,...,m$ the multi-bracket $[X_{a_1},[...,X_{a_q}]...]$. One can add, in the lexicographic order, $n-m$ elements to the set $\{X_1,...,X_m\}$ until we get a collection $\{X_1,...,X_n\}$ such that: for any $j = 1,...,k$ and for any point x the set $\left\{X_1(x),...,X_{\nu_j}(x)\right\}$ is a basis for $V^j(x)$.

Let $\{X_1,...,X_n\}$ be an adapted frame. For any $j = 1,...,n$ the degree $deg\, X_j$ of the vector field X_j is defined as the only positive integer p such that for any point x we have

$$X_j(x) \in V_x^p \setminus V^{p-1}(x).$$

According with Gromov suggestions in the last section of Bellaïche [3], the key details in the definition below are the uniform convergence assumptions.

Definition 6.5. An adapted frame $\{X_1,...,X_n\}$ is a **normal frame** if the following two conditions are satisfied:

(a) we have the limit $\lim_{\varepsilon\to 0_+} \frac{1}{\varepsilon} d\left(\exp\left(\sum_1^n \varepsilon^{deg\,X_i} a_i X_i\right)(y), y\right) = A(y,a) \in (0,+\infty)$, which is uniform with respect to y in compact sets and vector $a = (a_1,...,a_n) \in W$, with $W \subset \mathbb{R}^n$ compact neighbourhood of $0 \in \mathbb{R}^n$,

(b) for any compact set $K \subset M$ with diameter (with respect to the distance d) sufficiently small, and for any $i = 1,...,n$ there are functions $P_i(\cdot,\cdot,\cdot) : U_K \times U_K \times K \to \mathbb{R}$, with $U_K \subset \mathbb{R}^n$ a sufficiently small compact neighbourhood of $0 \in \mathbb{R}^n$ such that for any $x \in K$ and any $a,b \in U_K$ we have

$$\exp\left(\sum_1^n a_i X_i\right)(x) = \exp\left(\sum_1^n P_i(a,b,y) X_i\right) \circ \exp\left(\sum_1^n b_i X_i\right)(x)$$

and such that the following limit uniform with respect to $x \in K$ and $a,b \in U_K$:

$$\lim_{\varepsilon\to 0_+} \varepsilon^{-deg\,X_i} P_i(\varepsilon^{deg\,X_j} a_j, \varepsilon^{deg\,X_k} b_k, x) \in \mathbb{R}.$$

In order to understand normal frames let us look to the case of a Lie group G endowed with a left invariant distribution. The distribution is completely non-integrable if it is generated by the left translation of a vector subspace D of the algebra $\mathfrak{g} = T_e G$ which bracket generates the whole algebra $\mathfrak{g}$. Take $\{X_1,...,X_m\}$ a collection of $m = dim\, D$ left invariant independent vector fields and define with their help an adapted frame. Then the adapted frame $\{X_1,...,X_n\}$ is in fact normal.

With the help of a normal frame we can prove the existence of strong dilation structures on regular sub-riemannian manifolds (theorems 6.3, 6.4 [7]).

Theorem 6.6. *Let (M,D,g) be a regular sub-riemannian manifold, $U \subset M$ an open set which admits a normal frame. Define for any $x \in U$ and $\varepsilon > 0$ (sufficiently small if necessary), the dilation δ_ε^x given by:*

$$\delta_\varepsilon^x\left(\exp\left(\sum_{i=1}^n a_i X_i\right)(x)\right) = \exp\left(\sum_{i=1}^n a_i \varepsilon^{deg X_i} X_i\right)(x).$$

Then (U,d,δ) is a strong dilation structure.

Proof. Indeed, it is enough to check the axioms A3, A4 of a strong dilation structure, because the other axioms are obviously true. By theorem 6.3 [7] A3 is true and by theorem 6.4 [7] A4 is true. □

6.2 Carnot groups

Carnot groups appear in sub-riemannian geometry as models of tangent spaces, [3], [15], [20]. In particular such groups can be endowed with a structure of sub-riemannian manifold.

Carnot groups are particular cases of normed conical groups.

Definition 6.7. A **Carnot (or stratified homogeneous) group** (N, V_1) is a pair consisting of a real connected simply connected group N with a distinguished subspace V_1 of the Lie algebra $Lie(N)$, such that the following direct sum decomposition occurs:

$$n = \sum_{i=1}^{m} V_i \, , \quad V_{i+1} = [V_1, V_i].$$

The number m is the **step of the group**. The number $Q = \sum_{i=1}^{m} i \, dim V_i$ is called the **homogeneous dimension of the group**.

Because the group is nilpotent and simply connected, the exponential mapping is a diffeomorphism. We shall identify the group with the algebra, if is not locally otherwise stated.

The structure that we obtain is a set N endowed with a Lie bracket and a group multiplication operation, related by the Baker-Campbell-Hausdorff formula. Remark that the group operation is polynomial.

Any Carnot group admits a one-parameter family of dilations. For any $\varepsilon > 0$, the associated dilation is:

$$x = \sum_{i=1}^{m} x_i \mapsto \delta_\varepsilon x = \sum_{i=1}^{m} \varepsilon^i x_i.$$

Any such dilation is a group morphism and a Lie algebra morphism.

In a Carnot group N let us choose an euclidean norm $\|\cdot\|$ on V_1. We shall endow the group N with a structure of a sub-riemannian manifold. For this take the distribution obtained from left translates of the space V_1. The metric on that distribution is obtained by left translation of the inner product restricted to V_1.

Because V_1 generates (the algebra) N then any element $x \in N$ can be written as a product of elements from V_1, in a controlled way, described in the following useful lemma (slight reformulation of Lemma 1.40, Folland, Stein [14]).

Lemma 6.8. *Let N be a Carnot group and $X_1, ..., X_p$ an orthonormal basis for V_1. Then there is a a natural number M and a function $g : \{1, ..., M\} \to \{1, ..., p\}$ such that any element $x \in N$ can be written as:*

$$x = \prod_{i=1}^{M} \exp(t_i X_{g(i)}). \tag{6.1}$$

Moreover, if x is sufficiently close (in Euclidean norm) to 0 *then each t_i can be chosen such that $| t_i | \leq C \|x\|^{1/m}$.*

As a consequence we get:

Corollary 6.9. *The Carnot-Carathéodory distance*

$$d(x,y) = \inf\left\{ \int_0^1 \|c^{-1}\dot{c}\| \, dt \; : \; c(0) = x, \; c(1) = y, \right.$$

$$\left. c^{-1}(t)\dot{c}(t) \in V_1 \text{ for a.e. } t \in [0,1] \right\}$$

is finite for any two $x, y \in N$. The distance is obviously left invariant, thus it induces a norm on N.

The Carnot-Carathéodory distance induces a homogeneous norm on the Carnot group N by the formula: $\|x\| = d(0,x)$. From the invariance of the distance with respect to left translations we get: for any $x, y \in N$

$$\|x^{-1}y\| = d(x,y).$$

For any $x \in N$ and $\varepsilon > 0$ we define the dilation $\delta^x_\varepsilon y = x\delta_\varepsilon(x^{-1}y)$. Then (N,d,δ) is a dilation structure, according to theorem 5.2.

Such dilation structures have the Radon-Nikodym property (defined further), as proven several times, in [17], [20], or [23].

7 The Radon-Nikodym property

Let (X,d,δ) be a strong dilation structure or a length dilation structure. We have then a notion of differentiability for curves in X, seen as continuous functions from (a open interval in) $\mathbb{R}$, with the usual dilation structure, to X with the dilation structure (X,d,δ). Further we want to see what differentiability in the sense of definition 5.11 means for curves. In proposition 7.2 we shall arrive to a kind of intrinsic notion of a distribution in a dilation structure, with the geometrical meaning of a cone of all possible derivatives of curves passing through a point.

Definition 7.1. In a normed conical group N we shall denote by $D(N)$ the set of all $u \in N$ with the property that $\varepsilon \in ((0,\infty),+) \mapsto \delta_\varepsilon u \in N$ is a morphism of groups.

$D(N)$ is always non empty, because it contains the neutral element of N. $D(N)$ is also a cone, with dilations δ_ε, and a closed set.

Proposition 7.2. *Let (X,d,δ) be a strong dilation structure or a length dilation structure and let $c : [a,b] \to (X,d)$ be a continuous curve. For any $x \in X$ and any $y \in T_x(X,d,\delta)$ we denote by*

$$inv^x(y) = \Delta^x(y,x)$$

the inverse of y with respect to the group operation in $T_x(X,d,\delta)$. Then the following are equivalent:

(a) c is derivable in $t \in (a,b)$ with respect to the dilation structure (X,d,δ);

(b) there exists $\dot{c}(t) \in D(T_{c(t)}(X,d,\delta))$ such that

$$\frac{1}{\varepsilon} d(c(t+\varepsilon), \delta_\varepsilon^{c(t)} \dot{c}(t)) \to 0,$$

$$\frac{1}{\varepsilon} d(c(t-\varepsilon), \delta_\varepsilon^{c(t)} inv^{c(t)}(\dot{c}(t))) \to 0.$$

Proof. It is straightforward that a conical group morphism $f : \mathbb{R} \to N$ is defined by its value $f(1) \in N$. Indeed, for any $a > 0$ we have $f(a) = \delta_a f(1)$ and for any $a < 0$ we have $f(a) = \delta_a f(1)^{-1}$. From the morphism property we also deduce that

$$\delta v = \left\{ \delta_a v : a > 0, v = f(1) \text{ or } v = f(1)^{-1} \right\}$$

is a one parameter group and that for all $\alpha, \beta > 0$ we have $\delta_{\alpha+\beta} u = \delta_\alpha u \, \delta_\beta u$. We have therefore a bijection between conical group morphisms $f : \mathbb{R} \to (N, \delta)$ and elements of $D(N)$.

The curve $c : [a,b] \to (X,d)$ is derivable in $t \in (a,b)$ if and only if there is a morphism of normed conical groups $f : \mathbb{R} \to T_{c(t)}(X,d,\delta)$ such that for any $a \in \mathbb{R}$ we have

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d(c(t+\varepsilon a), \delta_\varepsilon^{c(t)} f(a)) \ = \ 0.$$

Take $\dot{c}(t) = f(1)$. Then $\dot{c}(t) \in D(T_{c(t}(X,d,\delta))$. For any $a > 0$ we have $f(a) = \delta_a^{c(t)} \dot{c}(t)$; otherwise if $a < 0$ we have $f(a) = \delta_a^{c(t)} inv^{c(t)} \dot{c}(t)$. This implies the equivalence stated on the proposition. □

Definition 7.3. A strong dilation structure or a length dilation structure (X,d,δ) has the **Radon-Nikodym property (or rectifiability property, or RNP)** if any Lipschitz curve $c : [a,b] \to (X,d)$ is derivable almost everywhere.

7.1 Two examples

The following two easy examples will show that not any strong dilation structure has the Radon-Nikodym property.

For $(X,d) = (\mathbb{V},d)$, a real, finite dimensional, normed vector space, with distance d induced by the norm, the (usual) dilations δ_ε^x are given by:

$$\delta_\varepsilon^x y \ = \ x + \varepsilon(y - x).$$

Dilations are defined everywhere.

There are few things to check: axioms 0,1,2 are obviously true. For axiom A3, remark that for any $\varepsilon > 0$, $x,u,v \in X$ we have:

$$\frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) \ = \ d(u,v),$$

therefore for any $x \in X$ we have $d^x = d$.

Finally, let us check the axiom A4. For any $\varepsilon > 0$ and $x,u,v \in X$ we have

$$\delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v \ = \ x + \varepsilon(u-x) + \frac{1}{\varepsilon}(x + \varepsilon(v-x) - x - \varepsilon(u-x)) \ =$$

$$= x + \varepsilon(u - x) + v - u,$$

therefore this quantity converges to

$$x + v - u = x + (v - x) - (u - x).$$

as $\varepsilon \to 0$. The axiom A4 is verified.

This dilation structure has the Radon-Nikodym property.

Further is an example of a dilation structure which does not have the Radon-Nikodym property. Take $X = \mathbb{R}^2$ with the euclidean distance d. For any $z \in \mathbb{C}$ of the form $z = 1 + i\theta$ we define dilations

$$\delta_\varepsilon x = \varepsilon^z x.$$

It is easy to check that $(\mathbb{R}^2, d, \delta)$ is a dilation structure, with dilations

$$\delta_\varepsilon^x y = x + \delta_\varepsilon(y - x).$$

Two such dilation structures (constructed with the help of complex numbers $1 + i\theta$ and $1 + i\theta'$) are equivalent if and only if $\theta = \theta'$ (mod 2π).

There are two other interesting properties of these dilation structures. The first is that if $\theta \neq 0$ then there are no non trivial Lipschitz curves in X which are differentiable almost everywhere. It means that such dilation structure does not have the Radon-Nikodym property.

The second property is that any holomorphic and Lipschitz function from X to X (holomorphic in the usual sense on $X = \mathbb{R}^2 = \mathbb{C}$) is differentiable almost everywhere, but there are Lipschitz functions from X to X which are not differentiable almost everywhere (enough to take a C^∞ function from $\mathbb{R}^2$ to $\mathbb{R}^2$ which is not holomorphic).

7.2 Length formula from Radon-Nikodym property

Theorem 7.4. *Let (X, d, δ) be a strong dilation structure with the Radon-Nikodym property, over a complete length metric space (X, d). Then for any $x, y \in X$ we have*

$$d(x,y) = \inf\left\{ \int_a^b d^{c(t)}(c(t), \dot{c}(t))\, dt : c : [a,b] \to X \textit{ Lipschitz }, \right.$$
$$\left. c(a) = x, c(b) = y \right\}.$$

Proof. From theorem 2.7 we deduce that for almost every $t \in (a,b)$ the upper dilation of c in t can be expressed as:

$$Lip(c)(t) = \lim_{s \to t} \frac{d(c(s), c(t))}{\mid s - t \mid}.$$

If the dilation structure has the Radon-Nikodym property then for almost every $t \in [a,b]$ there is $\dot{c}(t) \in D(T_{c(t)}X)$ such that

$$\frac{1}{\varepsilon} d(c(t+\varepsilon), \delta_\varepsilon^{c(t)} \dot{c}(t)) \to 0.$$

Therefore for almost every $t \in [a,b]$ we have

$$Lip(c)(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d(c(t+\varepsilon), c(t)) = d^{c(t)}(c(t), \dot{c}(t)).$$

The formula for length follows from here. □

A straightforward consequence is that the distance d is uniquely determined by the "distribution" $x \in X \mapsto D(T_x(X,d,\delta))$ and the function which associates to any $x \in X$ the "norm" $\| \cdot \|_x : D(T_x(X,d,\delta)) \to [0,+\infty)$.

Corollary 7.5. *Let (X,d,δ) and $(X,\bar{d},\bar{\delta})$ be two strong dilation structures with the Radon-Nikodym property , which are also complete length metric spaces, such that for any $x \in X$ we have $D(T_x(X,d,\delta)) = D(T_x(X,\bar{d},\bar{\delta}))$ and $d^x(x,u) = \bar{d}^x(x,u)$ for any $u \in D(T_x(X,d,\delta))$. Then $d = \bar{d}$.*

7.3 Equivalent dilation structures and their distributions

Definition 7.6. Two strong dilation structures (X,δ,d) and $(X,\overline{\delta},\overline{d})$ **are equivalent** if

(a) the identity map $id : (X,d) \to (X,\overline{d})$ is bilipschitz and

(b) for any $x \in X$ there are functions P^x, Q^x (defined for $u \in X$ sufficiently close to x) such that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \overline{d}\left(\delta^x_\varepsilon u, \overline{\delta}^x_\varepsilon Q^x(u)\right) = 0, \tag{7.1}$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d\left(\overline{\delta}^x_\varepsilon u, \delta^x_\varepsilon P^x(u)\right) = 0, \tag{7.2}$$

uniformly with respect to x, u in compact sets.

Proposition 7.7. *(X,δ,d) and $(X,\overline{\delta},\overline{d})$ are equivalent if and only if*

(a) the identity map $id : (X,d) \to (X,\overline{d})$ is bilipschitz,

(b) for any $x \in X$ there are conical group morphisms:

$$P^x : T_x(X,\overline{\delta},\overline{d}) \to T_x(X,\delta,d) \text{ and } Q^x : T_x(X,\delta,d) \to T_x(X,\overline{\delta},\overline{d})$$

such that the following limits exist

$$\lim_{\varepsilon \to 0} \left(\overline{\delta}^x_\varepsilon\right)^{-1} \delta^x_\varepsilon(u) = Q^x(u), \tag{7.3}$$

$$\lim_{\varepsilon \to 0} (\delta^x_\varepsilon)^{-1} \overline{\delta}^x_\varepsilon(u) = P^x(u), \tag{7.4}$$

and are uniform with respect to x, u in compact sets.

The next theorem shows a link between the tangent bundles of equivalent dilation structures.

Theorem 7.8. *Let (X,d,δ) and $(X,\overline{d},\overline{\delta})$ be equivalent strong dilation structures. Then for any $x \in X$ and any $u,v \in X$ sufficiently close to x we have:*

$$\overline{\Sigma}^{x}(u,v) = Q^{x}\left(\Sigma^{x}\left(P^{x}(u),P^{x}(v)\right)\right). \tag{7.5}$$

The two tangent bundles are therefore isomorphic in a natural sense.

As a consequence, the following corollary is straightforward.

Corollary 7.9. *Let (X,d,δ) and $(X,\overline{d},\overline{\delta})$ be equivalent strong dilation structures. Then for any $x \in X$ we have*

$$Q^{x}(D(T_x(X,\delta,d))) \ = \ D(T_x(X,\overline{\delta},\overline{d})).$$

If (X,d,δ) has the Radon-Nikodym property , then $(X,\overline{d},\overline{\delta})$ has the same property.

Suppose that (X,d,δ) and $(X,\overline{d},\overline{\delta})$ are complete length spaces with the Radon-Nikodym property . If the functions P^x, Q^x from definition 7.6 (b) are isometries, then $d = \overline{d}$.

8 Tempered dilation structures

The notion of a tempered dilation structure is inspired by the results from Venturini [22] and Buttazzo, De Pascale and Fragala [9].

The examples of length dilation structures from this section are provided by the extension of some results from [9] (propositions 2.3, 2.6 and a part of theorem 3.1) to dilation structures.

The following definition gives a class of distances $\mathcal{D}(\Omega,\bar{d},\bar{\delta})$, associated to a strong dilation structure $(\Omega,\bar{d},\bar{\delta})$, which generalizes the class of distances $\mathcal{D}(\Omega)$ from [9], definition 2.1.

Definition 8.1. For any strong dilation structure $(\Omega,\bar{d},\bar{\delta})$ we define the class $\mathcal{D}(\Omega,\bar{d},\bar{\delta})$ of all distance functions d on Ω such that

(a) d is a length distance,

(b) for any $\varepsilon > 0$ and any x,u,v sufficiently close the are constants $0 < c < C$ such that:

$$c\,\bar{d}^{x}(u,v) \ \leq \ \frac{1}{\varepsilon}\, d(\bar{\delta}^{x}_{\varepsilon}u,\bar{\delta}^{x}_{\varepsilon}v) \ \leq \ C\,\bar{d}^{x}(u,v). \tag{8.1}$$

The dilation structure $(\Omega,\bar{d},\bar{\delta})$ is **tempered** if $\bar{d} \in \mathcal{D}(\Omega,\bar{d},\bar{\delta})$.

On $\mathcal{D}(\Omega,\bar{d},\bar{\delta})$ we put the topology of uniform convergence (induced by distance $\bar{d}$) on compact subsets of $\Omega\times\Omega$.

To any distance $d \in \mathcal{D}(\Omega,\bar{d},\bar{\delta})$ we associate the function:

$$\phi_d(x,u) \ = \ \limsup_{\varepsilon\to 0} \frac{1}{\varepsilon}\, d(x,\delta^{x}_{\varepsilon}u),$$

defined for any $x,u \in \Omega$ sufficiently close. We have therefore

$$c\,\bar{d}^{x}(x,u) \ \leq \ \phi_d(x,u) \ \leq \ C\,\bar{d}^{x}(x,u). \tag{8.2}$$

Notice that if $d \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$ then for any x, u, v sufficiently close we have

$$-\bar{d}(x,u)\, O(\bar{d}(x,u)) + c\, \bar{d}^{x}(u,v) \leq$$

$$\leq d(u,v) \leq C\, \bar{d}^{x}(u,v) + \bar{d}(x,u)\, O(\bar{d}(x,u)).$$

If $c : [0,1] \to \Omega$ is a d-Lipschitz curve and $d \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$ then we may decompose it in a finite family of curves $c_1, ..., c_n$ (with n depending on c) such that there are $x_1, ..., x_n \in \Omega$ with c_k is $\bar{d}^{x_k}$-Lipschitz. Indeed, the image of the curve $c([0,1])$ is compact, therefore we may cover it with a finite number of balls $B(c(t_k), \rho_k, \bar{d}^{c(t_k)})$ and apply (8.1). If moreover $(\Omega, \bar{d}, \bar{\delta})$ is tempered then it follows that $c : [0,1] \to \Omega$ d-Lipschitz curve is equivalent with c $\bar{d}$-Lipschitz curve.

By using the same arguments as in the proof of theorem 7.4, we get the following extension of proposition 2.4 [9].

Proposition 8.2. *If the triple $(\Omega, \bar{d}, \bar{\delta})$ is tempered, with the Radon-Nikodym property, and $d \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$ then*

$$d(x,y) \ = \ \inf \left\{ \int_a^b \phi_d(c(t), \dot{c}(t))\, dt \, : \, c : [a,b] \to X \ \bar{d}\text{-Lipschitz} \, , \right.$$

$$\left. c(a) = x, c(b) = y \right\}.$$

The next theorem is a generalization of a part of theorem 3.1 [9].

Theorem 8.3. *Let $(\Omega, \bar{d}, \bar{\delta})$ be a strong dilation structure which is tempered, with the Radon-Nikodym property, and $d_n \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$ a sequence of distances converging to $d \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$. Denote by L_n, L the length functional induced by the distance d_n, respectively by d. Then L_n Γ-converges to L.*

Proof. This is the generalization of the implication (i) ⇒ (iii), theorem 3.1 [9]. The proof (p. 252-253) is almost identical, we only need to replace everywhere expressions like $|\, x - y\,|$ by $\bar{d}(x,y)$ and use proposition 8.2, relations (8.2) and (8.1) instead of respectively proposition 2.4 and relations (2.6) and (2.3) [9]. □

Using this result we obtain a large class of examples of length dilation structures.

Corollary 8.4. *If $(\Omega, \bar{d}, \bar{\delta})$ is strong dilation structure which is tempered and it has the Radon-Nikodym property then it is a length dilation structure.*

Proof. Indeed, from the hypothesis we deduce that $\bar{\delta}^x_\varepsilon \bar{d} \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$. For any sequence $\varepsilon_n \to 0$ we thus obtain a sequence of distances $d_n = \bar{\delta}^x_{\varepsilon_n} \bar{d}$ converging to $\bar{d}^x$. We apply now theorem 8.3 and we get the result. □

9 Coherent projections

For a given dilation structure with the Radon-Nikodym property, we shall give a procedure to construct another dilation structure, such that the first one looks down to the the second one.

This will be done with the help of coherent projections.

Definition 9.1. Let $(X,\bar{d},\bar{\delta})$ be a strong dilation structure. A **coherent projection** of $(X,\bar{d},\bar{\delta})$ is a function which associates to any $x \in X$ and $\varepsilon \in (0,1]$ a map $Q^x_\varepsilon : U(x) \to X$ such that:

(I) $Q^x_\varepsilon : U(x) \to Q^x_\varepsilon(U(x))$ is invertible and the inverse will be denoted by $Q^x_{\varepsilon^{-1}}$; for any $\varepsilon, \mu > 0$ and any $x \in X$ we have

$$Q^x_\varepsilon \bar{\delta}^x_\mu = \bar{\delta}^x_\mu Q^x_\varepsilon,$$

(II) the limit $\lim_{\varepsilon \to 0} Q^x_\varepsilon u = Q^x u$ is uniform with respect to x, u in compact sets,

(III) for any $\varepsilon, \mu > 0$ and any $x \in X$ we have $Q^x_\varepsilon Q^x_\mu = Q^x_{\varepsilon\mu}$. Also $Q^x_1 = id$ and $Q^x_\varepsilon x = x$.

(IV) Let us define $\Theta^x_\varepsilon(u,v) = \bar{\delta}^x_{\varepsilon^{-1}} Q^{\bar{\delta}^x_\varepsilon Q^x_\varepsilon u}_{\varepsilon^{-1}} \bar{\delta}^x_\varepsilon Q^x_\varepsilon v$. Then the limit exists

$$\lim_{\varepsilon \to 0} \Theta^x_\varepsilon(u,v) = \Theta^x(u,v)$$

and it is uniform with respect to x, u, v in compact sets.

Remark 9.2. Property (IV) is basically a smoothness condition on the coherent projection Q, relative to the strong dilation structure $(X,\bar{d},\bar{\delta})$.

Proposition 9.3. *Let $(X,\bar{d},\bar{\delta})$ be a strong dilation structure and Q a coherent projection. We define $\delta^x_\varepsilon = \bar{\delta}^x_\varepsilon Q^x_\varepsilon$. Then:*

(a) for any $\varepsilon, \mu > 0$ and any $x \in X$ we have $\delta^x_\varepsilon \bar{\delta}^x_\mu = \bar{\delta}^x_\mu \delta^x_\varepsilon$.

(b) for any $x \in X$ we have $Q^x Q^x = Q^x$ (thus Q^x is a projection).

(c) δ satisfies the conditions A1, A2, A4 from definition 3.1.

Proof. (a) this is a consequence of the commutativity condition (I) (second part). Indeed, we have $\delta^x_\varepsilon \bar{\delta}^x_\mu = \bar{\delta}^x_\varepsilon Q^x_\varepsilon \bar{\delta}^x_\mu = \bar{\delta}^x_\varepsilon \bar{\delta}^x_\mu Q^x_\varepsilon = \bar{\delta}^x_\mu \bar{\delta}^x_\varepsilon Q^x_\varepsilon = \bar{\delta}^x_\mu \delta^x_\varepsilon$.

(b) we pass to the limit $\varepsilon \to 0$ in the equality $Q^x_{\varepsilon^2} = Q^x_\varepsilon Q^x_\varepsilon$ and we get, based on condition (II), that $Q^x Q^x = Q^x$.

(c) Axiom A1 for δ is equivalent with (III). Indeed, the equality $\delta^x_\varepsilon \delta^x_\mu = \delta^x_{\varepsilon\mu}$ is equivalent with: $\bar{\delta}^x_{\varepsilon\mu} Q^x_{\varepsilon\mu} = \bar{\delta}^x_{\varepsilon\mu} Q^x_\varepsilon Q^x_\mu$. This is true because $Q^x_\varepsilon Q^x_\mu = Q^x_{\varepsilon\mu}$. We also have $\delta^x_1 = \bar{\delta}^x_1 Q^x_1 = Q^x_1 = id$. Moreover $\delta^x_\varepsilon x = \bar{\delta}^x_\varepsilon Q^x_\varepsilon x = Q^x_\varepsilon \bar{\delta}^x_\varepsilon x = Q^x_\varepsilon x = x$. Let us compute now:

$$\Delta^x_\varepsilon(u,v) = \delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \delta^x_\varepsilon v = \bar{\delta}^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} Q^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \delta^x_\varepsilon v =$$

$$= \bar{\delta}^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \bar{\delta}^x_\varepsilon \Theta^x_\varepsilon(u,v) = \bar{\Delta}^x_\varepsilon(Q^x_\varepsilon u, \Theta^x_\varepsilon(u,v)).$$

We can pass to the limit in the last term of this string of equalities and we prove that the axiom A4 is satisfied by δ: there exists the limit

$$\Delta^x(u,v) = \lim_{\varepsilon\to 0} \Delta^x_\varepsilon(u,v), \tag{9.1}$$

which is uniform as written in A4, moreover we have the equality

$$\Theta^x_\varepsilon(u,v) = \bar{\Sigma}^x_\varepsilon(Q^x_\varepsilon u, \Delta^x_\varepsilon(u,v)). \tag{9.2}$$

□

We collect two useful relations in the next proposition.

Proposition 9.4. *Let $(X,\bar{d},\bar{\delta})$ be a strong dilation structure and Q a coherent projection. We denote by δ the field of dilations induced by the coherent projection, as in the previous proposition. The expression Δ^x is defined by (9.1). Then:*

$$\Delta^x(u,v) = \bar{\Delta}^x(Q^x u, \Theta^x(u,v)), \tag{9.3}$$

$$Q^x\Delta^x(u,v) = \bar{\Delta}^x(Q^x u, Q^x v). \tag{9.4}$$

Proof. After passing to the limit with $\varepsilon \to 0$ in the relation (9.2) we get the formula (9.3). In order to prove (9.4) we notice that:

$$Q^{\delta^x_\varepsilon u}_\varepsilon \Delta^x_\varepsilon(u,v) = Q^{\delta^x_\varepsilon u}_\varepsilon \delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \delta^x_\varepsilon v =$$

$$= \bar{\delta}^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \bar{\delta}^x_\varepsilon Q^x_\varepsilon v = \bar{\Delta}^x_\varepsilon(Q^x_\varepsilon u, Q^x_\varepsilon v),$$

which gives (9.4) as we pass to the limit with $\varepsilon \to 0$ in this relation. □

Next is described the notion of Q-horizontal curve.

Definition 9.5. Let $(X,\bar{d},\bar{\delta})$ be a strong dilation structure and Q a coherent projection. A curve $c : [a,b] \to X$ is Q-**horizontal** if for almost any $t \in [a,b]$ the curve c is derivable and the derivative of c at t, denoted by $\dot{c}(t)$ has the property:

$$Q^{c(t)}\dot{c}(t) = \dot{c}(t). \tag{9.5}$$

A curve $c : [a,b] \to X$ is Q-**everywhere horizontal** if for all $t \in [a,b]$ the curve c is derivable and the derivative has the horizontality property (9.5).

We shall now use the notations from section 4. We look first at some induced dilation structures.

For any $x \in X$ and $\varepsilon \in (0,1)$ the dilation δ^x_ε can be seen as an isomorphism of strong dilation structures with coherent projections:

$$\delta^x_\varepsilon : (U(x), \delta^x_\varepsilon \bar{d}, \hat{\delta}^x_\varepsilon, \hat{Q}^x_\varepsilon) \to (\delta^x_\varepsilon U(x), \frac{1}{\varepsilon}\bar{d}, \bar{\delta}, Q),$$

which defines the dilations $\hat{\delta}^{x,\cdot}_{\varepsilon,\cdot}$ and coherent projection $\hat{Q}^x_\varepsilon$ by:

$$\hat{\delta}^{x,u}_{\varepsilon,\mu} = \delta^x_{\varepsilon^{-1}} \bar{\delta}^{\delta^x_\varepsilon u}_\mu \delta^x_\varepsilon,$$

$$\hat{Q}^{x,u}_{\varepsilon,\mu} = \delta^x_{\varepsilon^{-1}} \, Q^{\delta^x_\varepsilon u}_\mu \, \delta^x_\varepsilon .$$

Also the dilation $\bar{\delta}^x_\varepsilon$ is an isomorphism of strong dilation structures with coherent projections:

$$\bar{\delta}^x_\varepsilon : (U(x), \bar{\delta}^x_\varepsilon \bar{d}, \bar{\delta}^x_\varepsilon, \bar{Q}^x_\varepsilon) \to (\bar{\delta}^x_\varepsilon U(x), \frac{1}{\varepsilon}\bar{d}, \bar{\delta}, Q),$$

which defines the dilations $\bar{\delta}^{x,\cdot}_{\varepsilon,\cdot}$ and coherent projection $\bar{Q}^x_\varepsilon$ by:

$$\bar{\delta}^{x,u}_{\varepsilon,\mu} = \bar{\delta}^x_{\varepsilon^{-1}} \, \bar{\delta}^{\bar{\delta}^x_\varepsilon u}_\mu \, \bar{\delta}^x_\varepsilon ,$$

$$\bar{Q}^{x,u}_{\varepsilon,\mu} = \bar{\delta}^x_{\varepsilon^{-1}} \, Q^{\bar{\delta}^x_\varepsilon u}_\mu \, \bar{\delta}^x_\varepsilon .$$

Because $\delta^x_\varepsilon = \bar{\delta}^x_\varepsilon \, Q^x_\varepsilon$ we get that

$$Q^x_\varepsilon : (U(x), \delta^x_\varepsilon \bar{d}, \hat{\delta}^x_\varepsilon, \hat{Q}^x_\varepsilon) \to (Q^x_\varepsilon U(x), \bar{\delta}^x_\varepsilon d, \bar{\delta}^x_\varepsilon, \bar{Q}^x_\varepsilon)$$

is an isomorphism of strong dilation structures with coherent projections.

Further is a useful description of the coherent projection $\hat{Q}^x_\varepsilon$.

Proposition 9.6. *With the notations previously made, for any $\varepsilon \in (0,1]$, $x, u, v \in X$ sufficiently close and $\mu > 0$ we have:*

(i) $\hat{Q}^{x,u}_{\varepsilon,\mu} v = \Sigma^x_\varepsilon(u, Q^{\delta^x_\varepsilon u}_\mu \Delta^x_\varepsilon(u,v))$,

(ii) $\hat{Q}^{x,u}_{\varepsilon} v = \Sigma^x_\varepsilon(u, Q^{\delta^x_\varepsilon u} \Delta^x_\varepsilon(u,v))$.

Proof. (i) implies (ii) when $\mu \to 0$, thus it is sufficient to prove only the first point. This is the result of a computation:

$$\hat{Q}^{x,u}_{\varepsilon,\mu} v = \delta^x_{\varepsilon^{-1}} \, Q^{\delta^x_\varepsilon u}_\mu \, \delta^x_\varepsilon =$$

$$= \delta^x_{\varepsilon^{-1}} \, \delta^{\delta^x_\varepsilon u}_\varepsilon \, Q^{\delta^x_\varepsilon u}_\mu \, \delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \, \delta^x_\varepsilon = \Sigma^x_\varepsilon(u, Q^{\delta^x_\varepsilon u}_\mu \Delta^x_\varepsilon(u,v)).$$

□

Notation concerning derivatives. We shall denote the derivative of a curve with respect to the dilations $\hat{\delta}^x_\varepsilon$ by $\frac{\hat{d}^x_\varepsilon}{dt}$. Also, the derivative of the curve c with respect to $\bar{\delta}$ is denoted by $\frac{\bar{d}}{dt}$, and so on.

By computation we get: the curve c is $\hat{\delta}^x_\varepsilon$-derivable if and only if $\delta^x_\varepsilon c$ is $\bar{\delta}$-derivable and

$$\frac{\hat{d}^x_\varepsilon}{dt} c(t) = \delta^x_{\varepsilon^{-1}} \frac{\bar{d}}{dt} (\delta^x_\varepsilon c)(t).$$

With these notations we give a proposition which explains that the operator Θ^x_ε , from the definition of coherent projections, is a lifting operator.

Proposition 9.7. *If the curve $\delta^x_\varepsilon c$ is Q-horizontal then*

$$\frac{\bar{d}^x_\varepsilon}{dt} (Q^x_\varepsilon c)(t) = \Theta^x_\varepsilon(c(t), \frac{\hat{d}^x_\varepsilon}{dt} c(t)).$$

Proof. If the curve $Q^x_\varepsilon c$ is $\bar{\delta}^x_\varepsilon$ derivable and $\bar{Q}^x_\varepsilon$ horizontal. We have therefore:

$$\frac{\bar{d}^x_\varepsilon}{dt}(Q^x_\varepsilon c)(t) = \bar{\delta}^x_{\varepsilon^{-1}} Q^{\delta^x_\varepsilon c(t)} \bar{\delta}^x_\varepsilon \frac{\bar{d}^x_\varepsilon}{dt}(Q^x_\varepsilon c)(t),$$

which implies:

$$\bar{\delta}^x_\varepsilon \frac{\bar{d}^x_\varepsilon}{dt}(Q^x_\varepsilon c)(t) = Q^{\delta^x_\varepsilon c(t)}_{\varepsilon^{-1}} \bar{\delta}^x_\varepsilon \frac{\bar{d}^x_\varepsilon}{dt}(Q^x_\varepsilon c)(t) = Q^{\delta^x_\varepsilon c(t)}_{\varepsilon^{-1}} \delta^x_\varepsilon \frac{\hat{d}^x_\varepsilon}{dt} c(t),$$

which is the formula we wanted to prove. □

9.1 Distributions in sub-riemannian spaces

The inspiration for the notion of coherent projection comes from sub-riemannian geometry. We shall look to the section 6 with a fresh eye.

Further we shall work locally, just as in the mentioned section. Same notations are used. Let $\{Y_1, ..., Y_n\}$ be a frame induced by a parameterization $\phi : O \subset \mathbb{R}^n \to U \subset M$ of a small open, connected set U in the manifold M. This parameterization induces a affine dilation structure on U, by

$$\tilde{\delta}^{\phi(a)}_\varepsilon \phi(b) = \phi(a + \varepsilon(-a+b)).$$

We take the distance $\tilde{d}(\phi(a), \phi(b)) = \|b-a\|$.

Let $\{X_1, ..., X_n\}$ be a normal frame, cf. definition 6.5, d be the Carnot-Carathéodory distance and let

$$\delta^x_\varepsilon\left(\exp\left(\sum_{i=1}^n a_i X_i\right)(x)\right) = \exp\left(\sum_{i=1}^n a_i \varepsilon^{degX_i} X_i\right)(x)$$

be the dilation structure associated, cf. theorem 6.6.

We may take another dilation structure, constructed as follows: extend the metric g on the distribution D to a riemannian metric on M, denoted for convenience also by g. Let $\bar{d}$ be the riemannian distance induced by the riemannian metric g, and the dilations

$$\bar{\delta}^x_\varepsilon\left(\exp\left(\sum_{i=1}^n a_i X_i\right)(x)\right) = \exp\left(\sum_{i=1}^n a_i \varepsilon X_i\right)(x).$$

Then $(U, \bar{d}, \bar{\delta})$ is a strong dilation structure which is equivalent with the dilation structure $(U, \tilde{d}, \tilde{\delta})$.

From now we may define coherent projections associated either to the pair $(\tilde{\delta}, \delta)$ or to the pair $(\bar{\delta}, \delta)$. Because we put everything on the manifold (by the use of the chosen frames), we shall obtain different coherent projections, both inducing the same dilation structure (U, d, δ).

Let us define Q^x_ε by:

$$Q^x_\varepsilon\left(\exp\left(\sum_{i=1}^n a_i X_i\right)(x)\right) = \exp\left(\sum_{i=1}^n a_i \varepsilon^{degX_i - 1} X_i\right)(x). \tag{9.6}$$

Proposition 9.8. *Q is a coherent projection associated with the dilation structure $(U, \bar{d}, \bar{\delta})$.*

Proof. (I) definition 9.1 is true, because $\delta^x_\varepsilon u = Q^x_\varepsilon \bar{\delta}^x_\varepsilon$ and $\delta^x_\varepsilon \bar{\delta}^x_\varepsilon = \bar{\delta}^x_\varepsilon \delta^x_\varepsilon$. (II), (III) and (IV) are consequences of these facts and theorem 6.6, with a proof similar to the one of proposition 9.3. □

Definition (9.6) of the coherent projection Q implies that:

$$Q^x \left(\exp\left(\sum_{i=1}^{n} a_i X_i \right)(x) \right) = \exp\left(\sum_{degX_i=1} a_i X_i \right)(x). \tag{9.7}$$

Therefore Q^x can be seen as a projection onto the (classical differential) geometric distribution.

Remark 9.9. The projection Q^x has one more interesting feature: for any x and

$$u = \exp\left(\sum_{degX_i=1} a_i X_i \right)(x)$$

we have $Q^x u = u$ and the curve

$$s \in [0,1] \mapsto \delta^x_s u = \exp\left(s \sum_{degX_i=1} a_i X_i \right)(x)$$

is D-horizontal and joins x and u. This will be related to the supplementary condition (B) further.

We may equally define a coherent projection which induces the dilations δ from $\tilde{\bar{\delta}}$. Also, if we change the chosen normal frame with another of the same kind, we shall pass to a dilation structure which is equivalent to (U,d,δ). In conclusion, coherent projections are not geometrical objects per se, but in a natural way one may define a notion of equivalent coherent projections such that the equivalence class is geometrical, i.e. independent of the choice of a pair of particular dilation structures, each in a given equivalence class.

The bottom line is that $(U,\bar{d},\bar{\delta})$ is a dilation structure which belongs to an equivalence class which is independent on the distribution D, and also independent on the choice of parameterization ϕ. It is associated to the manifold M only. On the other hand $(U,\bar{d},\bar{\delta})$ belongs to an equivalence class which is depending only on the distribution D and metric g on D, thus intrinsic to the sub-riemannian manifold (M,D,g). The only advantage of choosing $\bar{\delta},\delta$ related by the normal frame $\{X_1,...,X_n\}$ is that they are associated with a coherent projection with a simple expression.

9.2 Length functionals associated to coherent projections

Definition 9.10. Let $(X,\bar{d},\bar{\delta})$ be a strong dilation structure with the Radon-Nikodym property and Q a coherent projection. We define the associated distance $d : X \times X \to [0,+\infty]$ by:

$$d(x,y) = \inf \left\{ \int_a^b \bar{d}^{c(t)}(c(t),\dot{c}(t))\, \mathrm{d}t : c : [a,b] \to X \ \bar{d}\text{-Lipschitz}, \right.$$
$$\left. c(a) = x, c(b) = y, \text{ and } \forall a.e.\ t \in [a,b] \quad Q^{c(t)}\dot{c}(t) = \dot{c}(t) \right\}.$$

The relation $x \equiv y$ if $d(x,y) < +\infty$ is an equivalence relation. The space X decomposes into a reunion of equivalence classes, each equivalence class being connected by horizontal curves.

It is easy to see that d is a finite distance on each equivalence class. Indeed, from theorem 7.4 we deduce that for any $x,y \in X$ $d(x,y) \geq \bar{d}(x,y)$. Therefore $d(x,y) = 0$ implies $x = y$. The other properties of a distance are straightforward.

Later we shall give a sufficient condition (the generalized Chow condition (Cgen)) on the coherent projection Q for X to be (locally) connected by horizontal curves.

Proposition 9.11. *Suppose that X is connected by horizontal curves and (X,d) is complete. Then d is a length distance.*

Proof. Because (X,d) is complete, it is sufficient to check that d has the approximate middle property: for any $\varepsilon > 0$ and for any $x,y \in X$ there exists $z \in X$ such that

$$\max\{d(x,z), d(y,z)\} \leq \frac{1}{2} d(x,y) + \varepsilon.$$

Given $\varepsilon > 0$, from the definition of d we deduce that there exists a horizontal curve $c : [a,b] \to X$ such that $c(a) = x$, $c(b) = y$ and $d(x,y) + 2\varepsilon \geq l(c)$ (where $l(c)$ is the length of c with respect to the distance $\bar{d}$). There exists then $\tau \in [a,b]$ such that

$$\int_a^\tau \bar{d}^{c(t)}(c(t), \dot{c}(t))\, \mathrm{d}t \ = \ \int_\tau^b \bar{d}^{c(t)}(c(t), \dot{c}(t))\, \mathrm{d}t \ = \ \frac{1}{2} l(c).$$

Let $z = c(\tau)$. We have then: $\max\{d(x,z), d(y,z)\} \ \leq \ \frac{1}{2} l(c) \ \leq \ \frac{1}{2} d(x,y) + \varepsilon$. Therefore d is a length distance. □

Notations concerning length functionals. The length functional associated to the distance $\bar{d}$ is denoted by $\bar{l}$. In the same way the length functional associated to the distance $\bar{\delta}^x_\varepsilon \bar{d}$ given by:

$$\left(\bar{\delta}^x_\varepsilon \bar{d}\right)(u,v) \ = \ \frac{1}{\varepsilon} \bar{d}\left(\bar{\delta}^x_\varepsilon u, \bar{\delta}^x_\varepsilon v\right)$$

is denoted by $\bar{l}^x_\varepsilon$.

We introduce the space $\mathcal{L}_\varepsilon(X,d,\delta) \subset X \times Lip([0,1],X,d)$:

$$\mathcal{L}_\varepsilon(X,d,\delta) \ = \ \{(x,c) \in X \times C([0,1],X) : c : [0,1] \in U(x)\,,$$
$$\delta^x_\varepsilon c \text{ is } \bar{d} - Lip, \quad Q - \text{horizontal and } Lip(\delta^x_\varepsilon c) \leq 2\varepsilon l_d(\delta^x_\varepsilon c)\Big\}.$$

For any $\varepsilon \in (0,1)$ we define the length functional

$$l_\varepsilon : \mathcal{L}_\varepsilon(X,d,\delta) \to [0,+\infty], \quad l_\varepsilon(x,c) \ = \ l^x_\varepsilon(c) \ = \ \frac{1}{\varepsilon} \bar{l}(\delta^x_\varepsilon c).$$

By theorem 7.4 we have:

$$l^x_\varepsilon(c) \ = \ \int_0^1 \frac{1}{\varepsilon} \bar{d}^{\delta^x_\varepsilon c(t)} \left(\delta^x_\varepsilon c(t), \frac{\bar{d}}{dt}(\delta^x_\varepsilon c)(t) \right) \mathrm{d}t \ =$$
$$= \ \int_0^1 \frac{1}{\varepsilon} \bar{d}^{\delta^x_\varepsilon c(t)} \left(\delta^x_\varepsilon c(t), \delta^x_\varepsilon \frac{\hat{d}^x_\varepsilon}{dt} c(t) \right) \mathrm{d}t.$$

Another description of the length functional l^x_ε is the following.

Proposition 9.12. *For any* $(x,c) \in \mathcal{L}_\varepsilon(X,d,\delta)$ *we have*

$$l^x_\varepsilon(c) = \bar{l}^x_\varepsilon\,(Q^x_\varepsilon c)\,.$$

Proof. Indeed, we shall use an alternate definition of the length functional. Let c be a curve such that $\delta^x_\varepsilon c$ is $\bar{d}$-Lipschitz and Q-horizontal. Then:

$$l^x_\varepsilon(c) = \sup\left\{\sum_{i=1}^{n} \frac{1}{\varepsilon}\bar{d}\,(\delta^x_\varepsilon c(t_i), \delta^x_\varepsilon c(t_{i+1}))\ :\ 0 = t_1 < ... < t_{n+1} = 1\right\} =$$

$$= \sup\left\{\sum_{i=1}^{n} \frac{1}{\varepsilon}\bar{d}\left(\bar{\delta}^x_\varepsilon Q^x_\varepsilon c(t_i), \bar{\delta}^x_\varepsilon Q^x_\varepsilon c(t_{i+1})\right)\ :\ 0 = t_1 < ... < t_{n+1} = 1\right\} =$$

$$= \bar{l}^x_\varepsilon\,(Q^x_\varepsilon c)\,.$$

□

9.3 Supplementary hypotheses

Definition 9.13. Let $(X,\bar{d},\bar{\delta})$ be a strong dilation structure and Q a coherent projection. Further is a list of supplementary hypotheses on Q:

(A) δ^x_ε is $\bar{d}$-bilipschitz in compact sets in the following sense: for any compact set $K \subset X$ and for any $\varepsilon \in (0,1]$ there is a number $L(K) > 0$ such that for any $x \in K$ and u,v sufficiently close to x we have:

$$\frac{1}{\varepsilon}\,\bar{d}\,(\delta^x_\varepsilon u, \delta^x_\varepsilon v) \leq L(K)\,\bar{d}(u,v),$$

(B) if $u = Q^x u$ then the curve $t \in [0,1] \mapsto Q^x\,\delta^x_t\,u = \bar{\delta}^x_t u = \delta^x_t u$ is Q-everywhere horizontal and for any $a \in [0,1]$ we have

$$\limsup_{a\to 0} \frac{\bar{l}\left(t \in [0,a] \mapsto \bar{\delta}^x_t u\right)}{\bar{d}(x, \bar{\delta}^x_a u)} = 1$$

uniformly with respect to x,u in compact set K.

Condition (A), as well as the property (IV) definition 9.1, is another smoothness condition on Q with respect to the strong dilation structure $(X,\bar{d},\bar{\delta})$.

The condition (A) has several useful consequences, among them the fact that for any $\bar{d}$-Lipschitz curve c, the curve $\delta^x_\varepsilon c$ is also Lipschitz. Another consequence is that Q^x_ε is locally $\bar{d}$-Lipschitz. More precisely, for any compact set $K \subset X$ and for any $\varepsilon \in (0,1]$ there is a number $L(K) > 0$ such that for any $x \in K$ and u,v sufficiently close to x we have:

$$\left(\bar{\delta}^x_\varepsilon \bar{d}\right)(Q^x_\varepsilon u, Q^x_\varepsilon v) \leq L(K)\,\bar{d}(u,v). \tag{9.8}$$

Indeed, we have:

$$\left(\bar{\delta}^x_\varepsilon \bar{d}\right)(Q^x_\varepsilon u, Q^x_\varepsilon v) = \frac{1}{\varepsilon}\bar{d}(\delta^x_\varepsilon u, \delta^x_\varepsilon v) \leq L(K)\bar{d}(u,v).$$

See the remark 9.9 for the meaning of the condition (B) for the case sub-riemannian geometry, where it is explained why condition (B) is a generalization of the fact that the "distribution" $x \mapsto Q^x U(x)$ is generated by horizontal one parameter flows.

Condition (B) will be useful later, along with the generalized Chow condition (Cgen).

10 The generalized Chow condition

Notations about words. For any set A we denote by A^* the collection of finite words $q = a_1...a_p$, $p \in \mathbb{N}$, $p > 0$. The empty word is denoted by $\emptyset$. The length of the word $q = a_1...a_p$ is $\mid q \mid= p$; the length of the empty word is 0.

The collection of words infinite at right over the alphabet A is denoted by A^ω. For any word $w \in A^\omega \cup A^*$ and any $p \in \mathbb{N}$ we denote by $[w]_p$ the finite word obtained from the first p letters of w (if $p = 0$ then $[w]_0 = \emptyset$ (in the case of a finite word q, if $p >\mid q \mid$ then $[q]_p = q$).

For any non-empty $q_1, q_2 \in A^*$ and $w \in A^\omega$ the concatenation of q_1 and q_2 is the finite word $q_1 q_2 \in A^*$ and the concatenation of q_1 and w is the (infinite) word $q_1 w \in A^\omega$. The empty word $\emptyset$ is seen both as an infinite word or a finite word and for any $q \in A^*$ and $w \in A^\omega$ we have $q\emptyset = q$ (as concatenation of finite words) and $\emptyset w = w$ (as concatenation of a finite empty word and an infinite word).

10.1 Coherent projections as transformations of words

To any coherent projection Q in a strong dilation structure $(X, \bar{d}, \bar{\delta})$ we associate a family of transformations as follows.

Definition 10.1. For any non-empty word $w \in (0,1]^\omega$ and any $\varepsilon \in (0,1]$ we define the transformation

$$\Psi_{\varepsilon w} : X^*_{\varepsilon w} \subset X^* \setminus \{\emptyset\} \to X^*,$$

given by: for any non-empty finite word $q = x x_1 ... x_p \in X^*_{\varepsilon w}$ we have

$$\Psi_{\varepsilon w}(x x_1 ... x_p) = \Psi^1_{\varepsilon w}(x) ... \Psi^{k+1}_{\varepsilon w}(x x_1 ... x_k) ... \Psi^{p+1}_{\varepsilon w}(x x_1 ... x_p).$$

The functions $\Psi^k_{\varepsilon w}$ are defined by: $\Psi^1_{\varepsilon w}(x) = x$, and for any $k \geq 1$ we have

$$\Psi^{k+1}_{\varepsilon w}([q]_{k+1}) = \delta^x_{\varepsilon^{-1}} Q^{\delta^x_\varepsilon \Psi^k_{\varepsilon w}([q]_k)}_{w_k} \delta^x_\varepsilon q_{k+1}. \tag{10.1}$$

If $w = \emptyset$ then $\Psi^k_{\varepsilon\emptyset}$ is defined as previously $\Psi^1_{\varepsilon\emptyset}(x) = x$, with the only difference that for any $k \geq 1$ we have

$$\Psi^{k+1}_{\varepsilon\emptyset}([q]_{k+1}) = \delta^x_{\varepsilon^{-1}} Q^{\delta^x_\varepsilon \Psi^k_{\varepsilon w}([q]_k)} \delta^x_\varepsilon q_{k+1}.$$

The domain $X^*_{\varepsilon w} \subset X^* \setminus \{\emptyset\}$ is such that the previous definition makes sense. By using the definition of a coherent projection, we may redefine $X^*_{\varepsilon w}$ as follows: for any compact set $K \subset X$ there is $\rho = \rho(K) > 0$ such that for any $x \in K$ the word $q = xx_1...x_p \in X^*_{\varepsilon w}$ if for any $k \geq 1$ we have

$$\bar{d}\Big(x_{k+1}, \Psi^k_{\varepsilon w}([q]_k)\Big) \leq \rho.$$

We shall explain the meaning of these transformations for $\varepsilon = 1$.

Proposition 10.2. *Suppose that condition (B) holds for the coherent projection Q. If*

$$y \;=\; \Psi^{k+1}_{1\emptyset}(xx_1...x_k)$$

then there is a Q-horizontal curve joining x and y.

Proof. By definition 10.1 for $\varepsilon = 1$ we have:

$$\Psi^1_{1w}(x) = x, \quad \Psi^2_{1w}(x, x_1) \;=\; Q^x_{w_1}\, x_1,$$

$$\Psi^3_{1w}(x, x_1, x_2) \;=\; Q^{Q^x_{w_1} x_1}_{w_2}\, x_2 \quad ...$$

Suppose now that condition (B) holds for the coherent projection Q. Then the curve $t \in [0,1] \mapsto \bar{\delta}^x_t Q^x u$ is a Q-horizontal curve joining x with $Q^x u$. Therefore by applying inductively the condition (B) we get that there is a Q-horizontal curve between $\Psi^k_{1\emptyset}(xx_1...x_{k-1})$ and $\Psi^{k+1}_{1\emptyset}(xx_1...x_k)$ for any $k > 1$ and a Q-horizontal curve joining x and $\Psi^2_{1\emptyset}(xx_1)$. □

There are three more properties of the transformations $\Psi_{\varepsilon w}$.

Proposition 10.3. *With the notations from definition 10.1, we have:*

(a) $\Psi_{\varepsilon w}\Psi_{\varepsilon\emptyset} \;=\; \Psi_{\varepsilon\emptyset}$. *Therefore we have the equality of sets:*

$$\Psi_{\varepsilon\emptyset}\left(X^*_{\varepsilon\emptyset} \cap xX^*\right) \;=\; \Psi_{\varepsilon w}\left(\Psi_{\varepsilon\emptyset}\left(X^*_{\varepsilon\emptyset} \cap xX^*\right)\right),$$

(b) $\Psi^{k+1}_{\varepsilon\emptyset}(xq_1...q_k) \;=\; \delta^x_{\varepsilon^{-1}}\,\Psi^{k+1}_{1\emptyset}(x\delta^x_\varepsilon q_1...\delta^x_\varepsilon q_k)$,

(c) $\lim_{\varepsilon\to 0}\delta^x_{\varepsilon^{-1}}\,\Psi^{k+1}_{1\emptyset}(x\delta^x_\varepsilon q_1...\delta^x_\varepsilon q_k) \;=\; \Psi^{k+1}_{0\emptyset}(xq_1...q_k)$ *uniformly with respect to $x, q_1, ..., q_k$ in compact set.*

Proof. (a) We use induction on k to prove that for any natural number k

$$\Psi^{k+1}_{\varepsilon w}\Big(\Psi^1_{\varepsilon\emptyset}(x)...\Psi^{k+1}_{\varepsilon\emptyset}(xq_1...q_k)\Big) \;=\; \Psi^{k+1}_{\varepsilon\emptyset}(xq_1...q_k). \tag{10.2}$$

For $k = 0$ we have to prove that $x = x$ which is trivial. For $k = 1$ we have to prove that

$$\Psi^2_{\varepsilon w}\Big(\Psi^1_{\varepsilon\emptyset}(x)\,\Psi^2_{\varepsilon\emptyset}(xq_1)\Big) \;=\; \Psi^2_{\varepsilon\emptyset}(xq_1).$$

This means:

$$\Psi^2_{\varepsilon w}\left(x\,\delta^x_{\varepsilon^{-1}}\,Q^x\delta^x_\varepsilon q_1\right) \;=\; \delta^x_{\varepsilon^{-1}}\,Q^x_{w_1}\,\delta^x_\varepsilon\,\delta^x_{\varepsilon^{-1}}\,Q^x\,\delta^x_\varepsilon x_1 \;=$$

$$=\; \delta^x_{\varepsilon^{-1}}\,Q^x\,\delta^x_\varepsilon x_1 \;=\; \Psi^2_{\varepsilon\emptyset}(xq_1).$$

Suppose now that $l \geq 2$ and for any $k \leq l$ the relations (10.2) are true. Then, as previously, it is easy to check (10.2) for $k = l+1$.

(b) is true by direct computation. The point (c) is a straightforward consequence of (b) and definition of coherent projections. □

Definition 10.4. Let $N \in \mathbb{N}$ be a strictly positive natural number and $\varepsilon \in (0,1]$. Then a point $x \in X$ is (ε, N, Q)**-nested** in a open neighbourhood $U \subset X$ if there is $\rho > 0$ such that for any finite word $q = x_1 ... x_N \in X^N$ with

$$\bar{\delta}^x_\varepsilon \bar{d}\left(x_{k+1}, \Psi^k_{\varepsilon\emptyset}([xq]_k)\right) \leq \rho$$

for any $k = 1, ..., N$, we have $q \in U^N$.

If $x \in U$ is (ε, N, Q)-nested then denote by $U(x, \varepsilon, N, Q, \rho) \subset U^N$ the collection of words $q \in U^N$ such that $\bar{\delta}^x_\varepsilon \bar{d}\left(x_{k+1}, \Psi^k_{\varepsilon\emptyset}([xq]_k)\right) < \rho$ for any $k = 1, ..., N$.

Definition 10.5. A coherent projection Q satisfies the **generalized Chow condition** if:

(Cgen) for any compact set K there are $\rho = \rho(K) > 0$, $r = r(K) > 0$, a natural number $N = N(Q, K)$ and a function $F(\eta) = O(\eta)$ such that for any $x \in K$ and $\varepsilon \in (0,1]$ there are neighbourhoods $U(x)$, $V(x)$ such that any $x \in K$ is (ε, N, Q)-nested in $U(x)$, such that $B(x, r, \bar{\delta}^x_\varepsilon \bar{d}) \subset V(x)$ and such that the mapping

$$x_1 ... x_N \in U(x, N, Q, \rho) \mapsto \Psi^{N+1}_{\varepsilon\emptyset}(x x_1 ... x_N)$$

is surjective from $U(x, \varepsilon, N, Q, \rho)$ to $V(x)$. Moreover for any $z \in V(x)$ there exist $y_1, ... y_N \in U(x, \varepsilon, N, Q, \rho)$ such that $z = \Psi^{N+1}_{\varepsilon\emptyset}(x y_1, ... y_N)$ and for any $k = 0, ..., N-1$ we have

$$\delta^x_\varepsilon \bar{d}\left(\Psi^{k+1}_{\varepsilon\emptyset}(x y_1 ... y_k), \Psi^{k+2}_{\varepsilon\emptyset}(x y_1 ... y_{k+1})\right) \leq F(\delta^x_\varepsilon \bar{d}(x, z)).$$

Condition (Cgen) is inspired from lemma 1.40 Folland-Stein [14]. If the coherent projection Q satisfies also (A) and (B) then in the space $(U(x), \bar{\delta}^x_\varepsilon)$, with coherent projection $\hat{Q}^{x,\cdot}_{\varepsilon,\cdot}$, we can join any two sufficiently close points by a sequence of at most N horizontal curves. Moreover there is a control on the length of these curves via condition (B) and condition (Cgen); in sub-riemannian geometry the function F is of the type $F(\eta) = \eta^{1/m}$ with m positive natural number.

Definition 10.6. Suppose that the coherent projection Q satisfies conditions (A), (B) and (Cgen). Let us consider $\varepsilon \in (0,1]$ and $x, y \in K$, K compact in X. With the notations from definition 10.5, suppose that there are numbers $N = N(Q, K)$, $\rho = \rho(Q, K) > 0$ and words $x_1 ... x_N \in U(x, \varepsilon, N, Q, \rho)$ such that

$$y = \Psi^{N+1}_{\varepsilon\emptyset}(x x_1 ... x_N).$$

For any $t \in [0, N]$ let $k = [t]$, where $[b]$ is the integer part of the real number b. We define then a **short curve** joining x and y, $c : [0, N] \to X$, by

$$c(t) = \bar{\delta}^{x, \Psi^{k+1}_{\varepsilon\emptyset}(x x_1 ... x_k)}_{\varepsilon, t+N-k} Q^{\Psi^{k+1}_{\varepsilon\emptyset}(x x_1 ... x_k)} x_{k+1}.$$

Any short curve joining x and y is a increasing linear reparameterization of a curve c described previously.

10.2 The candidate tangent space

Let $(X,\bar{d},\bar{\delta})$ be a strong dilation structure and Q a coherent projection. Then we have the induced dilations

$$\mathring{\delta}^{x,u}_{\mu} v \;=\; \Sigma^{x}(u, \delta^{x}_{\mu} \Delta^{x}(u,v))$$

and the induced projection

$$\mathring{Q}^{x,u}_{\mu} v \;=\; \Sigma^{x}(u, Q^{x}_{\mu} \Delta^{x}(u,v)).$$

For any point $x \in X$ we introduce an associated length functional, denoted by l^{x}, which is defined for any $\mathring{\delta}^{x}$-derivable and $\mathring{Q}^{x}$-horizontal almost everywhere curve $c : [0,1] \to U(x)$:

$$l^{x}(c) \;=\; \int_{0}^{1} \bar{d}^{x}\left(x, \Delta^{x}(c(t), \frac{\mathring{d}^{x}}{dt} c(t)) \right) \, \mathrm{d}t.$$

Associated to this length functional is the distance function:

$$\mathring{d}^{x}(u,v) \;=\; \inf \Big\{ l^{x}(c) : c : [0,1] \to U(x) \text{ is } \mathring{\delta}^{x}\text{-derivable},$$

$$\text{and } \mathring{Q}^{x}\text{-horizontal a.e.}, \; c(0) = u, c(1) = v \Big\}.$$

We want to prove that $(U(x), \mathring{d}^{x}, \mathring{\delta}^{x})$ is a strong dilation structure and $\mathring{Q}^{x}$ is a coherent projection. For this we need first the following proposition.

Proposition 10.7. *The curve $c : [0,1] \to U(x)$ is $\mathring{\delta}^{x}$-derivable, $\mathring{Q}^{x}$-horizontal almost everywhere, and $l^{x}(c) < +\infty$ if and only if the curve $Q^{x}c$ is $\bar{\delta}^{x}$-derivable almost everywhere and $\bar{l}^{x}(Q^{x}c) < +\infty$. Moreover, we have*

$$\bar{l}^{x}(Q^{x}c) \;=\; l^{x}(c).$$

Proof. The curve c is $\mathring{Q}^{x}$-horizontal almost everywhere if and only if for almost any $t \in [0,1]$ we have

$$Q^{x} \Delta^{x}(c(t), \frac{\mathring{d}^{x}}{dt} c(t)) \;=\; \Delta^{x}(c(t), \frac{\mathring{d}^{x}}{dt} c(t)).$$

We shall prove that c is $\mathring{Q}^{x}$-horizontal is equivalent with

$$\Theta^{x}(c(t), \frac{\mathring{d}^{x}}{dt} c(t)) \;=\; \frac{\bar{d}^{x}}{dt} (Q^{x}c)(t) \tag{10.3}$$

Indeed, (10.3) is equivalent with

$$\lim_{\varepsilon \to 0} \bar{\delta}^{x}_{\varepsilon^{-1}} \bar{\Delta}^{x}(Q^{x}c(t), Q^{x}c(t+\varepsilon)) \;=\; \bar{\Delta}^{x}(Q^{x}c(t), \Theta^{x}(c(t), \frac{\mathring{d}^{x}}{dt} c(t))),$$

which is equivalent with

$$\lim_{\varepsilon \to 0} \bar{\delta}^{x}_{\varepsilon^{-1}} \bar{\Delta}^{x}(Q^{x}c(t), Q^{x}c(t+\varepsilon)) \;=\; \Delta^{x}(c(t), \frac{\mathring{d}^{x}}{dt} c(t)).$$

But this is equivalent with:

$$\lim_{\varepsilon\to 0} \bar{\delta}^x_{\varepsilon^{-1}} \bar{\Delta}^x(Q^x c(t), Q^x c(t+\varepsilon)) = \lim_{\varepsilon\to 0} \delta^x_{\varepsilon^{-1}} \Delta^x(c(t), c(t+\varepsilon)). \quad (10.4)$$

The horizontality condition for the curve c can be written as:

$$\lim_{\varepsilon\to 0} Q^x \delta^x_{\varepsilon^{-1}} \Delta^x(c(t), c(t+\varepsilon)) = \lim_{\varepsilon\to 0} \delta^x_{\varepsilon^{-1}} \Delta^x(c(t), c(t+\varepsilon)).$$

We use now the properties of Q^x in the left hand side of the previous equality:

$$Q^x \delta^x_{\varepsilon^{-1}} \Delta^x(c(t), c(t+\varepsilon)) = \bar{\delta}^x_{\varepsilon^{-1}} Q^x \Delta^x(c(t), c(t+\varepsilon)) =$$

$$= \bar{\delta}^x_{\varepsilon^{-1}} \bar{\Delta}^x(Q^x c(t), Q^x c(t+\varepsilon)),$$

thus after taking the limit as $\varepsilon \to 0$ we prove that the limit

$$\lim_{\varepsilon\to 0} \bar{\delta}^x_{\varepsilon^{-1}} \bar{\Delta}^x(Q^x c(t), Q^x c(t+\varepsilon))$$

exists and we obtain:

$$\lim_{\varepsilon\to 0} \delta^x_{\varepsilon^{-1}} \Delta^x(c(t), c(t+\varepsilon)) = \lim_{\varepsilon\to 0} \bar{\delta}^x_{\varepsilon^{-1}} \bar{\Delta}^x(Q^x c(t), Q^x c(t+\varepsilon)).$$

This last equality is the same as (10.4), which is equivalent with (10.3).

As a consequence we obtain the following equality, for almost any $t \in [0,1]$:

$$\bar{d}^x\left(x, \Delta^x(c(t), \frac{\mathring{d}^x}{dt} c(t))\right) = \bar{\Delta}^x(Q^x c(t), \frac{\bar{d}^x}{dt}(Q^x c)(t)). \quad (10.5)$$

This implies that $Q^x c$ is absolutely continuous and by theorem 2.7, as in the proof of theorem 7.4 (but without using the Radon-Nikodym property property, because we already know that $Q^x c$ is derivable a.e.), we obtain the following formula for the length of the curve $Q^x c$:

$$\bar{l}^x(Q^x c) = \int_0^1 \bar{d}^x\left(x,, \bar{\Delta}^x(Q^x c(t), \frac{\bar{d}^x}{dt}(Q^x c)(t))\right) \mathrm{d}t.$$

But we have also:

$$l^x(c) = \int_0^1 \bar{d}^x\left(x, \Delta^x(c(t), \frac{\mathring{d}^x}{dt} c(t))\right) \mathrm{d}t.$$

By (10.5) we obtain $\bar{l}^x(Q^x c) = l^x(c)$. □

Proposition 10.8. *If $(X, \bar{d}, \bar{\delta})$ is a strong dilation structure, Q is a coherent projection and $\mathring{d}^x$ is finite then the triple $(U(x), \Sigma^x, \delta^x)$ is a normed conical group, with the norm induced by the left-invariant distance $\mathring{d}^x$.*

Proof. The fact that $(U(x), \Sigma^x, \delta^x)$ is a conical group comes directly from the definition 9.1 of a coherent projection. Indeed, it is enough to use proposition 9.3 (c) and the formalism of binary decorated trees in [5] section 4 (or theorem 11 [5]), in order to reproduce the part of the proof of theorem 10 (p.87-88) in that paper, concerning the conical group structure. There is one small subtlety though. In the proof of theorem 5.6(a) the same modification of proof has been done starting from the axiom A4+, namely the existence of the uniform limit $\lim_{\varepsilon\to 0}\Sigma^x_\varepsilon(u,v) = \Sigma^x(u,v)$. Here we need first to prove this limit, in a similar way as in the corollary 9 [5]. We shall use for this the distance $\mathring{d}^x$ instead of the distance in the metric tangent space of (X,d) at x denoted by d^x (which is not yet proven to exist). The distance $\mathring{d}^x$ is supposed to be finite by hypothesis. Moreover, by its definition and proposition 10.7 we have

$$\mathring{d}^x(u,v) \geq \bar{d}^x(u,v),$$

therefore the distance $\mathring{d}^x$ is non degenerate. By construction this distance is also left invariant with respect to the group operation $\Sigma^x(\cdot,\cdot)$. Therefore we may repeat the proof of corollary 9 [5] and obtain the result that A4+ is true for (X,d,δ).

What we need to prove next is that $\mathring{d}^x$ induces a norm on the conical group $(U(x),\Sigma^x,\delta^x)$. For this it is enough to prove that

$$\mathring{d}^x(\mathring{\delta}^{x,u}_\mu v,\mathring{\delta}^{x,u}_\mu w) = \mu\,\mathring{d}^x(v,w). \tag{10.6}$$

for any $v,w\in U(x)$. This is a direct consequence of relation (10.5) from the proof of the proposition 10.7. Indeed, by direct computation we get that for any curve c which is $\mathring{Q}^x$-horizontal a.e. we have:

$$l^x(\mathring{\delta}^{x,u}_\mu c) = \int_0^1 \bar{d}^x\left(x,\Delta^x\left(\mathring{\delta}^{x,u}_\mu c(t),\frac{\mathring{d}^x}{dt}\left(\mathring{\delta}^{x,u}_\mu c\right)(t)\right)\right)\,\mathrm{dt} =$$

$$= \int_0^1 \bar{d}^x\left(x,\delta^x_\mu\Delta^x\left(c(t),\frac{\mathring{d}^x}{dt}c(t)\right)\right)\,\mathrm{dt}.$$

But c is $\mathring{Q}^x$-horizontal a.e., which implies, via (10.5), that

$$\delta^x_\mu\Delta^x\left(c(t),\frac{\mathring{d}^x}{dt}c(t)\right) = \bar{\delta}^x_\mu\Delta^x\left(c(t),\frac{\mathring{d}^x}{dt}c(t)\right),$$

therefore we have

$$l^x(\mathring{\delta}^{x,u}_\mu c) = \int_0^1 \bar{d}^x\left(x,\bar{\delta}^x_\mu\Delta^x\left(c(t),\frac{\mathring{d}^x}{dt}c(t)\right)\right)\,\mathrm{dt} = \mu\, l^x(c).$$

This implies (10.6), therefore the proof is done. □

Theorem 10.9. *If the generalized Chow condition (Cgen) and condition (B) are true then $(U(x),\Sigma^x,\delta^x)$ is local conical group which is a neighbourhood of the neutral element of a Carnot group generated by $Q^xU(x)$.*

Proof. For any $\varepsilon\in(0,1]$, as a consequence of proposition 9.6 we can put the recurrence relations (10.1) in the form:

$$\Psi^{k+1}_{\varepsilon w}([q]_{k+1}) = \Sigma^x_\varepsilon\left(\Psi^k_{\varepsilon w}([q]_k),\, Q^{\delta^x_\varepsilon\Psi^k_{\varepsilon w}([q]_k)}_{w_k}\,\Delta^x_\varepsilon\left(\Psi^k_{\varepsilon w}([q]_k),q_{k+1}\right)\right). \tag{10.7}$$

This recurrence relation allows us to prove by induction that for any k the limit

$$\Psi^k_w([q]_k) = \lim_{\varepsilon\to 0}\Psi^k_{\varepsilon w}([q]_k)$$

exists and it satisfies the recurrence relation:

$$\Psi^{k+1}_{0w}([q]_{k+1}) \;=\; \Sigma^x\Big(\Psi^k_{0w}([q]_k), Q^x_{w_k}\,\Delta^x\Big(\Psi^k_{0w}([q]_k), q_{k+1}\Big)\Big). \tag{10.8}$$

and the initial condition $\Psi^1_{0w}(x) = x$. We pass to the limit in the generalized Chow condition (Cgen) and we thus obtain that a neighbourhood of the neutral element x is (algebraically) generated by $Q^x U(x)$. Then the distance $\mathring{d}^x$ is finite. Therefore by proposition 10.8 $(U(x), \Sigma^x, \delta^x)$ is a normed conical group generated by $Q^x U(x)$.

Let $c : [0,1] \to U(x)$ be the curve $c(t) = \delta^x_t u$, with $u \in Q^x U(x)$. Then we have $Q^x c(t) = c(t) = \bar{\delta}^x_t u$. From condition (B) we get that c is $\bar{\delta}$-derivable at $t = 0$. A short computation of this derivative shows that:

$$\frac{d\bar{\delta}}{dt} c(0) \;=\; u.$$

Another easy computation shows that the curve c is $\bar{\delta}^x$-derivable if and only if the curve c is $\bar{\delta}$-derivable at $t = 0$, which is true, therefore c is $\bar{\delta}^x$-derivable, in particular at $t = 0$. Moreover, the expression of the $\bar{\delta}^x$-derivative of c shows that c is also Q^x-everywhere horizontal (compare with the remark 9.9). We use the proposition 10.7 and relation (10.3) from its proof to deduce that $c = Q^x c$ is $\mathring{\delta}^x$-derivable at $t = 0$, thus for any $u \in Q^x U(x)$ and small enough $t, \tau \in (0,1)$ we have

$$\mathring{\delta}^{x,x}_{t+\tau} u \;=\; \bar{\Sigma}^x(\bar{\delta}^x_t u, \bar{\delta}^x_\tau u). \tag{10.9}$$

By previous proposition 10.8 and corollary 6.3 [6] (here proposition 5.8) the normed conical group $(U(x), \Sigma^x, \delta^x)$ is in fact locally a homogeneous group, i.e. a simply connected Lie group which admits a positive graduation given by the eigenspaces of δ^x. Indeed, corollary 6.3 [5] is originally about strong dilation structures, but the generalized Chow condition implies that the distances d, $\bar{d}$ and $\mathring{d}^x$ induce the same uniformity, which, along with proposition 10.8, are the only things needed for the proof of this corollary. The conclusion of corollary 6.3 [6] therefore is true, that is $(U(x), \Sigma^x, \delta^x)$ is locally a homogeneous group. Moreover it is locally Carnot if and only if on the generating space $Q^x U(x)$ any dilation $\mathring{\delta}^{x,x}_\varepsilon u \;=\; \bar{\delta}^x_\varepsilon$ is linear in ε. But this is true, as shown by relation (10.9). This ends the proof. □

10.3 Coherent projections induce length dilation structures

Theorem 10.10. *If $(X, \bar{d}, \bar{\delta})$ is a tempered strong dilation structure, has the Radon-Nikodym property and Q is a coherent projection, which satisfies (A), (B), (Cgen) then (X, d, δ) is a length dilation structure.*

Proof. We shall prove that:

(a) for any function $\varepsilon \in (0,1) \mapsto (x_\varepsilon, c_\varepsilon) \in \mathcal{L}_\varepsilon(X, d, \delta)$ which converges to (x, c) as $\varepsilon \to 0$, with $c : [0,1] \to U(x)$ $\mathring{\delta}^x$-derivable and $\mathring{Q}^x$-horizontal almost everywhere, we have:

$$l^x(c) \;\leq\; \liminf_{\varepsilon \to 0} l^{x_\varepsilon}(c_\varepsilon),$$

(b) for any sequence $\varepsilon_n \to 0$ and any (x,c), with $c : [0,1] \to U(x)$ $\mathring{\delta}^x$-derivable and $\mathring{Q}^x$-horizontal almost everywhere, there is a recovery sequence $(x_n, c_n) \in \mathcal{L}_{\varepsilon_n}(X,d,\delta)$ such that

$$l^x(c) \;=\; \lim_{n\to\infty} l^{x_n}(c_n).$$

Proof of (a). This is a consequence of propositions 10.7, 9.12 and definition 9.1 of a coherent projection. With the notations from (a) we see that we have to prove that

$$l^x(c) \;=\; \bar{l}^x(Q^x c) \;\leq\; \liminf_{\varepsilon\to 0} \bar{l}^{x_\varepsilon}(Q^{x_\varepsilon}_\varepsilon c_\varepsilon).$$

This is true because $(X,\bar{d},\bar{\delta})$ is a tempered dilation structure and because of condition (A). Indeed from the fact that $(X,\bar{d},\bar{\delta})$ is tempered and from (9.8) (which is a consequence of condition (A)) we deduce that Q_ε is uniformly continuous on compact sets in a uniform way: for any compact set $K \subset X$ there is are constants $L(K) > 0$ (from (A)) and $C > 0$ (from the tempered condition) such that for any $\varepsilon \in (0,1]$, any $x \in K$ and any u, v sufficiently close to x we have:

$$\bar{d}(Q^x_\varepsilon u, Q^x_\varepsilon v) \leq C\left(\bar{\delta}^x_\varepsilon \bar{d}\right)(Q^x_\varepsilon u, Q^x_\varepsilon v) \leq C\,L(K)\,\bar{d}(u,v).$$

The sequence Q^x_ε uniformly converges to Q^x as ε goes to 0, uniformly with respect to x in compact sets. Therefore if $(x_\varepsilon, c_\varepsilon) \in \mathcal{L}_\varepsilon(X,d,\delta)$ converges to (x,c) then $(x_\varepsilon, Q^{x_\varepsilon}_\varepsilon c_\varepsilon) \in \mathcal{L}_\varepsilon(X,\bar{d},\bar{\delta})$ converges to $(x, Q^x c)$. Use now the fact that by corollary 8.4 $(X,\bar{d},\bar{\delta})$ is a length dilation structure. The proof is done.

Proof of (b). We have to construct a recovery sequence. We are doing this by discretization of $c : [0,L] \to U(x)$. Recall that c is a curve which is $\mathring{\delta}^x$-derivable a.e. and $\mathring{Q}^x$-horizontal, that is for almost every $t \in [0,L]$ the limit

$$u(t) \;=\; \lim_{\mu\to 0} \delta^x_{\mu^{-1}}\,\Delta^x(c(t), c(t+\mu))$$

exists and $Q^x u(t) = u(t)$. Moreover we may suppose that for almost every t we have $\bar{d}^x(x,u(t)) \leq 1$ and $\bar{l}^x(c) \leq L$.

There are functions $\omega^1, \omega^2 : (0,+\infty) \to [0,+\infty)$ with $\lim_{\lambda\to 0} \omega^i(\lambda) = 0$, with the following property: for any $\lambda > 0$ sufficiently small there is a division $A_\lambda = \{0 < t_0 < ... < t_P < L\}$ such that

$$\frac{\lambda}{2} \leq \min\left\{\frac{t_0}{t_1 - t_0}, \frac{L - t_P}{t_P - t_{P-1}}, t_k - t_{k-1} : k = 1,...,P\right\}, \tag{10.10}$$

$$\lambda \geq \max\left\{\frac{t_0}{t_1 - t_0}, \frac{L - t_P}{t_P - t_{P-1}}, t_k - t_{k-1} : k = 1,...,P\right\}, \tag{10.11}$$

and such that $u(t_k)$ exists for any $k = 1,...,P$ and

$$\mathring{d}^x(c(0), c(t_0)) \;\leq\; t_0 \;\leq\; \lambda^2, \tag{10.12}$$

$$\mathring{d}^x(c(L), c(t_P)) \;\leq\; L - t_P \;\leq\; \lambda^2, \tag{10.13}$$

$$\mathring{d}^x(u(t_{k-1}), \Delta^x(c(t_{k-1}), c(t_k))) \;\leq\; (t_k - t_{k-1})\,\omega^1(\lambda), \tag{10.14}$$

$$\left|\int_0^L \bar{d}^x(x,u(t))\,\mathrm{d}t - \sum_{k=0}^{P-1}(t_{k+1} - t_k)\,\bar{d}^x(x,u(t_k))\right| \leq \omega^2(\lambda). \tag{10.15}$$

Indeed (10.12), (10.13) are a consequence of the fact that c is $\mathring{d}^x$-Lipschitz, (10.14) is a consequence of Egorov theorem applied to

$$f_\mu(t) \ = \ \delta^x_{\mu^{-1}} \Delta^x(c(t), c(t+\mu))$$

and (10.15) comes from the definition of the integral

$$l(c) \ = \ \int_0^L \bar{d}^x(x, u(t))\, \mathrm{d}t.$$

For each λ we shall choose $\varepsilon = \varepsilon(\lambda)$ and we shall construct a curve c_λ with the properties:

(i) $(x, c_\lambda) \in \mathcal{L}_{\varepsilon(\lambda)}(X, d, \delta)$,

(ii) $\lim_{\lambda \to 0} l^x_{\varepsilon(\lambda)}(c_\lambda) \ = \ l^x(c)$.

At almost every t the point $u(t)$ represents the velocity of the curve c seen as the the left translation of $\frac{\mathring{d}^x}{dt} c(t)$ by the group operation $\Sigma^x(\cdot, \cdot)$ to x (which is the neutral element for the mentioned operation). The derivative (with respect to $\mathring{\delta}^x$) of the curve c at t is

$$y(t) \ = \ \Sigma^x(c(t), u(t)).$$

Let us take $\varepsilon > 0$, arbitrary for the moment. We shall use the points of the division A_λ and for any $k = 0, ..., P-1$ we shall define the point:

$$y^\varepsilon_k \ = \ \hat{Q}^{x, c(t_k)}_\varepsilon \Sigma^x_\varepsilon(c(t_k), u(t_k)). \tag{10.16}$$

Thus y^ε_k is obtained as the "projection" by $\hat{Q}^{x,c(t_k)}_\varepsilon$ of the "approximate left translation" $\Sigma^x_\varepsilon(c(t_k), \cdot)$ by $c(t_k)$ of the velocity $u(t_k)$. Define also the point:

$$y_k \ = \Sigma^x(c(t_k), u(t_k)).$$

By construction we have:

$$y^\varepsilon_k \ = \ \hat{Q}^{x,c(t_k)}_\varepsilon y^\varepsilon_k \tag{10.17}$$

and by computation we see that y^ε_k can be expressed as:

$$y^\varepsilon_k \ = \ \delta^x_{\varepsilon^{-1}} Q^{\delta^x_\varepsilon c(t_k)} \delta^{\delta^x_\varepsilon c(t_k)}_\varepsilon u(t_k) \ = \tag{10.18}$$

$$= \ \Sigma^x_\varepsilon(c(t_k), Q^{\delta^x_\varepsilon c(t_k)} u(t_k)) \ = \ \delta^x_{\varepsilon^{-1}} \bar{\delta}^{\delta^x_\varepsilon c(t_k)}_\varepsilon Q^{\delta^x_\varepsilon c(t_k)} u(t_k).$$

Let us define the curve

$$c^\varepsilon_k(s) \ = \ \hat{\delta}^{x,c(t_k)}_{\varepsilon,s} y^\varepsilon_k, \quad s \in [0, t_{k+1} - t_k], \tag{10.19}$$

which is a $\hat{Q}^x_\varepsilon$-horizontal curve (by supplementary hypothesis (B)) which joins $c(t_k)$ with the point

$$z^\varepsilon_k \ = \ \hat{\delta}^{x,c(t_k)}_{\varepsilon, t_{k+1}-t_k} y^\varepsilon_k. \tag{10.20}$$

The point z_k^ε is an approximation of the point

$$z_k = \mathring{\delta}^{x,c(t_k)}_{t_{k+1}-t_k} y_k.$$

We shall also consider the curve

$$c_k(s) = \mathring{\delta}^{x,c(t_k)}_{s} y_k, \quad s \in [0, t_{k+1} - t_k]. \tag{10.21}$$

There is a short curve g_k^ε which joins z_k^ε with $c(t_{k+1})$, according to condition (Cgen). Indeed, for ε sufficiently small the points $\delta_\varepsilon^x z_k^\varepsilon$ and $\delta_\varepsilon^x c(t_{k+1})$ are sufficiently close.

Finally, take g_0^ε and g_{P+1}^ε "short curves" which join $c(0)$ with $c(t_0)$ and $c(t_P)$ with $c(L)$ respectively.

Correspondingly, we can find short curves g_k (in the geometry of the dilation structure $(U(x), \mathring{d}^x, \mathring{\delta}^x, \mathring{Q}^x)$) joining z_k with $c(t_{k+1})$, which are the uniform limit of the short curves g_k^ε as $\varepsilon \to 0$. Moreover this convergence is uniform with respect to k (and λ). Indeed, these short curves are made by N curves of the type $s \mapsto \hat{\delta}^{x,u_\varepsilon}_{\varepsilon,s} v_\varepsilon$, with $\hat{Q}^{x,u_\varepsilon} v_\varepsilon = v_\varepsilon$. Also, the short curves g_k are made respectively by N curves of the type $s \mapsto \mathring{\delta}^{x,u}_s v$, with $\mathring{Q}^{x,u} v = v$. Therefore we have:

$$\bar{d}(\mathring{\delta}^{x,u}_s v, \hat{\delta}^{x,u_\varepsilon}_{\varepsilon,s} y_k^\varepsilon) =$$

$$= \bar{d}(\Sigma^x(u, \bar{\delta}^x_s \Delta^x(u,v)), \Sigma^x_\varepsilon(u_\varepsilon, \bar{\delta}^{\delta^x_\varepsilon u_\varepsilon}_s \Delta^x_\varepsilon(u_\varepsilon, v_\varepsilon))).$$

By an induction argument on the respective ends of segments forming the short curves, using the axioms of coherent projections, we get the result.

By concatenation of all these curves we get two new curves:

$$c_\lambda^\varepsilon = g_0^\varepsilon \left(\prod_{k=0}^{P-1} c_k^\varepsilon g_k^\varepsilon \right) g_{P+1}^\varepsilon,$$

$$c_\lambda = g_0 \left(\prod_{k=0}^{P-1} c_k g_k \right) g_{P+1}.$$

From the previous reasoning we get that as $\varepsilon \to 0$ the curve c_λ^ε uniformly converges to c_λ, uniformly with respect to λ.

By theorem 10.9, specifically from relation (10.9) and considerations below, we notice that for any $u = Q^x u$ the length of the curve $s \mapsto \delta_s^x u$ is:

$$l^x(s \in [0,a] \mapsto \delta_s^x u) = a \bar{d}^x(x,u).$$

From here and relations (10.12), (10.13), (10.14), (10.15) we get that

$$l^x(c) = \lim_{\lambda \to 0} l^x(c_\lambda). \tag{10.22}$$

Condition (B) and the fact that $(X, \bar{d}, \bar{\delta})$ is tempered imply that there is a positive function $\omega^3(\varepsilon) = O(\varepsilon)$ such that

$$| l_\varepsilon^x(c_\lambda^\varepsilon) - l^x(c_\lambda) | \leq \frac{\omega^3(\varepsilon)}{\lambda}. \tag{10.23}$$

This is true because if $v \hat{Q}^{x,u}_{\varepsilon} v$ then $\delta^x_\varepsilon v = Q^{\delta^x_\varepsilon u} \delta^x_\varepsilon v$, therefore by condition (B)

$$\frac{l^x_\varepsilon(s \in [0,a] \mapsto \hat{\delta}^{x,u}_{\varepsilon,s} v)}{\delta^x_\varepsilon \bar{d}(u,v)} = \frac{\bar{l}(s \in [0,a] \mapsto \bar{\delta}^{\delta^x_\varepsilon u}_s \delta^x_\varepsilon v)}{\bar{d}(\delta^x_\varepsilon u, \delta^x_\varepsilon v)} \leq O(\varepsilon) + 1.$$

Since each short curve is made by N segments and the division A_λ is made by $1/\lambda$ segments, the relation (10.23) follows.

We shall choose now $\varepsilon(\lambda)$ such that $\omega^3(\varepsilon(\lambda)) \leq \lambda^2$ and we define:

$$c_\lambda = c^{\varepsilon(\lambda)}_\lambda .$$

These curves satisfy the properties (i), (ii). Indeed (i) is satisfied by construction and (ii) follows from the choice of $\varepsilon(\lambda)$, uniform convergence of c^ε_λ to c_λ, uniformly with respect to λ, and relations (10.23), and (10.22). □

11 Conclusion

In our opinion, the fact that sub-riemannian geometry may be described by about 12 axioms, **without using any a priori given differential structure**, is remarkable and it shows the power of the dilation structures approach. A geometry is not a simple object, for example euclidean geometry needs twice this number of axioms. It should be clear that renouncing to such a basic object as a differential structure is payed by the introduction of a number of axioms which might seem too high at the first view. It is not so high though; just for an example, the number of axioms for the euclidean geometry decreases dramatically once we use as basic objects the algebraic and topological structure of real numbers (or real vector spaces).

Let us go back to Gromov viewpoint that the only intrinsic object of a sub-riemannian space is the Carnot-Carathéodory distance. One of the most striking features of a regular sub-riemannian space is that it has at any point a metric tangent space, which algebraically is a Carnot group. This has been proved several times, by using the CC distance and lots of informations coming from the underlying differential structure of the manifold. Let us compare this with the result of Siebert, which characterizes homogeneous Lie groups as locally compact groups admitting a contracting and continuous one-parameter group of automorphisms. Siebert result has not a metric character.

In the work presented in this paper we tried to argue that we need more than only the CC distance in order to describe regular sub-riemannian manifolds, but less than the underlying differential structure: we need only dilation structures. Dilation structures bring forth the other intrinsic ingredient, namely the dilations, which are generalizations of Siebert' contracting group of automorphisms.

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