Deconstructing analysis I.
Dilatation structures for sub-Riemannian and fractal spaces

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Introduction

It is the purpose of this series of papers to identify and play with a set of axioms which can be seen as fundament of a significant part of analysis: differential calculus.

A similar deconstruction, concerning the notion of curvature, has been done in [...]. As in the mentioned paper, the conclusion is twofold:

1. it is a matter of taste to assume (or not) the axioms which give rise to differential calculus, because they visibly are products of the historical evolution of mathematics,

2. the understanding of point 1. leads us to a sea of unused possibilities, some of them in reach by a simple play with the axioms.

Better than anything else, the meaning of the word "deconstructing", as I understand and use it, can be found in "Le città invisibili" by Italo Calvino, namely in one of the discussions between Kublai Khan and Marco Polo (citation from the french translation [Calvino], page 55-56):

" Kublai Khan s'en était aperçu: les villes de Marco Polo se ressemblaient, comme si le passsage de l'une à l'autre n'eût pas impliqué un voyage mais un échange d'éléments. A présent, à partir de chaque ville que Marco lui décrivait, l'esprit du Grand Khan partait pour son propre compte et, la ville une fois démontée pièce à pièce, il la reconstruisait d'une autre façon, par substitutions, déplacements, interversions de ses ingrédients.

Marco cependant continuait à relater son voyage, mais l'empereur ne l'écoutait plus, il l'interrompait:

— A partir de maintenant, ce sera moi qui décrirai les villes et toi, tu vérifieras si elles existent et si elles sont bien telles que je les aurai pensées. [...]

— Sire, [...] du nombre des villes imaginables il faut exclure celles dont les éléments s'additionnent sans un fil qui les relie, sans règle interne, perspective ou discours. Il en est des villes comme des rêves: tout ce qui est imaginable peut être rêvé mais le rêve le plus surprenant est un rébus qui dissimule un désir, ou une peur, son contraire. [...]

— Moi, je n'ai ni désirs ni peurs, déclara le Khan, et mes rêves sont composés soit par mon esprit soit par le hasard.

— Les villes aussi se croient l'oeuvre de l'esprit ou du hasard [...]. Tu ne jouis pas d'une ville à cause de ses [...] merveilles, mais de la réponse qu'elle apporte à l'une de tes questions.

— Ou de la question qu'elle te pose, t'obligeant à répondre ... "

1 Length metric spaces and metric profiles

1.1 Length and distance

Definition 1.1 *(distance) A function $d : X \times X \to [0, +\infty)$ is a distance on X if:*

(a) $d(x, y) = 0$ if and only if $x = y$.

(b) for any x, y $d(x, y) = d(y, x)$.

(c) for any x, y, z $d(x, z) \leq d(x, y) + d(y, z)$.

(X, d) is called a metric space. The open ball with centre $x \in X$ and radius $r > 0$ is denoted by $B(x, r)$, or $B(x, r; d)$ if we want to emphasize the distance. The closed ball with centre $x \in X$ and radius $r \geq 0$ will be denoted by $\bar{B}(x, r; d)$.

If d ranges in $[0, +\infty]$ then it is called a pseudo-distance. The property of two points of being at finite distance is an equivalence relation. d is then a distance on each equivalence class (leaf).

Definition 1.2 *A map between metric spaces $f : X \to Y$ is Lipschitz if there is a positive constant C such that for any $x, y \in X$ we have*

$$d_Y(f(x), f(y)) \ \leq \ C \ d_X(x, y)$$

The least such constant is denoted by $Lip(f)$.

The (upper) dilatation of a map $f : X \to Y$ between metric spaces, in a point $u \in Y$ is

$$Lip(f)(u) = \limsup_{\varepsilon \to 0} \ \sup \left\{ \frac{d_Y(f(v), f(w))}{d_X(v, w)} \ : \ v \neq w \ , \ v, w \in B(u, \varepsilon) \right\}$$

The distorsion of a map $f : X \to Y$ is

$$dis \ f \ = \ \sup \left\{ |\ d_Y(f(y), f(y')) - d_X(y, y')\ | \ : y, y' \in X \right\}$$

In the particular case of a derivable function $f : \mathbb{R} \to \mathbb{R}^n$ the upper dilatation is $Lip(f)(t) \ = |\ \dot{f}(t)\ |$. For any Lipschitz function $f : X \to Y$ and for any $x \in X$ we have the obvious relation:

$$Lip(f)(x) \ \leq \ Lip(f) \ .$$

A curve is a function $f : [a, b] \to X$. The image of a curve is called path. Length measures paths. Therefore length does not depends on the reparametrisation of the path and it is additive with respect to concatenation of paths.

In a metric space (X, d) one can measure the length of curves in several ways.

Definition 1.3 *The length of a curve with L^1 dilatation $f : [a, b] \to X$ is*

$$L(f) = \int_a^b Lip(f)(t) \ dt$$

A different way to define a length of a curve is to consider its variation.

Definition 1.4 *The curve f has bounded variation if the quantity*

$$Var(f) \ = \ \sup\left\{\sum_{i=0}^{n} d(f(t_i), f(t_{i+1})) \ : \ a = t_0 < t_1 < ... < t_n < t_{n+1} = b\right\}$$

(called variation of f) is finite.

There is a third, more basic way to introduce the length of a curve in a metric space. For this we need first the definition of Hausdorff measures associated to a distance.

Definition 1.5 *Let $k > 0$ and (X, d) be a metric space. The k-Hausdorff measure is defined by:*

$$\mathcal{H}^k(A) \ = \ \mathcal{H}^k(A; d) \ = \ \lim_{\delta \to 0} \inf\left\{\sum_{i \in I} (diam \ E_i)^k \ : \ diam \ E_i < \delta \ , \quad A \subset \bigcup_{i \in I} E_i\right\}$$

Any k-Hausdorff measure is an outer measure. The Hausdorff dimension of a set A is defined by:

$$\mathcal{H} - dim \ A \ = \ \inf\left\{k > 0 \ : \ \mathcal{H}^k(A) \ = \ 0\right\}$$

Definition 1.6 *The length of the path $A = f([a, b])$ is the one-dimensional Hausdorff measure of the path. The definition is the following:*

$$l(A) \ = \ \lim_{\delta \to 0} \inf\left\{\sum_{i \in I} diam \ E_i \ : \ diam \ E_i < \delta \ , \quad A \subset \bigcup_{i \in I} E_i\right\}$$

The definitions are not equivalent. Let us see examples where the various notions of length give different results.

The variation $Var(f)$ of a curve f and the length of a path L(f) do not agree in general. Consider for example: $f : [-1, 1] \to R^2$, $f(t) \ = \ (t, \ sign(t))$. We have $Var(f) \ = \ 4$ and $L(f([-1, 1]) = 2$. Another example: the Cantor staircase function is continuous, but not Lipschitz. It has variation equal to 1 and length of the graph equal to 2.

Nevertheless, for Lipschitz functions, the first two definitions agree.

Theorem 1.7 *For each Lipschitz curve $f : [a, b] \to X$, we have $L(f) \ = \ Var(f)$.*

Variation of a curve and the Hausdorff measure of the associated path don't coincide. For example take a circle S^1 in R^n parametrised such that any point is covered two times. Then the variation is two times the length.

For injective Lipschitz functions (i.e. for simple Lipschitz curves) the last two definitions agree.

Theorem 1.8 *Suppose that $f : [a,b] \to X$ is a Lipschitz function and $A = f([a,b])$. Then $\mathcal{H}^1(A) \leq Var(f)$.*

If f is moreover injective then $\mathcal{H}^1(A) = Var(f)$.

An important tool used in the proof of the previous theorem is the geometrically obvious, but not straightforward to prove in this generality, Reparametrisation Theorem.

Theorem 1.9 *Any path $A \subset X$ with a Lipschitz parametrisation admits a reparametrisation $f : [a,b] \to A$ such that $Lip(f)(t) = 1$ for almost any $t \in [a,b]$.*

Another useful result relating various notions of length is the following.

Lemma 1.10 *If $f : [a,b] \to X$ is continuous then*

$$\mathcal{H}^1(f([a,b])) \ \geq \ d(f(a), f(b))$$

A good reference for the proofs, available online, is [Duda]. We shall not reproduce the proofs here.

We shall denote by l_d the length functional, defined only on Lipschitz curves, induced by the distance d.

The length induces a new distance d_l, say on any Lipschitz connected component of the space (X,d). The distance d_l is given by:

$$d_l(x,y) \ = \ \inf \ \{l_d(f([a,b])) : f : [a,b] \to X \ \text{ Lipschitz }, \ f(a) = x \ , \ f(b) = y\}$$

We have therefore two operators $d \mapsto l_d$ and $l \mapsto d_l$. Is one the inverse of another? The answer is no. This leads to the introduction of length metric spaces.

Definition 1.11 *A length metric space is a metric space (X,d) such that $d = d_l$.*

In terms of distances there is an easy criterion to decide if a metric space is length metric (Theorem 1.8., page 6-7, Gromov [12]).

Theorem 1.12 *A complete metric space is length metric if and only if (a) or (b) from above is true:*

(a) for any $x, y \in X$ and for any $\varepsilon > 0$ there is $z \in X$ such that

$$\max\{d(x,z), d(z,y)\} \ \leq \ \frac{1}{2} d(x,y) + \varepsilon$$

(b) for any $x, y \in X$ and for any $r_1, r_2 > 0$, if $r_1 + r_2 \leq d(x, y)$ then

$$d(B(x, r_1), B(y, r_2)) \ \leq \ d(x, y) - r_1 - r_2$$

Lenght metric spaces are geometrically more interesting, because one can define geodesics. Let c be any curve; denote the length of the restriction of c to the interval t, t' by $l_c(t, t')$.

Definition 1.13 *A (local) geodesic is a curve $c : [a, b] \to X$ with the property that for any $t \in (a, b)$ there is a small $\varepsilon > 0$ such that $c : [t - \varepsilon, t + \varepsilon] \to X$ is length minimising. A global geodesic is a length minimising curve.*

Therefore in a length metric space a local geodesic has the property that in the neighbourhood of any of it's points the relation

$$d(c(t), c(t')) = l_c(t, t')$$

holds. Any global geodesic is also local geodesic.

Can one join any two points with a geodesic?

The abstract Hopf-Rinow theorem (Gromov [12], page 9) states that:

Theorem 1.14 *If (X, d) is a connected locally compact length metric space then each pair of points can be joined by a global geodesic.*

Proof. It is sufficient to give the proof for compact length metric spaces. Given the points x, y, there is a sequence of curves f_h joining those points such that $l(f_h) \leq d(x, y) + 1/h$. The sequence, if parametrised by arclength, is equicontinuous; by Arzela-Ascoli theorem one can extract a subsequence (denoted also f_h) which converges uniformly to f. By construction the length function is lower semicontinuous hence:

$$l(f) \leq \liminf_{h \to \infty} l(f_h) \leq d(x, y)$$

Therefore f is a length minimising curve joining x and y. ■

The proofs of these results (excepting Lemma 1.10) have an important feature: they all involve reasoning with quantities of the type:

$$\frac{1}{\varepsilon} d(x_\varepsilon, y_\varepsilon) \to 0 \text{ as } \varepsilon \to 0 \ .$$

This fact will be of interest further.

1.2 Distances between metric spaces

The references for this subsection are Gromov [12], chapter 3, Gromov [10], and Burago & al. [8] section 7.4. There are several definitions of distances between metric spaces. The very fertile idea of introducing such distances belongs to Gromov.

In order to introduce the Hausdorff distance between metric spaces, recall the Hausdorff distance between subsets of a metric space.

Definition 1.15 *For any set $A \subset X$ of a metric space and any $\varepsilon > 0$ set the ε neighbourhood of A to be*

$$A_\varepsilon \ = \ \bigcup_{x \in A} B(x, \varepsilon)$$

The Hausdorff distance between $A, B \subset X$ is defined as

$$d_H^X(A, B) \ = \ \inf \{\varepsilon > 0 \ : \ A \subset B_\varepsilon \ , \ B \subset A_\varepsilon\}$$

By considering all isometric embeddings of two metric spaces X, Y into an arbitrary metric space Z we obtain the Hausdorff distance between X, Y (Gromov [12] definition 3.4).

Definition 1.16 *The Gromov-Hausdorff distance $d_{GH}(X, Y)$ between metric spaces X Y is the infimum of the numbers*

$$d_H^Z(f(X), g(Y))$$

for all isometric embeddings $f : X \to Z$, $g : Y \to Z$ in a metric space Z.

If X, Y are compact then $d_H(X, Y) < +\infty$.

The Hausdorff distance between isometric spaces equals 0. The converse is also true in the class of compact metric spaces (Gromov *op. cit.* proposition 3.6).

Likewise one can think about a notion of distance between pointed metric spaces. A pointed metric space is a triple (X, x, d), with $x \in X$. Gromov [10] introduced the distance between pointed metric spaces (X, x, d_X) and (Y, y, d_Y) to be the infimum of all $\varepsilon > 0$ such that there is a distance d on the disjoint sum $X \cup Y$, which extends the distances on X and Y, and moreover

- $d(x, y) < \varepsilon$,
- the ball $B(x, \frac{1}{\varepsilon})$ in X is contained in the ε neighbourhood of Y,
- the ball $B(y, \frac{1}{\varepsilon})$ in Y is contained in the ε neighbourhood of X.

Denote by $[X, x, d_X]$ the isometry class of (X, x, d_x), that is the class of spaces (Y, y, d_Y) such that it exists an isometry $f : X \to Y$ with the property $f(x) = y$.

The Gromov-Hausdorff distance between isometry classes of pointed metric spaces is almost a distance, in the sense that whenever two of the spaces $[X, x, d_X]$, $[Y, y, d_Y]$, $[Z, z, d_Z]$ have diameter at most equal to 2, then the triangle inequality for this distance is true. We shall use this distance and the induced convergence for isometry classes of the form $[X, x, d_X]$, with $diam\ X \ \leq 2$.

1.3 Metric profiles

The notion of metric profile was introduced in Buliga [6].

We shall denote by CMS the set of isometry classes of pointed compact metric spaces. The distance on this set is the Gromov distance between (isometry classes of) pointed metric spaces and the topology is induced by this distance.

To any locally compact metric space we can associate a metric profile.

Definition 1.17 *The metric profile associated to the locally metric space (M,d) is the assignment (for small enough $\varepsilon > 0$)*

$$(\varepsilon > 0,\ x \in M) \ \mapsto \ \mathbb{P}^m(\varepsilon, x) = [\bar{B}(x,1), x, \frac{1}{\varepsilon} d] \in CMS$$

We can define a notion of metric profile regardless to any distance.

Definition 1.18 *A metric profile is a curve $\mathbb{P} : [0,a] \to CMS$ such that:*

(a) it is continuous at 0,

(b) for any $b \in [0,a]$ and fixed $\varepsilon \in (0,1]$ we have

$$d_{GH}(\mathbb{P}(\varepsilon b), \mathbb{P}^m_{d_b}(\varepsilon, x)) \ = \ O(b)$$

We used here the notation $\mathbb{P}(b) = [\bar{B}(x,1), x, d_b]$ and $\mathbb{P}^m_{d_b}(\varepsilon, x) = [\bar{B}(x,1), x, \frac{1}{\varepsilon} d_b]$.

The metric profile is nice if

$$d_{GH}(\mathbb{P}(\varepsilon b), \mathbb{P}^m_{d_b}(\varepsilon, x)) \ = \ O(b\varepsilon)$$

We are using the notation $\mathcal{O}(\alpha)$ for any function $f(\alpha)$ with the property that $\lim_{\alpha \to 0^+} f(\alpha) \ = \ 0$.

A particular class of (pointed) metric spaces is formed by metric cones.

Definition 1.19 *A metric cone (X, x, d) is a locally compact metric space (X,d), with a marked point $x \in X$ such that for any $a, b \in (0,1]$ we have $\mathbb{P}^m(a,x) \ = \ \mathbb{P}^m(b,x)$.*

Metric cones are good candidates for being tangent spaces in the metric sense.

Definition 1.20 *A (locally compact) metric space (M,d) admits a (metric) tangent space in $x \in M$ if the metric profile $\varepsilon \mapsto \mathbb{P}^m(\varepsilon, x)$ admits a prolongation by continuity in $\varepsilon = 0$, i.e. if the following limit exists:*

$$[T_x M, x, d^x] \ = \ \lim_{\varepsilon \to 0} \mathbb{P}^m(\varepsilon, x) \ . \tag{1.3.1}$$

The connection between metric cones, tangent spaces and metric profiles in the abstract sense is made by the following proposition.

Proposition 1.21 *The metric profile $\varepsilon \mapsto \mathbb{P}^m(\varepsilon, x)$ of a metric space (M, d) for a fixed $x \in M$ is a metric profile in the sense of the definition 1.18 if and only if the space (M, d) admits a tangent space in x.*

In such a case the tangent space is a metric cone.

Proof. Indeed, a tangent space $[V, v, d_v]$ exists if and only if we have the limit from the relation (1.3.1). In this case the metric profile $\mathbb{P}^m(\cdot, x)$ can be prolonged to $\varepsilon = 0$. The prolongation is a metric profile in the sense of definition 1.18. Indeed, we have still to check the property (b). But this is trivial, because for any $\varepsilon, b > 0$, sufficiently small, we have

$$\mathbb{P}^m(\varepsilon b, x) \;=\; \mathbb{P}^m_{d_b}(\varepsilon, x)$$

where $d_b \;=\; (1/b)d$ and $\mathbb{P}^m_{d_b}(\varepsilon, x) = [\bar{B}(x, 1), \frac{1}{\varepsilon} d_b]$.

Finally, let us prove that the tangent space is a metric cone. Indeed, for any $a \in (0, 1]$ we have

$$[\bar{B}(x, 1), x, \frac{1}{a} d^x] \;=\; \lim_{\varepsilon \to 0} \mathbb{P}^m(a\varepsilon, x) \;.$$

Therefore $[\bar{B}(x, 1), x, \frac{1}{a} d^x] \;=\; [T_x M, x, d^x]$. ■

Metric cones have dilatations. By this we mean the following:

Definition 1.22 *Let (X, x, d) be a metric cone. For any $\varepsilon \in (0, 1]$ a dilatation is a function $\delta^x_\varepsilon : \bar{B}(x, 1) \to \bar{B}(x, \varepsilon)$ such that:*

- $\delta^x_\varepsilon(x) \;=\; x$,

- *for any $u, v \in X$ we have*

$$d(\delta^x_\varepsilon(u), \delta^x_\varepsilon(v) \;=\; \varepsilon \; d(u, v) \;.$$

The existence of dilatations for metric cones comes from the definition 1.19. Indeed, dilatations are just isometries from $(\bar{B}(x, 1), x, d)$ to $(\bar{B}, x, \frac{1}{a} d)$.

1.4 On the notion of curvature

The metric profile of a Riemannian homogeneous space is just a curve in the space CMS, continuous at 0. Likewise, if we look at a homogeneous regular sub-Riemannian manifold, the metric profile is not depending on points in the manifold.

More general, let us call a metric space (X, d) homogeneous if for any $x, y \in X$ there is an isometry $T : X \to X$ such that $T(x) = y$.

The metric profile of such a metric space is the same at any point. This fact allows to use homogeneous spaces for classifying the curvature of an (almost) arbitrary metric space in a point.

Before explaining this idea, let us take a look at a popular notion of curvature of a metric space: Alexandrov's. A locally compact length metric space (X, d) has Alexandrov curvature bounded from above by $c \in \mathbb{R}$ if triangles in X are thinner than triangles in $M(c)$, the homogeneous surface (of dimension 2) of constant curvature c. More precisely, $M(c)$ is the sphere of radius $1/c$, if $c > 0$, the plane if $c = 0$, and the hyperbolic plane of curvature c if $c < 0$. Equally one can define a notion of Alexander curvature bounded from below.

The comparison between triangles in (X, d) and $M(c)$ is made like this: to any 3 points A, B, C in X we associate 3 points A', B', C' in $(M(c), d')$ such that $d(A, B) = d'(A', B')$ and so on. The triangle $A'B'C'$ it is uniquely determined up to an isometry of $M(c)$. If the points A, B, C are close enough then there is at least a geodesic joining A and B, with a parametrisation by length $f(t)$, $t \in [0, d(A, B)]$. We represent then in $M(c)$ this geodesic by considering the congruent triangles $Af(t)B$ and $A'f'(t)B'$. This is leading us to the curve $f'(t)$ in $M(c)$. The space (X, d) has curvature bounded from above by c if for all $t \in [0, d(A, B)]$ the point $f'(t)$ lies inside the triangle $A'B'C'$, for any choice of points A, B, C and any choice of geodesic $f(t)$.

This is a property which can be written exclusively in terms of distances $d(A, B)$, $d(A, C), d(B, C), d(f(t), A), d(f(t), B), d(f(t), C)$.

Not all metric spaces are Alexandrov spaces in the sense just described. In fact, Alexandrov spaces have metric tangent spaces of a particular type and we know a lot of spaces which have different tangent spaces, hence they cannot be Alexandrov.

But the property of being Alexandrov is written in terms of distances between 4 points. This means that it is enough for us to have the table of distances between points, in order to check if a space is Alexander or not.

The metric profile of the space in a point gives us exactly the table of distances between points close to the base point, so the information that we have it is the same, only structured differently. We can imagine that the metric profile $\mathbb{P}^m(\varepsilon, x)$ is just the table of distances $\frac{1}{\varepsilon} d(u, v)$, $u, v \in \bar{B}(x, \varepsilon; d)$.

Different spaces which have the same table of distances, up to negligible terms, are said to have the same curvature. This can obviously be generalized to metric profiles.

Definition 1.23 *Two metric profiles $\mathbb{P}_1$, $\mathbb{P}_2$ are equivalent if*

$$\frac{1}{\varepsilon} d_{GH}((P_1(\varepsilon), \mathbb{P}_2(\varepsilon)) = \mathcal{O}(\varepsilon)$$

The curvature class of a metric profile $\mathbb{P}$ is the equivalence class of $\mathbb{P}$.

If it happens that the metric profile $\mathbb{P}^m(\varepsilon, x)$ is equivalent with a metric profile of a homogeneous space Y then we shall say that the space Y is the curvature of X in x.

These ideas were introduced and detailed in [buliga curvature] and [buliga srlie2].

2 From fractal sets to scaled metric spaces

In the previous chapter we have seen that for a length metric space there are several ways to define the length of a curve, all definitions being equivalent for simple Lipschitz curves. A key remark was that the proofs of the equivalences are all written in terms of scaled distances, that is, in terms of metric profiles $[\bar{B}(x,1), x, \frac{1}{\varepsilon}d]$.

We shall argue in this chapter that in the real world we don't work with a metric space (X,d), but merely we have access to incomplete informations concerning metric profiles functions

$$\varepsilon \in [a,b] \mapsto [\bar{B}(x,1), x, \frac{1}{\varepsilon}d], \quad , 0 < a < b \ .$$

There are two hypotheses that we use to make:

I. the metric profiles in different points correspond to a distance (we can make a map based on local measurements of the distance) up to experimental errors.

II. by an induction procedure, we guess the behaviour of the metric profile as $\varepsilon \to 0$. There can be no experimental check of this second hypothesis.

The accent that we put on metric profiles implies a new point of view on fractals. In few words and taking an example: the divergent (in length) series of measurements of distances between points on the coast of Britain is an experimental truth ; the fact that the coast of Britain can be seen as a fractal curve in euclidean space, of dimension 1.2, is less relevant (and impossible to check).

Taking fractals as example, it might make sense to think in terms of series of approximate distances, and not about sets in euclidean space. There are several reasons supporting this point of view:

1. the physical reason: what we have is the series of measured distances. Fractal sets in euclidean spaces indeed provide models of sets with the property that when we measure distances at different scales, we obtain series of such distances which behave qualitatively as the observed distances. (This corresponds to the previous hypothesis I.)

2. first mathematical reason: the realm of metric spaces is huge and we can produce metric spaces which are not euclidean at any scale. Is the geometry of the physical space (where the coast of Britain lies) euclidean at all scales? (This is to cast doubts about the soundness of the "guessing" part of hypothesis II.)

3. second mathematical reason: it is possible that the series of approximate distances are not in fact approximating any distance. Otherwise said: there might be no "exact" distance behind the series of measuring of approximate distances. (This seems an extreme situation. In fact I can't find reasons to disregard such a possibility).

One can easily reformulate the three definitions of lenght in terms of metric profiles. Then, these definitions become idealisations of procedures to approximate lenght in real world. It is then not very amazing that different procedures lead to different lenghts, in spaces that we call fractals.

2.1 Fractal sets and series of measurements

In this section we shall examine what happens when we are making photos of smaller and smaller regions of an object, using a microscope.

Here the words "microscope" and "object" could mean a lot of things. The microscope could be a real microscope and the object could be a piece of living tissue, or the microscope could be in fact a telescope used by an intelligent being from Mars, looking at the coast of Britain in greater and greater detail, trying to figure out what is the length of this coast.

There are other possibilities though. We can suppose that the image that we see using the microscope is an outcome of a set of actuators and the object is a region in the phase space of a physical system. Or, the image obtained with the microscope is a map in Euclidean space of a metric space which is non-euclidean.

We shall use a camera to make a series of photos of what we see when looking at the ocular of the microscope. When we take a photo by using a given microscope will shall say that we have made a measurement.

When we make a photo of the ocular, we note on the back of the photo the magnification which was used and the error bounds of measurements.

The outcome of the experiment is a series of photos, all of same dimensions (we use all the time the same kind and size of photographic paper).

We shall suppose that the ocular of the microscope has a cross-wire (which indicates a middle point in the ocular, called the center).

The following hypothesis will be made:

- The microscope is ideal in the sense that the magnification can be as big as we want.

- The microscope is ideal in the sense that the error bounds on the measurements can be as small as we want.

- The photo that we take after we change the magnification of after we re-center the microscope depends only on the magnification, error bounds of measurement and the region that we observe.

Each photo is therefore a metric space (we can measure distances between points) which is pointed (the center of the ocular always fall on the same spot in all photos). In fact we shall suppose that we model photos by a metric cone (Z, d, a) ("a" is the center). The points of the photo Z are called pixels.

The numerical value of $d(u, v)$ represents the distance measured on the photo between the pixels u and v, using a given ruler, the same for all photos. To make

things clearer, suppose that for all pixels $u \in Z$ we have $d(a,u) \leq 1$ and that there is at least one pixel $v \in Z$ such that $d(a,u) = 1$.

The object that we look at is a pointed space (X, x_0). Later, after some preparations, we shall add the hypothesis that:

- (X, x_0) is a scaled metric space.

- the photos are quasi-isometries.

Given a microscope, we shall neglect for the moment some of the error bounds of measurements. Namely we are going to suppose that in any photo, two different pixels represent different points in X.

We are going to take photos using the following iterative procedure:

The constant $C > 1$ is given.

Initial Step. Set the magnification $M = 1$. Position the object (or the microscope) such that we see the point x_0 at the pixel a. Set the counter $n = 1$. Take the photo $P(1, x_0)$.

Repeat:

i) Increment the counter $n \mapsto n+1$. Increment the magnification $M \mapsto CM$.

ii) Position the object (or the microscope) such that we see the point x_0 at the pixel a. Take the photo $P(n, x_0)$.

How can we know that the experiment is well done? We have to compare the photos. We restrict our attention to successive photos $P(n, x_0)$ and $P(n+1, x_0)$.

Notice that for the moment we keep the point x_0 fixed. For the moment we shall not mention the dependence of the photos on x_0.

$P(n)$ represents a photo of the region X_n of the object under study. When we modify the magnification from M to CM we see the region X_{n+1}. We shall have $X_{n+1} \subset X_n$. Any point $x \in X_{n+1}$ has two images:

- in the photo $P(n)$ it occupies the pixel $\alpha(x) \in Z$,

- in the photo $P(n+1)$ it occupies the pixel $\beta(x) \in Z$.

The experiment is well done if for any $x, y \in X_{n+1}$ we have

$$| \frac{1}{C} d(\beta(x), \beta(y)) - d(\alpha(x), \alpha(y)) | \leq \mu \quad , \tag{2.1.1}$$

where $\mu > 0$ is some very small positive number, considered to be negligible.

Otherwise said, let Z' be the image of the region X_{n+1}, in the photo $P(n)$. The criterion that the experiment is well done is that if we shrink the photo $P(n+1)$, using the inverse of the magnification coefficient, that is $1/C$, then what we see looks the same as Z'.

There are several facts neglected in this description. Indeed, let us not forget that:

1. a photo is not a function from the object X to the metric space Z. Indeed, we have no reason to suppose that two different but close pixels represents different points in X,

2. The microscope (and the camera) are not precise. There are everywhere tolerable errors, that are bounded by the constant μ which appears in relation (2.1.1).

3. We have not explained in what precise sense μ is a "very small, negligible" quantity.

We have to embed these errors into the description of the experiment, but before that let us see what we learn about our object X.

We shall call the inverse of the magnification $s = 1/M$ by the name "scale". After making a lot of photos of the object at the scale s we can basically construct maps of the object, at different scales. We shall not be too ambitious, i.e. we shall not claim that we can make an atlas (in differential geometric sense) of the object by using photos. Nevertheless we can suppose that we have (after the experiment) a way to measure distances comparable with s, simply by measuring distances between pixels in the photos made at the scale s.

Therefore we get an approximate distance d_s, at the scale s. The followings are natural questions, that we like to answer:

Q1. Are these distances approximate versions of a real distance?

Q2. How is the condition (2.1.1) modified by the introduction of error bounds of the type: two pixels sufficiently far apart represent different points in X?

Q3. How the distance d in the metric space Z influences the distances d_s ?

We shall try to answer these questions after a more precise description of the experiment.

2.2 Portraits of metric spaces

The purpose of this section is to explain what exactly is a photo and to relate this notion with the Gromov-Hausdorff distance between pointed metric spaces.

A photo is not a function from X to Z, but a relation in $X \times Z$.

We shall use the following convention: we denote by $f \subset X \times Z$ a relation and we write $f(x) = y$ if $(x, y) \in f$. Therefore we may have $f(x) = y$ and $f(x) = y'$ with $y \neq y'$, if $(x, y) \in f$ and $(x, y') \in f$.

The domain of f is the set of $x \in X$ such that there is $z \in Z$ with $f(x) = z$. We denote the domain by $dom\ f$. The image of f is the set of $z \in Z$ such that there is $x \in X$ with $f(x) = z$. We denote the image by $im\ f$.

By convention, when we state that a relation $R(f(x), f(y), ...)$ is true, it means that $R(x', y', ...)$ is true for any choice of $x', y', ...$, such that $(x, x'), (y, y'), ... \in f$.

Definition 2.1 *(Z, d, a) is a pointed metric space and $\mu > 0$. A μ-photo of (X, x_0) in (Z, d, a) is a relation $f \subset X \times Z$ which is almost a function. This means:*

1. *for any $x \in X$ there is $y \in Y$ such that $y = f(x)$,*
2. *if $f(x) = y$, $f(x') = y'$ and $d(y, y') > \mu$ then $x \neq x'$,*
3. *for any $z \in Z$ such that $(x_0, z) \in f$ we have $d(a, z) \leq \mu$.*

The photo is surjective if

4. *$Im\ f$ is μ-dense in Z.*

A choice of the photo f is by definition a function $\bar{f} : X' \to Z$ such that:

5. *for any $x' \in X'$ we have $(x', \bar{f}(x')) \in f$,*
6. *for any $x \in X$ and any $z \in Z$ with $f(x) = z$, there is $x' \in X'$ such that $d(z, \bar{f}(x')) \leq \varepsilon$.*

We explain now a well-known alternative definition of the Gromov-Hausdorff distance, up to a multiplicative factor.

Let (X_i, x_i, d_i), $i = 1, 2$, be a pair of pointed metric spaces and $\mu > 0$. We shall say that μ is admissible if there is a relation $\rho \subset X_1 \times X_2$ such that:

1'. *dom* ρ is μ-dense in X_1,

2'. *im* ρ is μ-dense in X_2,

3'. $(x_1, x_2) \in \rho$,

4' for all $x, y \in$ *dom* ρ we have :

$$| d_2(\rho(x), \rho_y) - d_1(x, y) | \leq 2\mu \ . \tag{2.2.2}$$

Proposition 2.2 *The Gromov-Hausdorff distance between (X_1, x_1, d_1) and (X_2, x_2, d_2) is (up to a multiplicative factor of 4) equal to the infimum of admissible numbers μ.*

This is well-known. In terms of photos, remark that for an admissible μ, the corresponding relation (or any corresponding relation) ρ is a surjective 2μ-photo (in the sense of definition 2.1) of (*dom* ρ, x_1) in (X_2, x_2, d_2). Indeed, this comes from relation (2.2.2).

Therefore the alternative description of Gromov-Hausdorff distance becomes: let us call $\mu > 0$ photo-admissible if there is a surjective 2μ-photo of (X_1, x_1) in (X_2, x_2, d_2) which satisfies (2.2.2). Then (3 times) Gromov-Hausdorff distance between (X_1, x_1, d_1) and (X_2, x_2, d_2) is equal to the infimum of all photo-admissible μ.

Consider now the definition of the Gromov-Hausdorff distance 1.16. We shall make a small modification of this definition in order to obtain the notion of portrait of a metric space.

Let (X, d_X) be a locally compact metric space and (Z, a, d) a pointed metric space. For any $x \in X$ and $\varepsilon > 0$, sufficiently small, we have defined the metric profile $\mathbb{P}^m(\varepsilon, x) = [\bar{B}(x, 1), \frac{1}{\varepsilon} d, x]$. How far is this space from a pointed subset of (Z, a, d)?

We are looking therefore only at photos of $\mathbb{P}^m(\varepsilon, x)$ in (Z, a, d). This justifies the following definition.

Definition 2.3 *A portrait of (X, d_X) in (Z, a, d) is a collection of photos*

$$\mathcal{P}(X, Z) = \left\{ f_\varepsilon^x \subset \bar{B}(x, \varepsilon; d_X) \times Z \, : \, x \in X, \ \varepsilon > 0 \right\}$$

such that for all $x \in X$ and $\varepsilon > 0$ (sufficiently small), we have

(a) dom $f_\varepsilon^x = \bar{B}(x, \varepsilon; d_X)$,

(b) $(x, a) \in f_\varepsilon^x$,

(c) for all $\mu > 0$ there is $\varepsilon(\mu) > 0$ such that for any $\varepsilon \in (0, \varepsilon(\mu)]$ and any $y, z \in \bar{B}(x, \varepsilon; d_X)$ we have

$$| \frac{1}{\varepsilon} d_X(y, z) - d(f_\varepsilon^x(y), f_\varepsilon^x(z)) | \leq \mu \, . \tag{2.2.3}$$

Given (Z, a, d), not any metric space (X, d_X) admits a portrait in Z. This is an important fact that we shall explain later.

A portrait of X in Z has an important property which can be expressed without using the distance d_X.

Proposition 2.4 *With the notations from definition 2.3, for any $\mu > 0$ there is $\varepsilon(\mu) > 0$ such that for any $x \in X$, any $\varepsilon \in (0, \varepsilon(\mu)]$, any $\alpha \in (0, 1]$ and any $y, z \in \bar{B}(x, \alpha\varepsilon; d_X)$ we have*

$$| \alpha \, d(f_{\alpha\varepsilon}^x(y), f_{\alpha\varepsilon}^x(z)) - d(f_\varepsilon^x(y), f_\varepsilon^x(z)) | \leq \mu(1 + \alpha) \, . \tag{2.2.4}$$

Proof. Indeed, using the estimate (2.2.3), with the notation from the proposition, we have

$$\begin{aligned} &| \alpha \, d(f_{\alpha\varepsilon}^x(y), f_{\alpha\varepsilon}^x(z)) - d(f_\varepsilon^x(y), f_\varepsilon^x(z)) | \leq \\ &\leq \alpha \, | \frac{1}{\alpha\varepsilon} d_X(y, z) - d(f_{\alpha\varepsilon}^x(y), f_{\alpha\varepsilon}^x(z)) | + \\ &\quad + | \frac{1}{\varepsilon} d_X(y, z) - d(f_\varepsilon^x(y), f_\varepsilon^x(z)) | \leq \\ &\leq \mu(1 + \alpha) \, . \quad \blacksquare \end{aligned}$$

2.3 Metric portraits and scaled metric spaces

The notions of metric portrait and scaled metric space, that we introduce in this section, are somehow dual. From a purist point of view, we don't need both notions, one of them is sufficient. But from a more realistic point of view, there is a need to have both notions. As you shall see, a metric portrait induces a scaled metric space. On the other hand, if we "have" a scaled metric space, we shall ask if we can make a metric portret of this space in a given metric space.

Scaled metric spaces are a generalisation of metric spaces. Do we need a generalisation, is the notion of distance not enough general? The point of view of this paper is that in many real cases we don't "have" a metric space, but merely partial informations that we interpret as distances between points in the space that we are studying. Based on this partial information, our imagination and historical context, we suppose that our space is a metric space of a special kind. While in classical mathematics it is accepted to claim that "we have" a metric space (or a manifold), or that a real space (configuration space, phase space, ...) "is" a metric space (or a manifold), recently in real life we have more and more to do with large and/or incomplete quantities of data that we work hard to fit in a metric space. The notions of scaled metric space and metric portrait are proposed as a small step ahead in this effort: what we are dealing with are metric portraits of scaled metric spaces.

Further we shall denote by $\mathcal{O}(\mu)$ any function $f = f(\mu)$ with the property that $\lim_{\mu \to 0} f(\mu) = 0$.

In the microscope interpretation, suppose that we have made photos for all scales (1/magnification) $\varepsilon \in (0, 1]$, centered at all points $x \in X$, with all precisions $\mu > 0$. If things were well done, we get a metric portrait.

Definition 2.5 *A homogeneous metric cone (Z, a, d) is the unit ball of center $a \in Z$, in a homogeneous metric space (Y, d) (i.e. $Isom(Y, d)$ acts transitively on Y), such that (Z, a, d) is a metric cone.*

Definition 2.6 *Let (Z, a, d) be a homogeneous metric cone and X be a space. A metric portrait of X in (Z, a, d) is a collection of photos f_ε^x with the properties: for any $\mu > 0$ there is $\varepsilon(\mu) > 0$ such that for any $x \in X$ and $\varepsilon \in (0, \varepsilon(\mu))$*

(a) $f_\varepsilon^x \subset X_\varepsilon^x \times Z$ is a relation with domain dom $f_\varepsilon^x = X_\varepsilon^x \subset X$,

(b) $(x, a) \in f_\varepsilon^x$,

(c) for all $\alpha \in (0, 1]$ we have $X_{\alpha\varepsilon}^x \subset X_\varepsilon^x$,

(d) for all $\alpha \in (0, 1]$ and $y, z \in X_{\alpha\varepsilon}^x$ we have

$$| \alpha \, d(f_{\alpha\varepsilon}^x(y), f_{\alpha\varepsilon}^x(z)) - d(f_\varepsilon^x(y), f_\varepsilon^x(z)) | \leq \mu \, . \tag{2.3.5}$$

Here comes the definition of a scaled metric space.

Definition 2.7 *A scaled metric space is a set X endowed with a family of distances d_ε^x, with the properties: for any $\mu > 0$ there is $\varepsilon(\mu) > 0$ such that for any $x \in X$ and $\varepsilon \in (0, \varepsilon(\mu))$ we have the distance*

$$d_\varepsilon^x : X_\varepsilon^x \times X_\varepsilon^x \to [0, +\infty)$$

such that

(a) $x \in X_\varepsilon^x$,

(b) for all $\alpha \in (0, 1]$ *we have* $X_{\alpha\varepsilon}^x \subset X_\varepsilon^x$,

(c) for all $\alpha \in (0, 1]$ *and* $y, z \in X_{\alpha\varepsilon}^x$ *we have*

$$| \alpha \ d_{\alpha\varepsilon}^x(y, z) \ - \ d_\varepsilon^x(y, z) | \leq \mu \ . \tag{2.3.6}$$

(d) there is a constant M which does not depend on x, ε, μ, such that diam $X_\varepsilon^x \leq M$ ($M = 3$ is a good constant, for example).

2.4 Examples, comments, justifications

Definitions 2.6 and 2.7 certainly need some explanations. We start with the first definition, concerning metric portraits.

First of all, notice that we have asked nothing about the set X. We shall always suppose that X is not empty, but otherwise it can be any set, without any extra structure.

This is not so weird. Using words, the definition says that at any scale ε, we can cover X with "approximate" charts f_ε^x, such that a compatibility between charts holds: the relation (2.3.5). Thus we arrive at our first example of metric portrait.

Example 1. Portraits in the sense of definition 2.3 are metric portraits. This is shown in proposition 2.4.

Example 2. Any metric space can be seen as a scaled metric space. Indeed, let (X, d) be a metric space, $\mu > 0$, $\varepsilon \in (0, 1]$. Define then for any $x \in X$ the scaled distance:

$$d_\varepsilon^x : X \times X \to [0, +\infty) \ , \ \ d_\varepsilon^x(y, z) \ = \ \frac{1}{\varepsilon} d(y, z) \ .$$

We compute the left hand side of relation (2.3.6):

$$| \alpha \ d_{\alpha\varepsilon}^x(y, z) \ - \ d_\varepsilon^x(y, z) \ | = | \ \alpha \frac{1}{\alpha\varepsilon} d(y, z) \ - \ \frac{1}{\varepsilon} d(y, z) \ | = \ 0 \ .$$

A metric space is an infinitely precise scaled metric space. ■

Example 3. A portrait of a scaled metric space is a metric portrait. Indeed, we start by making a modification of definition 2.3.

Definition 2.8 *A portrait of a scaled metric space (X, d_ε^x) in (Z, a, d) is a collection of photos*

$$\mathcal{P}(X, Z) \ = \ \{f_\varepsilon^x \subset X_\varepsilon^x \times Z \ : \ x \in X, \ \varepsilon > 0\}$$

such that for all $x \in X$ and $\varepsilon > 0$ (sufficiently small), we have

(a) dom $f_\varepsilon^x \ = \ X_\varepsilon^x$,

(b) $(x, a) \in f_\varepsilon^x$,

(c) for all $\mu > 0$ there is $\varepsilon(\mu) > 0$ such that for any $\varepsilon \in (0, \varepsilon(\mu)]$ and any $y, z \in X_\varepsilon^x$ we have

$$| \ d_\varepsilon^x(y, z) \ - \ d(f_\varepsilon^x(y), f_\varepsilon^x(z)) \ | \leq \ \mu \ . \tag{2.4.7}$$

The quivalent of proposition 2.4 is the following.

Proposition 2.9 *With the notations from definition 2.8, $\mathcal{P}(X, Z)$ is a metric portrait.*

Proof. We shall prove that for any $\mu > 0$ there is $\varepsilon(\mu) > 0$ such that for any $x \in X$, any $\varepsilon \in (0, \varepsilon(\mu)]$, any $\alpha \in (0, 1]$ and any $y, z \in X_{\alpha\varepsilon}^x$ we have

$$| \ \alpha \ d(f_{\alpha\varepsilon}^x(y), f_{\alpha\varepsilon}^x(z)) \ - \ d(f_\varepsilon^x(y), f_\varepsilon^x(z)) \ | \leq \ \mu(2 + \alpha) \ . \tag{2.4.8}$$

By using the estimates (2.4.7) and (2.3.6) we have (for a well chosen $\varepsilon(\mu)$)

$$\begin{aligned} &| \ \alpha \ d(f_{\alpha\varepsilon}^x(y), f_{\alpha\varepsilon}^x(z)) \ - \ d(f_\varepsilon^x(y), f_\varepsilon^x(z)) \ | \leq \\ &\leq \ \alpha \ | \ d_{\alpha\varepsilon}^x(y, z) \ - \ d(f_{\alpha\varepsilon}^x(y), f_{\alpha\varepsilon}^x(z)) \ | \ + \\ &+ \ | \ \alpha \ d_{\alpha\varepsilon}^x(y, z) \ - \ d^x(y, z) \ | \ + \\ &+ \ | \ d_\varepsilon^x(y, z) \ - \ d(f_\varepsilon^x(y), f_\varepsilon^x(z)) \ | \leq \\ &\leq \ \mu(2 + \alpha) \ . \quad \blacksquare \end{aligned}$$

Example 4. Let M be a real n dimensional manifold and

$$\mathcal{A} \;=\; \left\{\phi_x : U_x \subset M \to \bar{B}_1 \subset \mathbb{R}^N \,:\, x \in M,\; x \in U_x,\; \phi_x(x) \;=\; 0\right\}$$

an atlas of M made of charts ϕ_x such that ϕ_x sends $x \in U_x$ to $0 \in \mathbb{R}^n$.

$\bar{B}_1$ is the unit ball of center $a = 0$ in $\mathbb{R}^n$. It is a homogeneous metric cone in the sense of definition 2.5.

Let $\mu > 0$, $\varepsilon \in (0, 1]$. Denote by $\delta_\varepsilon : \bar{B}_1 \to \bar{B}_1$ the map $\delta_\varepsilon x = \varepsilon x$. It is a dilatation of coefficient ε.

Let $x \in M$. Define

$$X^x_\varepsilon \;=\; (\phi_x)^{-1}\, \delta_\varepsilon \bar{B}_1 \;,\quad f^x_\varepsilon(y) \;=\; (\delta_\varepsilon)^{-1}\, \phi_x(y) \;.$$

Therefore f^x_ε is a function.

Let $\alpha \in (0, 1]$ and $y, z \in X^x_{\alpha\varepsilon}$. We compute the left hand side of relation (2.3.5):

$$\begin{aligned} &| \;\alpha\; d\,(f^x_{\alpha\varepsilon}(y), f^x_{\alpha\varepsilon}(z)) \;-\; d\,(f^x_\varepsilon(y), f^x_\varepsilon(z)) \;| = \\ &= | \;\alpha\; \| \,(\delta_{\alpha\varepsilon})^{-1}\, \phi_x(y) - (\delta_{\alpha\varepsilon})^{-1}\, \phi_x(z)\| - \| \,(\delta_\varepsilon)^{-1}\, \phi_x(y) - (\delta_\varepsilon)^{-1}\, \phi_x(z)\| \;| = \\ &= | \;\alpha \| \frac{1}{\alpha\varepsilon}\phi_x(y) - \frac{1}{\alpha\varepsilon}\phi_x(z)\| - \| \frac{1}{\varepsilon}\phi_x(y) - \frac{1}{\varepsilon}\phi_x(y)\| \;| \;=\; 0 \;. \end{aligned}$$

Therefore trivially this is a metric portrait.

In particular let (M, g) be a riemannian manifold of dimension n. For any $x \in M$ we denote by $\exp_x : V_x \subset T_xM \to M$ the riemannian exponential. We shall suppose (or otherwise we shall reason locally on M) that $\bar{B}_1(x) \subset T_xM$, the unit ball in the tangent space at x to M, with respect to the riemannian norm on T_xM induced by the metric g_x.

We shall denote by d_M the distance induced by the riemannian metric on M. The closed balls with respect to this distance are denoted by:

$$\bar{B}(x, \varepsilon; d_M) \;=\; \{y \in M \,:\, d_M(x, y) \leq \varepsilon\} \;.$$

For any $x \in X$ choose a function $\phi_x :\; \bar{B}_1(x) \to \bar{B}_1 \subset \mathbb{R}^n$ which is an isometry, that is for all $u, v \in \bar{B}_1(x)$ we have

$$g_x(u, v) \;=\; \langle \phi_x(u), \phi_x(v)\rangle \;,\quad \phi_x(0) \;=\; 0 \;,$$

where $\langle \cdot, \dot{}\, \rangle$ is the "canonical" scalar product on $\mathbb{R}^n$: for any $X, Y \in \mathbb{R}^n$

$$\langle X, Y\rangle \;=\; \sum_{i=1}^n X_i Y_i \;.$$

We shall use the dilatations δ_ε in $\mathbb{R}^n$. Let us define, for $\varepsilon \in (0, 1]$, the sets

$$X^x_\varepsilon \;=\; \bar{B}(x, \varepsilon; d_M) \;.$$

The relations f_ε^x are functions in this example, too. They are defined by: for any $y \in barB(x, \varepsilon; d_M)$ we have

$$f_\varepsilon^x(y) \ = \ \Big(\exp_x \ (\phi_x)^{-1} \ \delta_\varepsilon\Big)^{-1}(y) \ . \tag{2.4.9}$$

We compute the left hand side of relation (2.3.5):

$$|\ \alpha \ d\,(f_{\alpha\varepsilon}^x(y), f_{\alpha\varepsilon}^x(z)) \ - \ d\,(f_\varepsilon^x(y), f_\varepsilon^x(z))\ | = \ 0 \ .$$

In few words, an atlas of a manifold can be seen as an infinitely precise metric portrait.

Example 5. Let us modify the previous construction in order to get a more interesting example. Instead of using the exponential associated to the metric g, for any point $x \in M$ we shall replace g by g_ε^x, a metric which is a second order approximation of g in x. We repeat the construction done before, only that we replace in (2.4.9) the exponential $\exp_x$ with respect to g, by the exponential $(\exp_\varepsilon^x)$ with respect to g_ε^x:

$$f_\varepsilon^x(y) \ = \ \Big((\exp_\varepsilon^x)_x \ (\phi_x)^{-1} \ \delta_\varepsilon\Big)^{-1}(y) \ .$$

Such constructions are explained, for the more general situation of sub-riemannian manifolds, in section ...

We compute the left hand side of relation (2.3.5):

$$|\ \alpha \ d\,(f_{\alpha\varepsilon}^x(y), f_{\alpha\varepsilon}^x(z)) \ - \ d\,(f_\varepsilon^x(y), f_\varepsilon^x(z))\ | =$$

$$= |\ \alpha \ \frac{1}{\alpha\varepsilon} \ \|\phi_x\Big((\exp_{\alpha\varepsilon}^x)_x^{-1}(y)\Big) \ - \ \phi_x\Big((\exp_{\alpha\varepsilon}^x)_x^{-1}(z)\Big)\| \ -$$

$$- \ \frac{1}{\varepsilon} \ \|\phi_x\Big((\exp_\varepsilon^x)_x^{-1}(y)\Big) \ - \ \phi_x\Big((\exp_\varepsilon^x)_x^{-1}(z)\Big)\| \ | \ \leq$$

$$\frac{1}{\varepsilon} \ \Big(\|\,(\exp_{\alpha\varepsilon}^x)_x^{-1}(y) \ - \ (\exp_\varepsilon^x)_x^{-1}(y)\| \ + \ \|\,(\exp_{\alpha\varepsilon}^x)_x^{-1}(z) \ - \ (\exp_\varepsilon^x)_x^{-1}(z)\|\Big) \ \leq$$

$$\leq 2(1-\alpha)K(x)\varepsilon \ ,$$

where $K(x)$ is a number with the meaning of a Gauss curvature.

Example 6. We shall look to fractals now. For the definition of IFS and the fractal associated to an IFS, go to section ... . Approximation by a dense set.

Example 7. Coast of Britain. In order to understand the relation (2.3.5), we shall repeat the microscope experiment and we shall give a more precise formulation of (2.1.1). We shall obtain a (discretized version of a) metric portrait after the microscope experiment.

The scale ε and the coefficients α will be discretized, in the sense that we shall consider a positive number $C > 1$ and ε, α will be powers of C^{-1}.

We shall use this experiment for understanding the definition of a scaled metric space as well.

The experiment works as following. Set $\lambda > 0$ to be the precision of the microscope. Set $x \in X$, arbitrary but fixed. Set the magnification $M = 1$.

Repeat:

Initial step. Take a photo f^x_M. Let $X^x_M = dom\ f^x_M$.

(i) make a choice $\bar{f}^x_M : X^x_M \to Z$ for the photo f^x_M and define the distance d^x_M on X^x_M by

$$d^x_M(z, y) \ = \ d(\bar{f}^x_M(z), \bar{f}^x_M(y)) \ \ ,$$

ii) Set the magnification $M \mapsto C^{-1}M$.

(iii) Take a photo f^x_M. Let $X^x_M = dom\ f^x_M$ such that $X^x_M \subset X^x_{CM}$ and there is a dilatation δ (composed with an isometry of Z), of coefficient C^{-1}, with the property: for any $y \in X^x_M$, any $u = f^x_M(y)$ and any $y' = f^x_{CM}(y)$ we have

$$d(\delta(u), y') \leq \ \mu \ \ . \tag{2.4.10}$$

Suppose now that we have performed the experiment with better and better microscopes, such that the error $\mu \to 0$.

Theorem 2.10 *Let $m, n \in \mathbb{N}$, $n, m \geq 1$ and $x \in X$. Set $\varepsilon = C^{-n}, \alpha = C^{-m}$. Then for any $y, z \in X^x_{\alpha\varepsilon}$ we have the relation similar to (2.3.5), that is:*

$$|\ \alpha\ d\,(f^x_{\alpha\varepsilon}(y), f^x_{\alpha\varepsilon}(z)) \ - \ d\,(f^x_{\varepsilon}(y), f^x_{\varepsilon}(z))\ | \leq \ \frac{2C}{C-1}\mu(1-\alpha)\ . \tag{2.4.11}$$

We equally have a relation similar to (2.3.6), that is:

$$|\ \alpha\ d^x_{\alpha\varepsilon}(y, z) \ - \ d^x_{\varepsilon}(y, z)\ | \leq \ \frac{3C+1}{C-1}\mu(1-\alpha)\ . \tag{2.4.12}$$

Before giving the proof of the theorem let us see what is nice in these estimates: the right hand term does not depend on scale. It means that the error is not proportional with the scale, but with the difference of scale α! The same phenomenon is encountered in the definition of metric profiles.

Proof. It is enough to prove the estimate (2.3.5) for $m = 1$ (hence $\alpha = C^{-1}$). The general estimates follow then by induction upon m and unwinding the step (i) in the experiment (the definition of the distances d^x_ε).

Take $y, z \in X^x_{\alpha\varepsilon}$. Denote by $u = f^x_{\alpha\varepsilon}(y)$, $v = f^x_{\alpha\varepsilon}(z)$, $u' = f^x_\varepsilon(y)$, $v' = f^x_\varepsilon(z)$. Let δ be a dilatation (composed with an isometry), of coefficient C^{-1}, such that

$$d(\delta(u), u') \leq \mu \ , \ d(\delta(u), u') \leq \mu \ .$$

Then we have

$$| \ \frac{1}{C} d(u, v) - d(u', v') \ | = | \ d(\delta(u), \delta(v)) - d(u', v') \ | \leq \ 2\mu \ .$$

For $m > 1$ remark that that $C^{-m} = \alpha$. As previously, for $i = 0, ..., m$, let us denote $u_i = f^x_{C^{-i}\varepsilon}(y)$, $v_i = f^x_{C^{-i}\varepsilon}(z)$. We shall have, for any $i = 0, ..., m-1$

$$| \ \frac{1}{C} d(u_{i+1}, v_{i+1}) - d(u_i, v_i) \ | \leq \ 2\mu \ ,$$

therefore

$$| \ \frac{1}{C^m} d(u_m, v_m) - d(u_0, v_0) \ | \leq \ \sum_{i=0}^{m-1} \frac{1}{C^i} \ | \ \frac{1}{C} d(u_{i+1}, v_{i+1}) - d(u_i, v_i) \ | \leq$$

$$\leq \ 2\mu \sum_{i=0}^{m-1} \frac{1}{C^i} \ = \ \frac{2C}{C-1} \mu \left(1 - \frac{1}{C^m} \right) \ .$$

But this is equivalent with (2.4.11).

The relation (2.4.12) has a similar proof. There is only one difference. For $m = 1$, with the notations used before, we have:

$$| \ \alpha d^x_{\alpha\varepsilon}(y, z) \ - \ d^x_\varepsilon(y, z) \ | \leq \ \frac{1}{C} \ | \ d^x_{C^{-1}\varepsilon}(y, z) - d(u, v) \ | + | \ d^x_\varepsilon(y, z) - d(u', v') \ | +$$

$$+ | \ \frac{1}{C} d(u, v) - d(u', v') \ | \leq \ \left(3 + \frac{1}{C} \right) \mu \ .$$

This inequality, applied repeatedly, will lead for $m > 1$, to

$$| \ \frac{1}{C^m} d^x_{C^{-m}\varepsilon}(y, z) \ - \ d^x_\varepsilon(y, z) \ | \leq \left(3 + \frac{1}{C} \right) \mu \sum_{i=0}^{m-1} \frac{1}{C^i} \ = \ \frac{3C+1}{C-1} \mu \left(1 - \frac{1}{C^m} \right) \ ,$$

which is equivalent to (2.4.12). ■

These inequalities justify the definitions of metric portrait and scaled metric space. There is however one question that naturally appears: the discretized version of metric portraits and scaled metric spaces involve a constant $C > 1$. Is this a parameter independent on μ?

The answer is "yes". If we deal with real microscopes, the notion which physicists use is "power of separation". This is a quantity d_0 related to the the magnification C by the relation

$$C\frac{d_0}{diam\ Z} \ = \ \mathcal{O}(\mu) \ .$$

This incidentally shows that in reality we cannot repeat indefinitely the steps of the experiment, for a given microscope.

From the definition 2.6 we have, for any $x \in X$ and $\alpha \in (0,1]$:

$$\lim_{\varepsilon \to 0} \sup \left\{ \mid d^x_{\alpha\varepsilon}(y_1,y_2) - \frac{1}{\alpha} d^x_\varepsilon(y_1,y_2) \mid \ : \ y_1,y_2 \in X^x_{\alpha\varepsilon} \right\} \ = \ 0 \ . \tag{2.4.13}$$

The relation (2.4.13) establishes the connection between metric portraits and metric profiles. Suppose that the photos f^x_ε are surjective. Then X^x_ε are almost unit balls (with center x), with respect to the distances d^x_ε. If any X^x_ε is compact with respect to the topology induced by the distance d^x_ε then the relation (2.4.13) shows that

$$\lim_{\varepsilon \to 0} d_{GH} \left([X^x_{\alpha\varepsilon}, d^x_{\alpha\varepsilon}, x] \, , \left[X^x_{\alpha\varepsilon}, \frac{1}{\alpha} d^x_\varepsilon, x \right] \right) \ = \ 0 \ .$$

This relation is closed to one of the conditions for $[X^x_\varepsilon, d^x_\varepsilon, x]$ to be a metric profile. There are still to things missing:

- an estimate of the Gromov-Hausdorff distance between

$$[X^x_\varepsilon, d^x_\varepsilon, x] \quad \text{and} \quad \left[\bar{B}(x,1), \frac{1}{a} d^x_\varepsilon, x \right] \ ,$$

- that the curve $\varepsilon \mapsto [X^x_\varepsilon, d^x_\varepsilon, x]$ converges when ε goes to 0. This may not be true in general. If it is true then the limit deserves the name "tangent space" in x to the scaled metric space X.

3 Quasi-isometric scaled metric spaces

Scaled metric spaces and their portraits might be notions which are too general. Further we collect several properties and links between scaled metric spaces or portraits to more standard subjects. We start with the relation of quasi-isometric equivalence between scaled metric spaces. After this we shortly examine some results concerning the Geometric Traveling Salesman Problem. We discuss then about the length notion in scaled metric spaces and about curves with infinite observed length, as measured with the help of a metric portrait. We suggest that the "fractal" nature of some measurements is a natural thing that may happen if we use metric portraits.

3.1 Quasi-isometry equivalence

Definition 3.1 *Two scaled metric spaces $(X, d_\varepsilon^{1,x})$ and $(X, d_\varepsilon^{2,x})$, defined over the same set X, are quasi-isometric if there exists $R > 0$ with the properties that for any $\mu > 0$ there is $\varepsilon(\mu) > 0$ such that for all $x \in X$ and all $\varepsilon \in (0, \varepsilon(\mu)]$:*

(a) we have the inclusions:

$$\bar{B}(x, R - \mu; d_\varepsilon^{1,x}) \subset \bar{B}(x, R; d_\varepsilon^{2,x}) \subset \bar{B}(x, R + \mu; d_\varepsilon^{1,x}) \ ,$$

$$\bar{B}(x, R - \mu; d_\varepsilon^{2,x}) \subset \bar{B}(x, R; d_\varepsilon^{1,x}) \subset \bar{B}(x, R + \mu; d_\varepsilon^{2,x}) \ ,$$

(b) for any $u, v \in X_\varepsilon^{1,x} \cap X_\varepsilon^{2,x}$ we have

$$\mid d_\varepsilon^{1,x}(u, v) \ - \ d_\varepsilon^{2,x}(u, v) \mid \leq \ \mu \ .$$

The following proposition describes an important property of scaled metric spaces quasi-isometric with metric spaces.

Proposition 3.2 *Let (X, d_ε^x) be a scaled metric space which is quasi-isometric with a metric space (X, d). Then there exists $R > 0$ such that for any $\alpha \in (0, \frac{1}{2})$ there is $\varepsilon(\alpha) > 0$ such that for any $\varepsilon \in (0, \varepsilon(\alpha))$ we have:*

(a) for any $y \in X$, if $d_\varepsilon^x(x, y) \ \leq \ \alpha R$ then

$$\bar{B}(y, \alpha R; d_\varepsilon^y) \subset \bar{B}(x, R; d_\varepsilon^x) \ ,$$

$$\bar{B}(x, \alpha R; d_\varepsilon^x) \subset \bar{B}(y, R; d_\varepsilon^y) \ ,$$

(b) Define the intrinsic distorsion

$$\Delta(x, R, \alpha, \varepsilon) \ = \ \sup \left\{ \mid d_\varepsilon^y(u, v) - d_\varepsilon^x(u, v) \mid \ : d_\varepsilon^x(x, y) \ \leq \ \alpha R \ , u, v \in \bar{B}(y, \alpha R; d_\varepsilon^y) \right\} \ .$$

We have then

$$\lim_{\varepsilon \to 0} \Delta(x, R, \alpha, \varepsilon) \ = \ 0$$

uniformly with respect to x.

Proof. Relations from point (a) are obvious for a true metric space. Recall that in example 2 we have described a metric space as a scaled metric space with scaled distances

$$d^x_\varepsilon(y,z) \ = \ \frac{1}{\varepsilon} d(y,z) \ .$$

The inclusions from point (a), written for the true metric space, are then:

$$\bar{B}(y, \varepsilon\alpha R; d) \subset \bar{B}(x, \varepsilon R; d) \ ,$$

$$\bar{B}(x, \varepsilon\alpha R; d) \subset \bar{B}(y, \varepsilon R; d) \ .$$

They are true by the triangle inequality for the distance d.

Let now consider the scaled metric space (X, d^x_ε) which is quasi-isometric with (X, d). According to point (a) of definition 3.1, we have that (for small enough ε, well chosen $R > 0$ and $0 < \mu < \frac{R}{4}$)

$$\bar{B}(y, \alpha R; d^y_\varepsilon) \subset \bar{B}(y, \varepsilon(\alpha R + \mu); d) \subset \bar{B}(x, \varepsilon(R - \mu); d) \subset \bar{B}(x, R; d^x_\varepsilon) \ .$$

The other inclusion is proved in the same way.

For the point (b) of the proposition 3.2 we use the point (b) of definition 3.1. Notice that the point (a) that we proved is needed in order to have a good definition of the intrinsic distorsion.

With the notations from the proposition, we have (for given $\mu > 0$ but small enough, for any $0 < \varepsilon < \varepsilon(\mu)$):

$$\mid d^y_\varepsilon(u,v) - d^x_\varepsilon(u,v) \mid \leq$$

$$\leq \mid d^y_\varepsilon(u,v) - \frac{1}{\varepsilon} d(u,v) \mid \ + \ \mid \frac{1}{\varepsilon} d(u,v) - d^x_\varepsilon(u,v) \mid \leq \ 2\mu \ .$$

The proof is done. ■

The property from the point (a) is interesting in general. We shall give it a name.

Definition 3.3 *We say that a scaled metric space is roughly symmetric if it has the property (a) from proposition 3.2.*

The next proposition has a proof similar to the one for the proposition 3.2.

Proposition 3.4 *(a) The property of being roughly symmetric is preserved by quasi-isometric equivalence.*

(b) Let $(X, d^{1,x}_\varepsilon)$ and $(X, d^{2,x}_\varepsilon)$ be two quasi-isometric spaces, defined over the same set X. Suppose that they are roughly symmetric. Define then the intrinsic distorsions $\Delta^i(x, R, \alpha, \varepsilon)$, $i = 1, 2$, like in point (b) of proposition 3.2. We have then

$$\lim_{\varepsilon \to 0} \mid \Delta^1(x, R, \alpha, \varepsilon) - \Delta^2(x, R, \alpha, \varepsilon) \mid = \ 0$$

uniformly with respect to x.

By a simple unwinding of definitions we obtain the following:

Proposition 3.5 *Two scaled metric spaces with the same portrait are quasi-isometric.*

Therefore a metric portrait induces an equivalence class of quasi-isometric scaled metric spaces. In particular, in the example 7, different choices associated to photos give quasi-isometric scaled metric spaces.

3.2 Reifenberg flat spaces

A locally compact metric space is Reinfenberg flat [Reifenberg] if locally small balls are quasi-isometric with euclidean balls. More precisely we have the following:

Definition 3.6 *Let $n \in \mathbb{N} \setminus \{0\}$ and $\alpha > 0$, A locally compact metric space (X, d) is (α, n) Reifenberg flat if for any compact set $K \subset X$ there is $r(K) > 0$ such that for any $x \in K$ and any $0 < r < r(K)$ we have $\alpha(x, r) \leq \alpha$.*

The numbers $\alpha(x, r)$ are defined as

$$\alpha(x, r) \ = \ \frac{1}{r} \inf_f \alpha(f) \ ,$$

where the infimum is taken over all functions

$$f : \bar{B}(x, r; d) \to \bar{B}(a, r) \subset \mathbb{R}^N$$

and $\alpha(f)$ is given by:

$$\alpha(f) \ = \ \sup \left\{ | \ \|f(y) - f(z)\| - d(y, z) \ | \ : y, z \in \bar{B}(x, r) \right\} \ +$$

$$+ \ \sup \left\{ d_{\mathbb{R}^n}(u, f(\bar{B}(x, r; d))) \ : \ u \in \bar{B}(a, r) \right\} \ .$$

The space (X, d) is n Reifenberg vanishing if for any $\varepsilon > 0$ and any compact $K \subset X$ there is $r(\varepsilon, K) > 0$ such that for all $x \in K$ and $0 < r < r(\varepsilon, K)$ we have $\alpha(x, r) \leq \varepsilon$.

Recall the definition of a photo in section 2.2 and the interpretation of Gromov-Hausdorff distance explained in proposition 2.2. It follows that in the previous definition we can replace the numbers $\alpha(x, r)$ measure Gromov-Hausdorff distances. Also, in the language of metric profiles and their curvature, a Reifenberg vanishing metric space is one with (approximately) the same curvature as $\mathbb{R}^n$.

We shall use the definition 2.6 of metric portraits to explain what is a scaled metric space which is Reinfenberg vanishing w.r.t. (Z, a, d), a pointed metric cone.

Definition 3.7 *Let (X, d_ε^x) be a scaled metric space and (Z, a, d), $a \in Z$, a homogeneous metric cone. We say that the scaled metric space X is (Z, a, d) Reinfenberg vanishing if it admits a quasi-isometric portrait in (Z, a, d).*

It is natural to ask if the converse of proposition 3.2 is true, namely: suppose that a scaled metric space is roughly symmetric and its intrinsic distorsion converges to 0 when the scale goes to 0; is then this space quasi-isometric with a true metric space?

I don't know the answer in general. However, "reverse engineering " of Corollary 2.21 in [DavidToro] provides the following weaker result.

Theorem 3.8 *Let (X, d_ε^x) be a roughly symmetric scaled metric space, with intrinsic distorsion converging to 0, which is also $\mathbb{R}^n$ Reinfenberg vanishing. Then there is a distance d on X such that ...*

3.3 Length of curves in scaled metric spaces

Having a scaled metric space does not mean that we are capable to measure length of curves. Indeed, let $x, y \in X$ and $\varepsilon > 0$. What can be considered as the ε-approximate distance between x and y? Here are some candidates:

- if x, y are close enough, that is $y \in X_\varepsilon^x$, then $d_\varepsilon^x(x, y)$ makes sense,
- one can have a symmetric situation: $x \in X_\varepsilon^y, y \in X_\varepsilon^x$. In this case we have two distances available: $d_\varepsilon^x(x, y)$ and $d_\varepsilon^y(x, y)$.
- suppose that there is $w \in X$ such that $x, y \in X_\varepsilon^w$. Then we can consider the distance $d_\varepsilon^w(x, y)$.
- suppose that there are intermediary points $z_1, ..., z_m$ and $w_1, ..., w_{m+1}$ such that $x, z_1 \in X_\varepsilon^{w_0}$, for all $i = 1, ..., m-1$ $z_i, z_{i+1} \in X_\varepsilon^{w_i}$ and $z_m, y \in X_\varepsilon^{w_{m+1}}$. Then we might look at

$$d_\varepsilon^{w_0}(x, z_1) + \sum_{i=1}^{m-1} d_\varepsilon^{w_i}(z_i, z_{i+1}) + d_\varepsilon^{w_{m+1}}(z_m, y) \ .$$

  Then we can try to minimize this number by taking a infimum for all choices of intermediary points and $m \in \mathbb{N}$.

But how to estimate the error the differences between these various choices? We need a basic thing: an estimate of

$$\mid d_\varepsilon^x(u, v) \ - \ d_\varepsilon^y(u, v) \mid$$

for $u, v X_\varepsilon^x \cap X_\varepsilon^y$. In principle there is no estimate of this kind available, for a general scaled metric space (definition 2.7).

We shall see later that such an estimate is related to a condition entering in one of the axioms of a dilatation structure (see [...]).

Definition 3.9 *With the notations from definition 2.6, a metric portrait is complete if there are numbers $\alpha_0, R \in (0, 1]$ and for any $x \in X$, $\varepsilon \in (0, 1]$ a set Y_ε^x, such that:*

(a) for all $x \in X$ *and* $\varepsilon \in (0,1]$ *we have* $f_\varepsilon^x Y_\varepsilon^x \ = \ \bar{B}(a,R)$,

(b) for all $x, y \in X$, *if* $d_\varepsilon^x(x,y) \leq R$ *then*

$$Y_\varepsilon^x \subset X_\varepsilon^x \cap X_\varepsilon^y \ ,$$

(c) for all $x \in X$, $\varepsilon > 0$, $\alpha \in (0, \alpha_0)$ *we have*

$$X_{\alpha\varepsilon}^x \subset Y_\varepsilon^x \ .$$

The following quantities will be important further. Each of them describes the behaviour of the metric portrait from one side.

Definition 3.10 *For a complete metric portrait, for any* $x \in X$, $\varepsilon \in (0,1]$ *and* $\alpha \in (0, \alpha_0]$, *we introduce the numbers:*

$$\Delta(x,\varepsilon,\alpha) \ = \ \sup\left\{ \frac{1}{\alpha} \mid d_\varepsilon^x(u,v) - d_\varepsilon^y(u,v) \mid \ : d_\varepsilon^y(x,y) \leq R \ , \ u,v \in X_{\varepsilon\alpha}^y \right\} \ ,$$

$$\bar{\Delta}(x,\varepsilon,\alpha) \ = \ \sup\left\{ \mid d\left(f_{\alpha\varepsilon}^y u, f_{\alpha\varepsilon}^y v\right) - \frac{1}{\alpha} d\left(f_\varepsilon^x u, f_\varepsilon^x v\right) \mid \ : d_\varepsilon^y(x,y) \leq R \ , \ u,v \in X_{\varepsilon\alpha}^y \right\} \ .$$

The first number $\Delta(x,\varepsilon,\alpha)$ *is called intrinsic distortion and the second number* $\bar{\Delta}(x,\varepsilon,\alpha)$ *is called extrinsic distortion.*

Theorem 3.11 *With the notation from definition 3.10 we have*

$$\mid \bar{\Delta}(x,\varepsilon,\alpha) - \Delta(x,\varepsilon,\alpha) \mid \leq \ 2\alpha\varepsilon + \frac{4\varepsilon}{\alpha} \ .$$

Proof.

3.4 Rigidity of quasi-isometric portraits

4 Dilatation structures

A dilatation structure on a uniform space (i.e. a set X endowed with a uniformity) is an assignment

$$x \in X \ \mapsto \delta(x) \ = \ \{\delta_\varepsilon^x : U(x) \in \mathcal{V}(x) \to V_\varepsilon(x) = \delta_\varepsilon^x U(x) : \varepsilon \in [0, a_x)\}$$

This means that we associate to any point $x \in X$ a set of dilatations (or homotheties) based on x.

In order for such an assignment to be interesting a set of axioms will be introduced. These axioms express a compatibility between the three objects: the distance, the measure and the dilatations.

Dilatation structures first appeared in [Buliga ...].

This theory contains the case of regular sub-Riemannian manifolds. Contrary to the classical treatment of the subject, ours is free of any differential geometric ingredient. This means that the theory is completely non-Euclidean. Others than the sub-Riemannian case are also considered.

The collection of axioms of dilation structures split into three parts.

The first part concerns only the interaction between dilatations and distance. The dilatations based at x generate a metric profile. We use them as an ideal microscope with infinite capacity of magnification. At the limit we see the (virtual) tangent space at x, as a set of "infinitesimal left translations", each being an isometry of the distance in the tangent metric space at x.

The second part concerns the operation of "addition" in the tangent space, seen as composition of infinitesimal left translations. It turns out that the virtual tangent space becomes either a conical group (if smooth, this is just a Carnot group), or a homogeneous space of a conical group.

The third part concerns the property of functions called "additivity almost everywhere", thus involving the dilatations, the distance and a measure.

4.1 Axioms of dilatation structures

The space (X, d) is a complete, locally compact metric space. This means that as a metric space (X, d) is complete and that small balls are compact. The space is a length metric space if the distance between points is realized as the infimum of length of curves joining these points. A necessary and sufficient condition for this is: for any two different points x, y and $\varepsilon > 0$ there exists an approximate middle point z with the property:

$$\max\{d(x,z), d(y,z)\} \leq \ \varepsilon + \frac{1}{2} d(x,y) \ .$$

We shall use from time to time the following convenient notation: by $\mathcal{O}(\varepsilon)$ we mean a function such that $\lim_{\varepsilon \to 0} \mathcal{O}(\varepsilon) \ = \ 0$.

The first group of axioms for the dilatation structure (X, d, δ) is listed further. The first axiom is merely a preparation for the next axioms. That is why we counted it as axiom 0.

A0. We assume that there is $1 < A$ such that for any $x \in X$ we have

$$\bar{B}_d(x, A) \subset U(x) \ .$$

The dilatations

$$\delta_\varepsilon^x : U(x) \to V_\varepsilon(x)$$

are defined for any $\varepsilon \in (0,1]$ and $\delta_1^x y \ = \ y$ for all $y \in U(x)$. All dilatations are homeomorphisms (invertible, continuous, with continuous inverse). We suppose that for all $\varepsilon \in (0,1)$ we have

$$B_d(x,\varepsilon) \subset \delta_\varepsilon^x B_d(x,A) \subset V_\varepsilon(x) \subset U(x) \ .$$

For $\varepsilon \in (1,+\infty)$ the associated dilatation is

$$\delta_\varepsilon^x : W_\varepsilon(x) \to B_d(x,B) \ ,$$

it injective, invertible on the image. We shall suppose that

$$V_\varepsilon(x) \subset W_\varepsilon(x)$$

and that for all $\varepsilon \in (0,1]$ and $u \in U(x)$ we have

$$\delta_{\varepsilon^{-1}}^x \ \delta_\varepsilon^x u \ = \ u \ .$$

A further technical condition on the sets $V_\varepsilon(x)$ and $W_\varepsilon(x)$ will be given just before the axiom A4.

A1. $(\varepsilon, x, y) \mapsto \delta_\varepsilon^x y$ is continuous (for $\varepsilon > 0$). We also have:

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d(x, \delta_\varepsilon^x x) \ = \ 0$$

uniformly with respect to $x \in K$ compact set.

A2. For any compact set $K \subset X$ there is a function $\varepsilon \mapsto \mathcal{O}(\varepsilon)$, $\lim_{\varepsilon \to 0} \mathcal{O}(\varepsilon) = 0$, such that for any $x, \in K$, $\varepsilon, \mu \in (0,1)$ and $u, v \in \bar{B}_d(x,A)$ we have:

$$\frac{1}{\varepsilon\mu} \mid d(\delta_\varepsilon^x \delta_\mu^x u, \delta_\varepsilon^x \delta_\mu^x v) \ - \ d(\delta_{\varepsilon\mu}^x u, \delta_{\varepsilon\mu}^x v) \mid \leq \ \mathcal{O}(\varepsilon\mu) \ .$$

A3. Let $\varepsilon \mapsto \mathcal{O}(\varepsilon)$ be any function such that $\lim_{\varepsilon \to 0} \mathcal{O}(\varepsilon) = 0$. For any x it exists the function $(u,v) \mapsto d^x(u,v)$, defined for any u, v in the closed ball (in distance d) $\bar{B}(x,A)$, such that

$$\lim_{\varepsilon \to 0} \max \left\{ \mid \frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) \ - \ d^x(u,v) \mid \ : \ u, v \in \bar{B}_d(x,A) \right\} \ = \ 0$$

uniformly with respect to x in compact set.

For the following axiom to make sense we impose a technical condition on the co-domains $V_\varepsilon(x)$: for any compact set $K \subset X$ there are $R = R(K) > 0$ and $\varepsilon_0 = \varepsilon(K) \in (0,1)$ such that for all $u, v \in \bar{B}_d(x,R)$ and all $\varepsilon \in (0, \varepsilon_0)$ we have

$$v \in W_\varepsilon(x) \cap V_\varepsilon(\delta_\varepsilon^x u) \ .$$

The next axiom can now be stated:

A4. We have the limit

$$\lim_{\varepsilon \to 0} \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v \ = \ L_u^x(v)$$

uniformly with respect to x, u, v in compact set.

Note that it could be assumed, without great modification of the axioms, that ε takes values in a topological separated commutative group Γ endowed with a continuous morphism

$$\nu : \Gamma \to (0, +\infty)$$

with $\inf \nu(\Gamma) \ = \ 0$.

It follows that for any $a > 0$ and for any $\varepsilon \in \Gamma$ there is a $b > 0$ such that

$$\nu^{-1}(b) \subset \ \varepsilon \nu^{-1}(a) \ .$$

The function ν defines an "absolute" on the group Γ, that is an invariant topological filter on Γ. Indeed, let us consider denote by $\tau(\Gamma)$ the collection of open sets in Γ. A topological filter $\mathcal{F}$ on Γ is a non empty subset of $\tau(\Gamma)$ such that:

i) if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$,

ii) if $A \in \mathcal{F}$ and $B \subset \Gamma$ open such that $A \subset B$ then $B \in \mathcal{F}$.

The filter $\mathcal{F}$ is called invariant if for any $\varepsilon \in \Gamma$ and any $A \in \mathcal{F}$ we have $\varepsilon A \in \mathcal{F}$. (This would be a "left invariant filter" but because Γ is supposed commutative, we throw away the unnecessary "left" word.)

The function ν defines the filter:

$$\mathcal{F}_\nu \ = \ \left\{ A \in \tau(\Gamma) \, : \, \exists a > 0 \text{ such that } \nu^{-1}(a) \subset A \right\} \ .$$

It is easy to check that this is an invariant filter.

We say that $\varepsilon \to \mathcal{F}_\nu$ if $\nu(\varepsilon) \to 0$. This is compatible with the general definition of convergence to a filter $\mathcal{F}$ (for topologically separated spaces).

For example we may have $\Gamma \ = \ \mathbb{C}^*$ and $\nu(\varepsilon) = \mid \varepsilon \mid$ or

$$\Gamma \ = \ \left\{ \frac{1}{n} \, : \, n \in \mathbb{N}^* \right\}$$

and $\nu = \ id$.

In this more general situation the axioms are:

GA0. We assume that there are numbers $1 < A < B$ such that for any $x \in X$ we have

$$\bar{B}_d(x, A) \subset U(x) \subset B_d(x, B) \ .$$

The dilatations

$$\delta_\varepsilon^x : U(x) \to V_\varepsilon(x)$$

are defined for any $\nu(\varepsilon) \in (0,1]$ and $\delta_1^x y \ = \ y$ for all $y \in U(x)$ (where 1 is the neutral element of Γ). All dilatations are homeomorphisms (invertible, continuous, with continuous inverse). We suppose that for all $\nu(\varepsilon) \in (0,1)$ we have

$$B_d(x,\varepsilon) \subset \delta_\varepsilon^x B_d(x,A) \subset V_\varepsilon(x) \subset U(x) \ .$$

For $(\nu(\varepsilon) \in (1,+\infty)$ the associated dilatation is

$$\delta_\varepsilon^x : W_\varepsilon(x) \to B_d(x,B) \ ,$$

it injective, invertible on the image. We shall suppose that

$$V_\varepsilon(x) \subset W_\varepsilon(x)$$

and that for all $\varepsilon \in (0,1]$ and $u \in U(x)$ we have

$$\delta_{\varepsilon^{-1}}^x \ \delta_\varepsilon^x u \ = \ u \ .$$

GA1. $(\varepsilon, x, y) \mapsto \delta_\varepsilon^x y$ is continuous (for $\varepsilon \in \Gamma$). We also have:

$$\lim_{\nu(\varepsilon)\to 0} \frac{1}{\nu(\varepsilon)} d(x, \delta_\varepsilon^x x) \ = \ 0$$

uniformly with respect to $x \in K$ compact set.

GA2. For any compact set $K \subset X$ there is a function $\varepsilon \mapsto \mathcal{O}(\varepsilon)$, $\lim_{\nu(\varepsilon)\to 0} \mathcal{O}(\varepsilon) = 0$, such that for any $x, \in K$, ε, μ with $\nu((\varepsilon), \nu(mu) \in (0,1)$ and $u, v \in \bar{B}_d(x,A)$ we have:

$$\frac{1}{\nu(\varepsilon\mu)} \mid d(\delta_\varepsilon^x \delta_\mu^x u, \delta_\varepsilon^x \delta_\mu^x v) \ - \ d(\delta_{\varepsilon\mu}^x u, \delta_{\varepsilon\mu}^x v) \mid \leq \ \mathcal{O}(\varepsilon\mu) \ .$$

GA3. Let $\varepsilon \mapsto \mathcal{O}(\varepsilon)$ be any function such that $\lim_{\nu(\varepsilon)\to 0} \mathcal{O}(\varepsilon) = 0$. For any x it exists the function $(u,v) \mapsto d^x(u,v)$, defined for any u, v in the closed ball (in distance d) $\bar{B}(x,A)$, such that

$$\lim_{\nu(\varepsilon)\to 0} \max \left\{ \mid \frac{1}{\nu(\varepsilon)} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) \ - \ d^x(u,v) \mid \ : \ u, v \in \bar{B}_d(x,A) \right\} \ = \ 0$$

uniformly with respect to x in compact set.

The new condition to be considered for the axiom 4 is: for any compact set $K \subset X$ there are $R = R(K) > 0$ and $\alpha_0 = \alpha(K) \in (0,1)$ such that for all $u, v \in \bar{B}_d(x,R)$ and all $\varepsilon \in \Gamma$ such that $\nu(\varepsilon) < \alpha_0$) we have

$$v \in W_\varepsilon(x) \cap V_\varepsilon(\delta_\varepsilon^x u) \ .$$

The next axiom can now be stated:

GA4. We have the limit

$$\lim_{\nu(\varepsilon)\to 0} \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v \;=\; L_u^x(v)$$

uniformly with respect to x, u, v in compact set.

We shall explain now what the axioms mean. The first axiom A1. is stating that the distance between $\delta_\varepsilon^x x$ and x is negligible with respect to ε. If $\delta_\varepsilon^x x \;=\; x$ then this axiom is trivially satisfied.

We preferred to leave some flexibility for the choice of base point of the dilatation, based on the considerations made in [tangent bundles], but also with the example of self-similar fractals in mind.

The second axiom A2. states that in an approximate sense the transformations δ_ε^x form an action of Γ on X. As previously, if we suppose that

$$\delta_\varepsilon^x \delta_\mu^x \;=\; \delta_{\varepsilon\mu}^x$$

then this axiom is trivially satisfied.

All these are natural properties of dilatations, easy to state and rather weak as assumptions. This changes starting with the axiom A3. We state the interpretation of this axiom as a theorem. But before a definition: we denote by (δ, ε) the distance on

$$\bar{B}_{d^x}(x, 1) \;=\; \{y \in X \,:\, d^x(x, y) \leq 1\}$$

given by

$$(\delta, \varepsilon)(u, v) \;=\; \frac{1}{\nu(\varepsilon)} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) \;.$$

Important remark: The "distance" d^x can be degenerated. That means: there might be $v, w \in \bar{B}_d(x, A)$ such that $d^x(v, w) = 0$ but $v \neq w$. We shall use further the name "distance" for d^x, essentially by commodity, but the reader is urged to keep in mind the possible degeneracy of d^x.

In the microscope interpretation, think about the ocular of the microscope being $\bar{B}_{d^x}(x, 1)$. The microscope is looking to a neighbourhood of x. Then with the magnification $1/\varepsilon$ we see the pointed metric space $(\bar{B}_{d^x}(x, 1), (\delta, \varepsilon), x)$.

Theorem 4.1 *Let (X, d, δ) be a dilatation structure with associated commutative group Γ and morphism $\nu : \Gamma \to (0, +\infty)$. Then the following are consequences of axioms GA0, ... , GA3 only:*

(a) for all $u, v \in X$ such that $d(x, u) \leq 1$ and $d(x, v) \leq 1$ and all $\mu \in (0, A)$ we have:

$$d^x(u, v) \;=\; \frac{1}{\mu} d^x(\delta_\mu^x u, \delta_\mu^x v) \;.$$

We shall say that d^x has the cone property with respect to dilatations.

(b) For all $\varepsilon \in (0, 1]$ we have

$$\delta_\varepsilon^x \bar{B}_{d^x}(x, 1) \;=\; \bar{B}_{d^x}(\delta_x^x, \varepsilon) \;.$$

(c) We have the following limit:

$$\lim_{\varepsilon\to 0}\ \frac{1}{\varepsilon}\sup\{\mid d(u,v)-d^x(u,v)\mid\ :\ d(x,u)\leq\varepsilon\ ,\ d(x,v)\leq\varepsilon\}\ =\ 0\ .$$

Therefore if d^x is a true (i.e. nondegenerate) distance, then (X,d) admits a metric tangent space in x.

Proof. Needs proof.
■

Theorem 4.2 *Let (X,d,δ) be a dilatation structure with associated commutative group Γ and morphism $\nu:\Gamma\to(0,+\infty)$. Then:*

(a) The "infinitesimal translations" L^x_u are d^x isometries.

(b) For any $\varepsilon,\mu\in\Gamma$ such that $\nu(\varepsilon),\nu(mu)\in(0,1]$ the associated transformations:

$$T^x_{\varepsilon,\mu}\ =\ \left(\delta^x_{\varepsilon\mu}\right)^{-1}\delta^x_\varepsilon\delta^x_\mu\ ,$$
$$C^x_{\varepsilon,\mu}\ =\ [\delta^x_\varepsilon,\delta^x_\mu]$$

are isometries. In the formula for $C^x_{\varepsilon,\mu}$ the bracket signifies the commutator of transformations.

Proof. The point (b) is a direct consequence of theorem 4.1. We have to prove the point (a). For this remark that the point (c) of theorem 4.1 can be written as:

$$\max\left\{\frac{1}{\nu(\varepsilon)}\mid d(u,v)\ -\ d^x(u,v)\mid\ :\ d(x,u)\leq\mathcal{O}(\nu(\varepsilon)),\ d(x,v)\leq\mathcal{O}(\nu(\varepsilon))\right\}\to 0 \tag{4.1.1}$$

as $\nu(\varepsilon)\to 0$.

For $\nu(\varepsilon)>0$ sufficiently small the points $x,\delta^x_\varepsilon u,\delta^x_\varepsilon v,\delta^x_\varepsilon w$ are close one to another, so we can apply the estimate (4.1.1) for the basepoint $\delta^x_\varepsilon u$ to get:

$$\frac{1}{\nu(\varepsilon)}\mid d(\delta^x_\varepsilon v,\delta^x_\varepsilon w)\ -\ d^{\delta^x_\varepsilon u}(\delta^x_\varepsilon v,\delta^x_\varepsilon w)\mid\to 0$$

when $\nu(\varepsilon)\to 0$. This can be written, using the cone property of the distance $d^{\delta^x_\varepsilon u}$, like this:

$$\mid\frac{1}{\nu(\varepsilon)}d(\delta^x_\varepsilon v,\delta^x_\varepsilon w)\ -\ d^{\delta^x_\varepsilon u}\left(\delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}}\delta^x_\varepsilon v,\delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}}\delta^x_\varepsilon w\right)\mid\to 0 \tag{4.1.2}$$

as $\nu(\varepsilon)\to 0$. By the axioms GA1, GA3, the function

$$(x,u,v)\mapsto d^x(u,v)$$

is an uniform limit of continuous functions, therefore uniformly continuous on compact sets. We can pass to the limit in the left hand side of the estimate (4.1.2), using this uniform continuity and axioms GA3, GA4, to get the result. ■

Theorem 4.3 *If the metric space (X,d) admits a dilatation structure δ which satisfies the axioms A1. – A3. then it admits a metric tangent cone in any point $x\in X$. The curve $\nu(\varepsilon)>0\mapsto \mathbb{P}^x(\varepsilon) \;=\; [\bar{B}_{d^x}(x,1),(\delta,\varepsilon),x]$ is a metric profile. The limit (in the Gromov-Hausdorff convergence) represents the metric tangent cone in x.*

Proof. We start by proving that d^x is a cone distance with respect to the dilatations δ^x_μ, that is: for all $u,v\in X$ such that $d(x,u)\leq 1$ and $d(x,v)\leq 1$ and all $\mu\in(0,A)$ we have:

$$d^x(u,v) \;=\; \frac{1}{\mu}d^x(\delta^x_\mu u,\delta^x_\mu v)\ .$$

Indeed, we have:

$$|\ \frac{1}{\varepsilon\mu}d(\delta^x_\varepsilon\delta^x_\mu u,\delta^x_\varepsilon\delta^x_\mu v)\ -\ d^x(u,v)\ |\leq\ \frac{1}{\varepsilon\mu}d(\delta^x_{\varepsilon\mu}u,\delta^x_\varepsilon\delta^x_\mu u)+\frac{1}{\varepsilon\mu}d(\delta^x_{\varepsilon\mu}v,\delta^x_\varepsilon\delta^x_\mu v)+$$

$$+\ |\ \frac{1}{\varepsilon\mu}d(\delta^x_{\varepsilon\mu}u,\delta^x_{\varepsilon\mu}v)\ -\ d^x(u,v)\ |\ .$$

Use now the axioms A2 and A3 and pass to the limit with $\varepsilon\to 0$. This gives the desired equality.

We have to prove that $\mathbb{P}^x$ is a metric profile. For this we have to compare two pointed metric spaces:

$$\big((\delta^x,\varepsilon\mu),\bar{B}_{d^x}(x,1),x\big)\quad\text{and}\quad\left(\frac{1}{\mu}(\delta^x,\varepsilon),\bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,1),x\right)\ .$$

Let $u\in X$ such that

$$\frac{1}{\mu}(\delta^x,\varepsilon)(x,u)\leq 1\ .$$

This means that:

$$\frac{1}{\varepsilon}d(\delta^x_\varepsilon x,\delta^x_\varepsilon u)\ \leq\ \mu\ .$$

Use further axioms A2 and the cone property proved before:

$$\frac{1}{\varepsilon}d^x(\delta^x_\varepsilon x,\delta^x_\varepsilon u)\ \leq\ (\mathcal{O}(\varepsilon)+1)\mu$$

therefore

$$d^x(x,u)\ \leq\ (\mathcal{O}(\varepsilon)+1)\mu\ .$$

It follows that for any $u\in\bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,1)$ we can choose $w(u)\in\bar{B}_{d^x}(x,1)$ such that

$$\frac{1}{\mu}d^x(u,\delta^x_\mu w(u))\ =\ \mathcal{O}(\varepsilon)\ .$$

We want to prove that

$$|\ \frac{1}{\mu}(\delta^x,\varepsilon)(u_1,u_2)\ -\ (\delta^x,\varepsilon\mu)(w(u_1),w(u_2))\ |\leq\ \mathcal{O}(\varepsilon\mu)+\frac{1}{\mu}\mathcal{O}(\varepsilon)+\mathcal{O}(\varepsilon)\ .$$

This goes as following:

$$| \frac{1}{\mu}(\delta^x,\varepsilon)(u_1,u_2) - (\delta^x,\varepsilon\mu)(w(u_1),w(u_2)) | = | \frac{1}{\varepsilon\mu} d(\delta^x_\varepsilon u_1, \delta^x_\varepsilon u_2) - \frac{1}{\varepsilon\mu} d(\delta^x_{\varepsilon\mu} w(u_1), \delta^x_{\varepsilon\mu} w(u_2)) | \leq$$

$$\leq \ \mathcal{O}(\varepsilon\mu) \ + \ | \frac{1}{\varepsilon\mu} d(\delta^x_\varepsilon u_1, \delta^x_\varepsilon u_2) \ - \ \frac{1}{\varepsilon\mu} d(\delta^x_\varepsilon \delta^x_\mu w(u_1), \delta^x_\varepsilon \delta^x_\mu w(u_2)) | \leq$$

$$\leq \ \mathcal{O}(\varepsilon\mu) \ + \ \frac{1}{\mu}\mathcal{O}(\varepsilon) \ + \ \frac{1}{\mu} \, | \, d^x(u_1,u_2) \ - \ d^x(\delta^x_\mu w(u_1), \delta^x_\mu w(u_2)) \ .$$

In order to obtain the last estimate we used twice axiom A3. We continue:

$$\mathcal{O}(\varepsilon\mu) \ + \ \frac{1}{\mu}\mathcal{O}(\varepsilon) \ + \ \frac{1}{\mu} \, | \, d^x(u_1,u_2) \ - \ d^x(\delta^x_\mu w(u_1), \delta^x_\mu w(u_2)) \ \leq$$

$$\leq \mathcal{O}(\varepsilon\mu) + \frac{1}{\mu}\mathcal{O}(\varepsilon) + \frac{1}{\mu} d^x(u_1, \delta^x_\mu w(u_1)) + \frac{1}{\mu} d^x(u_1, \delta^x_\mu w(u_2)) \ \leq \ \mathcal{O}(\varepsilon\mu) + \frac{1}{\mu}\mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon) \ .$$

This shows that we have a metric profile.

This metric profile is *almost nice*. Indeed, let $c \in (0,1)$. Then (by the previous reasoning) there is a function $\mathcal{O}(\varepsilon)$ such that for any

$$u \in \bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,c)$$

we have

$$d^x(x,u) \ \leq \ c\mu + \mathcal{O}(\varepsilon) \ .$$

Then there exists $\varepsilon(c) > 0$ such that for any $\varepsilon \in (0, \varepsilon(c))$ and u in the mentioned ball we have:

$$d^x(x,u) \leq \ \mu$$

In this case we can take directly $w(u) \ = \ \delta^x_{\mu^{-1}} u$ and simplify the previous string of inequalities, to get eventually that :

$$\delta^x_{\mu^{-1}} \left(\bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,c) \right) \ \subset \ \bar{B}_{d^x}(x,1) \ ,$$

that the Hausdorff difference between these two sets is of order $\mathcal{O}(\varepsilon)$ for μ fixed:

$$\mu \ d_{GH} \left([\bar{B}_{d^x}(x,1), (\delta^x,\varepsilon), x], \ [\delta^x_{\mu^{-1}} \left(\bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,c) \right), (\delta^x,\varepsilon\mu), x] \right) = \mathcal{O}(\varepsilon)$$

and

$$d_{GH} \left([\bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,c), \frac{1}{\mu}(\delta^x,\varepsilon), x] \ , \ [\delta^x_{\mu^{-1}} \left(\bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,c) \right), (\delta^x,\varepsilon\mu), x] \right) = \mathcal{O}(\varepsilon\mu)$$

when $\varepsilon \in (0, \varepsilon(c))$. ■

4.2 Virtual tangent bundle

Definition 4.4 *The virtual tangent space at X in x, denoted by VT_xX, is the (closure in the uniform convergence of the) group of transformations generated by all infinitesimal translations in x.*

Theorem 4.5 *The virtual tangent space VT_xX is a conical subgroup of the group of isometries of d^x, with the dilatations:*

$$\Delta_\varepsilon(T) \;=\; \delta^x_\varepsilon T \delta^x_{\varepsilon^{-1}} \;.$$

Proof. Let us start by the following simple remark: if T is an isometry of d^x then

$$\delta^x_\varepsilon T \delta^x_{\varepsilon^{-1}}$$

is another isometry. Indeed, we have

$$d^x(\delta^x_\varepsilon T \delta^x_{\varepsilon^{-1}}(u), \delta^x_\varepsilon T \delta^x_{\varepsilon^{-1}}(v)) \;=\; \varepsilon d^x(T\delta^x_{\varepsilon^{-1}}(u), T\delta^x_{\varepsilon^{-1}}(v)) \;=$$

$$=\; \varepsilon d^x(\delta^x_{\varepsilon^{-1}}(u), \delta^x_{\varepsilon^{-1}}(v)) \;=\; d^x(u,v) \;.$$

In order to prove the theorem, it is sufficient to show that for any L^x_u and small enough $\mu > 0$ there is w such that

$$\delta^x_\mu L^x_u \delta^x_{\mu^{-1}} \;=\; L^x_w \;. \tag{4.2.3}$$

We shall prove that $w = \delta^x_\mu u$ satisfies the previous relation.

Let us compute:

$$d\left(\delta^{\delta^x_\varepsilon \delta^x_{\mu^{-1}} u}_{\mu\varepsilon^{-1}} \delta^x_\varepsilon \delta^x_{\mu^{-1}} v, \delta^{\delta^x_{\varepsilon\mu^{-1}} u}_{\mu\varepsilon^{-1}} \delta^x_{\varepsilon\mu^{-1}} v\right) \;\leq$$

$$\leq\; d\left(\delta^{\delta^x_\varepsilon \delta^x_{\mu^{-1}} u}_{\mu\varepsilon^{-1}} \delta^x_\varepsilon \delta^x_{\mu^{-1}} v, \delta^{\delta^x_\varepsilon \delta^x_{\mu^{-1}} u}_{\mu\varepsilon^{-1}} \delta^x_{\varepsilon\mu^{-1}} v\right) \;+\; d\left(\delta^{\delta^x_\varepsilon \delta^x_{\mu^{-1}} u}_{\mu\varepsilon^{-1}} \delta^x_{\varepsilon\mu^{-1}} v, \delta^{\delta^x_{\varepsilon\mu^{-1}} u}_{\mu\varepsilon^{-1}} \delta^x_{\varepsilon\mu^{-1}} v\right) \;.$$

Denote by

$$A(\varepsilon) \;=\; d\left(\delta^{\delta^x_\varepsilon \delta^x_{\mu^{-1}} u}_{\mu\varepsilon^{-1}} \delta^x_\varepsilon \delta^x_{\mu^{-1}} v, \delta^{\delta^x_\varepsilon \delta^x_{\mu^{-1}} u}_{\mu\varepsilon^{-1}} \delta^x_{\varepsilon\mu^{-1}} v\right) \;,$$

$$B(\varepsilon) \;=\; d\left(\delta^{\delta^x_\varepsilon \delta^x_{\mu^{-1}} u}_{\mu\varepsilon^{-1}} \delta^x_{\varepsilon\mu^{-1}} v, \delta^{\delta^x_{\varepsilon\mu^{-1}} u}_{\mu\varepsilon^{-1}} \delta^x_{\varepsilon\mu^{-1}} v\right) \;.$$

From axioms A2, and relation (4.1.1) we get that

$$A(\varepsilon) \;\leq\; \mathcal{O}(\varepsilon) + \frac{\mu}{\varepsilon} d^{\delta^x_\varepsilon \delta^x_{\mu^{-1}} u}(\delta^x_\varepsilon \delta^x_{\mu^{-1}} v, \delta^x_{\varepsilon\mu^{-1}} v) \to 0$$

as $\varepsilon \to 0$ (μ is fixed).

From same axiom A2 and relation (4.1.1) we get that

$$\delta^x_{\mu\varepsilon^{-1}} \delta^x_\varepsilon \delta^x_{\mu^{-1}} u \to \;\; u$$

uniformly with respect to (x,u) in compact set. From axiom A4 we deduce that

$$\delta^{\delta^x_\varepsilon \delta^x_{\mu^{-1}} u}_{\mu\varepsilon^{-1}} \delta^x_{\varepsilon\mu^{-1}} v \ \to \ L^x_u(v)$$

as $\varepsilon \to 0$. On the other side

$$\delta^{\delta^x_{\varepsilon\mu^{-1}} u}_{\mu\varepsilon^{-1}} \delta^x_{\varepsilon\mu^{-1}} v \ \to \ L^x_u(v)$$

as $\varepsilon \to 0$. Therefore

$$B(\varepsilon) \ \to 0$$

as $\varepsilon \to 0$. This proves that

$$d\left(\delta^{\delta^x_\varepsilon \delta^x_{\mu^{-1}} u}_{\mu\varepsilon^{-1}} \delta^x_\varepsilon \delta^x_{\mu^{-1}} v, \delta^{\delta^x_{\varepsilon\mu^{-1}} u}_{\mu\varepsilon^{-1}} \delta^x_{\varepsilon\mu^{-1}} v\right) \ \to 0 \tag{4.2.4}$$

as $\varepsilon \to 0$. Finally we claim that:

$$\delta^x_\mu L^x_{\delta^x_{\mu^{-1}} u} \delta^x_{\mu^{-1}} v \ = \ \lim_{\varepsilon\to 0} \delta^{\delta^x_\varepsilon \delta^x_{\mu^{-1}} u}_{\mu\varepsilon^{-1}} \delta^x_\varepsilon \delta^x_{\mu^{-1}} v \ . \tag{4.2.5}$$

Indeed, from axiom A3 the limit

$$\delta^{\delta^x_\varepsilon u}_\mu w \to \delta^x_\mu w$$

as $\varepsilon \to 0$, is uniform. Then, from axioms A2, A4 we get:

$$\lim_{\varepsilon\to 0} \delta^x_\mu \delta^{\delta^x_\varepsilon \delta^x_{\mu^{-1}} u} \delta^x_\varepsilon \delta^x_{\mu^{-1}} v \ = \ \lim_{\varepsilon\to 0} \delta^{\delta^x_\varepsilon \delta^x_{\mu^{-1}} u}_{\mu\varepsilon^{-1}} \delta^x_\varepsilon \delta^x_{\mu^{-1}} v \ .$$

But this is the equality (4.2.5). The claim is proven.

To end the proof of the theorem, put together (4.2.4) and (4.2.5). They show that:

$$\delta^x_\mu L^x_{\delta^x_{\mu^{-1}}} \delta^x_{\mu^{-1}} \ = \ L^x_u \ ,$$

which is another way of writing the relation (4.2.3). The proof is done. ■

Let us define now the function:

$$T \in VT_xX \ \mapsto \|T\|_x \ = \ d^x(T(x), x) \ ,$$

defined on the set of infinitesimal translations in x. By a reasoning similar to the one in the previous proof we have:

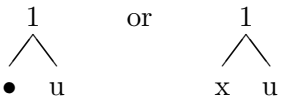
$$\|L^x_u\|_x \ = \ \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} d(x, \delta^{\delta^x_\varepsilon u}_\varepsilon x) \ . \tag{4.2.6}$$

4.3 Dilatation profile as a metric profile. The microscope

5 Binary labeled trees and dilatations

5.1 Coding dilatations by binary labeled trees

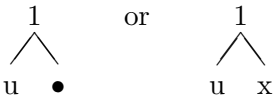
We want to explore what happens when we make compositions of dilatations (which depends also on $\varepsilon > 0$) and then we do the limit when $\varepsilon \to 0$. The ε variable apart, any dilatation $\delta_\varepsilon^x(y)$ is a function of two arguments: x and y. The functions we can obtain when composing dilatations are difficult to write, that is why we shall use a tree notation.

A dilatation: 

$= \delta_\varepsilon^x u$. The $\bullet$ sign means that the variable x is fixed (and known from the context) and we are interested in the uniform convergence with respect to $x = \bullet$ when $\varepsilon \to 0$. We call $\bullet$ the anchor. We can erase the anchor and we get a tree with one root, labeled with 1, and one leaf (which is not labeled). The only edge of the tree (the right edge in our drawing) is labeled by 1.

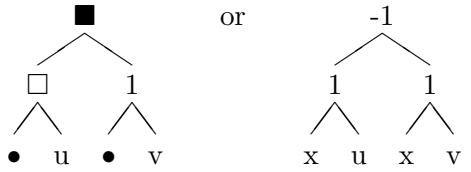
The inverse dilatation:

$= \delta_{\varepsilon^{-1}}^x u$. If we delete the anchor we are left with a tree with one root, labeled with -1, and one leaf (which is not labeled). The only edge of the tree (the right edge in our drawing) is labeled by 1.

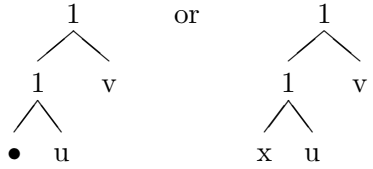
We can also consider the tree: 

$=$ $\delta_\varepsilon^u x$. If we delete the anchor we are left with a tree with one root, labeled with 1, and one leaf (which is not labeled). The only edge of the tree (the left edge in this case) is labeled by 0.

Other trees will be of use later. Some of them are given further. Delete the anchor and label the edges accordingly.

The difference tree: 

$= \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v$

The curvature tree: 

$= \delta_\varepsilon^{\delta_\varepsilon^x u} v$

We have five rules:

- all edges pointing to $\bullet$ are erased.

- any edge going to the left with be labeled with 0,

- any edge going to the right will be labeled with 1,

- leaves are not labeled,

- nodes which are not leaves are labeled with 1 or -1.

After these rules we are left with a class of binary labeled trees (with labeled edges). Conversely, we can put back the anchor $\bullet$: we add an edge pointing to $\bullet$ to any node which is not a leaf and has exactly one edge which exits; the new edge is labeled differently than the existing edge. The labeling of the edges tells us how to draw the tree. And to any such tree corresponds a composition of dilatations.

Further we describe the class of trees which will interest us (they correspond to trees with the anchor erased and they are labeled such that here is a chance that the associated composition of dilatations converges uniformly with respect to the anchor $x = \bullet$, as $\varepsilon \to 0$. For example the difference tree, the dilatation and the curvature trees (with the anchor erased) fit the description given further, but the inverse dilatation does not.

A forrest t is a graph, with nodes $nodes(t)$ and edges

$$edges(t) \subset \ nodes(t) \times \ nodes(t) \ .$$

There is an order relation $<$ on $nodes(t)$ such that for any $(x, y) \in edges(t)$ we have $x < y$. The order relation $<$ satisfies the tree condition:

$$\forall x, y, z \in \ nodes(t) \ \ x < y \text{ and } z < y \ \Rightarrow \ \ x < z \text{ or } z < x$$

The meaning of $x < y$ is: y is on a branch which starts from x.

Further we shall identify forrests which are isomorphic.

For any node $x \in \ nodes(t)$ we denote by

$$out(x) \ = \ \{y \in \ nodes(t) \ \mid \ (x, y) \in \ edges(t) \ , \ x < y\}$$

The forrests we are interested in are binary forrests: for any $x \in \ nodes(t)$ $card\, out(x) \in \{0, 1, 2\}$.

The roots of the forrest t is the set $root(t)$ of nodes which are not greater than any other node in t. The forrest is called a tree if it has only one root. In this case we denote the root of a tree by $root(t)$.

The leaves of a forrest t are the nodes in the class:

$$leaves(t) \ = \ \{z \in \ nodes(t) \ \mid \ \ card\ out(z) = 0\}$$

The interior nodes are the nodes which are not leaves:

$$int(t) \ = \ nodes(t) \setminus leaves(t)$$

Our trees (and forrests) are labeled: each interior node and each edge are labeled. This is described by two functions:

$$labn: \ int(t) \to \{-1, 1\}$$

$$labe : edges(t) \to \{0, 1\}$$

These labels satisfy some conditions explained further.

If $out(x) \ = \ \{y, z\}$ then $labe(x, y) \neq \ labe(z)$.

For any $x \in \ nodes(t)$ let $x_0 \in \ root(t)$ the only root such that $x_0 < x$ and

$$path(x) \ = \ \{x_0, ..., x_m = x\}$$

the only path which goes from x_0 to x (therefore $x_{i+1} \in \ out(x_i$ for any i between 1 and m). The number m is called the degree of x and denoted by $m = \ deg(x)$.

The power of x is the number:

$$\pi(x) \ = \ \ labn(x_0) + \sum_{i=1}^{deg(x)} \ labe(x_{i-1}, x_i) \ labn(x_i)$$

The relative power of a pair of nodes (x, y) such that $x < y$ is defined by

$$\pi(x, y) \ = \ \ labn(x_0) + \sum_{i=1}^{deg(x,y)} \ labe(x_{i-1}, x_i) \ labn(x_i)$$

with the notation

$$path(x, y) \ = \ \{x_0 = x, ..., x_m = x\} \quad , m = \ deg(x, y)$$

for the unique path which links x and y.

We say that a node x is alive if $\pi(x) \geq 0$. A node is admissible if is a member of the set:

$$adm(t) \ = \ \{x \in \ nodes(t) \ \mid \ \forall y \in \ nodes(t) \ x < y \Rightarrow \pi(x, y) \geq 0\}$$

We impose that all leaves are alive.

The set of cuts of a forrest t is

$$cut(t) \ = \ \{k \subset adm(t) \ \mid \ \forall x, y \in k \ x \neq y \Rightarrow \ x \not< y, y \not< x\}$$

Now we are ready to describe operations with trees.

1. Grafting (of first kind). Let $x \in leaves(t)$ and t' another tree. Then $t\hat{x}t'$ is the tree obtained by identification of $root(t')$ with x. The operation is well defined on the class of trees under consideration.

2. Grafting (of second kind). Let t, t' be trees and $x \in \ nodes(t)$ such that $out(x) = \{y\}$, $out(root(t')) = \{z\}$ and finally $labe(x, y) \neq labe(root('), z)$. Then $t\bar{x}t'$ is the tree obtained by identification of $root(t')$ with x. It can be checked that this operation is also well defined.

3. Cutting. Let $k \in \ cut(t)$ be a cut. Cut then t by eliminating the nodes $x \in \ cut(t)$. The forrest we get is in the class of trees that we described, due to the definition of the cut.

5.2 Distances between trees. Metric spaces from forrests

6 Examples of dilatation structures

This section serves as illustration of dilatations structures associated to sub-riemannian Lie groups and to sub-riemannian manifolds.

6.1 Euclidean and snowflake examples

The first example is known to everybody: take $(X, d) = (\mathbb{R}^n, d_E)$, with usual (euclidean) dilatations δ_ε^x, with:

$$d_E(x, y) = \|x - y\| \quad , \quad \delta_\varepsilon^x y \ = \ x + \varepsilon(y - x) \ .$$

Dilatations are defined everywhere. The group Γ is $(0, +\infty)$ and the function ν is the identity.

There are few things to check: axioms 0,1,2 are obviously true. For axiom A3, remark that for any $\varepsilon > 0$, $x, u, v \in X$ we have:

$$\frac{1}{\varepsilon} d_E(\delta_\varepsilon^x u, \delta_\varepsilon^x v) \ = \ d_E(u, v) \ ,$$

therefore for any $x \in X$ we have $d^x = d_E$.

Finally, let us check the axiom A4. For any $\varepsilon > 0$ and $x, u, v \in X$ we have

$$\delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v \ = \ x + \varepsilon(u - x) + \frac{1}{\varepsilon} \left(x + \varepsilon(v - x) - x - \varepsilon(u - x) \right) \ =$$

$$= \ x + \varepsilon(u - x) + v - u$$

therefore this quantity converges to

$$x + v - u \ = \ x + (v - x) - (u - x)$$

as $\varepsilon \to 0$. The axiom A4 is verified.

Take now $\phi : \mathbb{R}^n \to \mathbb{R}^n$ a bi-Lipschitz diffeomorphism. Then we can define the dilatation structure: $X \ = \ \mathbb{R}^n$,

$$d_\phi(x, y) \ = \ \|\phi(x) - \phi(y)\| \quad , \quad \delta_\varepsilon^x y \ = \ x + \varepsilon(y - x) \ ,$$

or the equivalent dilatation structure: $X = \mathbb{R}^n$,

$$d_\phi(x,y) = \|x-y\| \quad , \quad \delta_\varepsilon^x y = \phi^{-1}\left(\phi(x)+\varepsilon(\phi(y)-\phi(x))\right).$$

In this example (look at its first version) the distance d_ϕ is not equal to d^x. Indeed, a direct calculation shows that

$$d^x(u,v) = \|D\phi(x)(v-u)\| .$$

The axiom A4 gives the same result as previously.

Because dilatation structures are defined by local requirements, we can easily define dilatation structures on riemannian manifolds, using particular atlases of the manifold and the riemannian distance (infimum of length of curves joining two points). Note that any finite dimensional manifold can be endowed with a riemannian metric. This class of examples covers all dilatation structures used in differential geometry. The axiom A4 gives an operation of addition of vectors in the tangent space (compare with Bellaïche [bell] last section).

The next example is a snowflake variation of the euclidean case: $X = \mathbb{R}^n$ and for any $a \in (0,1]$ take

$$d_a(x,y) = \|x-y\|^\alpha \quad , \quad \delta_\varepsilon^x y = x + \varepsilon^{\frac{1}{a}}(y-x) .$$

We leave to the reader to verify the axioms.

Another example can be obtained from the euclidean dilatation structure and a function:

$$g : (0,+\infty) \to \mathbb{R}^n \quad , \quad \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} g(\varepsilon) = 0$$

Define then $X = \mathbb{R}^n$, $d = d_E$ and

$$\delta_\varepsilon^x y = g(\varepsilon) + x + \varepsilon(y-x) .$$

The axioms are verified but the dilatation flow δ_ε^x is not forming a group and

$$\delta_\varepsilon^x x = x + g(\varepsilon) \neq x .$$

This example is inspired from section 3.4, defintion 3.17 (ii), [buligatangent]. We see there that the euclidean and subriemannian cases are in sharp contrast concerning the result of the axiom A4. In the euclidean case just explained we have:

$$\lim_{\varepsilon \to 0} \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v = x + v - u ,$$

as before.

More general, if (X,d,δ) is a dilatation structure then $(X,d_a,\delta(a))$ is also a dilatation structure, for any $a \in (0,1]$, where

$$d_a(x,y) = (d(x,y))^a \quad , \quad \delta(a)_\varepsilon^x = \delta_{\varepsilon^{\frac{1}{a}}}^x .$$

6.2 Sub-Riemannian Lie groups

6.2.1 Uniform and conical groups

We start with the following setting: G is a topological group endowed with an uniformity such that the operation is uniformly continuous. More specifically, we introduce first the double of G, as the group $G^{(2)} = G \times G$ with operation

$$(x,u)(y,v) = (xy, y^{-1}uyv)$$

The operation on the group G, seen as the function

$$op : G^{(2)} \to G \ , \quad op(x,y) = xy$$

is a group morphism. Also the inclusions:

$$i' : G \to G^{(2)} \ , \quad i'(x) = (x,e)$$

$$i" : G \to G^{(2)} \ , \quad i"(x) = (x,x^{-1})$$

are group morphisms.

Definition 6.1 *1. G is an uniform group if we have two uniformity structures, on G and G^2, such that op, i', $i"$ are uniformly continuous.*

2. *A local action of a uniform group G on a uniform pointed space (X,x_0) is a function $\phi \in W \in \mathcal{V}(e) \mapsto \hat{\phi} : U_\phi \in \mathcal{V}(x_0) \to V_\phi \in \mathcal{V}(x_0)$ such that:*
   (a) *the map $(\phi,x) \mapsto \hat{\phi}(x)$ is uniformly continuous from $G \times X$ (with product uniformity) to X,*
   (b) *for any $\phi,\psi \in G$ there is $D \in \mathcal{V}(x_0)$ such that for any $x \in D$ $\phi\hat{\psi}^{-1}(x)$ and $\hat{\phi}(\hat{\psi}^{-1}(x))$ make sense and $\phi\hat{\psi}^{-1}(x) = \hat{\phi}(\hat{\psi}^{-1}(x))$.*

3. *Finally, a local group is an uniform space G with an operation defined in a neighbourhood of $(e,e) \subset G \times G$ which satisfies the uniform group axioms locally.*

Remark that a local group acts locally at left (and also by conjugation) on itself.

This definition deserves an explanation.

An uniform group, according to the definition (6.1), is a group G such that left translations are uniformly continuous functions and the left action of G on itself is uniformly continuous too. In order to precisely formulate this we need two uniformities: one on G and another on $G \times G$.

These uniformities should be compatible, which is achieved by saying that i', $i"$ are uniformly continuous. The uniformity of the group operation is achieved by saying that the *op* morphism is uniformly continuous.

The particular choice of the operation on $G \times G$ is not essential at this point, but it is justified by the case of a Lie group endowed with the CC distance induced

by a left invariant distribution. We shall construct a natural CC distance on $G \times G$, which is left invariant with respect to the chosen operation on $G \times G$. These distances induce uniformities which transform G into an uniform group according to definition (6.1).

In proposition (6.7) we shall prove that the operation function *op* is derivable, even if right translations are not "smooth", i.e. commutative smooth according to definition (6.5). This will motivate the choice of the operation on $G \times G$. It also gives a hint about what a sub-Riemannian Lie group should be.

We prepare now the path to this result. The "infinitesimal version" of an uniform group is a conical local uniform group.

Definition 6.2 *A conical local uniform group N is a local group with a local action of $(0, +\infty)$ by morphisms δ_ε such that $\lim_{\varepsilon \to 0} \delta_\varepsilon x = e$ for any x in a neighbourhood of the neutral element e.*

We shall make the following hypotheses on the local uniform group G: there is a local action of $(0, +\infty)$ (denoted by δ), on (G, e) such that

H0. the limit $\lim_{\varepsilon \to 0} \delta_\varepsilon x = e$ exists and is uniform with respect to x.

H1. the limit

$$\beta(x, y) = \lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \left((\delta_\varepsilon x)(\delta_\varepsilon y) \right)$$

is well defined in a neighbourhood of e and the limit is uniform.

H2. the following relation holds

$$\lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \left((\delta_\varepsilon x)^{-1} \right) = x^{-1}$$

where the limit from the left hand side exists in a neighbourhood of e and is uniform with respect to x.

These axioms are the prototype of a dilatation structure. Further comes a proposition which corresponds to theorem **??**.

Proposition 6.3 *Under the hypotheses H0, H1, H2 (G, β) is a conical local uniform group.*

Proof. All the uniformity assumptions permit to change at will the order of taking limits. We shall not insist on this further and we shall concentrate on the algebraic aspects.

We have to prove the associativity, existence of neutral element, existence of inverse and the property of being conical. The proof is straightforward. For the associativity $\beta(x, \beta(y, z)) = \beta(\beta(x, y), z)$ we compute:

$$\beta(x, \beta(y, z)) = \lim_{\varepsilon \to 0, \eta \to 0} \delta_\varepsilon^{-1} \left\{ (\delta_\varepsilon x) \delta_{\varepsilon/\eta} \left((\delta_\eta y)(\delta_\eta z) \right) \right\}$$

We take $\varepsilon = \eta$ and we get

$$\beta(x, \beta(y, z)) \;=\; \lim_{\varepsilon \to 0} \{(\delta_\varepsilon x)(\delta_\varepsilon y)(\delta_\varepsilon z)\}$$

In the same way:

$$\beta(\beta(x, y), z) \;=\; \lim_{\varepsilon \to 0, \eta \to 0} \delta_\varepsilon^{-1} \left\{(\delta_{\varepsilon/\eta} x) \left((\delta_\eta x)(\delta_\eta y)\right) (\delta_\varepsilon z)\right\}$$

and again taking $\varepsilon = \eta$ we obtain

$$\beta(\beta(x, y), z) \;=\; \lim_{\varepsilon \to 0} \{(\delta_\varepsilon x)(\delta_\varepsilon y)(\delta_\varepsilon z)\}$$

The neutral element is e, from H0 (first part): $\beta(x, e) \;= \beta(e, x) \;=\; x$. The inverse of x is x^{-1}, by a similar argument:

$$\beta(x, x^{-1}) \;=\; \lim_{\varepsilon \to 0, \eta \to 0} \delta_\varepsilon^{-1} \left\{(\delta_\varepsilon x) \left(\delta_{\varepsilon/\eta} (\delta_\eta x)^{-1}\right)\right\}$$

and taking $\varepsilon = \eta$ we obtain

$$\beta(x, x^{-1}) \;=\; \lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \left((\delta_\varepsilon x)(\delta_\varepsilon x)^{-1}\right) \;=\; \lim_{\varepsilon \to 0} \delta_\varepsilon^{-1}(e) \;=\; e$$

Finally, β has the property:

$$\beta(\delta_\eta x, \delta_\eta y) \;=\; \delta_\eta \beta(x, y)$$

which comes from the definition of β and commutativity of multiplication in $(0, +\infty)$. This proves that (G, β) is conical. ■

We arrive at a natural realization of the tangent space to the neutral element. Let us denote by $[f, g] \;= f \circ g \circ f^{-1} \circ g^{-1}$ the commutator of two transformations. For the group we shall denote by $L_x^G y \;= xy$ the left translation and by $L_x^N y \;= \beta(x, y)$. The preceding proposition tells us that (G, β) acts locally by left translations on G. We shall call the left translations with respect to the group operation β "infinitesimal". Those infinitesimal translations admit the very important representation:

$$\lim_{\lambda \to 0} [L^G_{(\delta_\lambda x)^{-1}}, \delta_\lambda^{-1}] \;=\; L_x^N \tag{6.2.1}$$

Definition 6.4 *The group VT_eG formed by all transformations L_x^N is called the virtual tangent space at e to G.*

The virtual tangent space VT_xG at $x \in G$ to G is obtained by translating the group operation and the dilatations from e to x. This means: define a new operation on G by

$$y \overset{x}{\cdot} z \;=\; yx^{-1}z$$

The group G with this operation is isomorphic to G with old operation and the left translation $L_x^G y = xy$ is the isomorphism. The neutral element is x. Introduce also the dilatations based at x by

$$\delta_\varepsilon^x y = x\delta_\varepsilon(x^{-1}y)$$

Then $G^x = (G, \overset{x}{\cdot})$ with the group of dilatations δ_ε^x satisfy the axioms Ho, H1, H2. Define then the virtual tangent space VT_xG to be: $VT_xG = VT_xG^x$. A short computation shows that

$$VT_xG = \left\{ L_y^{N,x} = L_x L^N_{x^{-1}y} L_x : y \in U_x \in \mathcal{V}(X) \right\}$$

where

$$L_y^{N,x} = \lim_{\lambda \to 0} \delta_\lambda^{-1,x} [\delta_\lambda^x, L^G_{(\delta_\lambda x)^{x,-1}}] \delta_\lambda^x$$

6.2.2 The dilatation structure and commutative derivative

We shall introduce the notion of commutative smoothness, which contains a derivative resembling with Pansu derivative. This definition is a little bit stronger than the one given by Vodopyanov & Greshnov [23], because their definition is good for a general CC space, when uniformities are taken according to the distances in CC spaces $G^{(2)}$ and G.

Definition 6.5 *A function $f : G_1 \to G_2$ is commutative smooth at $x \in G_1$, where G_1, G_2 are two groups satisfying H0, H1, H2, if the application*

$$u \in G_1 \mapsto (f(x), Df(x)u) \in G_2^{(2)}$$

exists, where

$$Df(x)u = \lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \left(f(x)^{-1} f(x\delta_\varepsilon u) \right)$$

and the convergence is uniform with respect to u in compact sets.

For example the left translations L_x are commutative smooth and the derivative equals identity. If we want to see how the derivative moves the virtual tangent spaces we have to give a definition.

Inspired by (6.2.1), we shall introduce the virtual tangent. We proceed as follows: to $f : G \to G$ and $x \in G$ let associate the function:

$$\hat{f}^x : G \times G \to G \ , \quad \hat{f}^x(y,z) = \hat{f}^x_y(z) = (f(x))^{-1} f(xy)z$$

To this function is associated a flow of left translations

$$\lambda > 0 \mapsto \hat{f}^x_{\delta_\lambda y} : G \to G$$

Definition 6.6 *The function $f : G \to G$ is virtually derivable at $x \in G$ if there is a virtual tangent $VDf(x)$ such that*

$$\lim_{\lambda \to 0} \left[\left(\hat{f}^x_{\delta_\lambda y} \right)^{-1}, \delta_\lambda^{-1} \right] \; = \; VDf(x)y \tag{6.2.2}$$

and the limit is uniform with respect to y in a compact set.

Remark that in principle the right translations are not commutative smooth. In Buliga [5], section 4, it is shown that right translations are smooth in the "mild" sense.

Now that we have a model for the tangent space to e at G, we can show that the operation is commutative smooth.

Proposition 6.7 *Let G satisfy H0, H1, H2 and $\delta^{(2)}_\varepsilon : G^{(2)} \to G^{(2)}$ be defined by*

$$\delta^{(2)}_\varepsilon (x, u) \; = \; (\delta_\varepsilon x, \delta_\varepsilon y)$$

Then $G^{(2)}$ satisfies H0, H1, H2, the operation (op function) is commutative smooth and we have the relation:

$$D \; op \; (x, u)(y, v) \; = \; \beta(y, v)$$

Proof. It is sufficient to use the morphism property of the operation. Indeed, the right hand side of the relation to be proven is

$$RHS \; = \; \lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \left(op(x, u)^{-1} op(x, u) op \left(\delta^{(2)}_\varepsilon (y, v) \right) \right) \; =$$

$$= \; \lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \left(op(\delta^{(2)}_\varepsilon (y, v)) \right) \; = \; \beta(y, v)$$

The rest is trivial. ■

This proposition justifies the choice of the operation on $G^{(2)} = G \times G$ and it is a quite surprising result.

6.3 Sub-Riemannian manifolds

Classical references to this subject are Bellaïche [2] and Gromov [11]. The interested reader is advised to look also to the references of these papers.

In the literature on sub-Riemannian manifolds not everything written can be trusted. The source of errors lies in the first notions and constructions, mostly in the fact that obvious properties connected to a Riemannian manifold are not true (or yet unproven) for a sub-Riemannian manifold. Of special importance is the difference between a normal frame and the frame induced by a privileged chart. Also,

there are many things in the sub-Riemannian realm which have no correspondent in the Riemannian case.

To close this introductory comments, let us remark that for the informed reader should be clear that in the sub-Riemannian realm there are problems even with the notion of manifold. Everybody agrees to define a sub-Riemannian manifold as in the definition 6.9. However, a better name for the object in this definition would be "model of sub-Riemannian manifold". We can illustrate this situation by the following comparison: the notion of sub-Riemannian manifold given in the definition 6.9 is to the real notion what the Poincaré disk is to the hyperbolic plane. Only that apparently nobody found the real notion yet.

Let M be a connected manifold. A distribution (or horizontal bundle) is a subbundle D of M. To any point $x \in M$ there is associated the vectorspace $D_x \subset T_xM$.

Given the distribution D, a point $x \in M$ and a sufficiently small open neighbourhood $x \in U \subset M$, one can define on U a filtration of bundles as follows. Define first the class of horizontal vectorfields on U:

$$\mathcal{X}^1(U,D) \ = \ \{X \in \Gamma^\infty(TU) : \forall y \in U \ , \ X(y) \in D_y\}$$

Next, define inductively for all positive integers k:

$$\mathcal{X}^{k+1}(U,D) \ = \ \mathcal{X}^k(U,D) \cup [\mathcal{X}^1(U,D), \mathcal{X}^k(U,D)]$$

Here $[\cdot,\cdot]$ denotes vectorfields bracket. We obtain therefore a filtration $\mathcal{X}^k(U,D) \subset \mathcal{X}^{k+1}(U,D)$. Evaluate now this filtration at x:

$$V^k(x,U,D) \ = \ \Big\{X(x) : X \in \mathcal{X}^k(U,D)\Big\}$$

There are $m(x)$, positive integer, and small enough U such that $V^k(x,U,D) = V^k(x,D)$ for all $k \geq m$ and

$$D_x = V^1(x,D) \subset V^2(x,D) \subset ... \subset V^{m(x)}(x,D)$$

We equally have

$$\nu_1(x) = \dim V^1(x,D) < \nu_2(x) = \dim V^2(x,D) < ... < n = \dim M$$

Generally $m(x)$, $\nu_k(x)$ may vary from a point to another.

The number $m(x)$ is called the step of the distribution at x.

Definition 6.8 *The distribution D is regular if $m(x)$, $\nu_k(x)$ are constant on the manifold M.*

The distribution is completely non-integrable if for any $x \in M$ we have $V^{m(x)} = T_xM$.

Definition 6.9 *A sub-Riemannian (SR) manifold is a triple (M, D, g), where M is a connected manifold, D is a completely non-integrable distribution on M, and g is a metric (Euclidean inner-product) on the distribution (or horizontal bundle) D.*

A horizontal curve $c : [a, b] \to M$ is a curve which is almost everywhere derivable and for almost any $t \in [a, b]$ we have

$$\dot{c}(t) \in D_{c(t)}$$

The class of horizontal curves will be denoted by $Hor(M, D)$.

The lenght of a horizontal curve is

$$l(c) \ = \ \int_a^b \sqrt{g(c(t))(\dot{c}(t), \dot{c}(t))} \ \mathrm{d}t$$

The length depends on the metric g.

The Carnot-Carathéodory (CC) distance associated to the sub-Riemannian manifold is the distance induced by the length l of horizontal curves:

$$d(x, y) \ = \ \inf \{l(c) : \ c \in Hor(M, D) \ , \ c(a) = x \ , \ c(b) = y\}$$

The Chow theorem ensures the existence of a horizontal path linking any two sufficiently closed points, therefore the CC distance it at least locally finite.

We shall work further only with regular sub-Riemannian manifolds, if not otherwise stated.

6.3.1 Normal frames and privileged charts

Bellaïche introduced the concept of privileged chart around a point $x \in M$.

Let $(x_1, ..., x_n) \mapsto \phi(x_1, ..., x_n) \in M$ be a chart of M around x (i.e. x has coordinates $(0, ..., 0)$). Denote by $X_1, ..., X_n$ the frame of vectorfields associated to the coordinate chart. The chart is called adapted at x if the following happens: $X_1(x), ..., X_{\nu_1}(x)$ form a basis of $V^1(x, D)$, $X_{\nu_1+1}(x), ..., X_{\nu_2}(x)$ span a space in $V^2(x, D)$ which is transversal to $V^1(x, D)$ and $X_1(x), ..., X_{\nu_2}(x)$ form a basis of $V^2(x, D)$, and so on.

Suppose that the frame $X_1, ..., X_n$ corresponds to a chart adapted at x. The degree of X_i at x is then k if $X_i \in V^k \setminus V^{k-1}$.

Definition 6.10 *A chart is privileged around the point $x \in M$ if it is adapted at x and for any $i = 1, ..., n$ the function*

$$t \mapsto d(x, \phi(..., t, ...))$$

(with t on the position i) is exactly of order deg X_i at $t = 0$.

Privileged charts always exist, as proved by Bellaïche [2] Theorem 4.15. As an example consider a Lie group G endowed with a left invariant distribution. The distribution is completely non-integrable if it is generated by the left translation of a vector subspace D of the algebra $\mathfrak{g} = T_eG$ which bracket generates the whole algebra $\mathfrak{g}$. Then the exponential map is a privileged chart at the identity $e \in G$, but generically not privileged at $x \neq e$.

Let X be a vectorfield on M and $x \in M$. The degree of X at x is the order of the function

$$t \in \mathbb{R} \to \mathbb{R} \quad , t \mapsto d(x, \exp(tX)(x))$$

and it is denoted by $deg_x X$. The vectorfield is called regular in an open set $U \subset M$ if $deg_x X$ is constant for all $x \in U$.

Consider now a frame of vectorfields $X_1, ..., X_n$ defined in $U \subset M$, open set. Let $x \in U$ and define:

$$V_i(x) \ = \ span \ \{X_k(x) : deg_x \ X_k = i\}$$

Definition 6.11 *A frame $X_1, ..., X_n$, defined in an open set $U \subset M$, is normal if all vectorfields X_k are regular in U and moreover at any $x \in U$ and any $i = 1, ..., n$ one has:*

$$V^i(x) \ = \ V_1(x) + ... + V_i(x)$$

(direct sum).

A normal frame transforms the filtration into a direct sum. Each tangent space decomposes as a direct sum of vectorspaces V_i. Moreover, each space V^i decomposes in a direct sum of spaces V_k with $k \leq i$.

Normal frames exist. Indeed, start with a frame $X_1, ..., X_r$ such that for any x $X_1(x), ..., X_r(x)$ form a basis for $D(x)$. Associate now to any word $a_1....a_q$ with letters in the alphabet $1, ..., r$ the multi-bracket

$$[X_{a_1}, [..., X_{a_q}]...]$$

One can add, in the lexicographic order, elements to the set $\{X_1, ..., X_r\}$, until a normal frame is obtained.

To a normal frame $X_1, ..., X_n$ and the point $x \in U$ one can associate a privileged chart. Inded, such a chart is defined by:

$$(a_1, ..., a_n) \in R^n \equiv T_xM \mapsto \phi_x(\sum a_i X_i(x)) = \exp\left(\sum_{i=1}^{n} a_i X_i\right)(x) \qquad (6.3.3)$$

Remark how the privileged chart changes with the base-point x.

The intrinsic dilatations associated to a normal frame, in a point x, are defined via a choice of a privileged chart based at x. In such a chart ϕ, for any $\varepsilon > 0$ (sufficiently small if necessary) the dilatation is defined by

$$\delta_\varepsilon(x_i) = (\varepsilon^{deg \ i} x_i)$$

With the use of the privileged charts (6.3.3), for any $\varepsilon > 0$ (sufficiently small if necessary) the dilatations are

$$\delta_\varepsilon^x \left(\exp \left(\sum_{i=1}^n a_i X_i \right) (x) \right) \; = \; \exp \left(\sum_{i=1}^n a_i \varepsilon^{deg X_i} X_i \right) (x)$$

In terms of vectorfields, one can use an intrinsic dilatation associated to the normal frame. This dilatation transforms X_i into

$$\Delta_\varepsilon X_i = \varepsilon^{deg \; X_i} X_i$$

One can define then, as in Gromov [11], section 1.4, or Vodop'yanov [24], deformed vectorfields with respect to fixed x by

$$\hat{X}_i^x(\varepsilon)(y) \; = \; \Delta_\varepsilon \left((\delta_\varepsilon^x)^{-1} * X_i \right) (y)$$

When $\varepsilon \to 0$ the vectorfields $\hat{X}_i^x(\varepsilon)$ converge uniformly to a vectorfield $X_i^{N,x}$, on small enough compact neighbourhoods of x.

The nilpotentization of the distribution with respect to the chosen normal frame, in the point x, is then the bracket

$$[X, Y]_N^x = \lim_{\varepsilon \to 0} [\hat{X}^x(\varepsilon), \hat{Y}^x(\varepsilon)] \tag{6.3.4}$$

We have the equality

$$[X_i, X_j]_N^x(x) \; = \; [X_i^{N,x}, X_j^{N,x}](x)$$

It is generically false that there are privileged coordinates around an open set in M. We can state this as a theorem.

Theorem 6.12 *Let (M, D, g) be a regular sub-riemannian manifold of topological dimension n. There are $\emptyset \neq U \subset M$ an open subset and $\phi : B \subset \mathbb{R}^n \to U$ such that ϕ is a privileged chart for any $x \in U$ if and only if (M, D, g) is a Riemannian manifold.*

Proof. If (M, D, g) is Riemannian then it is known that such privileged charts exist. We have to prove the converse. Suppose there is a map $\phi : B \to U$, B open set in $\mathbb{R}^n$, ϕ surjective, such that for any $x \in U$ ϕ is privileged. Consider the frame $X_1, ..., X_n$ of vectorfields tangent to coordinate lines induced by ϕ. Then this is a normal frame in U. Moreover

$$[X_i, X_j] \; = \; 0$$

for any $i, j = 1, ..., n$ therefore the nilpotentization bracket (6.3.4) in any point $x \in U$ is equal to 0. According to Mitchell [18] theorem 1 (in this paper theorem 6.13 section 6.3.3) the tangent cone in the metric sense to x is the Euclidean $\mathbb{R}^n$. But this implies that the manifold (which is supposed regular) is Riemannian. ■

6.3.2 Deformations of sub-Riemannian manifolds

One can use privileged charts or normal frames to define several deformations of a sub-Riemannian manifold around a point. These deformations can be described as curves in the metric space CMS of isometry classes of pointed compact metric spaces, with the Gromov-Hausdorff distance. For the isometry class of the pointed metric space (X, x, d) we shall use the notation $[X, x, d]$ or $[X, d]$ when the marked point is obvious. We shall work only with spaces (X, x, d) such that $X = \bar{B}(x, 1)$.

Consider a privileged chart around $x \in M$ with the associated dilatations δ_ε^x.

The dilatation flow δ_ε^x induces the following deformation: let $(D_\varepsilon, g_\varepsilon)$ be the pair distribution - metric on the distribution obtained by transport with $(\delta_\varepsilon^x)^{-1}$, namely:

$$D\delta_\varepsilon^x(y)D_\varepsilon(y) = D(\delta_\varepsilon^x y)$$

$$g(\delta_\varepsilon^x y)(D\delta_\varepsilon^x(y)u, D\delta_\varepsilon^x(y)v) = g_\varepsilon(y)(u, v)$$

for any $u, v \in T_x M$. The deformation associated is

$$[D, g, \varepsilon] = [\bar{B}(p, 1), (D, g, \varepsilon)] \tag{6.3.5}$$

where the notation (D, g, ε) is used for the CC distance in the sub-Riemannian manifold $(M, D_\varepsilon, g_\varepsilon)$.

A slightly different deformation induced by the dilatation flow δ_ε^x is given by

$$[\delta^x, \varepsilon] \;=\; [\overline{B}_d(x, \varepsilon), (\delta^x, \varepsilon)] \tag{6.3.6}$$

where the distance (δ^x, ε) is given by

$$(\delta^x, \varepsilon)(\delta_\varepsilon^x y, \delta_\varepsilon^x z) = d(y, z)$$

and the ball $\overline{B}_d(x, \varepsilon)$ is taken with respect to the original distance d.

It is not trivial to remark that there is no reason for the equality $[D, g, \varepsilon] = [\delta^x, \varepsilon]$.

Another deformation is associated to the dilatations Δ_ε and pairs normal frame - Riemannian metric g. This is simply

$$[\Delta, \varepsilon] = [\bar{B}(x, 1), (\Delta, \varepsilon)] \tag{6.3.7}$$

where (Δ, ε) is the Riemannian distance induced by the Riemannian metric g_ε given by:

$$g_\varepsilon(y)(\Delta_\varepsilon X(y), \Delta_\varepsilon Y(y)) = g(y)(X(y), Y(y))$$

for any pair of vectorfields X, Y.

6.3.3 Meaning of Mitchell theorem 1

A key result in sub-Riemannian geometry is Mitchell [18] theorem 1:

Theorem 6.13 *For a regular sub-Riemannian space (M, D, g), the tangent cone of (M, d) at $x \in M$ exists and it is isometric to $(N(x), d_N^x)$.*

Recall that the limit in the Gromov-Hausdorff sense is defined up to isometry. This means it this case that $N(D)$ is a model for the tangent space at x to (M, d_{CC}). In the Riemannian case $D = TM$ and $N(D) = R^n$, as a group with addition.

This theorem tells us nothing about the tangent bundle.

One can identify in the literature several proofs of Mitchell theorem 1. Exactly what is proven in each available variant of proof? The answer is: each proof basically shows that various deformations, such as the ones introduced previously, are metric profiles which can be prolonged to 0. Each of this metric profiles are close in the GH distance to the original metric profile of the CC distance. More precisely:

Lemma 6.14 *Let $\mathbb{P}'(t)$ be any of the previously introduced deformations $[\delta, \varepsilon]$, $[\Delta, \varepsilon]$, $[D, g, \varepsilon]$. Then*

$$d_{GH}(\mathbb{P}^m_\varepsilon, \mathbb{P}'(\varepsilon)) = O(\varepsilon)$$

The proof of this lemma reduces to a control problem. In the case of the profile $[\delta, \varepsilon]$, this is Mitchell [18] lemma 1.2.

Mitchell [18] and Bellaïche [2] theorem 5.21, proposition 5.22, proved the following:

Theorem 6.15 *The deformation $\varepsilon \mapsto [D, g, \varepsilon]$ can be prolonged by continuity to $\varepsilon = 0$ and the prolongation is a metric profile. We have*

$$[D, g, 0] = [\bar{B}(0, 1), d_N]$$

The corresponding result of Gromov [11] section 1.4B and Vodop'yanov [24] is:

Theorem 6.16 *The deformation $\varepsilon \mapsto [\Delta, \varepsilon]$ can be prolonged by continuity to $\varepsilon = 0$ and the prolongation is a metric profile. We have*

$$[\Delta, 0] = [\bar{B}(0, 1), d_N]$$

Finally, a similar theorem for the metric profile $[\delta^x, \varepsilon]$ is true.

Any of these theorems imply the Mitchell theorem 1, with the use of the approximation lemma 6.14. But in fact these theorems are different statements in terms of metric profiles.

6.4 Kirchheim counterexample to metric differentiability

6.5 Fractal spaces

We shall be concerned with attractor sets for iterated functions systems (IFS). For clarity we shall remind here what IFS are. For more informations see[Barnsley, Fractals Everywhere].

Let (X, D) be a complete metric space and $H(X)$ be the class of closed sets in X. Denote by d_H the Hausdorff distance associated to (X, d).

Theorem 6.17 *$(H(X), d_H)$ is a complete metric space.*

A function $w : X \to X$ is called a contraction if it is s-Lipschitz with $s < 1$.

Definition 6.18 *An iterated functions system (IFS) is a finite set $\{w_i : i = 1, ..., N\}$ of contractions with $\max\{Lip(w_i) : i = 1, ..., N\} < 1$.*

Theorem 6.19 *Let $W : H(X) \to H(X)$ be defined by: for any $B \in H(X)$*

$$W(B) \ = \ \Cup_{i=1}^{N} w_i(B) \ ,$$

where $\{w_i : i = 1, ..., N\}$ is an IFS. Then W is a contraction, therefore it has a fixed point, denoted by A_w.

We want to put a dilatation structure on . Let M be the set of fixed points of the contractions w_i:

$$M \ = \ \{x_i : w_i(x_i) \ = \ x_i \ , i = 1, ..., N\} \ .$$

Then M is generating the fractal A_w. It follows that the set

$$M^* \ = \ \{\}$$

of iterates of points in M, is dense in A_w. It is enough therefore to put a dilatation structure on M^*, because this will provide a dilatation structure in A_w.

The dilatations based at x_i are generated by w_i. Th eoher dilatations are conjugated with these. In this way we get a discrete dilatation structure.

The only thing that we need to check is the axiom .... (aditivity).

6.6 Other examples

7 Rademacher theorems

7.1 Transfer of Radon-Nycodim property

We shall prove in this section the Rademacher theorem for Lipschitz curves

$$c : [a, b] \to (X, d_A) \quad , \quad \forall s, t \in [a, b] \ \ d_A(c(s), c(t)) \ \leq \ M \mid s - t \mid$$

Definition 7.1 *A triple (X, d, δ) has the Radon-Nycodim property if for any Lipschitz curve $c : [a, b] \to (X, d)$ the following is true: for almost every $t \in [a, b]$ there is $\dot{c}(t)$ such that*

$$\frac{1}{\varepsilon} d(c(t + \varepsilon), \delta_\varepsilon^{c(t)} \dot{c}(t)) \to 0$$

$$\frac{1}{\varepsilon} d(c(t - \varepsilon), \delta_\varepsilon^{c(t)} \dot{c}(t)^{-1}) \to 0$$

Equivalently, for almost every $t \in [a,b]$ there is a conical group morphism

$$\dot{c}(t) : \mathbb{R} \to T_{c(t)}X$$

such that for any $a \in \mathbb{R}$ we have

$$\frac{1}{\varepsilon} d(c(t+\varepsilon a), \delta_\varepsilon^{c(t)} \dot{c}(t)a) \to 0$$

The space (X, d_A) is endowed with a dilatation structure δ and has some extra structure on it. This extra structure generalizes what we have on a sub-riemannian manifold. We don't use differential geometric reasoning at this level of generality.

The main assumption, other that the space (X, d_A) has a dilatation structure, is that the space X admits another distance d_B and it has another dilatation structure $\bar{\delta}$ such that:

(a) the identity $id \ : (X, d_A) \to (X, d_B)$ is 1-Lipschitz,

(b) $(X, d_B, \bar{\delta})$ has the Radon-Nycodim property,

(c) the identity $id \ : (X, d_A) \to (X, d_B)$ is derivable everywhere and for any point $x \in X$ the derivative $Did(x)$ is a projector,

(d) for any $x \in X$, $\varepsilon > 0$ and $\lambda > 0$ let

$$F(x, \varepsilon, \lambda) \ = \ \left\{ z \in X \ \mid \ d_A^x(x,z) \leq 2 \ , \ d_A^x(x,z) - \frac{1}{\varepsilon} d_B^x(x, \delta_\varepsilon^x z) \leq \lambda \right\} \ .$$

Then, as $(\varepsilon, \lambda) \to (0,0)$ we have

$$\max \left\{ d_A^x(P_\varepsilon^x z, z) \ \mid \ z \in F(x, \varepsilon, \lambda) \right\} \ \to 0 \ ,$$

where $P_\varepsilon^x = \ \bar{\delta}_{\varepsilon^{-1}}^x \delta_\varepsilon^x$.

A sufficient condition to have (a) is the following (true in the case of sub-Riemannian manifolds):

(a') for any Lipschitz curve c, if $l_A(c) < +\infty$ then $l_B(c) = l_A(c)$. Here l_A and l_B denote the length functional associated to distance d_A, distance d_B respectively.

The theorem we want to prove is

Theorem 7.2 *Under the assumptions (a'), (b), (c), (d), the space (X, d_A) has the Rademacher property on lines.*

Proof. Let $c : [0,1] \to (X, d_A)$ be a Lipschitz curve. Because of hypothesis (a) it follows that $c : [0,1] \to (X, d_B)$ is also Lipschitz. Moreover, we can reparametrize the curve c with the d_A lenght and so we can suppose that c is d_A 1-Lipschitz. Therefore we can suppose that c is d_B Lipschitz.

We shall use now the hypothesis (b): for almost any $t \in [0,1]$ there is $\dot{c}(t)$ such that

$$\frac{1}{\varepsilon} d_B(c(t+\varepsilon), \bar{\delta}_\varepsilon^x \dot{c}(t)) \to 0 \quad \text{as } \varepsilon \to 0 \tag{7.1.1}$$

$$\frac{1}{\varepsilon} d_B(c(t-\varepsilon), \bar{\delta}_\varepsilon^x \dot{c}(t)^{-1}) \to 0 \quad \text{as } \varepsilon \to 0 \tag{7.1.2}$$

Further we shall give only the half of the proof, namely we shall use only relation (7.1.1). To get a complete proof, one has to repeat the reasoning starting from (7.1.2).

Because c is d_A 1- Lipschitz, it follows that

$$d_A^{c(t)}(\delta_{\varepsilon^{-1}}^{c(t)} c(t+\varepsilon), \dot{c}(t)) \leq 2$$

for any $\varepsilon < \varepsilon(t) \in (0, +\infty)$. From the locally compactness with respect to $d_A^{c(t)}$ we find that for any $t \in [0,1]$ there is a sequence $(\varepsilon_h)_h \subset (0, +\infty)$, converging t 0 as $h \to \infty$, and $u(t) \in X$ such that:

$$\lim_{h\to\infty} \delta_{\varepsilon_h^{-1}}^{c(t)} c(t+\varepsilon_h) \;=\; u(t)$$

Use equation (7.1.1) to get that

$$\lim_{h\to\infty} \bar{\delta}_{\varepsilon_h^{-1}}^{c(t)} c(t+\varepsilon_h) \;=\; \dot{c}(t)$$

Re-write this latter equation as:

$$\lim_{h\to\infty} \delta_{\varepsilon_h^{-1}}^{c(t)} \delta_{\varepsilon_h}^{c(t)} \delta_{\varepsilon_h^{-1}}^{c(t)} c(t+\varepsilon_h) \;=\; \dot{c}(t)$$

and use the hypothesis (c) to get

$$D\ id(c(t))u(t) \;=\; \dot{c}(t)$$

But according to the second part of the hypothesis (c) the operator $D\ id(c(t))$ is a projector, hence

$$D\ id(c(t))\dot{c}(t) \;=\; \dot{c}(t)$$

Because of the fact that the derivative commutes with dilatations we get the important fact that for any $\varepsilon > 0$

$$\delta_\varepsilon^{c(t)} \dot{c}(t) \;=\; \bar{\delta}_\varepsilon^{c(t)} \dot{c}(t) \tag{7.1.3}$$

We wish to prove

$$\frac{1}{\varepsilon} d_A(\delta_{\varepsilon}^{c(t)} \bar{\delta}_{\varepsilon^{-1}}^{c(t)} c(t+\varepsilon), c(t+\varepsilon)) \to 0 \quad \text{as } \varepsilon \to 0 \tag{7.1.4}$$

Suppose that (7.1.4) is true. Then we would have

$$d_A^{c(t)}(\bar{\delta}_{\varepsilon^{-1}}^{c(t)} c(t+\varepsilon), \delta_{\varepsilon^{-1}}^{c(t)} c(t+\varepsilon)) \to 0 \quad \text{as } \varepsilon \to 0$$

But relations (7.1.1) and (7.1.3) imply that

$$d_A^{c(t)}(\bar{\delta}_{\varepsilon^{-1}}^{c(t)} c(t+\varepsilon), \dot{c}(t)) \to 0 \quad \text{as } \varepsilon \to 0$$

therefore we would finally get

$$d_A^{c(t)}(\delta_{\varepsilon^{-1}}^{c(t)} c(t+\varepsilon), \dot{c}(t)) \to 0 \quad \text{as } \varepsilon \to 0$$

which implies that the curve c is δ derivable in t.

We try to prove now (7.1.4). According to hypothesis (a') we have:

$$0 \leq \frac{1}{\varepsilon} d_A(c(t+\varepsilon), c(t)) - \frac{1}{\varepsilon} d_B(c(t+\varepsilon), c(t)) \ \leq \ \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mid \dot{c}(\tau) \mid_B \ \mathrm{d}\tau - \frac{1}{\varepsilon} d_B(c(t+\varepsilon), c(t))$$

where the quantity

$$\mid \dot{c}(s) \mid_B \ = \ \lim_{\varepsilon \to 0} \frac{d_B((c(s+\varepsilon), c(s))}{\varepsilon} \ = \ d_B^{c(s)}(c(s), \dot{c}(s))$$

exists for almost every $s \in [0,1]$ (see for the integral representation the reparametrisation theorem ...).

We obtain therefore the relation:

$$\frac{1}{\varepsilon} d_A(c(t+\varepsilon), c(t)) - \frac{1}{\varepsilon} d_B(c(t+\varepsilon), c(t)) \ \to 0 \quad \text{as } \varepsilon \to 0 \tag{7.1.5}$$

Here is the moment to use the last hypothesis (d). Indeed, the relation (7.1.5) implies that

$$d_A^{c(t)}(c(t), \delta_{\varepsilon^{-1}}^{c(t)} c(t+\varepsilon)) - \frac{1}{\varepsilon} d_B^{c(t)}(c(t+\varepsilon), c(t)) \ \to 0 \quad \text{as } \varepsilon \to 0 \tag{7.1.6}$$

Denote by $z(t) \ = \ \delta_{\varepsilon^{-1}}^{c(t)} c(t+\varepsilon)$. (In this notation we don't put the ε variable, but remember that all depends also on th ε.) The relation (7.1.7) becomes:

$$d_A^{c(t)}(c(t), z(t)) - \frac{1}{\varepsilon} d_B^{c(t)}(c(t), \delta_{\varepsilon}^{x} z(t)) \ \to 0 \quad \text{as } \varepsilon \to 0 \tag{7.1.7}$$

We also have

$$d_A^{c(t)}(c(t), z(t)) \ = \ \frac{1}{\varepsilon} d_A^{c(t)}(c(t), c(t+\varepsilon)) \leq 2$$

for ε sufficiently small, because we supposed that c was reparametrized with the length. Therefore, for a $\lambda > 0$ given there is $\varepsilon(\lambda) > 0$ such that for any $\varepsilon \in (0, \varepsilon(\lambda))$ we have

$$z(t) \in F(c(t), \varepsilon, \lambda) \ .$$

From the hypothesis (d) we deduce that

$$\lim_{\varepsilon \to 0} d_A^{c(t)}(z(t), P_\varepsilon^{c(t)} z(t)) \ = \ 0$$

Let us see what this means:

$$\lim_{\varepsilon \to 0} d_A^{c(t)}(\bar{\delta}_\varepsilon^{c(t)} c(t+\varepsilon), \delta_\varepsilon^{c(t)} c(t+\varepsilon)) \ = \ 0$$

This relation is equivalent with (7.1.4). ■

7.2 Additivity almost everywhere for Lipschitz functions

7.3 Proofs for Rademacher theorems

References

[1] A. Agrachev, E. C. El Alaoui,J.-P. Gauthier, I. Kupka, Generic singularities of sub-Riemannian metrics on R^3, C. R. Acad. Sci. Paris Sr. I Math. 322 (1996), no. 4, 377–384

[2] A. Bellaïche, The tangent space in sub-Riemannian geometry, in: *Sub-Riemannian Geometry*, A. Bellaïche, J.-J. Risler eds., Progress in Mathematics, **144**, Birkhäuser, (1996), 4 – 78

[3] P. Bieliavsky, E. Falbel, C. Gorodski, The classification of simply-connected contact sub-Riemannian symmetric spaces,*Pacific J. Math.* , **188** (1999), no. 1, 65–82

[4] M. Buliga, Sub-Riemannian geometry and Lie groups. Part I, e-print arXiv:math.MG/0210189, (2002)

[5] M. Buliga, Tangent bundles to sub-Riemannian groups, e-print arXiv:mathMG/0307342, (2003)

[6] M. Buliga, Curvature of sub-Riemannian spaces, e-print arXiv:math.MG/0311482, (2003)

[7] M. Buliga, S.K. Vodop'yanov, Differential structures for sub-Riemannian spaces, in preparation (2004)

[8] D. Burago, Y. Burago, S. Ivanov, A Course in Metric Geometry, *Graduate Studies in Mathematics*, **33**, AMS Providence, Rhode Island, (2000)

[9] J. Cheeger, Differentiability of Lipschitz functions on metric measure spaces,*Geom. Funct. Anal.*,**9** (1999), no. 3, 428–517

[10] M. Gromov, Groups of polynomial growth and expanding maps. Inst. Hautes Études Sci. Publ. Math. No. 53, 53–73, (1981)

[11] M. Gromov, Carnot-Caratheodory spaces seen from within, in: *Sub-Riemannian Geometry*, A. Bellaïche, J.-J. Risler eds., Progress in Mathematics, **144**, Birkhäuser, (1996), 79 – 323

[12] M. Gromov, Metric structures for Riemannian and non-Riemannian spaces, *Progress in Math.*, **152**, Birchäuser (1999), chapter 3.

[13] K. Hughen, The sub-Riemannian geometry of 3-manifolds, preprint DUK-M-95-13, (1995), available at http://www.math.duke.edu/preprints/1995.html

[14] A.A. Kirillov, Elements of the theory of representations. Translated from the Russian by Edwin Hewitt. Grundlehren der Mathematischen Wissenschaften, Band 220. Springer-Verlag, Berlin-New York, (1976)

[15] A. A. Kirillov, Merits and demerits of the orbit method. Bull. Amer. Math. Soc. (N.S.) 36 (1999), **4**, 433–488

[16] G.A. Margulis, G.D. Mostow, The differential of a quasi-conformal mapping of a Carnot-Carathéodory space, *Geom. Funct. Analysis*, **8** (1995), 2, 402–433

[17] G.A: Margulis, G.D. Mostow, Some remarks on the definition of tangent cones in a Carnot-Carathéodory space, *J. D'Analyse Math.*, **80** (2000), 299–317

[18] J. Mitchell, On Carnot-Carathéodory metrics, *J. Diff. Geometry*, **21** (1985), 35–45

[19] R. Montgomery, A tour of subriemannian geometries, their geodesics and applications. Mathematical Surveys and Monographs, 91. American Mathematical Society, Providence, RI, (2000)

[20] P. Pansu, Métriques de Carnot-Carathéodory et quasiisometries des espaces symetriques de rang un, *Ann. of Math.*, **129** (1989), 1–60

[21] S. K. Vodop'yanov, $\mathcal{P}$-Differentiability on Carnot Groups in Different Topologies and Related Topics, *Proc. on Analysis ans Geometry*, Novosibirsk: Sobolev Institute Press, (2000), 603 – 670

[22] S. K. Vodop'yanov, On differentiability of mappings of Sobolev classes on Carnot groups, *Matematicheskiy Sbornik* **194** (2003), 6, 67–86.

[23] S. K. Vodop'yanov, A. V. Greshnov, On differentiability of mappings of Carnot-Carathéodory spaces, *Dokl. R.A.S.*, t. **389**, 5, (2003)

[24] S. K. Vodop'yanov, A. V. Greshnov, Geometry of Carnot-Carathéodory spaces and differentiability of weakly contact mappings, *Izvestiya R.A.S., Matematicheskaya seriya*, to appear