Metric spaces with dilations

Marius Buliga

Institute of Mathematics, Romanian Academy
P.O. BOX 1-764, RO 014700
Bucureşti, Romania
Marius.Buliga@imar.ro

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1 Introduction

The point of view that dilations can be taken as fundamental objects which induce a differential calculus is relatively well known. The idea is simple: in a vector space $\mathbb{V}$ define the dilation based at x and of coefficient $\varepsilon > 0$ as the function which associates to y the value

$$\delta_{\varepsilon}^{x} y \ = \ x + \varepsilon(y - x) \quad .$$

Then for a function $f : \mathbb{V} \rightarrow \mathbb{W}$ between vector spaces $\mathbb{V}$ and $\mathbb{W}$ we have:

$$\left(\delta_{\varepsilon^{-1}}^{f(x)} f \delta_{\varepsilon}^{x} \right)(u) \ = \ f(x) + \frac{1}{\varepsilon} \left[f(x + \varepsilon(u - x)) - f(x) \right] \quad ,$$

thus the directional derivative of f at x, along $u - x$ appears as:

$$f(x) \ + \ D\, f(x)(u - x) \ = \ \lim_{\varepsilon \rightarrow 0} \left(\delta_{\varepsilon^{-1}}^{f(x)} f \delta_{\varepsilon}^{x} \right)(u) \quad .$$

Until recently there was not much interest into the generalization of such a differential calculus, based on other dilations than the usual ones, probably because nobody knew any fundamentally different example.

This changed gradually due to different lines of research, like the study of hypoelliptic operators Hörmander [42], harmonic analysis on homogeneous groups Folland, Stein [28], probability theory on groups Hazod [40], Siebert [54], studies in geometric analysis in metric spaces in relation with sub-riemannian geometry Bellaïche [5], groups with polynomial growth Gromov [33], or Margulis type rigidity results Pansu [53].

Another line of research concerns the differential calculus over general base fields and rings, Bertram, Glöckner and Neeb [6]. As the authors explain, it is possible to construct such a differential calculus without using the specific properties of the base field (or ring). In their approach it is not made a distinction between real and ultrametric differential calculus (and even not between finite dimensional and infinite dimensional differential calculus). They point out that differential calculus (integral calculus not included) seems to be a part of analysis which is completely general, based only on elementary results in linear algebra and topology.

The differential calculus proposed by Bertram, Glöckner and Neeb is a generalization of "classical" calculus in topological vector spaces over general base fields, and even over rings. The operation of vector addition is therefore abelian, modifications being made in relation with the multiplication by scalars.

A different idea, emergent in the studies concerning geometric analysis in metric spaces, is to establish a differential calculus in homogeneous groups, in particular in Carnot groups. These are noncommutative versions of topological vector spaces, in the sense that the operation of addition (of "vectors") is replaced by a noncommutative group operation and there is a replacement of multiplication by scalars in a general base field with a multiplicative action of $(0, +\infty)$ by group automorphisms.

In fact this is only a part of the nonsmooth calculus encountered in geometric analysis on metric spaces. For a survey see the paper by Heinonen [41]. The objects of interest in nonsmooth calculus as described by Heinonen are spaces of homogeneous type, or metric measured spaces where a generalization of Poincaré inequality is true. In such spaces the differential calculus goes a long way: Sobolev spaces, differentiation theorems, Hardy spaces. It is noticeable that in such a general situation we don't have enough structure to define differentials, but only various constructions corresponding to the norm of a differential of a function. Nevertheless see the remarkable result of Cheeger [23], who proves that to a metric measure space satisfying a Poincaré inequality we can associate an L^∞ cotangent bundle with finite dimensional fibers. Other important works which might also be relevant in relation to this paper are David, Semmes [25], where spaces with arbitrary small neighbouhoods containing similar images of the whole space are studied, or David, Semmes [26], where they study rectifiability properties of subsets of $\mathbb{R}^n$ with arbitrary small neighbourhoods containing "big pieces of bi-Lipschitz images" of the whole subset.

A particular case of a space of homogeneous type where more can be said is a normed homogeneous group, definition 5.3. According to [28] p. 5, a homogeneous group is a connected and simply connected Lie group whose Lie algebra is endowed with a family of dilations $\{\delta_\varepsilon : \varepsilon \in (0, +\infty)\}$, which are algebra automorphisms, simultaneously diagonalizable. As in this case the exponential of the group is a bijective mapping, we may transform dilations of the algebra into dilations of the group, therefore homogeneous groups are conical groups. Also, they can be described as nilpotent Lie groups positively graded.

Carnot groups are homogeneous groups which are stratified, meaning that the first non-trivial element of the graduation generates the whole group (or algebra). The interest into such groups come from various sources, related mainly to the study of hypo-elliptic operators Hörmander [42], and to extensions of harmonic analysis Folland, Stein [28].

Pansu introduced the first really new example of such a differential calculus based on other than usual dilations: the ones which are associated to a Carnot group. He proved in [53] the potential of what is now called Pansu derivative, by providing an alternative proof of a Margulis rigidity type result, as a corollary of the Rademacher theorem for Lipschitz functions on Carnot groups. Rademacher theorem, stating that a Lipschitz function is derivable almost everywhere, is a mathematical crossroad, because there meet measure theory, differential calculus and metric geometry. In [53] Pansu proves a generalization of this theorem for his new derivative.

The challenge to extend Pansu results to general regular sub-riemannian manifolds, taken by Margulis, Mostow [48] [49], Vodopyanov [56] and others, is difficult because on such general metric space there is no natural underlying algebraic structure, as in the case of Carnot groups, where we have the group operation as a non commutative replacement of the operation of addition in vector spaces.

On a regular sub-riemannian manifold we have to construct simultaneously several objects: tangent spaces to a point in the sub-riemannian space, an operation of addition of "vectors" in the tangent space, and a derivative of the type considered by Pansu. Dedicated to the first two objects is a string of papers, either directly related to the subject, as Bellaïche [5], or growing on techniques which appeared in the paper dedicated to groups of polynomial growth of Gromov [33], continuing in the big paper Gromov [39].

In these papers dedicated to sub-riemannian geometry the lack of a underlying algebraic structure was supplanted by using techniques of differential geometry. At a closer look, this means that in order to construct the fundamentals of a non standard differential calculus, the authors used the classical one. This seems to me comparable to efforts to study hyperbolic geometry on models, like the Poincaré disk, instead of intrinsically explore the said geometry.

Dilation structures on metric spaces, introduced in [11], describe the approximate self-similarity properties of a metric space. A dilation structure is a notion related, but more general, to groups and differential structures.

The basic objects of a dilation structure are dilations (or contractions). The axioms of a dilation structure set the rules of interaction between different dilations.

The point of view of dilation structures is that dilations are really fundamental objects, not only for defining a notion of derivative, but as well for all algebraic structures that we may need.

This viewpoint is justified by the following results obtained in [11], explained in a condensed and improved presentation, in the first part of this paper. A metric space (X, d) which admits a strong dilation structure (definition 6.1) has a metric tangent space at any point $x \in X$ (theorem 13.1), and any such metric tangent space has an algebraic structure of a conical group (theorem 13.5).

Conical groups are generalizations of homogeneous Lie groups, but also of p-adic nilpotent groups, or of general contractible groups. A conical group is a locally compact group endowed with a family of dilations $\{\delta_\varepsilon : \varepsilon \in \Gamma\}$. Here Γ is a locally compact abelian group with an associated morphism $\nu : \Gamma \to (0, +\infty)$ which distinguishes an end of Γ, namely the filter generated by the pre-images $\nu^{-1}(0, r)$, $r > 0$. This end, is denoted by 0 and $\varepsilon \in \Gamma \to 0$ means $\nu(\varepsilon) \to 0$ in $(0, +\infty)$. Any contractible group is a conical group and to any conical group we can associate a family of contractible groups.

The structure of contractible groups is known in some detail, due to Siebert [54], Wang [65], Glöckner and Willis [30], Glöckner [29] and references therein.

By a classical result of Siebert [54] proposition 5.4, we can characterize the algebraic structure of the metric tangent spaces associated to dilation structures of a certain kind: they are homogeneous groups (corollary 13.8). The corollary 13.8 thus represents a generalization of difficult results in sub-riemannian geometry concerning the structure of the metric tangent space at a point of a regular sub-riemannian manifold. This line of research is pursued further in the paper [15].

Morphisms of dilation structures generalize the notion of affine transformation. A dilation structure on a metric space induce a family of dilation structures on the same space, at different scales. We explain that canonical morphisms between these induced dilation structures lead us to a kind of emergent affinity on smaller and smaller scale.

Finally we characterize contractible groups in terms of dilation structures. To a normed contractible group we can naturally associate a linear dilation structure (proposition 15.11). Conversely, by theorem 15.12 any linear and strong dilation structure comes from the dilation structure of a normed contractible group.

We are thus led to the introduction of a noncommutative affine geometry, in the spirit of Bertram "affine algebra", which is commutative according to our point of view. In such a geometry incidence relations are no longer relevant, being replaced by algebraic axioms concerning dilations. We define a version of the ratio of three collinear points (replaced by a "ratio function" which associates to a pair of points and two positive numbers the third point) and we prove that it is the basic invariant of this geometry. Moreover, it turns out that this is the geometry of normed affine group spaces, a notion which is to conical groups as a normed affine space is to a normed topological vector space (theorem 5.5).

2 Metric spaces, distances, norms

Definition 2.1 *A metric space (X,d) is a set X endowed with a distance function $d : X \times X \to [0,+\infty)$. In the metric space (X,d), the distance between two points $x,y \in X$ is $d(x,y) \geq 0$. The distance d satisfies the following axioms:*

(i) $d(x,y) = 0$ *if and only if* $x = y$,

(ii) (symmetry) for any $x,y \in X$ $d(x,y) = d(y,x)$,

(iii) (triangle inequality) for any $x,y,z \in X$ $d(x,z) \leq d(x,y) + d(y,z)$.

The ball of radius $r > 0$ and center $x \in X$ is the set

$$B(x,r) \, = \, \{ y \in X \, : \, d(x,y) < r \} \quad .$$

Sometimes we shall use the notation $B_d(x,r)$ for the ball of center x and radius r with respect to the distance d, in order to emphasize the dependence on the distance d. Any metric space (X,d) is endowed with the topology generated by balls. The notations $\bar{B}(x,r)$ and $\bar{B}_d(x,r)$ are used for the closed ball centered at x, with radius r.

A pointed metric space (X,x,d) is a metric space (X,d) with a chosen point $x \in X$.

The notion of a metric space is not very old: it has been introduced by Fréchet in the paper [Sur quelques points du calcul fonctionnel, *Rendic. Circ. Mat. Palermo 22* (1906), 1-74].

2.1 Metric spaces, normed groups and normed groupoids

An obvious example of a metric space is $\mathbb{R}^n$ endowed with an euclidean distance, that is with a distance function induced by an euclidean norm:

$$d(x,y) \, = \, \| x - y \| \quad .$$

In fact any normed vector space can be seen as a metric space. In order to define a distance from a norm, in a normed vector space, we only need the norm function and the abelian group structure of the vector space. (Later in this paper, he multiplication by scalars will provide us with the first example of a metric space with dilations). This leads us to the introduction of normed groups. Let us give, in increasing generality, the definition of a normed group, then the definition of a normed groupoid.

Definition 2.2 *A normed group (G,ρ) is a pair formed by:*

- *a group G, with the operation $(x,y) \in G \times G \mapsto xy$, inverse denoted by $x \in G \mapsto x^{-1}$ and neutral element denoted by e,*
- *a norm function $\rho : G \to [0,+\infty)$, which satisfies the following axioms:*
  - *(i)* $\rho(x) = 0$ *if and only if* $x = e$,
  - *(ii) (symmetry) for any* $x \in G$ $\rho(x^{-1}) = \rho(x)$,
  - *(iii) (sub-additivity) for any* $x,y \in G$ $\rho(xy) \leq \rho(x) + \rho(y)$.

Proposition 2.3 *Any normed group (G,ρ) can be seen as a metric space, with any of the distances*

$$d_L(x,y) \, = \, \rho(x^{-1}y) \quad , \quad d_R(x,y) \, = \, \rho(xy^{-1}) \quad .$$

The function d_L is left-*invariant, i.e. for any $x,y,z \in G$ we have $d_L(zx,zy) \, = \, d_L(x,y)$. Similarly d_R is* right-*invariant, that is for any $x,y,z \in G$ we have $d_R(xz,yz) \, = \, d_R(x,y)$.*

proof It suffices to give the proof for the distance d_L. Indeed, the first axiom of a distance is a consequence of the first axiom of a norm, the symmetry axiom for distances is a consequence of the symmetry axiom of the norm and the triangle inequality comes from the group identity

$$x^{-1}z = \left(x^{-1}y\right)\left(y^{-1}z\right)$$

(which itself is a consequence of the associativity of the group operation and of the existence of inverse) and from the sub-additivity of the norm. The left-invariance of d_L comes from the group identity $(zx)^{-1}(zy) = x^{-1}y$. $\square$

Groupoids are generalization of groups. A groupoid can be seen as a small category such that any arrow is invertible. Alternatively, if we look at the set of arrows of such a category, it is a set with a partially defined binary operation and a unary operation (the inverse function), which satisfy several properties. A norm is then a function defined on the set of arrows of a groupoid, with properties similar with the ones of a norm over a group. This is the definition which we give further.

Definition 2.4 *A normed groupoid (G, ρ) is a pair formed by:*

- *a groupoid G, which is a set with two operations $inv : G \rightarrow G$, $m : G^{(2)} \subset G \times G \rightarrow G$, which satisfy a number of properties. With the notations $inv(a) = a^{-1}$, $m(a, b) = ab$, these properties are: for any $a, b, c \in G$*
  - *(i) if $(a, b) \in G^{(2)}$ and $(b, c) \in G^{(2)}$ then $(a, bc) \in G^{(2)}$ and $(ab, c) \in G^{(2)}$ and we have $a(bc) = (ab)c$,*
  - *(ii) $(a, a^{-1}) \in G^{(2)}$ and $(a^{-1}, a) \in G^{(2)}$,*
  - *(iii) if $(a, b) \in G^{(2)}$ then $abb^{-1} = a$ and $a^{-1}ab = b$.*

  The set $X = Ob(G)$ is formed by all products $a^{-1}a$, $a \in G$. For any $a \in G$ we let $\alpha(a) = a^{-1}a$ and $\omega(a) = aa^{-1}$.
- *a norm function $d : G \rightarrow [0, +\infty)$ which satisfies the following axioms:*
  - *(i) $d(g) = 0$ if and only if $g \in Ob(G)$,*
  - *(ii) (symmetry) for any $g \in G$, $d(g^{-1}) = d(g)$,*
  - *(iii) (sub-additivity) for any $(g, h) \in G^{(2)}$, $d(gh) \leq d(g) + d(h)$,*

If $Ob(G)$ is a singleton then G is just a group and the previous definition corresponds exactly to the definition 2.2 of a normed group. As in the case of normed groups, normed groupoids induce metric spaces too.

Proposition 2.5 *Let (G, d) be a normed groupoid and $x \in Ob(G)$. Then the space $(\alpha^{-1}(x), d_x)$ is a metric space, with the distance d_x defined by: for any $g, h \in G$ with $\alpha(g) = \alpha(h) = x$ we have $d_x(g, h) = d(gh^{-1})$.*

Therefore a normed groupoid can be seen as a disjoint union of metric spaces

$$G = \bigcup_{x \in Ob(G)} \alpha^{-1}(x) \quad , \tag{2.1.1}$$

with the property that right translations in the groupoid are isometries, that is: for any $u \in G$ the transformation

$$R_u : \alpha^{-1}(\omega(u)) \rightarrow \alpha^{-1}(\alpha(u)) \quad , \quad R_u(g) = gu$$

has the property for any $g, h \in \alpha^{-1}(\omega(u))$

$$d_{\omega(u)}(g,h) \,=\, d_{\alpha(u)}(R_u(g), R_u(h)) \quad .$$

proof We begin by noticing that if $\alpha(g) = \alpha(h)$ then $(g, h^{-1}) \in G^{(2)}$, therefore the expression gh^{-1} makes sense. The rest of the proof of the first part of the proposition is identical with the proof of the previous proposition.

For the proof of the second part of the proposition remark first that R_u is well defined and that

$$R_u(g)\,(R_u(h))^{-1} \,=\, gh^{-1} \quad .$$

Then we have:

$$d_{\alpha(u)}(R_u(g), R_u(h)) \,=\, d\left(R_u(g)\,(R_u(h))^{-1}\right) \,=$$

$$=\, d(gh^{-1}) \,=\, d_{\omega(u)}(g,h) \quad .$$

□

Therefore normed groupoids provide examples of (disjoint unions of) metric spaces. Are there metric spaces more general than these? No, in fact we have the following.

Proposition 2.6 *Any metric space can be constructed from a normed groupoid, as in proposition 2.5. Precisely, let* (X, d) *be a metric space and consider the* trivial groupoid $G = X \times X$ *with multiplication*

$$(x,y)(y,z) \,=\, (x,z)$$

and inverse $(x,y)^{-1} = (y,x)$. *Then* (G, d) *is a normed groupoid and moreover any component of the decomposition (2.1.1) of* G *is isometric with* (X, d).

Conversely, if $G = X \times X$ *is the trivial groupoid associated to the set* X *and* d *is a norm on* G *then* (X, d) *is a metric space.*

proof We begin by noticing that $\alpha(x,y) = (y,y)$, $\omega(x,y) = (x,x)$, therefore $Ob(G) = \{(x,x) : x \in X\}$ can be identified with X by the bijection $(x,x) \mapsto x$. Moreover, for any $x \in X$ we have

$$\alpha^{-1}((x,x)) \,=\, X \times \{x\} \quad .$$

Because $d : X \times X \to [0, +\infty)$ and $G = X \times X$ it follows that $d : G \to [0, +\infty)$. We have to check the properties of a norm over a groupoid. But these are straightforward. The statement (i) ($d(x,y) = 0$ if and only if $(x,y) \in Ob(G)$) is equivalent with $d(x,y) = 0$ if and only if $x = y$. The symmetry condition (ii) is just the symmetry of the distance: $d(x,y) = d(y,x)$. Finally the sub-additivity of d seen as defined on the groupoid G is equivalent with the triangle inequality:

$$d((x,y)(y,z)) \,=\, d(x,z) \,\leq\, d(x,y) + d(y,z) \quad .$$

In conclusion (G, d) is a normed groupoid if and only if (X, d) is a metric space.

For any $x \in X$ the distance $d_{(x,x)}$ on the space $\alpha^{-1}((x,x))$ has the expression:

$$d_{(x,x)}((u,x),(v,x)) \,=\, d((u,x)(v,x)^{-1}) \,=\, d((u,x)(x,v)) \,=\, d(u,v)$$

therefore the metric space $(\alpha^{-1}((x,x)), d_{(x,x)})$ is isometric with (X, d) by the isometry $(u,x) \mapsto u$, for any $u \in X$. □

In conclusion normed groups give particular examples of metric spaces and metric spaces are particular examples of normed groupoids. For this reason normed groups make good examples of metric spaces. It is also interesting to extend the theory of metric spaces to normed groupoids (other than trivial normed groupoids). This is done in [16].

3 Maps of metric spaces

Imagine that the metric space (X, d) represents a territory. We want to make maps of (X, d) in the metric space (Y, D) (a piece of paper, or a scaled model).

In fact, in order to understand the territory (X, d), we need many maps, at many scales. For any point $x \in X$ and any scale $\varepsilon > 0$ we shall make a map of a neighbourhood of x, ideally. In practice, a good knowledge of a territory amounts to have, for each of several scales $\varepsilon_1 > \varepsilon_2 > ... > \varepsilon_n$ an atlas of maps of overlapping parts of X (which together form a cover of the territory X). All the maps from all the atlasses have to be compatible one with another.

The ideal model of such a body of knowledge is embodied into the notion of a manifold. To have X as a manifold over the model space Y means exactly this.

Examples from metric geometry (like sub-riemannian spaces) show that the manifold idea could be too rigid in some situations. We shall replace it with the idea of a dilation structure.

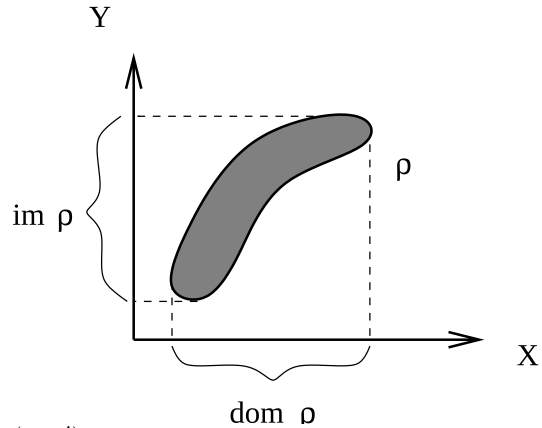
We shall see that a dilation structure (the right generalization of a smooth space, like a manifold), represents an idealization of the more realistic situation of having at our disposal many maps, at many scales, of the territory, with the property that the accuracy, precision and resolution of such maps, and of relative maps deduced from them, are controlled by the scale (as the scale goes to zero, to infinitesimal).

There are two facts which I need to stress. First is that such a generalization is necessary. Indeed, by looking at the large gallery of metric spaces which we now know, the metric spaces with a manifold structure form a tiny and very very particular class. Second is that we tend to take for granted the body of knowledge represented by a manifold structure (or by a dilation structure). Think as an example at the manifold structure of the Earth. It is an idealization of the collection of all cartographic maps of parts of the Earth. This is a huge data basis and it required a huge amount of time and energy in order to be constructed. To know, understand the territory is a huge task, largely neglected. We "have" a manifold, "let X be a manifold". And even if we do not doubt that the physical space (whatever that means) is a boring $\mathbb{R}^3$, it is nevertheless another task to determine with the best accuracy possible a certain point in that physical space, based on the knowledge of the coordinates. For example GPS costs money and time to build and use. Or, it is rather easy to collide protons, but to understand and keep the territory fixed (more or less) with respect to the map, that is where most of the effort goes.

A model of such a map of (X, d) in (Y, D) is a relation $\rho \subset X \times Y$, a subset of a cartesian product $X \times Y$ of two sets. A particular type of relation is the graph of a function $f : X \to Y$, defined as the relation

$$\rho = \{(x, f(x)) : x \in X\}$$

but there are many relations which cannot be described as graphs of functions.



Imagine that pairs $(u, u') \in \rho$ are pairs

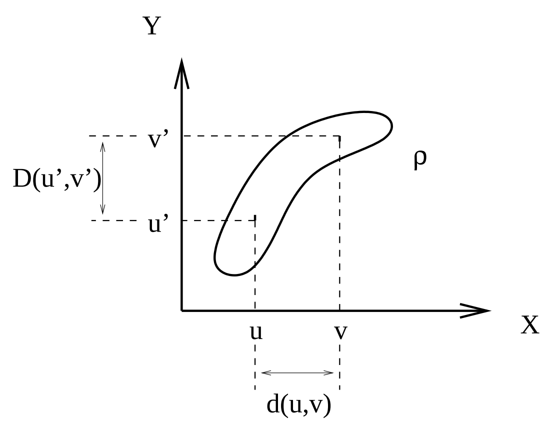
(point in the space X, pixel in the "map space" Y)

with the meaning that the point u in X is represented as the pixel u' in Y.

I don't suppose that there is a one-to-one correspondence between points in X and pixels in Y, for various reasons, for example: due to repeated measurements there is no unique way to associate pixel to a point, or a point to a pixel. The relation ρ represents the cloud of pairs point-pixel which are compatible with all measurements.

I shall use this model of a map for simplicity reasons. A better, more realistic model could be one using probability measures, but this model is sufficient for the needs of this paper.

For a given map ρ the point $x \in X$ in the space X is associated the set of points $\{y \in Y : (x, y) \in \rho\}$ in the "map space" Y. Similarly, to the "pixel" $y \in Y$ in the "map space" Y is associated the set of points $\{x \in X : (x, y) \in \rho\}$ in the space X.

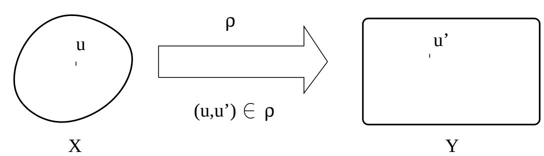


A good map is one which does not distort distances too much. Specifically, considering any two points $u, v \in X$ and any two pixels $u', v' \in Y$ which represent these points, i.e. $(u, u'), (v, v') \in \rho$, the distortion of distances between these points is measured by the number

$$| \, d(u,v) - D(u',v') \, |$$

3.1 Accuracy, precision, resolution, Gromov-Hausdorff distance

Notations concerning relations. Even if relations are more general than (graphs of) functions, there is no harm to use, if needed, a functional notation. For any relation $\rho \subset X \times Y$ we shall write $\rho(x) = y$ or $\rho^{-1}(y) = x$ if $(x,y) \in \rho$. Therefore we may have $\rho(x) = y$ and $\rho(x) = y'$ with $y \neq y'$, if $(x,y) \in f$ and $(x,y') \in f$. In some drawings, relations will be figured by a large arrow, as shown further.



The domain of the relation ρ is the set $dom \ \rho \ \subset X$ such that for any $x \in \ dom \ \rho$ there is $y \in Y$ with $\rho(x) = y$. The image of ρ is the set of $im \ \rho \ \subset Y$ such that for any $y \in \ im \ \rho$ there is $x \in X$ with $\rho(x) = y$. By convention, when we write that a statement $R(f(x), f(y), ...)$ is true, we mean that $R(x', y', ...)$ is true for any choice of $x', y', ...$, such that $(x,x'), (y,y'), ... \in f$.

The inverse of a relation $\rho \subset X \times Y$ is the relation

$$\rho^{-1} \subset Y \times X \, , \quad \rho^{-1} \, = \, \{(u',u) \, : \, (u,u') \in \rho\}$$

and if $\rho' \subset X \times Y$, $\rho" \subset Y \times Z$ are two relations, their composition is defined as

$$\rho = \rho" \circ \rho' \subset X \times Z$$

$$\rho \, = \, \{(u,u") \in X \times Z \, : \, \exists u' \in Y \; (u,u') \in \rho' \; (u',u") \in \rho"\}$$

I shall use the following convenient notation: by $\mathcal{O}(\varepsilon)$ we mean a positive function such that $\lim\limits_{\varepsilon \to 0} \mathcal{O}(\varepsilon) = 0$.

In metrology, by definition, accuracy is [50] 2.13 (3.5) "closeness of agreement between a measured quantity value and a true quantity value of a measurand". (Measurement) precision is [50] 2.15 "closeness of agreement between indications or measured quantity values obtained by replicate measurements on the same or similar objects under specified conditions". Resolution is [50] 2.15 "smallest change in a quantity being measured that causes a perceptible change in the corresponding indication".

For our model of a map, if $(u,u') \in \rho$ then u' represent the measurement of u. Moreover, because we see a map as a relation, the definition of the resolution can be restated as the supremum of distances between points in X which are represented by the same pixel. Indeed, if the distance between two points in X is bigger than this supremum then they cannot be represented by the same pixel.

Definition 3.1 *Let $\rho \subset X \times Y$ be a relation which represents a map of dom $\rho \subset (X,d)$ into im $\rho \subset (Y,D)$. To this map are associated three quantities: accuracy, precision and resolution.*

The accuracy of ρ is defined by:

$$acc(\rho) \,=\, \sup\{\mid D(y_1, y_2) - d(x_1, x_2) \mid \,:\, (x_1, y_1) \in \rho\,,\, (x_2, y_2) \in \rho\} \tag{3.1.1}$$

The resolution of ρ at $y \in im\ \rho$ is

$$res(\rho)(y) \,=\, \sup\{d(x_1, x_2) \,:\, (x_1, y) \in \rho\,,\, (x_2, y) \in \rho\} \tag{3.1.2}$$

and the resolution of ρ is given by:

$$res(\rho) \,=\, \sup\{res(\rho)(y) \,:\, y \in\ im\ \rho\} \tag{3.1.3}$$

The precision of ρ at $x \in dom\ \rho$ is

$$prec(\rho)(x) \,=\, \sup\{D(y_1, y_2) \,:\, (x, y_1) \in \rho\,,\, (x, y_2) \in \rho\} \tag{3.1.4}$$

and the precision of ρ is given by:

$$prec(\rho) \,=\, \sup\{prec(\rho)(x) \,:\, x \in\ dom\ \rho\} \tag{3.1.5}$$

After measuring (or using other means to deduce) the distances $d(x', x")$ between all pairs of points in X (we may have several values for the distance $d(x', x")$), we try to represent the collection of these distances in (Y, D). When we make a map ρ we are not really measuring the distances between all points in X, then representing them as accurately as possible in Y.

What we do is that we consider a relation ρ, with domain $M = dom(\rho)$ which is ε-dense in (X, d), then we perform a " cartographic generalization"[1] of the relation ρ to a relation $\bar{\rho}$, a map of (X, d) in (Y, D), for example as in the following definition.

Definition 3.2 *A subset $M \subset X$ of a metric space (X, d) is ε-dense in X if for any $u \in X$ there is $x \in M$ such that $d(x, u) \leq \varepsilon$.*

Let $\rho \subset X \times Y$ be a relation such that dom ρ is ε-dense in (X, d) and im ρ is μ-dense in (Y, D). We define then $\bar{\rho} \subset X \times Y$ by: $(x, y) \in \bar{\rho}$ if there is $(x', y') \in \rho$ such that $d(x, x') \leq \varepsilon$ and $D(y, y') \leq \mu$.

If ρ is a relation as described in definition 3.2 then accuracy $acc(\rho)$, ε and μ control the precision $prec(\rho)$ and resolution $res(\rho)$. Moreover, the accuracy, precision and resolution of $\bar{\rho}$ are controlled by those of ρ and ε, μ, as well. This is explained in the next proposition.

Proposition 3.3 *Let ρ and $\bar{\rho}$ be as described in definition 3.2. Then:*

(a) $res(\rho) \,\leq\, acc(\rho)$,

(b) $prec(\rho) \,\leq\, acc(\rho)$,

(c) $res(\rho) + 2\varepsilon \leq\, res(\bar{\rho}) \leq\, acc(\rho) + 2(\varepsilon + \mu)$,

(d) $prec(\rho) + 2\mu \leq\, prec(\bar{\rho}) \leq\, acc(\rho) + 2(\varepsilon + \mu)$,

(e) $\mid acc(\bar{\rho}) -\, acc(\rho) \mid\leq 2(\varepsilon + \mu)$.

[1] `http://en.wikipedia.org/wiki/Cartographic_generalization`, "Cartographic generalization is the method whereby information is selected and represented on a map in a way that adapts to the scale of the display medium of the map, not necessarily preserving all intricate geographical or other cartographic details.

Proof. Remark that (a), (b) are immediate consequences of definition 3.1 and that (c) and (d) must have identical proofs, just by switching ε with μ and X with Y respectively. I shall prove therefore (c) and (e).

For proving (c), consider $y \in Y$. By definition of $\bar{\rho}$ we write

$$\{x \in X \, : \, (x,y) \in \bar{\rho}\} \, = \bigcup_{(x',y') \in \rho, y' \in \bar{B}(y,\mu)} \bar{B}(x',\varepsilon)$$

Therefore we get

$$res(\bar{\rho})(y) \, \geq \, 2\varepsilon + \sup\left\{ res(\rho)(y') \, : \, y' \in \, im(\rho) \cap \bar{B}(y,\mu) \right\}$$

By taking the supremum over all $y \in Y$ we obtain the inequality

$$res(\rho) + 2\varepsilon \leq \, res(\bar{\rho})$$

For the other inequality, let us consider $(x_1, y), (x_2, y) \in \bar{\rho}$ and $(x_1', y_1'), (x_2', y_2') \in \rho$ such that $d(x_1, x_1') \leq \varepsilon, d(x_2, x_2') \leq \varepsilon, D(y_1', y) \leq \mu, D(y_2', y) \leq \mu$. Then:

$$d(x_1, x_2) \leq 2\varepsilon + d(x_1', x_2') \leq 2\varepsilon + \, acc(\rho) + d(y_1', y_2') \leq 2(\varepsilon + \mu) + \, acc(\rho)$$

Take now a supremum and arrive to the desired inequality.

For the proof of (e) let us consider for $i = 1, 2$ $(x_i, y_i) \in \bar{\rho}, (x_i', y_i') \in \rho$ such that $d(x_i, x_i') \leq \varepsilon, D(y_i, y_i') \leq \mu$. It is then enough to take absolute values and transform the following equality

$$d(x_1, x_2) - D(y_1, y_2) = d(x_1, x_2) - d(x_1', x_2') + d(x_1', x_2') - D(y_1', y_2') +$$

$$+ D(y_1', y_2') - D(y_1, y_2)$$

into well chosen, but straightforward, inequalities. □

The following definition of the Gromov-Hausdorff distance for metric spaces is natural, owing to the fact that the accuracy (as defined in definition 3.1) controls the precision and resolution.

Definition 3.4 *Let (X, d), (Y, D), be a pair of metric spaces and $\mu > 0$. We shall say that μ is admissible if there is a relation $\rho \subset X \times Y$ such that dom $\rho = X$, im $\rho = Y$, and $acc(\rho) \leq \mu$. The Gromov-Hausdorff distance between (X, d) and (Y, D) is the infimum of admissible numbers μ.*

As introduced in definition 3.4, the Gromov-Hausdorff (GH) distance is not a true distance, because the GH distance between two isometric metric spaces is equal to zero. In fact the GH distance induces a distance on isometry classes of compact metric spaces.

The GH distance thus represents a lower bound on the accuracy of making maps of (X, d) into (Y, D). Surprising as it might seem, there are many examples of pairs of metric spaces with the property that the GH distance between any pair of closed balls from these spaces, considered with the distances properly rescaled, is greater than a strictly positive number, independent of the choice of the balls. Simply put: *there are pairs of spaces X, Y such that is impossible to make maps of parts of X into Y with arbitrarily small accuracy.*

Any measurement is equivalent with making a map, say of X (the territory of the phenomenon) into Y (the map space of the laboratory). The possibility that there might a physical difference (manifested as a strictly positive GH distance) between these two spaces, even if they both might be topologically the same (and with trivial topology, say of a $\mathbb{R}^n$), is ignored in physics, apparently. On one side, there is no experimental way to confirm that a territory is the same at any scale (see the section dedicated to the notion of scale), but much of physical explanations are based on differential calculus, which has as the most basic assumption that locally and infinitesimally the territory is the same. On the other side the imposibility of making maps of the phase space of a quantum object into the macroscopic map space of the laboratory might be a manifestation of the fact that there is a difference (positive GH distance between maps of the territory realised with the help of physical phenomena) between "small" and "macroscopic" scale.

3.2 Scale

Let $\varepsilon > 0$. A map of (X, d) into (Y, D), at scale ε is a map of $(X, \frac{1}{\varepsilon} d)$ into (Y, D). Indeed, if this map would have accuracy equal to 0 then a value of a distance between points in X equal to L would correspond to a value of the distance between the corresponding points on the map in (Y, D) equal to εL.

In cartography, maps of the same territory done at smaller and smaller scales (smaller and smaller ε) must have the property that, at the same resolution, the accuracy and precision (as defined in definition 3.1) have to become smaller and smaller.

In mathematics, this could serve as the definition of the metric tangent space to a point in (X, d), as seen in (Y, D).

Definition 3.5 *We say that (Y, D, y) $(y \in Y)$ represents the (pointed unit ball in the) metric tangent space at $x \in X$ of (X, d) if there exist a pair formed by:*

- *a "zoom sequence", that is a sequence*

$$\varepsilon \in (0, 1] \mapsto \rho^x_\varepsilon \subset (\bar{B}(x, \varepsilon), \frac{1}{\varepsilon} d) \times (Y, D)$$

  such that dom $\rho^x_\varepsilon = \bar{B}(x, \varepsilon)$, im $\rho^x_\varepsilon = Y$, $(x, y) \in \rho^x_\varepsilon$ for any $\varepsilon \in (0, 1]$ and

- *a "zoom modulus" $F : (0, 1) \to [0, +\infty)$ such that $\lim_{\varepsilon \to 0} F(\varepsilon) = 0$,*

such that for all $\varepsilon \in (0, 1)$ we have $acc(\rho^x_\varepsilon) \leq F(\varepsilon)$.

Using the notation proposed previously, we can write $F(\varepsilon) = \mathcal{O}(\varepsilon)$, if there is no need to precisely specify a zoom modulus function.

Let us write again the definition of resolution, accuracy, precision, in the presence of scale. The accuracy of ρ^x_ε is defined by:

$$acc(\rho^x_\varepsilon) \ = \ \sup \left\{ \mid D(y_1, y_2) - \frac{1}{\varepsilon} d(x_1, x_2) \mid \ : \ (x_1, y_1), \, (x_2, y_2) \in \rho^x_\varepsilon \right\} \tag{3.2.6}$$

The resolution of ρ^x_ε at $z \in Y$ is

$$res(\rho^x_\varepsilon)(z) \ = \ \frac{1}{\varepsilon} \sup \left\{ d(x_1, x_2) \, : \, (x_1, z) \in \rho^x_\varepsilon \, , \, (x_2, z) \in \rho^x_\varepsilon \right\} \tag{3.2.7}$$

and the resolution of ρ^x_ε is given by:

$$res(\rho^x_\varepsilon) \ = \ \sup \left\{ res(\rho^x_\varepsilon)(y) \, : \, y \in Y \right\} \tag{3.2.8}$$

The precision of ρ_ε^x at $u \in \bar{B}(x,\varepsilon)$ is

$$prec(\rho_\varepsilon^x)(u) \, = \, \sup \left\{ D(y_1,y_2) \, : \, (u,y_1) \in \rho_\varepsilon^x \, , \, (u,y_2) \in \rho_\varepsilon^x \right\} \tag{3.2.9}$$

and the precision of ρ_ε^x is given by:

$$prec(\rho_\varepsilon^x) \, = \, \sup \left\{ prec(\rho_\varepsilon^x)(u) \, : \, u \in \bar{B}(x,\varepsilon) \right\} \tag{3.2.10}$$

If (Y, D, y) represents the (pointed unit ball in the) metric tangent space at $x \in X$ of (X, d) and ρ_ε^x is the sequence of maps at smaller and smaller scale, then we have:

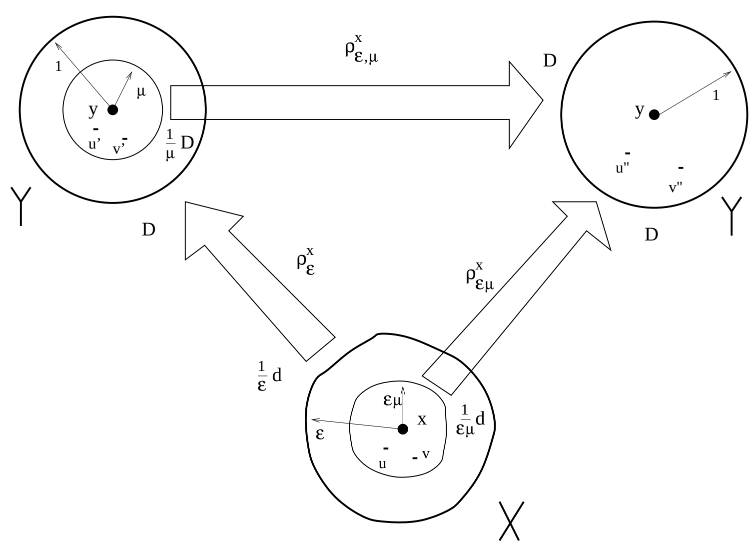
$$\sup \left\{ \mid D(y_1,y_2) - \frac{1}{\varepsilon} d(x_1,x_2) \mid \, : \, (x_1,y_1), \, (x_2,y_2) \in \rho_\varepsilon^x \right\} \, = \, \mathcal{O}(\varepsilon) \tag{3.2.11}$$

$$\sup \left\{ D(y_1,y_2) \, : \, (u,y_1) \in \rho_\varepsilon^x \, , \, (u,y_2) \in \rho_\varepsilon^x \, , \, u \in \bar{B}(x,\varepsilon) \right\} \, = \, \mathcal{O}(\varepsilon) \tag{3.2.12}$$

$$\sup \left\{ d(x_1,x_2) \, : \, (x_1,z) \in \rho_\varepsilon^x \, , \, (x_2,z) \in \rho_\varepsilon^x \, , \, z \in Y \right\} \, = \, \varepsilon \, \mathcal{O}(\varepsilon) \tag{3.2.13}$$

Of course, relation (3.2.11) implies the other two, but it is interesting to notice the mechanism of rescaling.

3.3 Scale stability. Viewpoint stability



I shall suppose further that there is a metric tangent space at $x \in X$ and I shall work with a zoom sequence of maps described in definition 3.5.

Let $\varepsilon, \mu \in (0,1)$ be two scales. Suppose we have the maps of the territory X, around $x \in X$, at scales ε and $\varepsilon\mu$,

$$\rho_\varepsilon^x \subset \bar{B}(x,\varepsilon) \times \bar{B}(y,1)$$

$$\rho_{\varepsilon\mu}^x \subset \bar{B}(x,\varepsilon\mu) \times \bar{B}(y,1)$$

made into the tangent space at x, $(\bar{B}(y,1), D)$. The ball $\bar{B}(x, \varepsilon\mu) \subset X$ has then two maps. These maps are at different scales: the first is done at scale ε, the second is done at scale $\varepsilon\mu$.

What are the differences between these two maps? We could find out by defining a new map

$$\rho^x_{\varepsilon,\mu} \; = \; \big\{ (u', u") \in \bar{B}(y,\mu) \times \bar{B}(y,1) \, : \\ \exists u \in \bar{B}(x, \varepsilon\mu) \, (u,u') \in \rho^x_\varepsilon \, , \, (u, u") \in \rho^x_{\varepsilon\mu} \big\} \tag{3.3.14}$$

and measuring its accuracy, with respect to the distances $\frac{1}{\mu} D$ (on the domain) and D (on the image).

Let us consider $(u,u'), (v,v') \in \rho^x_\varepsilon$ and $(u,u"), (v,v") \in \rho^x_{\varepsilon\mu}$ such that $(u',u"), (v',v") \in \rho^x_{\varepsilon,\mu}$. Then:

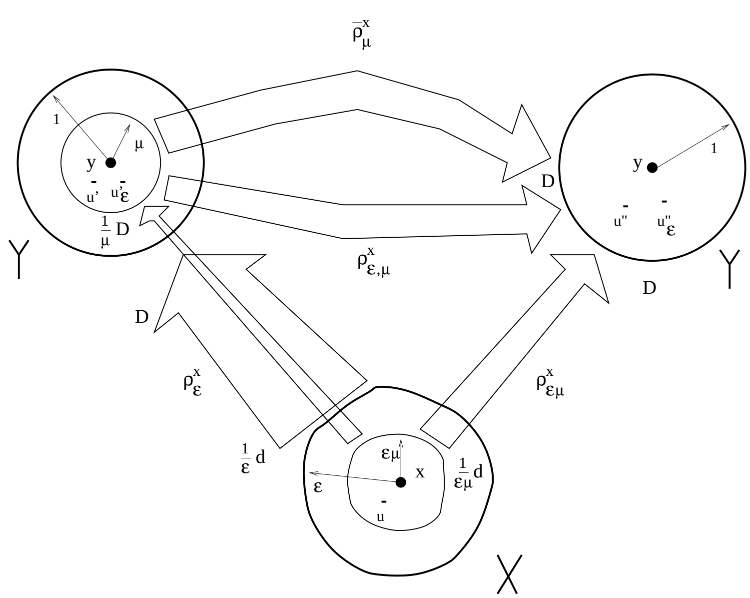
$$\mid D(u",v") - \frac{1}{\mu} D(u',v') \mid \leq \mid \frac{1}{\mu} D(u',v') - \frac{1}{\varepsilon\mu} d(u,v) \mid \, + \, \mid \frac{1}{\varepsilon\mu} d(u,v) - D(u",v") \mid$$

We have therefore an estimate for the accuracy of the map $\rho^x_{\varepsilon,\mu}$, coming from estimate (3.2.11) applied for ρ^x_ε and $\rho^x_{\varepsilon\mu}$:

$$acc(\rho^x_{\varepsilon,\mu}) \; \leq \; \frac{1}{\mu} \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon\mu) \tag{3.3.15}$$

This explains the cascading of errors phenomenon, namely, for fixed μ, as ε goes to 0 the accuracy of the map $\rho^x_{\varepsilon,\mu}$ becomes smaller and smaller, meaning that the maps of the ball $\bar{B}(x, \varepsilon\mu) \subset X$ at the scales $\varepsilon, \varepsilon\mu$ (properly rescaled) are more and more alike. On the contrary, for fixed ε, as μ goes to 0, the bound on the accuracy becomes bigger and bigger, meaning that by using only the map at scale ε, magnifications of a smaller scale region of this map may be less accurate than the map of this smaller region done at the smaller scale.

I shall add a supplementary hypothesis to the one concerning the existence of the metric tangent space. It is somehow natural to suppose that as ε converges to 0 the map $\rho^x_{\varepsilon,\mu}$ converges to a map $\bar{\rho}^x_\mu$. This is described further.



Definition 3.6 *Let the zoom sequence ρ^x_ε be as in definition 3.5 and for given $\mu \in (0,1)$, the map $\rho^x_{\varepsilon,\mu}$ be defined as in (3.3.14). We say that the zoom sequence ρ^x_ε is scale stable at scale μ if there is a relation $\bar{\rho}^x_\mu \subset \bar{B}(y,\mu) \times \bar{B}(y,1)$ such that the Haussorff distance between $\rho^x_{\varepsilon,\mu}$ and $\bar{\rho}^x_\mu$, in the metric space $\bar{B}(y,\mu) \times \bar{B}(y,1)$ with the distance*

$$D_\mu \left((u', u"), (v', v") \right) \, = \, \frac{1}{\mu} D(u', v') + D(u", v")$$

can be estimated as:

$$D^{Hausdorff}_\mu \left(\rho^x_{\varepsilon,\mu}, \bar{\rho}^x_\mu \right) \, \leq \, F_\mu(\varepsilon)$$

with $F_\mu(\varepsilon) = \mathcal{O}_\mu(\varepsilon)$. Such a function $F_\mu(\cdot)$ is called a scale stability modulus of the zoom sequence ρ^x_ε.

This means that for any $(u', u") \in \bar{\rho}^x_\mu$ there is a sequence $(u'_\varepsilon, u"_\varepsilon) \in \rho^x_{\varepsilon,\mu}$ such that

$$\lim_{\varepsilon \rightarrow 0} u'_\varepsilon \, = \, u' \quad \lim_{\varepsilon \rightarrow 0} u"_\varepsilon \, = \, u"$$

Proposition 3.7 *If there is a scale stable zoom sequence ρ^x_ε as in definitions 3.5 and 3.6 then the space (Y, D) is self-similar in a neighbourhood of point $y \in Y$, namely for any $(u', u"), (v', v") \in \bar{\rho}^x_\mu$ we have:*

$$D(u", v") \, = \, \frac{1}{\mu} D(u', v')$$

In particular $\bar{\rho}^x_\mu$ is the graph of a function (the precision and resolution are respectively equal to 0).

Proof. Indeed, for any $\varepsilon \in (0,1)$ let us consider $(u'_\varepsilon, u"_\varepsilon), (v'_\varepsilon, v"_\varepsilon) \in \rho^x_{\varepsilon,\mu}$ such that

$$\frac{1}{\mu} D(u', u'_\varepsilon) + D(u", u"_\varepsilon) \, \leq \, \mathcal{O}_\mu(\varepsilon)$$

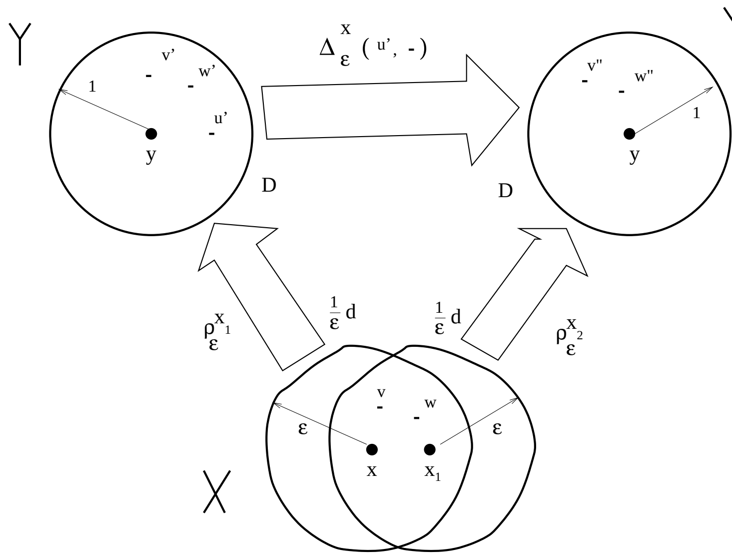
$$\frac{1}{\mu} D(v', v'_\varepsilon) + D(v", v"_\varepsilon) \, \leq \, \mathcal{O}_\mu(\varepsilon)$$

Then we get the following inequality, using also the cascading of errors inequality (3.3.15),

$$\mid D(u", v") - \frac{1}{\mu} D(u', v') \mid \leq \, 2\mathcal{O}_\mu(\varepsilon) + \frac{1}{\mu}\mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon\mu)$$

We pass with ε to 0 in order to obtain the conclusion. □

Instead of changing the scale (i.e. understanding the scale stability of the zoom sequence), we could explore what happens when we change the point of view.



This time we have a zoom sequence, a scale $\varepsilon \in (0,1)$ and two points: $x \in X$ and $u' \in \bar{B}(y,1)$. To the point u' from the map space Y corresponds a point $x_1 \in \bar{B}(x,\varepsilon)$ such that

$$(x_1, u') \in \rho_\varepsilon^x$$

The points x, x_1 are neighbours, in the sense that $d(x, x_1) < \varepsilon$. The points of X which are in the intersection

$$\bar{B}(x,\varepsilon) \cap \bar{B}(x_1,\varepsilon)$$

are represented by both maps, ρ_ε^x and $\rho_\varepsilon^{x_1}$. These maps are different; the relative map between them is defined as:

$$\Delta_\varepsilon^x(u',\cdot) \;=\; \big\{(v', v") \in \bar{B}(y,1) \,:\, \exists v \in \bar{B}(x,\varepsilon) \cap \bar{B}(x_1,\varepsilon) \tag{3.3.16}$$

$$(v, v') \in \rho_\varepsilon^x \,,\, (v, v") \in \rho_\varepsilon^{x_1}\}$$

and it is called "difference at scale ε, from x to x_1, as seen from u'".

The viewpoint stability of the zoom sequence is expressed as the scale stability: the zoom sequence is stable if the difference at scale ε converges in the sense of Hausdorff distance, as ε goes to 0.

Definition 3.8 *Let the zoom sequence ρ_ε^x be as in definition 3.5 and for any $u' \in \bar{B}(y,1)$, the map $\Delta_\varepsilon^x(u',\cdot)$ be defined as in (3.3.16). The zoom sequence ρ_ε^x is viewpoint stable if there is a relation $\Delta^x(u',\cdot) \subset \bar{B}(y,1) \times \bar{B}(y,1)$ such that the Haussorff distance can be estimated as:*

$$D_\mu^{Hausdorff}\left(\Delta_\varepsilon^x(u',\cdot), \Delta^x(u',\cdot)\right) \;\leq\; F_{diff}(\varepsilon)$$

with $F_{diff}(\varepsilon) = \mathcal{O}(\varepsilon)$. Such a function $F_{diff}(\cdot)$ is called a viewpoint stability modulus of the zoom sequence ρ_ε^x.

There is a proposition analoguous with proposition 3.7, stating that the difference relation $\Delta^x(u',\cdot)$ is the graph of an isometry of (Y,D).

3.4 Foveal maps

The following proposition shows that if we have a scale stable zoom sequence of maps ρ_ε^x as in definitions 3.5 and 3.6 then we can improve every member of the sequence such that all maps from the new zoom sequence have better accuracy near the "center" of the map $x \in X$, which justifies the name "foveal maps".

Definition 3.9 *Let ρ_ε^x be a scale stable zoom sequence. We define for any $\varepsilon \in (0,1)$ the μ-foveal map ϕ_ε^x made of all pairs $(u,u') \in \bar{B}(x,\varepsilon) \times \bar{B}(y,1)$ such that*

- *if $u \in \bar{B}(x,\varepsilon\mu)$ then $(u, \bar{\rho}_\mu^x(u')) \in \rho_{\varepsilon\mu}^x$,*
- *or else $(u,u') \in \rho_\varepsilon^x$.*

Proposition 3.10 *Let ρ_ε^x be a scale stable zoom sequence with associated zoom modulus $F(\cdot)$ and scale stability modulus $F_\mu(\cdot)$. The sequence of μ-foveal maps ϕ_ε^x is then a scale stable zoom sequence with zoom modulus $F(\cdot) + \mu F_\mu(\cdot)$. Moreover, the accuracy of the restricted foveal map $\phi_\varepsilon^x \cap \left(\bar{B}(x,\varepsilon\mu) \times \bar{B}(y,\mu)\right)$ is bounded by $\mu F(\varepsilon\mu)$, therefore the right hand side term in the cascading of errors inequality (3.3.15), applied for the restricted foveal map, can be improved to $2F(\varepsilon\mu)$.*

Proof. Let $u \in \bar{B}(x,\varepsilon\mu)$. Then there are $u', u'_\varepsilon \in \bar{B}(y,\mu)$ and $u", u"_\varepsilon \in \bar{B}(y,1)$ such that $(u,u') \in \phi_\varepsilon^x$, $(u,u") \in \rho_\varepsilon^x\mu)$, $(u',u") \in \bar{\rho}_\mu^x$, $(u'_\varepsilon, u"_\varepsilon) \in \rho_{\varepsilon,\mu}^x$ and

$$\frac{1}{\mu} D(u', u'_\varepsilon) + D(u", u"_\varepsilon) \leq F_\mu(\varepsilon)$$

Let $u,v \in \bar{B}(x,\varepsilon\mu)$ and $u',v' \in \bar{B}(y,\mu)$ such that $(u,u'),(v,v') \in \phi_\varepsilon^x$. According to the definition of ϕ_ε^x, it follows that there are uniquely defined $u",v" \in \bar{B}(y,1)$ such that $(u,u"),(v,v") \in \rho_{\varepsilon\mu}^x$ and $(u',u"),(v',v") \in \bar{\rho}_\mu^x$. We then have:

$$\mid \frac{1}{\varepsilon} d(u,v) - D(u',v') \mid =$$

$$= \mid \frac{1}{\varepsilon} d(u,v) - \mu D(u",v") \mid =$$

$$= \mu \mid \frac{1}{\varepsilon\mu} d(u,v) - D(u",v") \mid \leq \mu F(\varepsilon\mu)$$

Thus we proved that the accuracy of the restricted foveal map

$$\phi_\varepsilon^x \cap \left(\bar{B}(x,\varepsilon\mu) \times \bar{B}(y,\mu)\right)$$

is bounded by $\mu F(\varepsilon\mu)$:

$$\mid \frac{1}{\varepsilon} d(u,v) - D(u',v') \mid \leq \mu F(\varepsilon\mu) \tag{3.4.17}$$

If $u,v \in \bar{B}(x,\varepsilon) \setminus \bar{B}(x,\mu)$ and $(u,u'),(v,v') \in \phi_\varepsilon^x$ then $(u,u'),(v,v') \in \rho_\varepsilon^x$, therefore

$$\mid \frac{1}{\varepsilon} d(u,v) - D(u',v') \mid \leq F(\varepsilon)$$

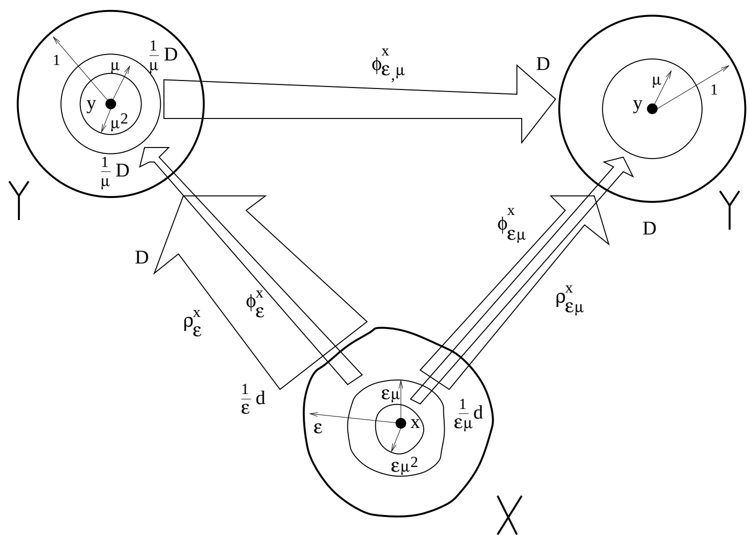
Suppose now that $(u,u'),(v,v') \in \phi_\varepsilon^x$ and $u \in \bar{B}(x,\varepsilon\mu)$ but $v \in \bar{B}(x,\varepsilon) \setminus \bar{B}(x,\mu)$. We have then:

$$\mid \frac{1}{\varepsilon} d(u,v) - D(u',v') \mid \leq$$

$$\leq \mid \frac{1}{\varepsilon} d(u,v) - D(u'_\varepsilon, v') \mid + D(u', u'_\varepsilon) \; \leq \; F(\varepsilon) + \mu F_\mu(\varepsilon)$$

We proved that the sequence of μ-foveal maps ϕ^x_ε is a zoom sequence with zoom modulus $F(\cdot) + \mu F_\mu(\cdot)$.

In order to prove that the sequence is scale stable, we have to compute $\phi^x_{\varepsilon,\mu}$, graphically shown in the next figure.



We see that $(u', u") \in \phi^x_{\varepsilon,\mu}$ implies that $(u', u") \in \rho^x_{\varepsilon,\mu}$ or $(u', u") \in \rho^x_{\varepsilon\mu,\mu}$. From here we deduce that the sequence of foveal maps is scale stable and that

$$\varepsilon \mapsto \max \left\{ F_\mu(\varepsilon), \mu F_\mu(\varepsilon\mu) \right\}$$

is a scale stability modulus for the foveal sequence.

The improvement of the right hand side for the cascading of errors inequality (3.3.15), applied for the restricted foveal map is then straightforward if we use (3.4.17). □

3.5 Metric profiles. Metric tangent space

We shall denote by CMS the set of isometry classes of pointed compact metric spaces. The distance on this set is the Gromov distance between (isometry classes of) pointed metric spaces and the topology is induced by this distance.

To any locally compact metric space we can associate a metric profile [17, 18].

Definition 3.11 *The metric profile associated to the locally metric space (M,d) is the assignment (for small enough $\varepsilon > 0$)*

$$(\varepsilon > 0, \; x \in M) \;\; \mapsto \;\; \mathbb{P}^m(\varepsilon, x) = \left[\bar{B}(x,1), \frac{1}{\varepsilon} d, x \right] \in CMS$$

We can define a notion of metric profile regardless to any distance.

Definition 3.12 *A metric profile is a curve $\mathbb{P} : [0, a] \to CMS$ such that*

(a) it is continuous at 0,

(b) for any $b \in [0, a]$ and $\varepsilon \in (0, 1]$ we have

$$d_{GH}(\mathbb{P}(\varepsilon b), \mathbb{P}^m_{d_b}(\varepsilon, x_b)) \ = \ O(\varepsilon)$$

The function $\mathcal{O}(\varepsilon)$ may change with b. We used the notations

$$\mathbb{P}(b) = [\bar{B}(x,1), d_b, x_b] \quad \text{ and } \quad \mathbb{P}^m_{d_b}(\varepsilon, x) = \left[\bar{B}(x,1), \frac{1}{\varepsilon} d_b, x_b\right]$$

The metric profile is nice if

$$d_{GH}\left(\mathbb{P}(\varepsilon b), \mathbb{P}^m_{d_b}(\varepsilon, x)\right) = O(b\varepsilon)$$

Imagine that $1/b$ represents the magnification on the scale of a microscope. We use the microscope to study a specimen. For each $b > 0$ the information that we get is the table of distances of the pointed metric space $(\bar{B}(x,1), d_b, x_b)$.

How can we know, just from the information given by the microscope, that the string of "images" that we have corresponds to a real specimen? The answer is that a reasonable check is the relation from point (b) of the definition of metric profiles 3.12.

Really, this point says that starting from any magnification $1/b$, if we further select the ball $\bar{B}(x, \varepsilon)$ in the snapshot $(\bar{B}(x,1), d_b, x_b)$, then the metric space $(\bar{B}(x,1), \frac{1}{\varepsilon} d_b, x_b)$ looks approximately the same as the snapshot $(\bar{B}(x,1), d_{b\varepsilon}, x_b)$. That is: further magnification by ε of the snapshot (taken with magnification) b is roughly the same as the snapshot $b\varepsilon$. This is of course true in a neighbourhood of the base point x_b.

The point (a) from the Definition 3.12 has no other justification than Proposition 3.16 in next subsection.

We rewrite definition 3.4 with more details, in order to clearly understand what is a metric profile. For any $b \in (0, a]$ and for any $\mu > 0$ there is $\varepsilon(\mu, b) \in (0, 1)$ such that for any $\varepsilon \in (0, \varepsilon(\mu, b))$ there exists a relation $\rho = \rho_{\varepsilon,b} \subset \bar{B}_{d_b}(x_b, \varepsilon) \times \bar{B}_{d_{b\varepsilon}}(x_{b\varepsilon}, 1)$ such that

1. *dom* $\rho_{\varepsilon,b}$ is μ-dense in $\bar{B}_{d_b}(x_b, \varepsilon)$,
2. *im* $\rho_{\varepsilon,b}$ is μ-dense in $\bar{B}_{d_{b\varepsilon}}(x_{b\varepsilon}, 1)$,
3. $(x_b, x_{b\varepsilon}) \in \rho_{\varepsilon,b}$,
4. for all $x, y \in$ *dom* $\rho_{\varepsilon,b}$ we have

$$\left| \frac{1}{\varepsilon} d_b(x,y) - d_{b\varepsilon}\left(\rho_{\varepsilon,b}(x), \rho_{\varepsilon,b}(y)\right) \right| \ \leq \ \mu \tag{3.5.18}$$

In the microscope interpretation, if $(x, u) \in \rho_{\varepsilon,b}$ means that x and u represent the same "real" point in the specimen.

Therefore a metric profile gives two types of information:

- a distance estimate like (3.5.18) from point 4,
- an "approximate shape" estimate, like in the points 1–3, where we see that two sets, namely the balls $\bar{B}_{d_b}(x_b, \varepsilon)$ and $\bar{B}_{d_{b\varepsilon}}(x_{b\varepsilon}, 1)$, are approximately isometric.

The simplest metric profile is one with $(\bar{B}(x_b, 1), d_b, x_b) = (X, d_b, x)$. In this case we see that $\rho_{\varepsilon,b}$ is approximately an ε dilation with base point x.

This observation leads us to a particular class of (pointed) metric spaces, namely the metric cones.

Definition 3.13 *A metric cone (X, d, x) is a locally compact metric space (X, d), with a marked point $x \in X$ such that for any $a, b \in (0, 1]$ we have*

$$\mathbb{P}^m(a, x) = \mathbb{P}^m(b, x)$$

Metric cones have dilations. By this we mean the following

Definition 3.14 *Let (X, d, x) be a metric cone. For any $\varepsilon \in (0, 1]$ a dilation is a function $\delta^x_\varepsilon : \bar{B}(x, 1) \to \bar{B}(x, \varepsilon)$ such that*

- $\delta^x_\varepsilon(x) = x$,
- *for any $u, v \in X$ we have*

$$d\left(\delta^x_\varepsilon(u), \delta^x_\varepsilon(v)\right) = \varepsilon\, d(u, v)$$

The existence of dilations for metric cones comes from the definition 3.13. Indeed, dilations are just isometries from $(\bar{B}(x, 1), d, x)$ to $(\bar{B}, \frac{1}{a} d, x)$.

Metric cones are good candidates for being tangent spaces in the metric sense.

Definition 3.15 *A (locally compact) metric space (M, d) admits a (metric) tangent space in $x \in M$ if the associated metric profile $\varepsilon \mapsto \mathbb{P}^m(\varepsilon, x)$ (as in definition 3.11) admits a prolongation by continuity in $\varepsilon = 0$, i.e if the following limit exists:*

$$[T_x M, d^x, x] = \lim_{\varepsilon \to 0} \mathbb{P}^m(\varepsilon, x) \tag{3.5.19}$$

The connection between metric cones, tangent spaces and metric profiles in the abstract sense is made by the following proposition.

Proposition 3.16 *The associated metric profile $\varepsilon \mapsto \mathbb{P}^m(\varepsilon, x)$ of a metric space (M, d) for a fixed $x \in M$ is a metric profile in the sense of the definition 3.12 if and only if the space (M, d) admits a tangent space in x. In such a case the tangent space is a metric cone.*

proof A tangent space $[V, d_v, v]$ exists if and only if we have the limit from the relation (3.5.19). In this case there exists a prolongation by continuity to $\varepsilon = 0$ of the metric profile $\mathbb{P}^m(\cdot, x)$. The prolongation is a metric profile in the sense of definition 3.12. Indeed, we have still to check the property (b). But this is trivial, because for any $\varepsilon, b > 0$, sufficiently small, we have

$$\mathbb{P}^m(\varepsilon b, x) = \mathbb{P}^m_{d_b}(\varepsilon, x)$$

where $d_b = (1/b)d$ and $\mathbb{P}^m_{d_b}(\varepsilon, x) = [\bar{B}(x, 1), \frac{1}{\varepsilon} d_b, x]$.

Finally, let us prove that the tangent space is a metric cone. For any $a \in (0, 1]$ we have

$$\left[\bar{B}(x, 1), \frac{1}{a} d^x, x\right] = \lim_{\varepsilon \to 0} \mathbb{P}^m(a\varepsilon, x)$$

Therefore

$$\left[\bar{B}(x, 1), \frac{1}{a} d^x, x\right] = [T_x M, d^x, x]$$

$\square$

4 Length in metric spaces

For a detailed introduction into the subject see for example [2], chapter 1.

Definition 4.1 *The* **(upper) dilation of a map** $f : X \to Y$ *between metric spaces, in a point* $u \in Y$ *is*

$$Lip(f)(u) = \limsup_{\varepsilon \to 0} \ \sup \left\{ \frac{d_Y(f(v), f(w))}{d_X(v,w)} \ : \ v \neq w \ , \ v, w \in B(u, \varepsilon) \right\}$$

In the particular case of a derivable function $f : \mathbb{R} \to \mathbb{R}^n$ the upper dilation is $Lip(f)(t) = \|\dot{f}(t)\|$.

A function $f : (X,d) \to (Y,d')$ is Lipschitz if there is a positive constant C such that for any $x, y \in X$ we have $d'(f(x), f(y)) \leq C\, d(x,y)$. The number $Lip(f)$ is the smallest such positive constant. Then for any $x \in X$ we have the obvious relation $Lip(f)(x) \ \leq \ Lip(f)$.

A curve is a continuous function $c : [a,b] \to X$. The image of a curve is called path. Length measures paths. Therefore length does not depends on the reparameterization of the path and it is additive with respect to concatenation of paths.

Definition 4.2 *In a metric space* (X,d) *there are several ways to define the length:*

(a) The **length of a curve with** L^1 **upper dilation** $c : [a,b] \to X$ *is*

$$L(f) = \int_a^b Lip(c)(t) \ dt$$

(b) The **variation of a curve** $c : [a,b] \to X$ *is the quantity* $Var(c) =$

$$= \sup \left\{ \sum_{i=0}^{n} d(c(t_i), c(t_{i+1})) \ : a = t_0 < t_1 < ... < t_n < t_{n+1} = b \right\}$$

(c) The **length of the path** $A = c([a,b])$ *is the one-dimensional Hausdorff measure of the path.:*

$$l(A) \ = \ \lim_{\delta \to 0} \inf \left\{ \sum_{i \in I} diam \ E_i \ : diam \ E_i < \delta \ , \quad A \subset \bigcup_{i \in I} E_i \right\}$$

The definitions are not equivalent. To see this consider the following easy example: $f : [-1,1] \to R^2$, $f(t) \ = \ (t, \ sign(t))$. We have $Var(f) \ = \ 4$ and $L(f([-1,1]) = 2$. Another example: the Cantor staircase function is continuous, but not Lipschitz. It has variation equal to 1 and length of the graph equal to 2. For Lipschitz curves the first two definitions agree. For simple Lipschitz curves all definitions agree.

Theorem 4.3 *For each Lipschitz curve* $f : [a,b] \to X$*, we have* $L(f) \ = \ Var(f)$*.*

Proof. We prove the thesis by double inequality. f is continuous therefore $f([a,b])$ is a compact metric space. Let $\{x_n : n \in N\}$ be a dense sequence in $f([a,b])$. All the functions

$$t \mapsto \phi_n(t) \;=\; d(f(t), x_n)$$

are Lipschitz and $Lip(\phi_n) \leq Lip(f)$, because of the general property:

$$Lip(f \circ g) \;\leq\; Lip(f)\; Lip(g)$$

if f, g are Lipschitz. In the same way we see that :

$$dil(f)(t) \;=\; \sup\{dil(\phi_n)(t) : n \in N\}$$

We have then, for $t < s$ in $[a,b]$:

$$d(f(t), f(s)) \;=\; \sup\{\mid d(f(t), x_n) - d(f(s), x_n) \mid \; : n \in N\} \;\leq$$

$$\leq \; \int_s^t dil(f)(\tau)\; \mathrm{d}\tau$$

From the definition of the variation we get

$$Var(f) \;\leq\; L(f)$$

For the converse inequality let $\varepsilon > 0$ and $n \geq 2$ natural number such that $h = (b-a)/n \; < \varepsilon$. Set $t_i \;=\; a + ih$. Then

$$\frac{1}{h}\int_a^{b-\varepsilon} d(f(t+h), f(t))\; \mathrm{d}t \;\leq\; \frac{1}{h}\sum_{i=0}^{n-2} d(f(\tau + t_{i+1}), f(\tau + t_i))\; \mathrm{d}\tau \;\leq$$

$$\leq\; \frac{1}{h}\int_0^h Var(f)\; \mathrm{d}\tau \;=\; Var(f)$$

Fatou lemma and definition of dilation lead us to the inequality

$$\int_a^{b-\varepsilon} dil(f)(t)\; \mathrm{d}t \;\leq\; Var(f)$$

This finishes the proof because ε is arbitrary. ■

Theorem 4.4 *For each Lipschitz curve $c : [a,b] \to X$, we have $L(c) \;=\; Var(c) \;\geq\; \mathcal{H}^1(c([a,b]))$.*

If c is moreover injective then $\mathcal{H}^1(c([a,b])) \;=\; Var(f)$.

An important tool used in the proof of the previous theorem is the geometrically obvious, but not straightforward to prove in this generality, Reparametrisation Theorem.

Theorem 4.5 *Any Lipschitz curve admits a reparametrisation $c : [a,b] \to A$ such that $Lip(c)(t) = 1$ for almost any $t \in [a,b]$.*

Lemma 4.6 *If $f : [a,b] \to X$ is continuous then*

$$\mathcal{H}^1(f([a,b]) \;\leq\; d(f(a), f(b))$$

Proof. Let us consider the Lipschitz function

$$\phi : X \to R \ , \quad \phi(x) \ = \ d(x, f(a))$$

It has the property $Lip(\phi) \leq 1$ therefore by the definition of Hausdorff measure we have

$$\mathcal{H}^1(\phi \circ f([a,b])) \ \leq \ \mathcal{H}^1(f([a,b])$$

On the left hand side of the inequality we have the Hausdorff measure on R, which coincides with the usual (outer) Lebesgue measure. Moreover $\phi \circ f([a,b]) \ = \ [0,\alpha]$, therefore we obtain

$$\mathcal{H}^1(\phi \circ f([a,b])) \ = \ \sup \{d(f(t), f(a)) \, : \, t \in [a,b]\} \ \geq \ d(f(a), f(b))$$

■

The proof of theorem 4.5 follows.

Proof. It is not restrictive to suppose that $A \ = \ f([a,b])$ can be parametrised by $f : [a,b] \to A$ such that $dil(f)(t) \ = \ 1$ for all $t \in [a,b]$. Due to theorem 4.4 and again the reparametrisation theorem, we can choose $[a,b] = [0, Var(f)]$.

For an arbitrary $\delta > 0$ we choose $n \in N$ such that $h = Var(f)/n < \delta$ and we divide the interval $[0, Var(f)]$ in intervals $J_i \ = \ [ih, (i+1)h]$. The function f has Lipschitz constant equal to 1 therefore (see definition of Hausdorff measure and notations therein)

$$\mathcal{H}^1_\delta(A) \ \leq \ \sum_{i=0}^{n} diam \ (J_i) \ = \ Var(f)$$

δ is arbitrary therefore $\mathcal{H}^1(A) \ \leq \ Var(f)$. This is a general inequality which does not use the injectivity hypothesis.

We prove the converse inequality from injectivity hypothesis. Let us divide the interval $[a,b]$ by $a \leq t_0 < ... < t_n \leq b$. From lemma 4.6 and sub-additivity of Hausdorff measure we have:

$$\sum_{i=0}^{n-1} d(f(t_i), f(t_{i+1})) \ \leq \ \sum_{i=0}^{n-1} \mathcal{H}^1(f([t_i), t_{i+1}])) \ \leq \ H^1(A)$$

The partition of the interval was arbitrary, therefore $Var(f) \leq \mathcal{H}^1(A)$. ■

Definition 4.7 *We shall denote by l_d the* **length functional induced by the distance** *d, defined only on the family of Lipschitz curves. If the metric space (X,d) is connected by Lipschitz curves, then the length induces a new distance d_l, given by:*

$$d_l(x,y) \ = \ \inf \ \{l_d(c([a,b])) \, : \, c : [a,b] \to X \ \ \textit{Lipschitz} \ ,$$
$$c(a) = x \ , \ c(b) = y\}$$

A **length metric space** *is a metric space (X,d), connected by Lipschitz curves, such that $d = d_l$.*

In terms of distances there is an easy criterion to decide if a metric space is path metric (Theorem 1.8., page 6-7, Gromov [38]).

Theorem 4.8 *A complete metric space is path metric if and only if (a) or (b) from above is true:*

(a) *for any $x, y \in X$ and for any $\varepsilon > 0$ there is $z \in X$ such that*

$$\max\{d(x,z), d(z,y)\} \ \leq \ \frac{1}{2} d(x,y) + \varepsilon$$

(b) *for any $x, y \in X$ and for any $r_1, r_2 > 0$, if $r_1 + r_2 \leq d(x,y)$ then*

$$d(B(x,r_1), B(y,r_2)) \ \leq \ d(x,y) - r_1 - r_2$$

Proof. We shall prove only that (a) implies that X is path metric. (b) implies (a) is straightforward, path metric space implies (b) likewise.

Set $\delta = d(x,y)$ and take a sequence $\varepsilon_k > 0$, with finite sum. We shall recursively define a function $z = z_\varepsilon$ from the dyadic numbers in $[0,1]$ to X. $z_\varepsilon(1/2) \ = \ z_{1/2}$ is a point such that

$$\max\left\{d(x, z_{1/2}), d(z_{1/2}, y)\right\} \ \leq \ \frac{1}{2}\delta(1+\varepsilon_1)$$

Suppose now that all points $z_\varepsilon(p/2^n) \ = \ z_{p/2^n}$ were defined, for $p = 1, ..., 2^n - 1$. Then $z_{2p+1/2^{n+1}}$ is a point such that

$$\max\left\{d(z_{p/2^n}, z_{2p+1/2^{n+1}}), d(z_{2p+1/2^{n+1}}, z_{p+1/2^n})\right\} \ \leq \ \frac{\delta}{2^n}\prod_{k=1}^{n+1}(1+\varepsilon_k)$$

Because (X,d) is a complete metric space it follows that z_ε can be prolonged to a Lipschitz curve c_ε, defined on the whole interval $[0,1]$, such that $c(0) = x$, $c(1) = y$ and

$$d(x,y) \ \leq \ l(c_\varepsilon \ \leq \ d(x,y) \ \prod_{k\geq 1}(1+\varepsilon_k)$$

But the product $\prod_{k\geq 1}(1+\varepsilon_k)$ can be made arbitrarily close to 1, which proves the thesis. ■

From theorem 4.4 we deduce that Lipschitz curves in complete length metric spaces are absolutely continuous. Indeed, here is the definition of an absolutely continuous curve (definition 1.1.1, chapter 1, [2]).

Definition 4.9 *Let (X,d) be a complete metric space. A curve $c : (a,b) \to X$ is* **absolutely continuous** *if there exists $m \in L^1((a,b))$ such that for any $a < s \leq t < b$ we have*

$$d(c(s), c(t)) \leq \int_s^t m(r) \ dr.$$

Such a function m is called a **upper gradient** *of the curve c.*

According to theorem 4.4, for a Lipschitz curve $c : [a,b] \to X$ in a complete length metric space such a function $m \in L^1((a,b))$ is the upper dilation $Lip(c)$. More can be said about the expression of the upper dilation. We need first to introduce the notion of metric derivative of a Lipschitz curve.

Definition 4.10 *A curve $c : (a,b) \to X$ is* **metrically derivable** *in $t \in (a,b)$ if the limit*

$$md(c)(t) = \lim_{s\to t} \frac{d(c(s), c(t))}{\mid s - t \mid}$$

exists and it is finite. In this case $md(c)(t)$ is called the **metric derivative** *of c in t.*

For the proof of the following theorem see [2], theorem 1.1.2, chapter 1.

Theorem 4.11 *Let (X,d) be a complete metric space and $c : (a,b) \rightarrow X$ be an absolutely continuous curve. Then c is metrically derivable for $\mathcal{L}^1$-a.e. $t \in (a,b)$. Moreover the function $md(c)$ belongs to $L^1((a,b))$ and it is minimal in the following sense: $md(c)(t) \leq m(t)$ for $\mathcal{L}^1$-a.e. $t \in (a,b)$, for each upper gradient m of the curve c.*

Definition 4.12 *A (local) geodesic is a curve $c : [a,b] \rightarrow X$ with the property that for any $t \in (a,b)$ there is a small $\varepsilon > 0$ such that $c : [t-\varepsilon, t+\varepsilon] \rightarrow X$ is length minimising. A global geodesic is a length minimising curve.*

Therefore in a path metric space a local geodesic has the property that in the neighbourhood of any of it's points the relation

$$d(c(t), c(t')) = l_c(t, t')$$

holds. Any global geodesic is also local geodesic.

Can one join any two points with a geodesic? The abstract Hopf-Rinow theorem (Gromov [38], page 9) states that:

Theorem 4.13 *If (X,d) is a connected locally compact path metric space then each pair of points can be joined by a global geodesic.*

Proof. It is sufficient to give the proof for compact path metric spaces. Given the points x, y, there is a sequence of curves f_h joining those points such that $l(f_h) \leq d(x,y) + 1/h$. The sequence, if parametrised by arclength, is equicontinuous; by Arzela-Ascoli theorem one can extract a subsequence (denoted also f_h) which converges uniformly to f. By construction the length function is lower semicontinuous hence:

$$l(f) \leq \liminf_{h \rightarrow \infty} l(f_h) \leq d(x,y)$$

Therefore f is a length minimising curve joining x and y. ■

5 Affine structure in terms of dilations

5.1 Affine algebra

Bertram [7] Theorem 1.1 (here theorem 5.1) and paragraph 5.2, proposes the following algebraic description of affine geometry and of affine metric geometry over a field $\mathbb{K}$ of characteristic different from 2, which is not based on incidence notions, but on algebraic relations concerning "product maps". He then pursues to the development of generalized projective geometries and their relations to Jordan algebras. For our purposes, we changed the name of "product maps" (see the theorem below) from "π" to "δ", more precisely:

$$\pi_r(x, y) \ = \ \delta^x_r y$$

Further, in theorem 5.1 and definition 5.2 is explained this point of view.

Theorem 5.1 *The category of affine spaces over a field $\mathbb{K}$ of characteristic different from 2 is equivalent with the category of sets $\mathbb{V}$ equipped with a family δ_r, $r \in \mathbb{K}$, of "product maps"*

$$\delta_r : \mathbb{V} \times \mathbb{V} \to \mathbb{V} \, , \quad (x, y) \mapsto \delta^x_r y$$

satisfying the following properties (Af1) - (Af4):

(Af1) The map $r \mapsto \delta^x_r$ is a homomorphism of the unit group $\mathbb{K}^\times$ into the group of bijections of $\mathbb{V}$ fixing x, that is

$$\delta^x_1 y = y \ , \ \delta^x_r \, \delta^x_s y = \delta^x_{rs} y \ , \ \delta^x_r x = x$$

(Af2) For all $r \in \mathbb{K}$ and $x \in \mathbb{V}$ the map δ^x_r is an endomorphism of δ_s, $s \in \mathbb{K}$:

$$\delta^x_r \, \delta^y_s z = \delta^{\delta^x_r y}_s \delta^x_r z$$

(Af3) The "barycentric condition": $\delta^x_r \, y \ = \delta^y_{1-r} \, x$

(Af4) The group generated by the $\delta^x_r \, \delta^y_{r^{-1}}$ ($r \in \mathbb{K}^\times$, $x, y \in \mathbb{V}$) is abelian, that is for all $r, s \in \mathbb{K}^\times$, $x, y, u, v \in \mathbb{V}$

$$\delta^x_r \, \delta^y_{r^{-1}} \, \delta^u_s \, \delta^v_{s^{-1}} \ = \ \delta^u_s \, \delta^v_{s^{-1}} \, \delta^x_r \, \delta^y_{r^{-1}}$$

More precisely, in every affine space over $\mathbb{K}$, the maps

$$\delta^x_r y = (1 - r)x + ry \tag{5.1.1}$$

with $r \in \mathbb{K}$, satisfy (Af1) - (Af4). Conversely, if product maps with the properties (Af1) - (Af4) are given and $x \in \mathbb{V}$ is an arbitrary point then

$$u \, +_x \, v := \delta^x_2 \, \delta^u_{\frac{1}{2}} v \quad , \quad r \, u \, := \delta^x_r u$$

defines on $\mathbb{V}$ the structure of a vector space over $\mathbb{K}$ with zero vector x, and this construction is inverse to the preceding one. Affine maps $g : \mathbb{V} \to \mathbb{V}'$ in the usual sense are precisely the homomorphisms of product maps, that is maps $g : \mathbb{V} \to \mathbb{V}'$ such that $g \, \pi_r(x, y) \ = \ \pi'_r(gx, gy)$ for all $x, y \in \mathbb{V}$, $r \in \mathbb{K}$.

We shall use the name "real normed affine space" in the following sense.

Definition 5.2 *A real normed affine space is an affine space* $\mathbb{V}$ *over* $\mathbb{R}$ *together with a distance function* $d : \mathbb{V} \times \mathbb{V} \to \mathbb{K}$ *such that:*

(Af5) for all $x \in \mathbb{V}$ $\| \cdot \|_x := d(x, \cdot) : \mathbb{V} \to \mathbb{K}$ *is a norm on the vector space* $(\mathbb{V}, x)$ *with zero vector* x.

(Af6) the distance d *is translation invariant: for any* $x, y, u, v \in \mathbb{V}$ *we have:*

$$d(x +_u v, y +_u v) = d(x, y)$$

We remark that the field of product maps δ^x_r (together with the distance function d for the metric case) is the central object in the construction of affine geometry over a general field.

5.2 Focus on dilations

There is another, but related, way of generalizing the affine geometry, which is the one of dilation structures [11]. In this approach product maps of Bertram are replaced by "dilations".

For this we have to replace the field $\mathbb{K}$ by a commutative group Γ (instead of the multiplicative group $\mathbb{K}^{\times}$) endowed with a "valuation map" $\nu : \Gamma \to (0, +\infty)$, which is a group morphism. We write $\varepsilon \to 0$, $\varepsilon \in \Gamma$, for $\nu(\varepsilon) \to 0$ in $(0, +\infty)$. We keep axioms like (Af1), (Af2) (from Theorem 5.1), but we modify (Af5) (from Definition 5.2). There will be one more axiom concerning the relations between the distance and dilations. This is explained in theorem 5.5.

The conditions appearing in theorem 5.5 are a particular case of the system of axioms of dilation structures, introduced in [11]. Dilation structures are also a generalization of homogeneous groups, definition 5.3, in fact we arrived to dilation structures after an effort to find a common algebraic and analytical ground for homogeneous groups and sub-riemannian manifolds.

The axioms of a dilation structure are partly algebraic and partly of an analytical nature (by using uniform limits). Metric spaces endowed with dilation structures have beautiful properties. The most important is that for any point in such a space there is a tangent space (in the metric sense) realized as a "normed conical group". Any normed conical group has an associated dilation structure which is "linear" in the sense that it satisfies (Af2). However, conical groups form a family much larger than affine spaces (in the usual sense, over $\mathbb{R}$ or $\mathbb{C}$). Building blocks of conical groups are homogeneous groups (graded Lie groups) or p-adic versions of them. By renouncing to (Af3) and (Af4) we thus allow noncommutativity of the "vector addition" operation.

Let us explain how we can recover the usual affine geometry from the viewpoint of dilation structures. For simplicity we take here $\Gamma = (0, +\infty)$ and $\mathbb{V}$ is a real, finite dimensional vector space.

Here is the definition of a normed homogeneous group. See section 8.2 for more details on the particular case of stratified homogeneous groups.

Definition 5.3 *A normed homogeneous group is a connected and simply connected Lie group whose Lie algebra is endowed with a family of dilations* $\{\delta_\varepsilon : \varepsilon \in (0, +\infty)\}$, *which are algebra automorphisms, simultaneously diagonalizable, together with a homogeneous norm.*

Since the Lie group exponential is a bijection we shall identify the Lie algebra with the Lie group, thus a normed homogeneous group is a group operation on a finite dimensional *vector space $\mathbb{V}$. The operation will be denoted multiplicatively, with 0 as neutral element, as in Folland, Stein [28]. We thus have a linear action $\delta : (0,+\infty) \to Lin(\mathbb{V},\mathbb{V})$ on $\mathbb{V}$, and a homogeneous norm $\| \cdot \| : \mathbb{V} \to [0,+\infty)$, such that:*

(a) for any $\varepsilon \in (0,+\infty)$ the transformation δ_ε is an automorphism of the group operation: for any $x,y \in \mathbb{V}$ we have $\delta_\varepsilon(x \cdot y) \ = \ \delta_\varepsilon x \cdot \delta_\varepsilon y$

(b) the family $\{\delta_\varepsilon \, : \, \varepsilon \in (0,+\infty)\}$ is simultaneously diagonalizable: there is a finite direct sum decomposition of the vector space $\mathbb{V}$

$$\mathbb{V} \ = \ V_1 + ... + V_m$$

such that for any $\varepsilon \in (0,+\infty)$ we have:

$$x = \sum_{i=1}^{m} x_i \ \in \mathbb{V}\, m \mapsto \ \delta_\varepsilon x \ = \ \sum_{i=1}^{m} \varepsilon^i x_i \quad .$$

(c) the homogeneous norm has the properties:

(c1) $\|x\| = 0$ if and only if $x = 0$,

(c2) $\|x \cdot y\| \leq \|x\| + \|y\|$ for any $x,y \in \mathbb{V}$,

(c3) for any $x \in \mathbb{V}$ and $\varepsilon > 0$ we have $\|\delta_\varepsilon x\| \ = \ \varepsilon \, \|x\|$

Definition 5.4 *To a normed homogeneous group $(\mathbb{V}, \delta, \cdot, \| \cdot \|)$ we associate a normed affine group space $(\mathbb{V}, +., \delta^{\cdot}_{\cdot}, d)$. Here we use the sign "+" for an operation which was denoted multiplicatively, for compatibility with the previous approach of Bertram , see theorem 5.1. The normed affine group space $(\mathbb{V}, +., \delta^{\cdot}_{\cdot}, d)$ is described by the following points:*

- *for any $u \in \mathbb{V}$ the function $+_u : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$, $x +_u v = x \cdot u^{-1} \cdot v$ is the left translation of the group operation $\cdot$ with the zero element u. In particular we have $x +_0 y = x \cdot y$.*
- *for any $x, y \in \mathbb{V}$ and $\varepsilon \in (0,+\infty)$ we define*

$$\delta^x_\varepsilon y \ = \ x \cdot \delta_\varepsilon (x^{-1} \cdot y)$$

and remark that the definition is invariant with the choice of the base point for the operation in the sense: for any $u \in \mathbb{V}$ we have:

$$\delta^x_\varepsilon y \ = \ x +_u \delta^u_\varepsilon \left(\, inv^u(x) +_u y \right)$$

where $inv^u(x)$ is the inverse of x with respect to the operation $+_u$, (by computation we get $inv^u(x) = u \cdot x^{-1} \cdot u$),

- *the distance d is defined as: for any $x,y \in \mathbb{V}$ we have $d(x,y) = \|x^{-1} \cdot y\|$. As previously, remark that the definition does not depend on the choice of the base point for the operation, that is: for any $u \in \mathbb{V}$ we have*

$$d(x,y) \ = \ \|inv^u(x) +_u y\|_u \, , \ \|x\|_u := \|u^{-1} \cdot x\|$$

Equally, this is a consequence of the invariance of the norm with respect to left translations (by any group operation $+_u$, $u \in \mathbb{V}$).

Theorem 5.5 *The category of normed affine group spaces is equivalent with the category of locally compact metric spaces (X,d) equipped with a family δ_ε, $\varepsilon \in (0,+\infty)$, of dilations*

$$\delta_\varepsilon : X \times X \to X \, , \quad (x,y) \mapsto \delta^x_\varepsilon y$$

satisfying the following properties:

(Af1') The map $\varepsilon \mapsto \delta\varepsilon^x$ is a homomorphism of the multiplicative group $(0,+\infty)$ into the group of continuous, with continuous inverse functions of X fixing x, that is

$$\delta^x_1 y = y \ , \ \delta^x_r \, \delta^x_s y = \delta^x_{rs} y \ , \ \delta^x_r x = x$$

(A2) the function $\delta : (0,+\infty) \times X \times X \to X$ defined by $\delta(\varepsilon, x, y) = \delta^x_\varepsilon y$ is continuous. Moreover, it can be continuously extended to $[0,+\infty) \times X \times X$ by $\delta(0,x,y) = x$ and the limit

$$\lim_{\varepsilon \to 0} \delta^x_\varepsilon y \ = \ x$$

is uniform with respect to x, y in compact set.

(A3') for any $x \in X$ and for any $u, v \in X$, $\varepsilon \in (0,+\infty)$ we have

$$\frac{1}{\varepsilon} \, d\left(\delta^x_\varepsilon u, \delta^x_\varepsilon v\right) \ = \ d(u,v)$$

(A4) for any $x, u, v \in X$, $\varepsilon \in (0,+\infty)$ let us define

$$\Delta^x_\varepsilon(u,v) = \delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \delta^x_\varepsilon v.$$

Then we have the limit

$$\lim_{\varepsilon \to 0} \Delta^x_\varepsilon(u,v) = \Delta^x(u,v)$$

uniformly with respect to x, u, v in compact set.

(Af2') For all $\varepsilon \in (0,+\infty)$ and $x \in X$ the map δ^x_ε is an endomorphism of δ_s, $s \in (0,+\infty)$:

$$\delta^x_r \, \delta^y_s z = \delta^{\delta^x_r y}_s \delta^x_r z$$

More precisely, in every normed affine group space , the maps δ^x_ε and distance d satisfy (Af1'), (A2), (A3'), (A4), (Af2'). Conversely, if dilations δ^x_ε and distance d are given, such that they satisfy the collection (Af1'), (A2), (A3'), (A4), (Af2'), for an arbitrary point $x \in \mathbb{V}$ the following expression

$$\Sigma^x(u,v) := \lim_{\varepsilon \to 0} \delta^x_{\varepsilon^{-1}} \, \delta^{\delta^x_\varepsilon u}_\varepsilon v$$

together with δ^x_ε and distance d defines on $\mathbb{V}$ the structure of a normed affine group space, and this construction is inverse to the preceding one. The arrows of this category are bilipschitz invertible homomorphisms of dilations, that is maps $g : \mathbb{V} \to \hat{\mathbb{V}}$ such that $g \, \delta^x_\varepsilon y \ = \ \hat{\delta}^{gx}_r gy$ for all $x, y \in \mathbb{V}$, $\varepsilon \in (0,+\infty)$.

Moreover, the category of real normed affine spaces is a subcategory of the previous one, namely the category of locally compact metric spaces (X,d) equipped with a family δ_ε, $\varepsilon \in (0,+\infty)$, of dilations satisfying (Af1'), (A2), (A3'), (A4), (Af2') and

(Af3) the "barycentric condition": for all $\varepsilon \in (0,1)$ $\delta^x_\varepsilon \, y \ = \delta^y_{1-\varepsilon} \, x$

The arrows of this category are exactly the affine, invertible maps.

Proof. Here we shall prove the easy implication, namely why the conditions (Af1'), (A2), (A3'), (A4), (Af2') and (Af3) are satisfied in a real normed affine space.

For the real normed affine space space $\mathbb{V}$ let us fix for simplicity a point $0 \in \mathbb{V}$ and work with the vector space $\mathbb{V}$ with zero vector 0. Since a real normed affine space is a particular example of a homogeneous group, definition 5.3 and observations inside apply. The dilation based at $x \in \mathbb{V}$, of coefficient $\varepsilon > 0$, is the function

$$\delta_\varepsilon^x : \mathbb{V} \to \mathbb{V} \quad , \quad \delta_\varepsilon^x y = x + \varepsilon(-x + y) \quad .$$

For fixed x the dilations based at x form a one parameter group which contracts any bounded neighbourhood of x to a point, uniformly with respect to x. Thus (Af1'), (A2) are satisfied. (A3') is also obvious.

The meaning of (A4) is that using dilations we can recover the operation of addition and multiplication by scalars. We shall explain this in detail since this will help the understanding of the axioms of dilation structures, described in section 6.

For $x, u, v \in \mathbb{V}$ and $\varepsilon > 0$ we define the following compositions of dilations:

$$\Delta_\varepsilon^x(u, v) = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v \quad , \tag{5.2.2}$$

$$\Sigma_\varepsilon^x(u, v) = \delta_{\varepsilon^{-1}}^x \delta_\varepsilon^{\delta_\varepsilon^x u}(v) \quad , \quad inv_\varepsilon^x(u) = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} x \quad .$$

The meaning of this functions becomes clear if we compute:

$$\Delta_\varepsilon^x(u, v) = x + \varepsilon(-x + u) + (-u + v) \quad ,$$

$$\Sigma_\varepsilon^x(u, v) = u + \varepsilon(-u + x) + (-x + v) \quad ,$$

$$inv_\varepsilon^x(u) == x + \varepsilon(-x + u) + (-u + x) \quad .$$

As $\varepsilon \to 0$ we have the limits:

$$\lim_{\varepsilon \to 0} \Delta_\varepsilon^x(u, v) = \Delta^x(u, v) = x + (-u + v) \quad ,$$

$$\lim_{\varepsilon \to 0} \Sigma_\varepsilon^x(u, v) = \Sigma^x(u, v) = u + (-x + v) \quad ,$$

$$\lim_{\varepsilon \to 0} inv_\varepsilon^x(u) = inv^x(u) = x - u + x \quad ,$$

uniform with respect to x, u, v in bounded sets. The function $\Sigma^x(\cdot, \cdot)$ is a group operation, namely the addition operation translated such that the neutral element is x:

$$\Sigma^x(u, v) \ = \ u +_x v \quad .$$

The function $inv^x(\cdot)$ is the inverse function with respect to the operation $+_x$

$$inv^x(u) +_x u \ = \ u +_x inv^x(u) \ = \ x$$

and $\Delta^x(\cdot, \cdot)$ is the difference function

$$\Delta^x(u, v) \ = \ inv^x(u) +_x v$$

Notice that for fixed x, ε the function $\Sigma_\varepsilon^x(\cdot, \cdot)$ is not a group operation, first of all because it is not associative. Nevertheless, this function satisfies a "shifted" associativity property, namely

$$\Sigma_\varepsilon^x(\Sigma_\varepsilon^x(u, v), w) = \Sigma_\varepsilon^x(u, \Sigma_\varepsilon^{\delta_\varepsilon^x u}(v, w)) \quad .$$

Also, the inverse function inv^x_ε is not involutive, but shifted involutive:

$$inv_\varepsilon^{\delta^x_\varepsilon u}\left(inv^x_\varepsilon u\right) = u \quad .$$

Affine continuous transformations $A : \mathbb{V} \rightarrow \mathbb{V}$ admit the following description in terms of dilations. (We could dispense of continuity hypothesis in this situation, but we want to illustrate a general point of view, described further in the paper).

Proposition 5.6 *A continuous transformation $A : \mathbb{V} \rightarrow \mathbb{V}$ is affine if and only if for any $\varepsilon \in (0,1)$, $x, y \in \mathbb{V}$ we have*

$$A\delta^x_\varepsilon y \ = \ \delta^{Ax}_\varepsilon Ay \quad . \tag{5.2.3}$$

The proof is a straightforward consequence of representation formulæ (5.2.2) for the addition, difference and inverse operations in terms of dilations.

In particular any dilation is an affine transformation, hence for any $x, y \in \mathbb{V}$ and $\varepsilon, \mu > 0$ we have

$$\delta^y_\mu \, \delta^x_\varepsilon \ = \ \delta^{\delta^y_\mu x}_\varepsilon \delta^y_\mu \quad . \tag{5.2.4}$$

Thus we recover (Af2') (see also condition (Af2)). The barycentric condition (Af3) is a consequence of the commutativity of the addition of vectors. The easy part of the theorem 5.5 is therefore proven.

The second, difficult part of the theorem is to prove that axioms (Af1'), (A2), (A3'), (A4), (Af2') describe normed affine group spaces. This is a direct consequence of several general results from this paper: theorem 8.4 and proposition 15.11 show that normed affine group spaces satisfy the axioms, corollary 13.8, theorem 15.12, proposition 16.9 and theorem 16.14 show that conversely a space where the axioms are satisfied is a normed affine group space, moreover that in the presence of the barycentric condition (Af3) we get real normed affine spaces. $\square$

Some compositions of dilations are dilations. This is precisely stated in the next theorem, which is equivalent with the Menelaos theorem in euclidean geometry.

Theorem 5.7 *For any $x, y \in \mathbb{V}$ and $\varepsilon, \mu > 0$ such that $\varepsilon\mu \neq 1$ there exists an unique $w \in \mathbb{V}$ such that*

$$\delta^y_\mu \, \delta^x_\varepsilon \ = \ \delta^w_{\varepsilon\mu} \quad .$$

For the proof see Artin [3]. A straightforward consequence of this theorem is the following result.

Corollary 5.8 *The inverse semigroup generated by dilations of the space $\mathbb{V}$ is made of all dilations and all translations in $\mathbb{V}$.*

Proof. Indeed, by theorem 5.7 a composition of two dilations with coefficients ε, μ with $\varepsilon\mu \neq 1$ is a dilation. By direct computation, if $\varepsilon\mu = 1$ then we obtain translations. This is in fact compatible with (5.2.2), but is a stronger statement, due to the fact that dilations are affine in the sense of relation (5.2.4).

Any composition between a translation and a dilation is again a dilation. The proof is done. $\square$

The corollary 5.8 allows us to describe the ratio of three collinear points in a way which will be generalized to normed affine group spaces. Indeed, in a real normed affine space $\mathbb{V}$, for any $x, y \in \mathbb{V}$ and $\alpha, \beta \in (0, +\infty)$ such that $\alpha\beta \neq 1$, there is an unique $z \in \mathbb{V}$ and $\gamma = 1/\alpha\beta$ such that

$$\delta^x_\alpha \, \delta^y_\beta \, \delta^z_\gamma = \, id$$

We easily find that x, y, z are collinear

$$z = \frac{1-\alpha}{1-\alpha\beta} x \, + \, \frac{\alpha(1-\beta)}{1-\alpha\beta} y \tag{5.2.5}$$

the ratio of these three points, named $r(x^\alpha, y^\beta, z^\gamma)$ is:

$$r(x^\alpha, y^\beta, z^\gamma) \; = \; \frac{\alpha}{1-\alpha\beta}$$

Conversely, let $x, y, z \in \mathbb{V}$ which are collinear, such that z is in between x and y. Then we can easily find (non unique) $\alpha, \beta, \gamma \in (0, +\infty)$ such that $\alpha\beta\gamma = 1$ and $\delta^x_\alpha \, \delta^y_\beta \, \delta^z_\gamma = \, id$.

It is then almost straightforward to prove the well known fact that any affine transformation is also geometrically affine, in the sense that it transforms triples of collinear points into triples of collinear points (use commutation with dilations) and it preserves the ratio of collinear points. (The converse is also true).

6 Dilation structures

A dilation structure (X, d, δ) over a metric space (X, d) is an assignment to any point $x \in X$ of a group of "dilations" $\{\delta_\varepsilon^x : \varepsilon \in \Gamma\}$, together with some compatibility conditions between the distance and the dilations and between dilations based in different points.

A basic difficulty in stating the axioms of a dilation structure is related to the domain of definition and the image of a dilation. In this subsection we shall neglect the problems raised by domains and codomains of dilations.

The axioms state that some combinations between dilations and the distance converge uniformly, with respect to some finite families of points in an arbitrary compact subset of the metric space (X, d), as $\nu(\varepsilon)$ converges to 0.

We present here an introduction into the subject of dilation structures. For more details see Buliga [11].

6.1 Notations

Let Γ be a topological separated commutative group endowed with a continuous group morphism $\nu : \Gamma \to (0, +\infty)$ with $\inf \nu(\Gamma) = 0$. Here $(0, +\infty)$ is taken as a group with multiplication. The neutral element of Γ is denoted by 1. We use the multiplicative notation for the operation in Γ.

The morphism ν defines an invariant topological filter on Γ (equivalently, an end). Indeed, this is the filter generated by the open sets $\nu^{-1}(0, a)$, $a > 0$. From now on we shall name this topological filter (end) by "0" and we shall write $\varepsilon \in \Gamma \to 0$ for $\nu(\varepsilon) \in (0, +\infty) \to 0$.

The set $\Gamma_1 = \nu^{-1}(0, 1]$ is a semigroup. We note $\bar{\Gamma}_1 = \Gamma_1 \cup \{0\}$ On the set $\bar{\Gamma} = \Gamma \cup \{0\}$ we extend the operation on Γ by adding the rules $00 = 0$ and $\varepsilon 0 = 0$ for any $\varepsilon \in \Gamma$. This is in agreement with the invariance of the end 0 with respect to translations in Γ.

The space (X, d) is a complete, locally compact metric space. For any $r > 0$ and any $x \in X$ we denote by $B(x, r)$ the open ball of center x and radius r in the metric space X.

On the metric space (X, d) we work with the topology (and uniformity) induced by the distance. For any $x \in X$ we denote by $\mathcal{V}(x)$ the topological filter of open neighbourhoods of x.

The dilation structures, which will be introduced soon, are invariant to the operation of multiplication of the distance by a positive constant. They should also be seen, as examples show, as local objects, therefore we may safely suppose, without restricting the generality, that all closed balls of radius at most 5 are compact.

6.2 Axioms of dilation structures

We shall list the axioms of a dilation structure (X, d, δ), in a simplified form, without concerning about domains and codomains of functions. In the next subsection we shall add the supplementary conditions concerning domains and codomains of dilations.

A1. *For any point $x \in X$ there is an action $\delta^x : \Gamma \to End(X, d, x)$, where $End(X, d, x)$ is the collection of all continuous, with continuous inverse transformations $\phi : (X, d) \to (X, d)$ such that $\phi(x) = x$.*

This axiom (the same as (A1) from theorem 5.1 or theorem 5.5) tells us that $\delta_\varepsilon^x x = x$ for any $x \in X$, $\varepsilon \in \Gamma$, also $\delta_1^x y = y$ for any $x, y \in X$, and $\delta_\varepsilon^x \delta_\mu^x y = \delta_{\varepsilon\mu}^x y$ for any $x, y \in X$ and $\varepsilon, \mu \in \Gamma$.

A2. *The function $\delta : \Gamma \times X \times X \rightarrow X$ defined by $\delta(\varepsilon, x, y) = \delta_\varepsilon^x y$ is continuous. Moreover, it can be continuously extended to $\bar{\Gamma} \times X \times X$ by $\delta(0, x, y) = x$ and the limit*

$$\lim_{\varepsilon \to 0} \delta_\varepsilon^x y \; = \; x$$

is uniform with respect to x, y in compact set.

We may alternatively put that the previous limit is uniform with respect to $d(x, y)$.

A3. *There is $A > 1$ such that for any x there exists a function $(u, v) \mapsto d^x(u, v)$, defined for any u, v in the closed ball (in distance d) $\bar{B}(x, A)$, such that*

$$\lim_{\varepsilon \to 0} \quad \sup \left\{ \mid \frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) \; - \; d^x(u, v) \mid \; : u, v \in \bar{B}_d(x, A) \right\} \; = \; 0$$

uniformly with respect to x in compact set.

It is easy to see that:

(a) The function d^x is continuous as an uniform limit of continuous functions on a compact set,

(b) d^x is symmetric $d^x(u, v) = d^x(v, u)$ for any $u, v \in \bar{B}(x, A)$,

(c) d^x satisfies the triangle inequality, but it can be a degenerated distance function: there might exist v, w such that $d^x(v, w) = 0$.

We make the following notation which generalizes the notation from (5.2.2):

$$\Delta_\varepsilon^x(u, v) = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v.$$

The next axiom can now be stated:

A4. We have the limit

$$\lim_{\varepsilon \to 0} \Delta_\varepsilon^x(u, v) = \Delta^x(u, v)$$

uniformly with respect to x, u, v in compact set.

Definition 6.1 *A triple (X, d, δ) which satisfies A1, A2, A3, but d^x is degenerate for some $x \in X$, is called degenerate dilation structure.*

If the triple (X, d, δ) satisfies A1, A2, A3 and d^x is non-degenerate for any $x \in X$, then we call it a dilation structure.

If a dilation structure satisfies A4 then we call it strong dilation structure.

6.3 Axiom 0: domains and codomains of dilations

The problem of domains and codomains of dilation cannot be neglected. In the section dedicated to examples of dilation structures we present the particular case of an ultrametric space which is also a ball of radius one. As dilations approximately contract distances, it follows that the codomain of a dilation δ_ε^x with $\nu(\varepsilon) < 1$ can not be the whole space. There are other examples showing that we can not always take the domain of a dilation to be the whole space. That is because the topology of small balls can be different from the topology of big ones (like in the case of a sphere).

For all these reasons we need to impose some minimal conditions on the domains and codomains of dilations. These conditions will be explained in the following. They will be considered as part of a new axiom, called Axiom 0.

For any $x \in X$ there is an open neighbourhood $U(x)$ of x such that for any $\varepsilon \in \Gamma_1$ the dilations are functions

$$\delta_\varepsilon^x : U(x) \to V_\varepsilon(x) \quad .$$

The sets $V_\varepsilon(x)$ are open neighbourhoods of x.

There is a number $1 < A$ such that for any $x \in X$ we have $\bar{B}_d(x, A) \subset U(x)$. There is a number $B > A$ such that for any $\varepsilon \in \Gamma$ with $\nu(\varepsilon) \in (1, +\infty)$ the associated dilation is a function

$$\delta_\varepsilon^x : W_\varepsilon(x) \to B_d(x, B) \quad .$$

We have the following string of inclusions, for any $\varepsilon \in \Gamma_1$, and any $x \in X$:

$$B_d(x, \nu(\varepsilon)) \subset \delta_\varepsilon^x B_d(x, A) \subset V_\varepsilon(x) \subset W_{\varepsilon^{-1}}(x) \subset \delta_\varepsilon^x B_d(x, B) \quad .$$

In relation with the axiom A4 we need the following condition on the co-domains $V_\varepsilon(x)$: for any compact set $K \subset X$ there are $R = R(K) > 0$ and $\varepsilon_0 = \varepsilon(K) \in (0, 1)$ such that for all $u, v \in \bar{B}_d(x, R)$ and all $\varepsilon \in \Gamma$, $\nu(\varepsilon) \in (0, \varepsilon_0)$, we have

$$\delta_\varepsilon^x v \in W_{\varepsilon^{-1}}(\delta_\varepsilon^x u) \ .$$

These conditions are important for describing dilation structures on the boundary of the dyadic tree, for example. In the first formulation of the axioms given in [11] some of these assumptions are part of the Axiom 0, others can be found in the initial formulation of the Axioms 1, 2, 3.

7 Colorings of tangle diagrams

The idempotent right quasigroups are related to algebraic structures appearing in knot theory. J.C. Conway and G.C. Wraith, in their unpublished correspondence from 1959, used the name ”wrack” for a self-distributive right quasigroup generated by a link diagram. Later, Fenn and Rourke [37] proposed the name ”rack” instead. Quandles are particular case of racks, namely self-distributive idempotent right quasigroups. They were introduced by Joyce [45], as a distillation of the Reidemeister moves.

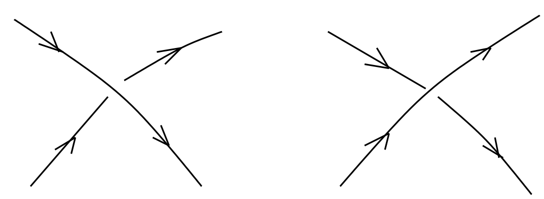
The axioms of a (rack ; quandle ; irq) correspond respectively to the (2,3 ; 1,2,3 ; 1,2) Reidemeister moves. That is why we shall use decorated braids diagrams in order to explain what emergent algebras are.

The basic idea of racks and quandles is that these are algebraic operations related to the coloring of tangles diagrams.

7.1 Oriented tangle diagrams and trivalent graphs

Visually, a oriented tangle diagram is the result of a regular projection on a plane of a properly embedded in the 3-dimensional space, oriented, one dimensional manifold, together with additional over- and under-information at crossings (adapted from the ”Tangle, relative link” article from Encyclopaedia of mathematics. Supplement. Vol. III. Edited by M. Hazewinkel. Kluwer Academic Publishers, Dordrecht, 2001, page 395).

Because the tangle diagram is oriented, there are two types of crossings, indicated in the next figure.



The tangle which projects to the tangle diagram is to be seen as a ”parameterization” of the tangle diagram. In this sense, by using the image of the tangle diagram with over- and under-crossings, it is easy to define an ”arc” of the diagram as the projection of a (part of a) 1-dimensional embedded manifold from the tangle. Arcs can be open or closed. An arc decomposes into connected parts (again by using the first image of a tangle diagram) which are called segments.

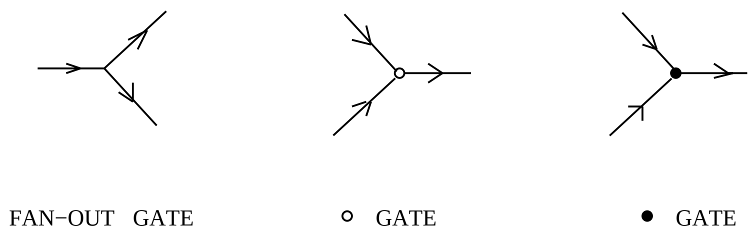
An input segment is a segment which enters (with respect to its orientation) into a crossing but it does not exit from a crossing. Likewise, an output segment is one which exits from a crossing but there is no crossing where this segment enters.

Oriented tangle diagrams are considered only up to continuous deformations of the plane.

Further I shall use, if necessary, the name ”first image of a tangle diagram”, if the oriented tangle diagram is represented with this convention. There is a ”second image”, which I explain next. In a sense, the true image of a tangle diagram is the second one, mainly because in this paper the fundamental object is the oriented tangle diagram, NOT the tangle which projects on the plane to the oriented tangle diagram.

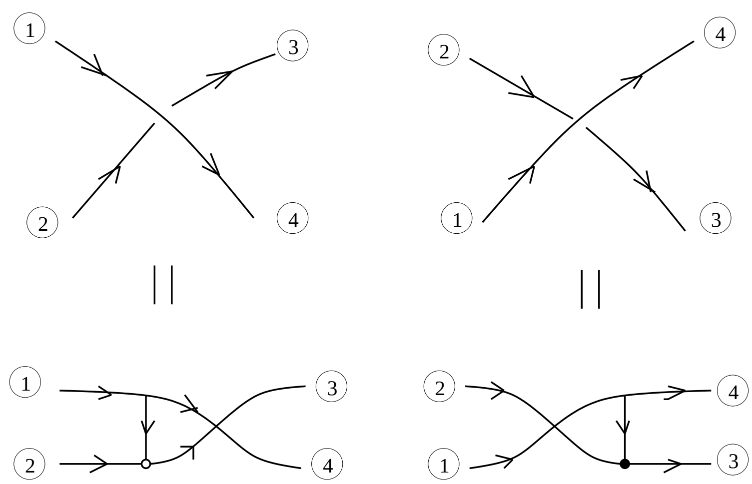
For the second image I use a chord diagram, or a Gauss diagram type of indication of crossings. See [36] [46] for more on the mathematical aspects of chord diagrams.

We may see a crossing like a gate, or black box with two inputs and two outputs. We open the black box and inside we find a combination of two simpler gates, among the following three available: the FAN-OUT, the $\circ$ and the $\bullet$ gate.



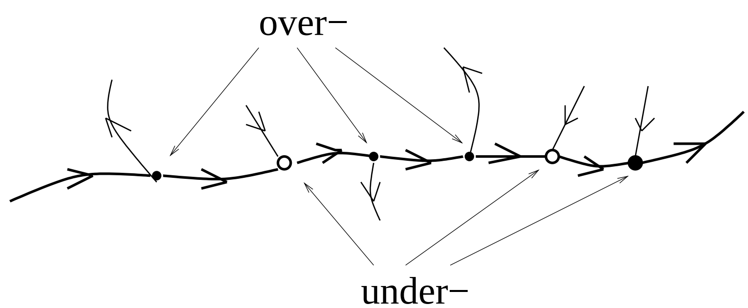
By using these gates we may transform the oriented tangle diagram into a oriented planar graph with 3-valent 2-valent or 1-valent nodes, that is into a circuit made only with these gates, connected by wires which could cross (crossings of wires in this graph has no meaning). The graph is planar in the sense that each trivalent node, which represents one of these 3 gates, inherits an orientation coming from the plane. A trivalent node is either undecorated, if it belongs to a FAN-OUT gate, or decorated by a $\circ$ or a $\bullet$, if it belongs to a $\circ$ gate or a $\bullet$ gate respectively.

This graph is obtained by the following procedure. 1-valent nodes represent input or output tangle segments. 2-valent nodes are used for closing an arc. These 2-valent nodes can be replaced by 3-valent nodes (corresponding to FAN-OUT gates), with the price of introducing also a 1-valent node. Crossings are replaced by combinations of trivalent nodes, as explained in the next figure.



In the second image of a tangle diagram, all segments are wires connecting the nodes of the graph, but some wires are not segments, namely the ones which connect a FAN-OUT with the corresponding $\circ$ gate or $\bullet$ gate (i.e. with the gate which constitutes together with the respective FAN-OUT a coding for a over- or under-crossing). The wires which are not segments are called "chords".

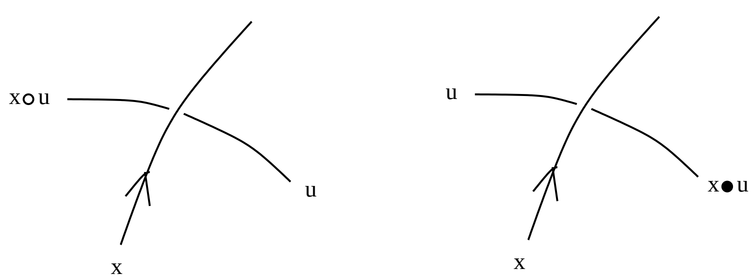
Arcs appear as connected unions of segments with compatible orientations (such that we can choose a segment of the arc and then walk along the whole arc, by following the local directions indicated by each segment). By walking along an arc, we can recognize the crossings: undecorated nodes correspond to over-crossings and decorated nodes to under-crossings.



As a circuit made by gates (in the second image), a tangle diagram appears as an ordered list of its crossings gates, each crossing gate being given as a pair of FAN-OUT and one of the other two gates. 1-valent nodes appear as input nodes (in a separate INPUT list), or as output nodes (in the OUTPUT list). The wiring is given as a matrix M of connectivity, namely the element M_{ij} corresponding to the pair of nodes (i, j) (from the trivalent and 1-valent graph, independently on the pairing of nodes given by the crossings) is equal to 1 if there is a wire oriented from i to to j, otherwise is equal to 0. The definition is unambiguous because from i to j can be at most one oriented wire.

7.2 Colorings with idempotent right quasigroups

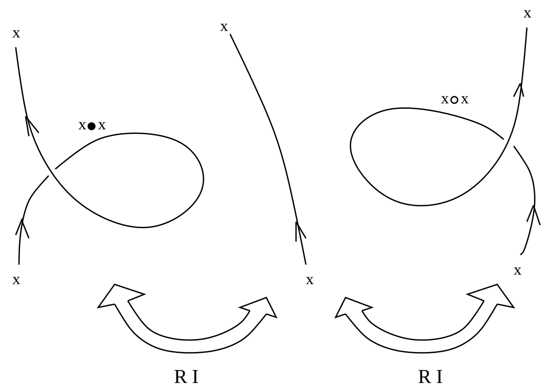
Let X be a set of colors which will be used to decorate the segments in a (oriented) tangle diagram. There are two binary operations on S related to the coloring, as shown in the next figure.



Notice that only matters the orientation of the arc which passes over.

We have therefore a set X endowed with two operations $\circ$ and $\bullet$. We want these operations to satisfy some conditions which ensure that the decoration of the segments of the tangle diagram rest unchanged after performing a Reidemeister move I or II on the tangle diagram. This is explained further. We show only the part of a (larger) diagram which changes during a Reidemeister move, with the convention that what is not shown will not change after the Reidemeister move is done.

The first condition, related to the Reidemeister move I, is depicted in the next figure.



It means that we can decorate ”tadpoles” such that we may remove them (by using the Reidemeister move I) afterwards. In algebraic terms, this condition means that we want the operations $\circ$ and $\bullet$ to be idempotent:

$$x \circ x \, = \, x \bullet x \, = \, x$$

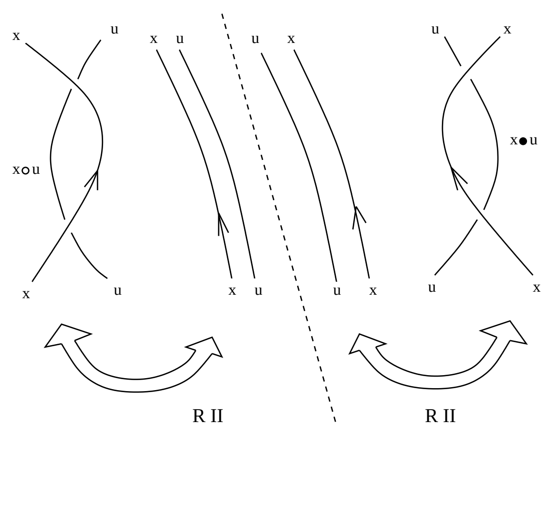
for all $x \in X$.

The second condition is related to the Reidemeister II move. It means that we can decorate the segments of a pair of arcs as shown in the following picture, in such a way that we can perform the Reidemeister II move and eliminate a pair of ”opposite” crossings.

This condition translates in algebraic terms into saying that $(X, \circ, \bullet)$ is a right quasigroup. Namely we want that

$$x \circ (x \bullet y) \, = \, x \bullet (x \circ y) \, = \, y$$

for all $x, y \in X$. This is the same as asking that for any a and b in X, the equation $a \circ x = b$ has a solution, which is unique, then denote the solution by $x = a \bullet b$. All in all, a set $(X, \circ, \bullet)$ which has the properties related to the first two Reidemeister moves is called an idempotent right quasigroup, or irq for short.



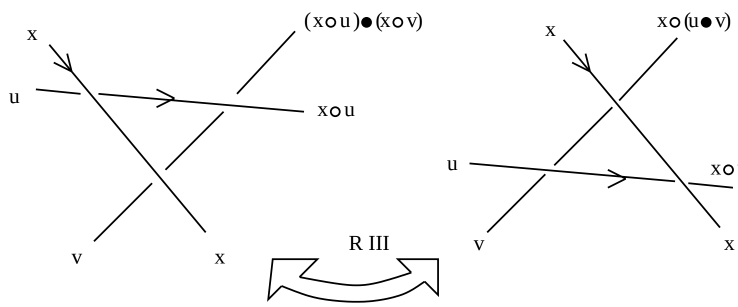
Definition 7.1 *A right quasigroup is a set X with a binary operation $\circ$ such that for each $a, b \in X$ there exists a unique $x \in X$ such that $a \circ x = b$. We write the solution of this equation $x = a \bullet b$.*

An idempotent right quasigroup (irq) is a right quasigroup $(X, \circ)$ such that for any $x \in X$ $x \circ x = x$. Equivalently, it can be seen as a set X endowed with two operations $\circ$ and $\bullet$, which satisfy the following axioms: for any $x, y \in X$

(R1) $$x \circ x = x \bullet x = x$$

(R2) $$x \circ (x \bullet y) = x \bullet (x \circ y) = y$$

The Reidemeister III move concerns the sliding of an arc (indifferent of orientation) under a crossing. In the next figure it is shown only one possible sliding movement.



Such a sliding move is possible, without modifying the coloring, if and only if the operation $\circ$ is left distributive with respect to the operation $\bullet$. (For the other possible choices of crossings, the "sliding" movement corresponding to the Reidemeister III move is possible if and only if $\bullet$ is distributive with respect to $\circ$ and also the operations $\circ$ and $\bullet$ are self distributive).

With this self-distributivity property, $(X, \circ, \bullet)$ is called a quandle. A well known quandle (therefore also an irq) is the Alexander quandle: consider $X = \mathbb{Z}[\varepsilon, \varepsilon^{-1}]$ with the operations

$$x \circ y = x + \varepsilon\,(-x + y) \quad , \quad x \bullet y = x + \varepsilon^{-1}\,(-x + y)$$

The operations in the Alexander quandle are therefore dilations in euclidean spaces.

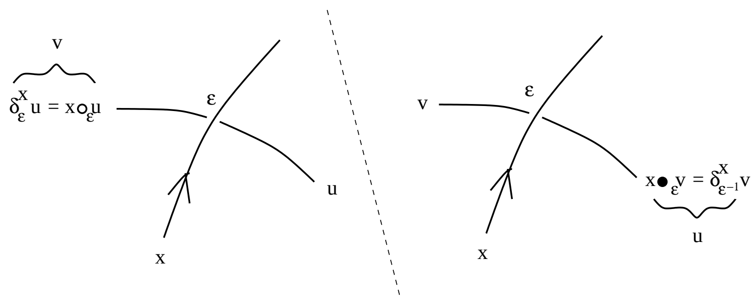
Important remark. Further I shall NOT see oriented tangle diagrams as objects associated to a tangle in three-dimensional space. That is because I am going to renounce to the Reidemeister III move. This interpretation, of being projections of tangles in space, is only for keeping a visually based vocabulary, like "over", "under", "sliding an arc under another" and so on.

7.3 Emergent algebras and tangles with decorated crossings

I shall adapt the tangle diagram coloring, presented in the previous section, for better understanding of the formalism of dilation structures. In fact we shall arrive to a more algebraic concept, more basic in some sense that the one of dilation structures, named "emergent algebra".

The first step towards this goal is to consider richer decorations as previously. We could decorate not only the connected components of tangle diagrams but also the crossings. I use for crossing decorations the scale parameter. Formally the scale parameter belongs to a commutative group Γ. In this paper is comfortable to think that $\Gamma = (0, +\infty)$ with the operation of multiplication or real numbers.

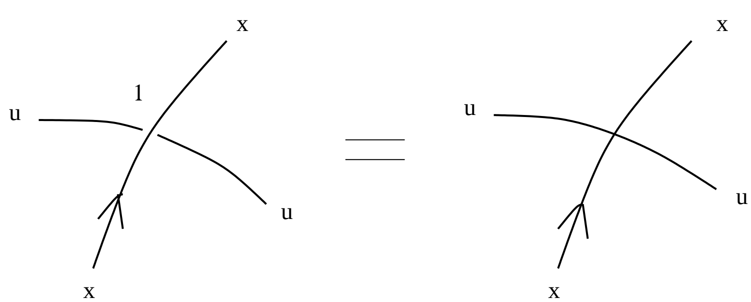
Here is the rule of decoration of tangle diagrams, by using a dilation structure:



In terms of idempotent right quasigroups, instead of one $(X, \circ, \bullet)$, we have a family $(X, \circ_\varepsilon, \bullet_\varepsilon)$, for all $\varepsilon \in \Gamma$. In terms of dilation structures, the operations are:

$$x \circ_\varepsilon u \,=\, \delta^x_\varepsilon u \quad , \quad x \bullet_\varepsilon u \,=\, \delta^x_{\varepsilon^{-1}} u$$

This implies that virtual crossings are allowed. A virtual crossing is just a crossing where nothing happens, a crossing with decoration $\varepsilon = 1$.

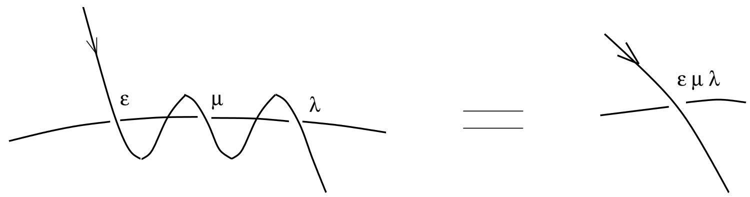


$$u = x \circ_1 u = x \bullet_1 u$$

Equivalent with the first two axioms of dilation structures, is that for all $\varepsilon \in \Gamma$ the triples $(X, \circ_\varepsilon, \bullet_\varepsilon)$ are idempotent right quasigroups (irqs), moreover we want that for any $x \in X$ the mapping

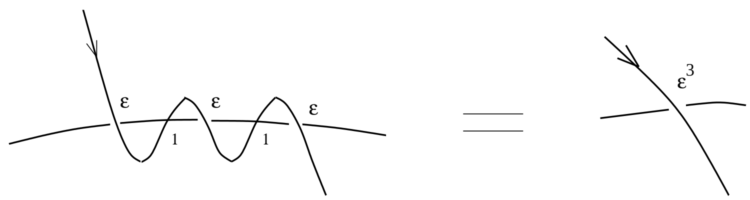
$$\varepsilon \in \Gamma \,\mapsto\, x \circ_\varepsilon (\cdot)$$

to be an action of Γ on X. This reflects into the following rules for combinations of decorated crossings.



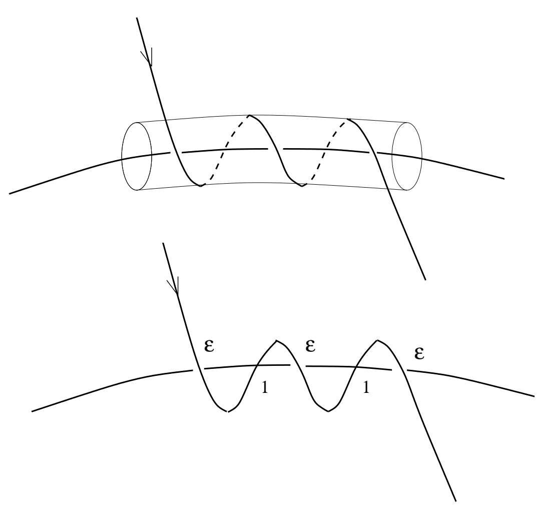
The equality sign means that we can replace one tangle diagram by the other.

In particular, we get an interpretation for the crossing decorated by a scale parameter. Look first to this equality of tangle diagrams.



If we fix the ε, take for example $\varepsilon = 1/2$, then any crossing decorated by a power of this ε is equivalent with a chain of crossings decorated with ε, with virtual crossings inserted in between.

The usual interpretation of virtual crossings is that these are crossings which are not really there. Alternatively, but only to get an intuitive image, we may imagine that a crossing decorated with ε^n is equivalent with the projection of a helix arc with n turns around an imaginary cylinder.



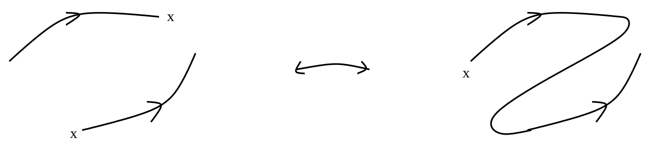
Thus, for example if we take $\varepsilon = 1/2$ as a "basis", then a crossing decorated with the scale parameter μ could be imagined as the projection of a helix arc with $-\log_2 \mu$ turns.

The sequence of irqs is the same as the algebraic object called a Γ-irq. The definition is given further (remember that for the needs of this paper $\Gamma = (0, +\infty)$).

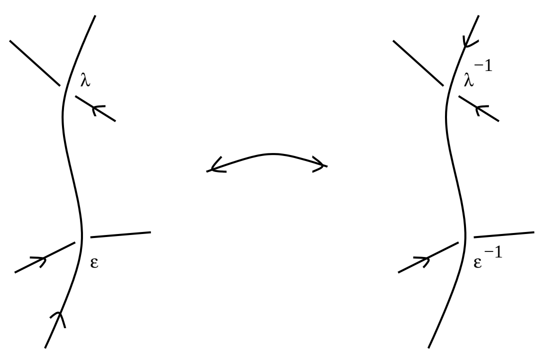
Definition 7.2 *Let Γ be a commutative group. A Γ-idempotent right quasigroup is a set X with a function $\varepsilon \in \Gamma \mapsto \circ_\varepsilon$ such that for any $\varepsilon \in \Gamma$ the pair $(X, \circ_\varepsilon)$ is a irq and moreover for any $\varepsilon, \mu \in \Gamma$ and any $x, y \in X$ we have*

$$x \circ_\varepsilon (x \circ_\mu y) = x \circ_{\varepsilon\mu} y$$

Rules concerning wires. (W1) We may join two wires decorated by the same element of the Γ-irq and with the same orientation.



(W2) We may change the orientation in a wire which passes over others, but we must invert (power "-1") the decoration of each crossing.

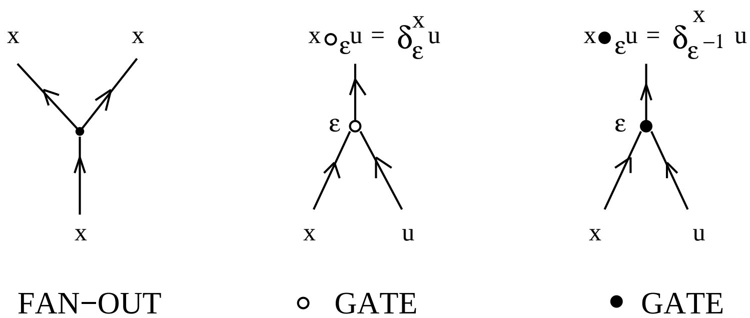


7.4 Decorated binary trees

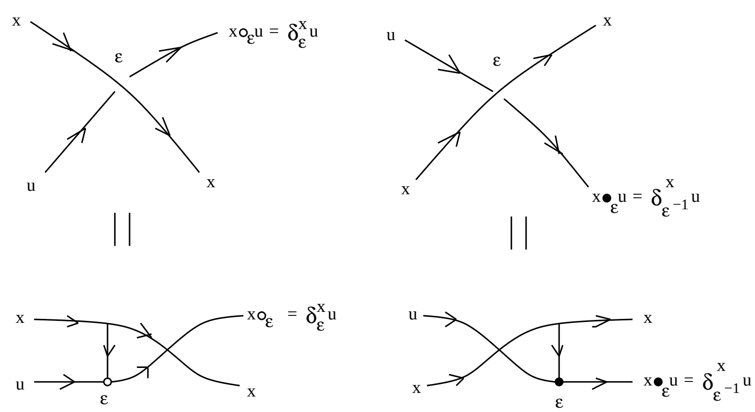
Here I use the second image of a oriented tangle diagram in order to understand the rules of decoration and movements described in the previous section.

In this interpretation an oriented tangle diagram is an oriented planar graph with 3-valent and 1-valent nodes (input or exit nodes), connected by wires which could cross (crossings of wires in this graph has no meaning).

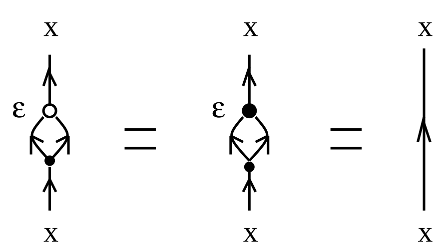
In fact we consider trivalent oriented planar graphs (together with 1-valent nodes representing inputs and outputs), with wires (and input, output nodes) decorated by elements of a Γ-irq $\varepsilon \in \Gamma \mapsto (X, \circ_\varepsilon, \bullet_\varepsilon)$. The nodes are either undecorated (corresponding to FAN-OUT gates) or decorated by pairs $(\circ, \varepsilon)$ or $(\bullet, \varepsilon)$, with $\varepsilon \in \Gamma$. The rules of decoration are the following.



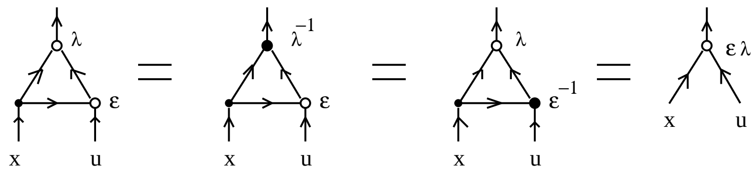
The trivalent graph is obtained from the tangle diagram by the procedure of replacing crossings with pairs of gates consisting of one FAN-OUT and one of the $\circ$ or $\bullet$ gates, explained in the next figure.



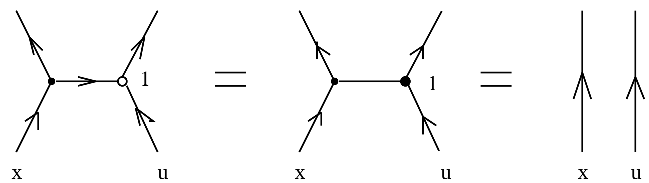
In terms of trivalent graphs, the condition (R1) from definition 7.1 applied for the irq $(X, \circ_\varepsilon, \bullet_\varepsilon)$ is graphically translated into the following identity (passing from one term of the identity to another is a "Reidemeister I move").



There are two more groups of identities (or moves), which describe the mechanisms of coloring trivalent graphs Γ-irqs. The first group consists in "triangle moves". This corresponds to Reidemeister move II and to the condition from the end of definition 7.2.

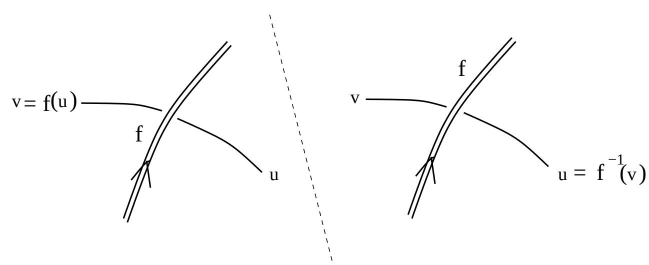


The second groups is equivalent with the re-wiring move (W1) and the relation $x \circ_1 u = x \bullet_1 u = u$.

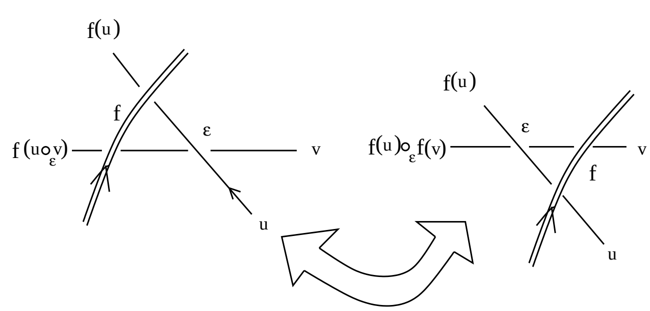


7.5 Linearity, self-similarity, Reidemeister III move

Let $f : Y \to X$ be an invertible function. We can use the tangle formalism for picturing the function f. To f is associated a special curved segment, figured by a double line. The crossings passing under this double line are colored following the rules explained in this figure.



Suppose that X and Y are endowed with a Γ-irq structure (in particular, we may suppose that they are endowed with dilation structures). Consider the following sliding movement.



The crossing decorated with ε from the left hand side diagram is in X (as well as the rules of decoration with the Γ-irq of X). Similarly, the crossing decorated with ε from the right hand side diagram is in Y. Therefore these two diagrams are equal (or we may pass from one to another by a sliding movement) if and only if f transforms an operation into another, equivalently if f is a morphism of Γ-irqs.

This sliding movement becomes the Reidemeister III move in the case of $X = Y$ and f equal to a dilation of X, $f = \delta^x_\mu$.

Definition 7.3 *A function $f : X \to Y$ is linear if and only if it is a morphism of Γ-irqs (of X and Y respectively). Moreover, if X and Y are endowed with dilation structures then f is linear if it is a morphism, written in terms of dilations notation as: for any $u, v \in X$ and any $\varepsilon \in \Gamma$*

$$f(\delta^u_\varepsilon v) \, = \, \delta^{f(u)}_\varepsilon f(v)$$

which is also a Lipschitz map from X to Y as metric spaces.

A dilation structure (X, d, δ) is (x, μ) self-similar (for a $x \in X$ and $\mu \in \Gamma$, different from 1, the neutral element of $\Gamma = (0, +\infty)$) if the dilation $f = \delta^x_\mu$ is linear from (X, d, δ) to itself and moreover for any $u, v \in X$ we have

$$d(\delta^x_\mu u, \delta^x_\mu v) \, = \, \mu \, d(u, v)$$

A dilation structure is linear if it is self-similar with respect to any $x \in X$ and $\mu \in \Gamma$.

Thus the Reidemeister III move is compatible with the tangle coloring by a dilation structure if and only if the dilation structure is linear.

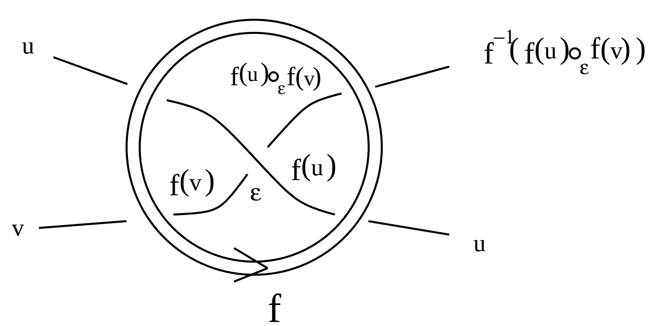
Conical groups are groups endowed with a one-parameter family of dilation morphisms. From the viewpoint of Γ-irqs, they are equivalent with linear dilation structures (theorem 6.1 [20], see also the Appendix).

A real vector space is a particular case of Carnot group. It is a commutative (hence nilpotent) group with the addition of vectors operation and it has a one-parameter family of dilations defined by the multiplication of vectors by positive scalars.

Carnot groups which are not commutative provide therefore a generalization of a vector space. Noncommutative Carnot groups are aplenty, in particular the simplest noncommutative Carnot groups are the Heisenberg groups, that is the simply connected Lie groups with the Lie algebra defined by the Heisenberg noncommutativity relations.

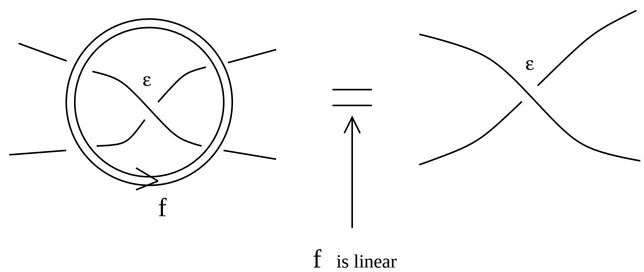
For me Carnot groups, or conical groups, are just linear objects. (By extension, manifolds, which are assemblies of open subsets of vector spaces, are locally linear objects as well. Moreover, they are "commutative", because the model of the tangent space at a point is a commutative Carnot group.)

It is also easy to explain graphically the transport of a dilation structure, or of a Γ-irq from X to Y, by using f^{-1}.



The transport operation amounts to adding a double circle decorated by f, which over-crosses the whole diagram (in this case a diagram containing only one crossing). If we use the map-territory distinction, then inside the circle we are in Y, outside we are in X.

It is obvious that f is linear if and only if the transported dilation structure on X (by f^{-1}) coincides with the dilation structure on X. Shortly said, encirclings by linear functions can be removed from the diagram.



In this tangle decoration formalism we have no reason to suppose that the dilations structures which we use are linear. This would be an unnecessary limitation of the dilation structure (or emergent algebra) formalism. That is why the Reidemeister III move is not an acceptable move in this formalism.

The last axiom (A4) of dilation structures can be translated into an algebraic statement which will imply a weak form of the Reidemeister III move, namely that this move can be done IN THE LIMIT.

7.6 Acceptable tangle diagrams

Consider a tangle diagram with decorated crossings, but with undecorated segments.

A notation for such a diagram is $T[\varepsilon, \mu, \eta, ...]$, where $\varepsilon, \mu, \eta, ...$ are decorations of the 3-valent nodes (in the second image) or decorations of the crossings (in the first image of a tangle diagram).

A tangle diagram $T[\varepsilon, \mu, \eta, ...]$ is "acceptable" if there exists at least a decoration of the input segments such that all the segments of the diagram can be decorated according to the rules specified previously, maybe non-uniquely.

A set of parameters of an acceptable tangle diagram is any coloring of a part of the segments of the diagram such that any coloring of the input segments which are already not colored by parameters, can be completed in a way which is unique for the output segments (which are not already colored by parameters).

Given an acceptable tangle diagram which admits a set of parameters, we shall see it as a function from the colorings of the input to the colorings of the output, with parameters from the set of parameters and with "scale parameters" the decorations of the crossings.

Given one acceptable tangle diagram which admits a set of parameters, given a set of parameters for it, we can choose one or more crossings and their decorations (scale parameters) as "scale variables". This is equivalent to considering a sequence of acceptable tangle diagrams, indexed by a multi-index of scale variables (i.e. taking values in some cartesian power of Γ). Each member of the sequence has the same tangle diagram, with the same set of parameters, with the same decorations of crossings which are not variables; the only difference is in the decoration of the crossings chosen as variables.

Associated to such a sequence is the sequence of input-output functions of these diagrams.

We shall consider uniform convergence of these functions with respect to compact sets of inputs.

All this is needed to formulate the emergent algebra correspondent of axiom A4 of dilation structures.

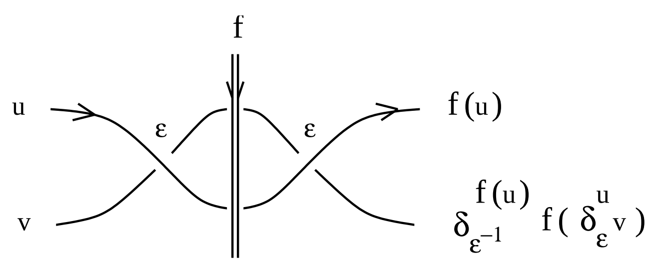
7.7 Going to the limit: emergent algebras

Basically, I see a decorated tangle diagram as an expression dependent on the decorations of the crossings. More precisely, I shall reserve the letter ε for an element of Γ which will be conceived as going to zero. This is the same kind of reasoning as for the zoom sequences in the section dedicated to maps.

Why is such a thing interesting? Let me give some examples.

Finite differences. We use the convention of adding to the tangle diagram supplementary arcs decorated by homeomorphisms. Let $f : X \to Y$ such a homeomorphism. I want to be able to differentiate the homeomorphism f, in the sense of dilation structures.

For this I need a notion of finite differences. These appear as the following diagram.



Indeed, suppose for simplicity that X and Y are finite dimensional normed vector spaces, with distance given by the norm and dilations

$$\delta^u_\varepsilon u \,=\, u + \varepsilon(-u+v)$$

Then we have:

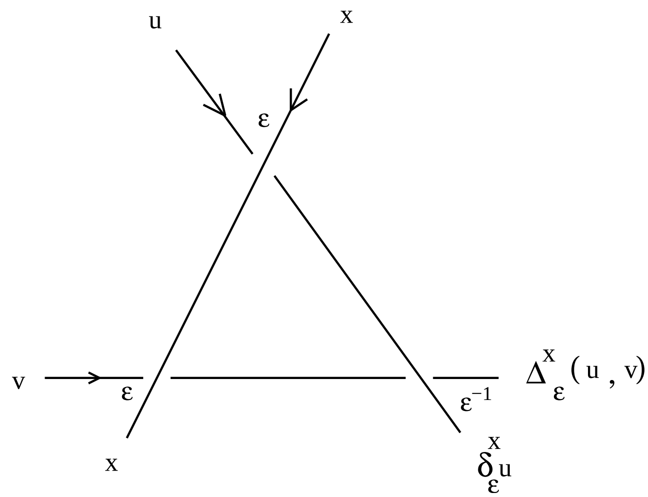
$$\delta^{f(u)}_{\varepsilon^{-1}} f\,(\delta^u_\varepsilon v) \,=\, f(u) \,+\, \frac{1}{\varepsilon}\,(f(u+\varepsilon(-u+v)) - f(u))$$

Pansu [53] generalized this definition of finite differences from real vector spaces to Carnot groups, which are nilpotent graduated simply connected Lie groups, a particular example of conical groups.

It is true that the diagram which encodes finite differences is not, technically speaking, of the type explain previously, because it has a segment (the one decorated by f), which is different from the other segments. But it is easy to see that all the mathematical formalism can be modified easily in order to accommodate such edges decorated with homeomorphism.

The notion of differentiability of f is obtained by asking that the sequence of input-output functions associated to the "finite difference" diagram, with parameter "x" and variable "ε", converges uniformly on compact sets.

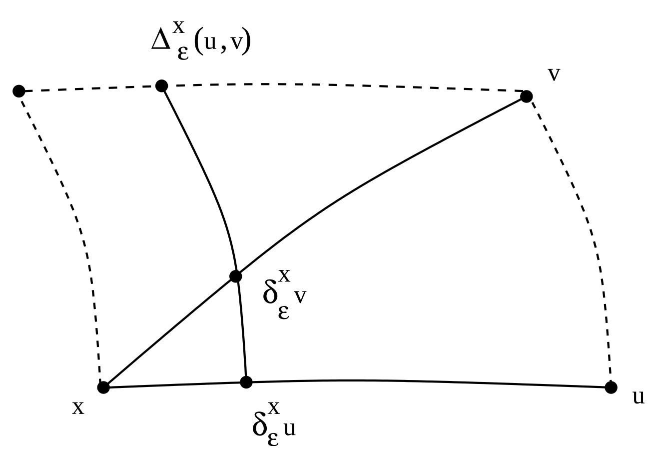
Difference gates. For any $\varepsilon \in \Gamma$, the ε-difference gate, is described by the next figure.



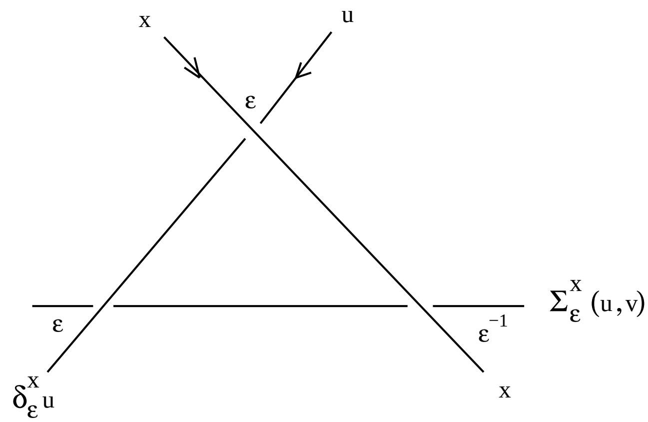
Here $\Delta^x_\varepsilon(u,v)$ is a construct made from operations $\circ_\varepsilon$, $\bullet_\varepsilon$. It corresponds to the difference coming from changing the viewpoint, in the map-territory frame. In terms of dilation structures, is the approximate difference which appears in axiom A4. In terms of notations of a Γ-irq, from the figure we can compute $\Delta^x_\varepsilon(u,v)$ as

$$\Delta^x_\varepsilon(u,v) \,=\, (x \circ_\varepsilon u) \bullet_\varepsilon \, (x \circ_\varepsilon v)$$

The geometric meaning of $\Delta^x_\varepsilon(u,v)$ is that it is indeed a kind of approximate difference between the vectors $\vec{xu}$ and $\vec{xv}$, by means of a generalization of the parallelogram law of vector addition. This is shown in the following figure, where straight lines have been replaced by slightly curved ones in order to suggest that this construction has meaning in settings far more general than euclidean spaces, like in Carnot-Caratheodory or sub-riemannian geometry, as shown in [11], or generalized (noncommutative) affine geometry [12], for length metric spaces with dilations [19] or even for normed groupoids [16].

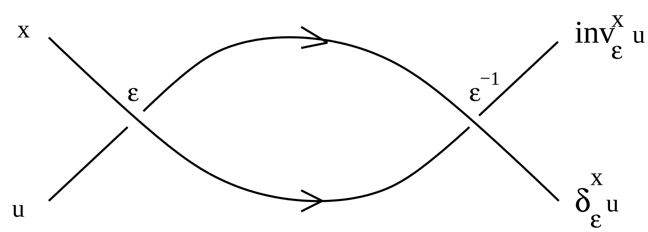


The ε-sum gate is described in the next figure.



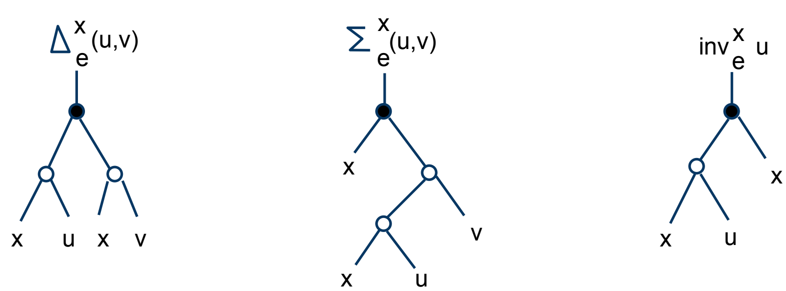
Similar comments can be made, concerning the sum gate. It is the approximate sum appearing in the axiom A4 of dilation structures.

Finally, there is another important tangle diagram, called ε-inverse gate. It is, at closer look, a particular case of a difference gate (take $x = u$ in the difference diagram).



The relevant outputs of the previously introduced gates, namely the approximate difference, sum and inverse functions, are described in the next definition, in terms of decorated binary trees (trivalent graphs). I am going to ignore the trees constructed from FAN-OUT gates, replacing them by patterns of decorations (of leaves of the binary trees). In the following all tree nodes are decorated with the same label ε and edges are oriented upwards.

Definition 7.4 *We define the difference, sum and inverse trees given by:*



The following proposition contains the main relations between the difference, sum and inverse gates. They can all be proved by this tangle diagram formalism. In [11] I explained these relations as appearing from the equivalent formalism using binary decorated trees.

Proposition 7.5 *Let $(X, \circ_\varepsilon)_{\varepsilon \in \Gamma}$ be a Γ-irq. Then we have the relations:*

(a) $\Delta^x_\varepsilon(u, \Sigma^x_\varepsilon(u,v)) = v$ *(difference is the inverse of sum)*

(b) $\Sigma^x_\varepsilon(u, \Delta^{x_\varepsilon}(u,v)) = v$ *(sum is the inverse of difference)*

(c) $\Delta^x_\varepsilon(u,v) = \Sigma^{x \circ_\varepsilon u}_\varepsilon(inv^x_\varepsilon u, v)$ *(difference approximately equals the sum of the inverse)*

(d) $inv^{x \circ u}_\varepsilon \, inv^x_\varepsilon \, u = u$ *(inverse operation is approximatively an involution)*

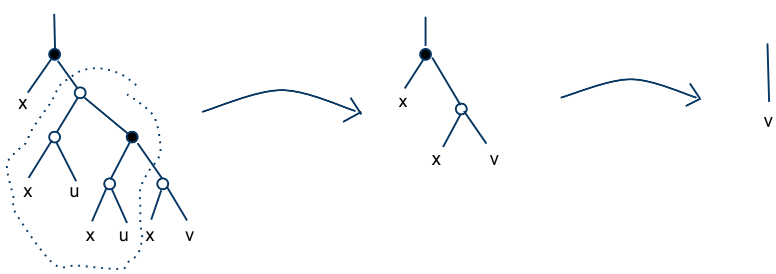
(e) $\Sigma^x_\varepsilon(u, \Sigma^{x \circ_\varepsilon u}_\varepsilon(v,w)) = \Sigma^x_\varepsilon(\Sigma^x_\varepsilon(u,v), w)$ *(approximate associativity of the sum)*

(f) $inv^x_\varepsilon \, u = \Delta^x_\varepsilon(u, x)$

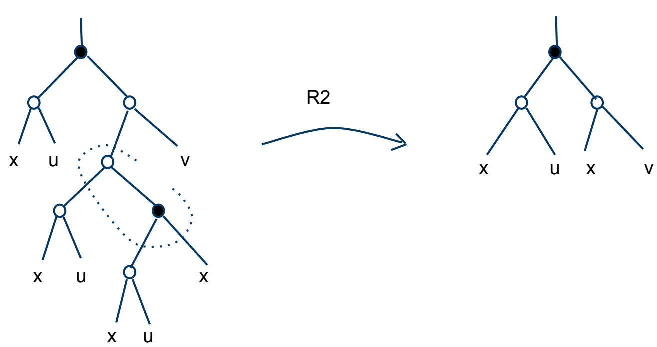
(g) $\Sigma^x_\varepsilon(x,u) = u$ *(neutral element at right).*

We shall use the tree formalism to prove some of these relations. For complete proofs see [11].

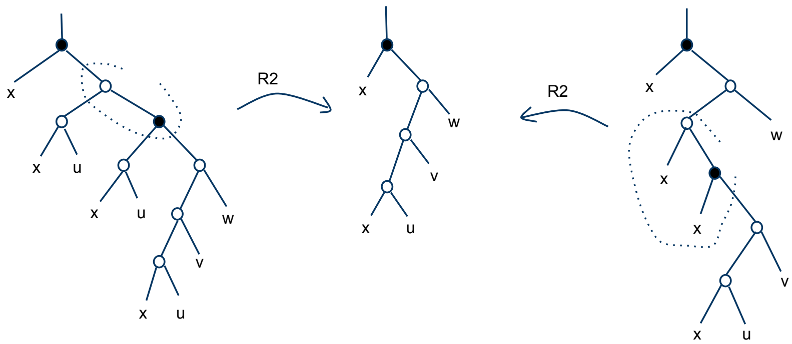
For example, in order to prove (b) we do the following calculus:



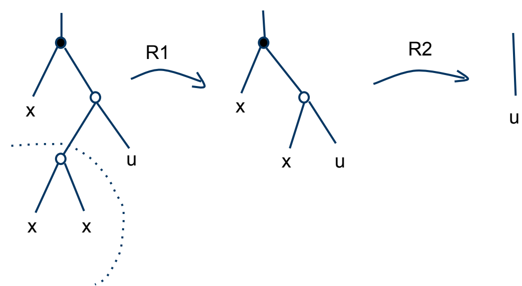
The relation (c) is obtained from:



Relation (e) (which is a kind of associativity relation) is obtained from:



Finally, for proving relation (g) we use also the rule (R1).



Emergent algebras. See [21] for all details.

Definition 7.6 *A Γ-uniform irq, or emergent algebra $(X, \circ, \bullet)$ is a separable uniform space X which is also a Γ-irq, with continuous operations, such that:*

(C) the operation $\circ$ is compactly contractive: for each compact set $K \subset X$ and open set $U \subset X$, with $x \in U$, there is an open set $A(K, U) \subset \Gamma$ with $\mu(A) = 1$ for any $\mu \in Abs(\Gamma)$ and for any $u \in K$ and $\varepsilon \in A(K, U)$, we have $x \circ_\varepsilon u \in U$;

(D) the following limits exist for any $\mu \in Abs(\Gamma)$

$$\lim_{\varepsilon \to \mu} \Delta^x_\varepsilon(u, v) = \Delta^x(u, v) \quad , \quad \lim_{\varepsilon \to \mu} \Sigma^x_\varepsilon(u, v) = \Sigma^x(u, v)$$

and are uniform with respect to x, u, v in a compact set.

Dilation structures are also emergent algebras. In fact, emergent algebras are generalizations of dilation structures, where the distance is no longer needed.

The main property of a uniform irq is the following. It is a consequence of relations from proposition 7.5.

Theorem 7.7 *Let $(X, \circ, \bullet)$ be a uniform irq. Then for any $x \in X$ the operation $(u, v) \mapsto \Sigma^x(u, v)$ gives X the structure of a conical group with the dilation $u \mapsto x \circ_\varepsilon u$.*

Proof. Pass to the limit in the relations from proposition 7.5. We can do this exactly because of the uniformity assumptions. We therefore have a series of algebraic relations which can be used to get the conclusion. □

8 Groups with dilations

For a dilation structure the metric tangent spaces have a group structure which is compatible with dilations. This structure, of a normed group with dilations, is interesting by itself. The notion has been introduced in [10], [11]; we describe it further.

We shall work further with local groups. We start with the following setting: G is a topological group endowed with an uniformity such that the operation is uniformly continuous. The description that follows is slightly non canonical, but is nevertheless motivated by the case of a Lie group endowed with a Carnot-Caratheodory distance induced by a left invariant distribution.

We introduce first the double of G, as the group $G^{(2)} \ = \ G \times G$ with operation

$$(x, u)(y, v) \ = \ (xy, y^{-1}uyv) \quad .$$

The operation on the group G, seen as the function

$$op : G^{(2)} \to G \ , \quad op(x, y) \ = \ xy \quad ,$$

is a group morphism. Also the inclusions:

$$i' : G \to G^{(2)} \ , \quad i'(x) \ = \ (x, e)$$

$$i" : G \to G^{(2)} \ , \quad i"(x) \ = \ (x, x^{-1})$$

are group morphisms.

Definition 8.1 *1. G is an uniform group if we have two uniformity structures, on G and G^2, such that op, i', $i"$ are uniformly continuous.*

2. A local action of a uniform group G on a uniform pointed space (X, x_0) is a function $\phi \in W \in \mathcal{V}(e) \mapsto \hat{\phi} : U_\phi \in \mathcal{V}(x_0) \to V_\phi \in \mathcal{V}(x_0)$ such that:

(a) the map $(\phi, x) \mapsto \hat{\phi}(x)$ is uniformly continuous from $G \times X$ (with product uniformity) to X,

(b) for any $\phi, \psi \in G$ there is $D \in \mathcal{V}(x_0)$ such that for any $x \in D$ $\widehat{\phi\psi^{-1}}(x)$ and $\hat{\phi}(\hat{\psi}^{-1}(x))$ make sense and $\widehat{\phi\psi^{-1}}(x) = \hat{\phi}(\hat{\psi}^{-1}(x))$.

3. Finally, a local group is an uniform space G with an operation defined in a neighbourhood of $(e, e) \subset G \times G$ which satisfies the uniform group axioms locally.

Remark that a local group acts locally at left (and also by conjugation) on itself.

This definition deserves an explanation. An uniform group, according to the definition (8.1), is a group G such that left translations are uniformly continuous functions and the left action of G on itself is uniformly continuous too. In order to precisely formulate this we need two uniformities: one on G and another on $G \times G$.

These uniformities should be compatible, which is achieved by saying that i', $i"$ are uniformly continuous. The uniformity of the group operation is achieved by saying that the op morphism is uniformly continuous.

Definition 8.2 *A group with dilations (G, δ) is a local group G with a local action of Γ (denoted by δ), on G such that*

H0. *the limit* $\lim_{\varepsilon \to 0} \delta_\varepsilon x = e$ *exists and is uniform with respect to* x *in a compact neighbourhood of the identity* e.

H1. *the limit*

$$\beta(x, y) = \lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \left((\delta_\varepsilon x)(\delta_\varepsilon y) \right)$$

is well defined in a compact neighbourhood of e *and the limit is uniform.*

H2. *the following relation holds*

$$\lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \left((\delta_\varepsilon x)^{-1} \right) = x^{-1}$$

where the limit from the left hand side exists in a neighbourhood of e *and is uniform with respect to* x.

Definition 8.3 *A normed group with dilations* $(G, \delta, \| \cdot \|)$ *is a group with dilations* (G, δ) *endowed with a continuous norm function* $\| \cdot \| : G \to \mathbb{R}$ *which satisfies (locally, in a neighbourhood of the neutral element* e*) the properties:*

(a) *for any* x *we have* $\|x\| \geq 0$*; if* $\|x\| = 0$ *then* $x = e$,

(b) *for any* x, y *we have* $\|xy\| \leq \|x\| + \|y\|$,

(c) *for any* x *we have* $\|x^{-1}\| = \|x\|$,

(d) *the limit* $\lim_{\varepsilon \to 0} \frac{1}{\nu(\varepsilon)} \|\delta_\varepsilon x\| = \|x\|^N$ *exists, is uniform with respect to* x *in compact set,*

(e) *if* $\|x\|^N = 0$ *then* $x = e$.

In a normed group with dilations we have a natural left invariant distance given by

$$d(x, y) = \|x^{-1} y\| \quad . \tag{8.0.1}$$

Any normed group with dilations has an associated dilation structure on it. In a group with dilations (G, δ) we define dilations based in any point $x \in G$ by

$$\delta_\varepsilon^x u = x \delta_\varepsilon (x^{-1} u). \tag{8.0.2}$$

The following result is theorem 15 [11].

Theorem 8.4 *Let* $(G, \delta, \| \cdot \|)$ *be a locally compact normed local group with dilations. Then* (G, d, δ) *is a dilation structure, where* δ *are the dilations defined by (8.0.2) and the distance* d *is induced by the norm as in (8.0.1).*

Proof. The axiom A0 is straightforward from definition 8.1, definition 8.2, axiom H0, and because the dilation structure is left invariant, in the sense that the transport by left translations in G preserves the dilations δ. We also trivially have axioms A1 and A2 satisfied.

For the axiom A3 remark that

$$d(\delta_{\varepsilon}^{x} u, \delta_{\varepsilon}^{x} v) = d(x\delta_{\varepsilon}(x^{-1}u), x\delta_{\varepsilon}(x^{-1}u)) = d(\delta_{\varepsilon}(x^{-1}u), \delta_{\varepsilon}(x^{-1}v)).$$

Denote $U = x^{-1}u$, $V = x^{-1}v$ and for $\varepsilon > 0$ let

$$\beta_{\varepsilon}(u, v) = \delta_{\varepsilon}^{-1}\left((\delta_{\varepsilon}u)(\delta_{\varepsilon}v)\right).$$

We have then:

$$\frac{1}{\varepsilon} d(\delta_{\varepsilon}^{x} u, \delta_{\varepsilon}^{x} v) = \frac{1}{\varepsilon} \| \delta_{\varepsilon} \beta_{\varepsilon} \left(\delta_{\varepsilon}^{-1}\left((\delta_{\varepsilon} V)^{-1}\right), U\right) \| \quad .$$

Define the function

$$d^{x}(u, v) = \| \beta(V^{-1}, U) \|^{N}.$$

From definition 8.2 axioms H1, H2, and from definition 8.3 (d), we obtain that axiom A3 is satisfied.

For the axiom A4 we have to compute:

$$\Delta^{x}(u, v) = \delta_{\varepsilon^{-1}}^{\delta_{\varepsilon}^{x} u} \delta_{\varepsilon}^{x} v = (\delta_{\varepsilon}^{x} u)\, (\delta_{\varepsilon})^{-1} \Big((\delta_{\varepsilon}^{x} u)^{-1}\, (\delta_{\varepsilon}^{x} v) \Big) =$$

$$= (x \delta_{\varepsilon} U)\, \beta_{\varepsilon} \left(\delta_{\varepsilon}^{-1}\left((\delta_{\varepsilon} V)^{-1}\right), U\right) \to x \beta \left(V^{-1}, U \right)$$

as $\varepsilon \to 0$. Therefore axiom A4 is satisfied. $\square$

8.1 Conical groups

Definition 8.5 *A normed conical group N is a normed group with dilations such that for any $\varepsilon \in \Gamma$ the dilation δ_{ε} is a group morphism and such that for any $\varepsilon > 0$ $\|\delta_{\varepsilon} x\| = \nu(\varepsilon) \|x\|$.*

A conical group is the infinitesimal version of a group with dilations ([11] proposition 2).

Proposition 8.6 *Under the hypotheses H0, H1, H2 $(G, \beta, \delta, \| \cdot \|^{N})$ is a local normed conical group, with operation β, dilations δ and homogeneous norm $\| \cdot \|^{N}$.*

Proof. All the uniformity assumptions allow us to change at will the order of taking limits. We shall not insist on this further and we shall concentrate on the algebraic aspects.

We have to prove the associativity, existence of neutral element, existence of inverse and the property of being conical.

For the associativity $\beta(x, \beta(y, z)) = \beta(\beta(x, y), z)$ we compute:

$$\beta(x, \beta(y, z)) = \lim_{\varepsilon \to 0, \eta \to 0} \delta_{\varepsilon}^{-1} \left\{ (\delta_{\varepsilon} x) \delta_{\varepsilon/\eta} \left((\delta_{\eta} y)(\delta_{\eta} z) \right) \right\} \quad .$$

We take $\varepsilon = \eta$ and we get

$$\beta(x, \beta(y, z)) = \lim_{\varepsilon \to 0} \left\{ (\delta_{\varepsilon} x)(\delta_{\varepsilon} y)(\delta_{\varepsilon} z) \right\} \quad .$$

In the same way:

$$\beta(\beta(x, y), z) = \lim_{\varepsilon \to 0, \eta \to 0} \delta_{\varepsilon}^{-1} \left\{ (\delta_{\varepsilon/\eta} x) \left((\delta_{\eta} x)(\delta_{\eta} y) \right) (\delta_{\varepsilon} z) \right\} \quad .$$

and again taking $\varepsilon = \eta$ we obtain

$$\beta(\beta(x,y),z) = \lim_{\varepsilon \to 0} \{(\delta_\varepsilon x)(\delta_\varepsilon y)(\delta_\varepsilon z)\} = \beta(x, \beta(y,z)) \quad .$$

The neutral element is e, from H0 (first part): $\beta(x,e) = \beta(e,x) = x$. The inverse of x is x^{-1}, by a similar argument:

$$\beta(x, x^{-1}) = \lim_{\varepsilon \to 0, \eta \to 0} \delta_\varepsilon^{-1} \left\{ (\delta_\varepsilon x) \left(\delta_{\varepsilon/\eta} (\delta_\eta x)^{-1} \right) \right\} \quad ,$$

and taking $\varepsilon = \eta$ we obtain

$$\beta(x, x^{-1}) = \lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \left((\delta_\varepsilon x)(\delta_\varepsilon x)^{-1} \right) = \lim_{\varepsilon \to 0} \delta_\varepsilon^{-1}(e) = e \quad .$$

Finally, β has the property:

$$\beta(\delta_\eta x, \delta_\eta y) = \delta_\eta \beta(x,y) \quad ,$$

which comes from the definition of β and commutativity of multiplication in $(0, +\infty)$. This proves that (G, β, δ) is conical. □

8.2 Carnot groups

Carnot groups appear in sub-riemannian geometry as models of tangent spaces, [5], [33], [53]. In particular such groups can be endowed with a structure of sub-riemannian manifold.

Definition 8.7 *A Carnot (or stratified homogeneous) group is a pair (N, V_1) consisting of a real connected simply connected group N with a distinguished subspace V_1 of the Lie algebra $Lie(N)$, such that the following direct sum decomposition occurs:*

$$n \ = \ \sum_{i=1}^{m} V_i \ , \quad V_{i+1} \ = \ [V_1, V_i]$$

The number m is the step of the group. The number $Q \ = \ \sum_{i=1}^{m} i \ dim V_i$ is called the homogeneous dimension of the group.

Because the group is nilpotent and simply connected, the exponential mapping is a diffeomorphism. We shall identify the group with the algebra, if is not locally otherwise stated.

The structure that we obtain is a set N endowed with a Lie bracket and a group multiplication operation, related by the Baker-Campbell-Hausdorff formula. Remark that the group operation is polynomial.

Any Carnot group admits a one-parameter family of dilations. For any $\varepsilon > 0$, the associated dilation is:

$$x \ = \ \sum_{i=1}^{m} x_i \ \mapsto \ \delta_\varepsilon x \ = \ \sum_{i=1}^{m} \varepsilon^i x_i$$

Any such dilation is a group morphism and a Lie algebra morphism.

In a Carnot group N let us choose an euclidean norm $\| \cdot \|$ on V_1. We shall endow the group N with a structure of a sub-riemannian manifold. For this take the distribution obtained from left translates of the space V_1. The metric on that distribution is obtained by left translation of the inner product restricted to V_1.

Because V_1 generates (the algebra) N then any element $x \in N$ can be written as a product of elements from V_1, in a controlled way, described in the following useful lemma (slight reformulation of Lemma 1.40, Folland, Stein [28]).

Lemma 8.8 *Let N be a Carnot group and $X_1, ..., X_p$ an orthonormal basis for V_1. Then there is a a natural number M and a function $g : \{1, ..., M\} \to \{1, ..., p\}$ such that any element $x \in N$ can be written as:*

$$x = \prod_{i=1}^{M} \exp(t_i X_{g(i)}) \tag{8.2.3}$$

Moreover, if x is sufficiently close (in Euclidean norm) to 0 then each t_i can be chosen such that $\mid t_i \mid \leq C\|x\|^{1/m}$

As a consequence we get:

Corollary 8.9 *The Carnot-Carathéodory distance*

$$d(x,y) = \inf \left\{ \int_0^1 \|c^{-1}\dot{c}\| \, dt \ : \ c(0) = x, \ c(1) = y, \right.$$
$$\left. c^{-1}(t)\dot{c}(t) \in V_1 \text{ for a.e. } t \in [0,1] \right\}$$

is finite for any two $x, y \in N$. The distance is obviously left invariant, thus it induces a norm on N.

The Carnot-Carathéodory distance induces a homogeneous norm on the Carnot group N by the formula: $\|x\| = d(0, x)$. From the invariance of the distance with respect to left translations we get: for any $x, y \in N$

$$\|x^{-1}y\| = d(x,y)$$

For any $x \in N$ and $\varepsilon > 0$ we define the dilation $\delta^x_\varepsilon y = x\delta_\varepsilon(x^{-1}y)$. Then (N, d, δ) is a dilation structure, according to theorem 8.4.

8.3 Contractible groups

Definition 8.10 *A contractible group is a pair (G, α), where G is a topological group with neutral element denoted by e, and $\alpha \in Aut(G)$ is an automorphism of G such that:*

- *α is continuous, with continuous inverse,*
- *for any $x \in G$ we have the limit $\lim_{n\to\infty} \alpha^n(x) = e$.*

For a contractible group (G, α), the automorphism α is compactly contractive (Lemma 1.4 (iv) [54]), that is: for each compact set $K \subset G$ and open set $U \subset G$, with $e \in U$, there is $N(K, U) \in \mathbb{N}$ such that for any $x \in K$ and $n \in \mathbb{N}$, $n \geq N(K, U)$, we have $\alpha^n(x) \in U$.

If G is locally compact then α compactly contractive is equivalent with: each identity neighbourhood of G contains an α-invariant neighbourhood. Further on we shall assume without mentioning that all groups are locally compact.

Any conical group can be seen as a contractible group. Indeed, it suffices to associate to a conical group (G, δ) the contractible group (G, δ_ε), for a fixed $\varepsilon \in \Gamma$ with $\nu(\varepsilon) < 1$.

Conversely, to any contractible group (G, α) we may associate the conical group (G, δ), with $\Gamma = \left\{ \frac{1}{2^n} \ : \ n \in \mathbb{N} \right\}$ and for any $n \in \mathbb{N}$ and $x \in G$

$$\delta_{\frac{1}{2^n}} x = \alpha^n(x) \quad .$$

(Finally, a local conical group has only locally the structure of a contractible group.)

The structure of contractible groups is known in some detail, due to Siebert [54], Wang [65], Glöckner and Willis [30], Glöckner [29] and references therein.

For this paper the following results are of interest. We begin with the definition of a contracting automorphism group [54], definition 5.1.

Definition 8.11 *Let G be a locally compact group. An automorphism group on G is a family $T = (\tau_t)_{t>0}$ in $Aut(G)$, such that $\tau_t \, \tau_s = \tau_{ts}$ for all $t, s > 0$.*

The contraction group of T is defined by

$$C(T) \ = \ \left\{ x \in G \, : \, \lim_{t \to 0} \tau_t(x) = e \right\} \quad .$$

The automorphism group T is contractive if $C(T) = G$.

It is obvious that a contractive automorphism group T induces on G a structure of conical group. Conversely, any conical group with $\Gamma = (0, +\infty)$ has an associated contractive automorphism group (the group of dilations based at the neutral element).

Further is proposition 5.4 [54].

Proposition 8.12 *For a locally compact group G the following assertions are equivalent:*

(i) G admits a contractive automorphism group;

(ii) G is a simply connected Lie group whose Lie algebra admits a positive graduation.

9 Some examples of dilation structures

9.1 Snowflakes

The next example is a snowflake variation of the euclidean case: $X = \mathbb{R}^n$ and for any $a \in (0,1]$ take

$$d_a(x,y) \ = \ \|x-y\|^{\alpha} \quad , \quad \delta_\varepsilon^x y \ = \ x + \varepsilon^{\frac{1}{a}}(y-x) \ .$$

We leave to the reader to verify the axioms.

More general, if (X,d,δ) is a dilation structure then $(X, d_a, \delta(a))$ is also a dilation structure, for any $a \in (0,1]$, where

$$d_a(x,y) \ = \ (d(x,y))^a \quad , \quad \delta(a)_\varepsilon^x \ = \ \delta^x_{\varepsilon^{\frac{1}{a}}} \ .$$

9.2 Nonstandard dilations in the euclidean space

Take $X = \mathbb{R}^2$ with the euclidean distance. For any $z \in \mathbb{C}$ of the form $z = 1 + i\theta$ we define dilations

$$\delta_\varepsilon x = \varepsilon^z x \ .$$

It is easy to check that $(X, \delta, +, d)$ is a conical group, equivalenty that the dilations

$$\delta_\varepsilon^x y = x + \delta_\varepsilon(y-x) \ .$$

form a dilation structure with the euclidean distance.

Two such dilation structures (constructed with the help of complex numbers $1+i\theta$ and $1+i\theta'$) are equivalent if and only if $\theta = \theta'$.

There are two other surprising properties of these dilation structures. The first is that if $\theta \neq 0$ then there are no non trivial Lipschitz curves in X which are differentiable almost everywhere. The second property is that any holomorphic and Lipschitz function from X to X (holomorphic in the usual sense on $X = \mathbb{R}^2 = \mathbb{C}$) is differentiable almost everywhere, but there are Lipschitz functions from X to X which are not differentiable almost everywhere (suffices to take a $\mathcal{C}^\infty$ function from $\mathbb{R}^2$ to $\mathbb{R}^2$ which is not holomorphic).

9.3 Riemannian manifolds

The following interesting quotation from Gromov book [32], pages 85-86, motivates some of the ideas underlying dilation structures, especially in the very particular case of a riemannian manifold:

"**3.15. Proposition:** *Let (V,g) be a Riemannian manifold with g continuous. For each $v \in V$ the spaces $(V, \lambda d, v)$ Lipschitz converge as $\lambda \to \infty$ to the tangent space $(T_v V, 0)$ with its Euclidean metric g_v.*

Proof$_+$: *Start with a C^1 map $(\mathbb{R}^n, 0) \to (V, v)$ whose differential is isometric at 0. The λ-scalings of this provide almost isometries between large balls in $\mathbb{R}^n$ and those in λV for $\lambda \to \infty$.* **Remark:** *In fact we can define Riemannian manifolds as locally compact path metric spaces that satisfy the conclusion of Proposition 3.15.*"

The problem of domains and codomains left aside, any chart of a Riemannian manifold induces locally a dilation structure on the manifold. Indeed, take (M,d) to be a n-dimensional Riemannian manifold with d the distance on M induced by the Riemannian structure. Consider a diffeomorphism ϕ of an open set $U \subset M$ onto $V \subset \mathbb{R}^n$ and transport the dilations from V to U (equivalently, transport the distance d from U to V). There is only one thing to check in order to see that we got a dilation structure: the axiom A3, expressing the compatibility of the distance d with the dilations. But this is just a metric way to express the distance on the tangent space of M at x as a limit of rescaled distances (see Gromov Proposition 3.15, [32], p. 85-86). Denoting by g_x the metric tensor at $x \in U$, we have:

$$\left[d^x(u,v)\right]^2 =$$
$$= g_x \left(\frac{d}{d\,\varepsilon}_{|\varepsilon=0} \phi^{-1}\left(\phi(x) + \varepsilon(\phi(u) - \phi(x))\right), \frac{d}{d\,\varepsilon}_{|\varepsilon=0} \phi^{-1}\left(\phi(x) + \varepsilon(\phi(v) - \phi(x))\right) \right)$$

A basically different example of a dilation structure on a riemannian manifold will be explained next. Let M be a n dimensional riemannian manifold and exp be the geodesic exponential. To any point $x \in M$ and any vector $v \in T_x M$ the point $\exp_x(v) \in M$ is located on the geodesic passing thru x and tangent to v; if we parameterize this geodesic with respect to length, such that the tangent at x is parallel and has the same direction as v, then $\exp_x(v) \in M$ has the coordinate equal with the length of v with respect to the norm on $T_x M$. We define implicitly the dilation based at x, of coefficient $\varepsilon > 0$ by the relation:

$$\delta^x_\varepsilon \exp_x(u) \;=\; \exp_x(\varepsilon u) \quad .$$

It is not straightforward to check that we obtain a strong dilation structure, but it is true. There are interesting facts related to the numbers A, B and the minimal regularity required for the riemannian manifold. This example is different from the first because instead of using a chart (same for all x) we use a family of charts indexed with respect to the basepoint of the dilations.

10 Dilation structures in ultrametric spaces

Here we are concerned with dilation structures on ultrametric spaces. The special case considered is the boundary of the infinite dyadic tree, topologically the same as the middle-thirds Cantor set. This is also the space of infinite words over the alphabet $X = \{0, 1\}$. Self-similar dilation structures are introduced and studied on this space.

We show that on the boundary of the dyadic tree, any self-similar dilation structure is described by a web of interacting automata. This is achieved in theorems 10.8 and 10.13. These theorems are analytical in nature, but they admit an easy interpretation in terms of automata by using classical results as theorem 10.2 and proposition 10.3.

The subject is relevant for applications to the hot topic of self-similar groups of isometries of the dyadic tree (for an introduction into self-similar groups see [4]).

10.1 Words and the Cantor middle-thirds set

Let X be a finite, non empty set. The elements of X are called letters. The collection of words of finite length in the alphabet X is denoted by X^*. The empty word $\emptyset$ is an element of X^*.

The length of any word $w \in X^*$, $w = a_1 ... a_m$, $a_k \in X$ for all $k = 1, ..., m$, is denoted by $\mid w \mid = m$. The set of words which are infinite at right is denoted by

$$X^\omega = \{ f \quad | \quad f : \mathbb{N}^* \to X \} = X^{\mathbb{N}^*} \quad .$$

Concatenation of words is naturally defined. If $q_1, q_2 \in X^*$ and $w \in X^\omega$ then $q_1 q_2 \in X^*$ and $q_1 w \in X^\omega$.

The shift map $s : X^\omega \to X^\omega$ is defined by $w = w_1 \, s(w)$, for any word $w \in X^\omega$. For any $k \in \mathbb{N}^*$ we define $[w]_k \in X^k \subset X^*$, $\{w\}_k \in X^\omega$ by

$$w = [w]_k \, s^k(w) \quad , \quad \{w\}_k = s^k(w) \quad .$$

The topology on X^ω is generated by cylindrical sets qX^ω, for all $q \in X^*$. The topological space X^ω is compact.

To any $q \in X^*$ is associated a continuous injective transformation $\hat{q} : X^\omega \to X^\omega$, $\hat{q}(w) = qw$. The semigroup X^* (with respect to concatenation) can be identified with the semigroup (with respect to function composition) of these transformations. This semigroup is obviously generated by X. The empty word $\emptyset$ corresponds to the identity function.

The dyadic tree $\mathcal{T}$ is the infinite rooted planar binary tree. Any node has two descendants. The nodes are coded by elements of X^*, $X = \{0, 1\}$. The root is coded by the empty word and if a node is coded by $x \in X^*$ then its left hand side descendant has the code $x0$ and its right hand side descendant has the code $x1$. We shall therefore identify the dyadic tree with X^* and we put on the dyadic tree the natural (ultrametric) distance on X^*. The boundary (or the set of ends) of the dyadic tree is then the same as the compact ultrametric space X^ω.

10.2 Automata

In this section we use the same notations as [31].

Definition 10.1 *An (asynchronous) automaton is an oriented set* $(X_I, X_O, Q, \pi, \lambda)$*, with:*

(a) X_I, X_O *are finite sets, called the input and output alphabets,*

(b) Q is a set of internal states of the automaton,

(c) π is the transition function, $\pi : X_I \times Q \to Q$,

(d) λ is the output function, $\lambda : X_I \times Q \to X_O^*$.

If λ takes values in X_O then the automaton is called synchronous.

The functions λ and π can be continued to the set $X_I^* \times Q$ by: $\pi(\emptyset, q) = q$, $\lambda(\emptyset, q) = \emptyset$,

$$\pi(xw, q) = \pi(w, \pi(x, q)) \quad , \quad \lambda(xw, q) = \lambda(x, q)\lambda(w, \pi(x, q))$$

for any $x \in X_I, q \in Q$ and any $w \in X_I^*$.

An automaton is nondegenerate if the functions λ and π can be uniquely extended by the previous formulæto $X_I^\omega \times Q$.

To any nondegenerated automaton $(X_I, X_O, Q, \pi, \lambda)$ and any $q \in Q$ is associated the function $\lambda(\cdot, q) : X_I^\omega \to X_O^\omega$. The following is theorem 2.4 [31].

Theorem 10.2 *The mapping $f : X_I^\omega \to X_O^\omega$ is continuous if and only if it is defined by a certain nondegenerate asynchronous automaton.*

The proof given in [31] is interesting to read because it provides a construction of an automaton which defines the continuous function f.

10.3 Isometries of the dyadic tree

An isomorphism of $\mathcal{T}$ is just an invertible transformation which preserves the structure of the tree. It is well known that isometries of (X^ω, d) are the same as isometries of $\mathcal{T}$.

Let $A \in Isom(X^\omega, d)$ be such an isometry. For any finite word $q \in X^*$ we may define $A_q \in Isom(X^\omega, d)$ by

$$A(qw) = A(q)\, A_q(w)$$

for any $w \in X^\omega$. Note that in the previous relation $A(q)$ makes sense because A is also an isometry of $\mathcal{T}$.

The following description of isometries of the dyadic tree in terms of automata can be deduced from an equivalent formulation of proposition 3.1 [31] (see also proposition 2.18 [31]).

Proposition 10.3 *A function $X^\omega \to X^\omega$ is an isometry of the dyadic tree if and only if it is generated by a synchronous automaton with $X_I = X_O = X$.*

10.4 Dilation structures on the boundary of the dyadic tree

Dilation structures on the boundary of the dyadic tree will have a simpler form than general, mainly because the distance is ultrametric.

We shall take the group Γ to be the set of integer powers of 2, seen as a subset of dyadic numbers. Thus for any $p \in \mathbb{Z}$ the element $2^p \in \mathbb{Q}_2$ belongs to Γ. The operation is the multiplication of dyadic numbers and the morphism $\nu : \Gamma \to (0, +\infty)$ is defined by

$$\nu(2^p) = d(0, 2^p) = \frac{1}{2^p} \in (0, +\infty) \quad .$$

Axiom A0. This axiom states that for any $p \in \mathbb{N}$ and any $x \in X^{\omega}$ the dilation

$$\delta^x_{2^p} : U(x) \to V_{2^p}(x)$$

is a homeomorphism, the sets $U(x)$ and $V_{2^p}(x)$ are open and there is $A > 1$ such that the ball centered in x and radius A is contained in $U(x)$. But this means that $U(x) = X^{\omega}$, because $X^{\omega} = B(x, 1)$.

Further, for any $p \in \mathbb{N}$ we have the inclusions:

$$B(x, \frac{1}{2^p}) \subset \delta^x_{2^p} X^{\omega} \subset V_{2^p}(x) \quad . \tag{10.4.1}$$

For any $p \in \mathbb{N}^*$ the associated dilation $\delta^x_{2^{-p}} : W_{2^{-p}}(x) \to B(x, B) = X^{\omega}$, is injective, invertible on the image. We suppose that $W_{2^{-p}}(x)$ is open, that

$$V_{2^p}(x) \subset W_{2^{-p}}(x) \tag{10.4.2}$$

and that for all $p \in \mathbb{N}^*$ and $u \in X^{\omega}$ we have $\delta^x_{2^{-p}} \ \delta^x_{2^p} u \ = \ u$. We leave aside for the moment the interpretation of the technical condition before axiom A4.

Axioms A1 and A2. Nothing simplifies.

Axiom A3. Because d is an ultrametric distance and X^{ω} is compact, this axiom has very strong consequences, for a non degenerate dilation structure.

In this case the axiom A3 states that there is a non degenerate distance function d^x on X^{ω} such that we have the limit

$$\lim_{p \to \infty} 2^p d(\delta^x_{2^p} u, \delta^x_{2^p} v) = d^x(u, v) \tag{10.4.3}$$

uniformly with respect to $x, u, v \in X^{\omega}$.

We continue further with first properties of dilation structures.

Lemma 10.4 *There exists $p_0 \in \mathbb{N}$ such that for any $x, u, v \in X^{\omega}$ and for any $p \in \mathbb{N}$, $p \geq p_0$, we have*

$$2^p d(\delta^x_{2^p} u, \delta^x_{2^p} v) = d^x(u, v) \quad .$$

Proof. From the limit (10.4.3) and the non degeneracy of the distances d^x we deduce that

$$\lim_{p \to \infty} \log_2 \left(2^p d(\delta^x_{2^P} u, \delta^x_{2^P} v) \right) = \log_2 d^x(u, v) \quad ,$$

uniformly with respect to $x, u, v \in X^{\omega}$, $u \neq v$. The right hand side term is finite and the sequence from the limit at the left hand side is included in $\mathbb{Z}$. Use this and the uniformity of the convergence to get the desired result. $\square$

In the sequel p_0 is the smallest natural number satisfying lemma 10.4.

Lemma 10.5 *For any $x \in X^{\omega}$ and for any $p \in \mathbb{N}$, $p \geq p_0$, we have $\delta^x_{2^p} X^{\omega} = [x]_p X^{\omega}$. Otherwise stated, for any $x, y \in X^{\omega}$, any $q \in X^*$, $\mid q \mid \geq p_0$ there exists $w \in X^{\omega}$ such that*

$$\delta^{qx}_{2^{|q|}} w = qy$$

and for any $z \in X^{\omega}$ there is $y \in X^{\omega}$ such that $\delta^{qx}_{2^{|q|}} z = qy$. Moreover, for any $x \in X^{\omega}$ and for any $p \in \mathbb{N}$, $p \geq p_0$ the inclusions from (10.4.1), (10.4.2) are equalities.

Proof. From the last inclusion in (10.4.1) we get that for any $x, y \in X^{\omega}$, any $q \in X^{*}$, $\mid q \mid \geq p_0$ there exists $w \in X^{\omega}$ such that $\delta^{qx}_{2^{|q|}} w = qy$. For the second part of the conclusion we use lemma 10.4 and axiom A1. From there we see that for any $p \geq p_0$ we have

$$2^p d(\delta^x_{2^p} x, \delta^x_{2^p} u) = 2^p d(x, \delta^x_{2^p} u) = d^x(x, u) \leq 1 \quad .$$

Therefore $2^p d(x, \delta^x_{2^p} u) \leq 1$, which is equivalent with the second part of the lemma.

Finally, the last part of the lemma has a similar proof, only that we have to use also the last part of axiom A0. $\square$

The technical condition before the axiom A4 turns out to be trivial. Indeed, from lemma 10.5 it follows that for any $p \geq p_0$, $p \in \mathbb{N}$, and any $x, u, v \in X^{\omega}$ we have $\delta^x_{2^p} u = [x]_p w$, $w \in X^{\omega}$. It follows that

$$\delta^x_{2^p} v \in [x]_p X^{\omega} = W_{2^{-p}}(x) = W_{2^{-p}}(\delta^x_{2^p} u) \quad .$$

Lemma 10.6 *For any $x, u, v \in X^{\omega}$ such that $2^{p_0} d(x, u) \leq 1$, $2^{p_0} d(x, v) \leq 1$ we have $d^x(u, v) = d(u, v)$. Moreover, under the same hypothesis, for any $p \in \mathbb{N}$ we have*

$$2^p d(\delta^x_{2^p} u, \delta^x_{2^p} v) = d(u, v) \quad .$$

Proof. By lemma 10.4, lemma 10.5 and axiom A2. Indeed, from lemma 10.4 and axiom A2, for any $p \in \mathbb{N}$ and any $x, u', v' \in X^{\omega}$ we have

$$d^x(u', v') = 2^{p_0 + p} d(\delta^x_{2^{p+p_0}} u', \delta^x_{2^{p+p_0}} v') =$$

$$= 2^p \, 2^{p_0} d(\delta^x_{2^{p_0}} \delta^x_{2^p} u', \delta^x_{2^{p_0}} \delta^x_{2^p} v') = 2^p d^x(\delta^x_{2^p} u', \delta^x_{2^p} v') \quad .$$

This is just the cone property for d^x. From here we deduce that for any $p \in \mathbb{Z}$ we have $d^x(u', v') = 2^p d^x(\delta^x_{2^p} u', \delta^x_{2^p} v')$. If $2^{p_0} d(x, u) \leq 1$, $2^{p_0} d(x, v) \leq 1$ then write $x = qx'$, $\mid q \mid = p_0$, and use lemma 10.5 to get the existence of $u', v' \in X^{\omega}$ such that $\delta^x_{2^{p_0}} u' = u$, $\delta^x_{2^{p_0}} v' = v$. Therefore, by lemma 10.4, we have

$$d(u, v) = 2^{-p_0} d^x(u', v') = d^x(\delta^x_{2^{-p_0}} u', \delta^x_{2^{-p_0}} v') = d^x(u, v) \quad .$$

The first part of the lemma is proven. For the proof of the second part write again

$$2^p d(\delta^x_{2^p} u, \delta^x_{2^p} v) = 2^p d^x(\delta^x_{2^p} u, \delta^x_{2^p} v) = d^x(u, v) = d(u, v)$$

which finishes the proof. $\square$

The space X^{ω} decomposes into a disjoint union of 2^{p_0} balls which are isometric. There is no connection between the dilation structures on these balls, therefore we shall suppose further that $p_0 = 0$.

Our purpose is to find the general form of a dilation structure on X^{ω}, with $p_0 = 0$.

Definition 10.7 *A function $W : \mathbb{N}^* \times X^{\omega} \to Isom(X^{\omega})$ is smooth if for any $\varepsilon > 0$ there exists $\mu(\varepsilon) > 0$ such that for any $x, x' \in X^{\omega}$ such that $d(x, x') < \mu(\varepsilon)$ and for any $y \in X^{\omega}$ we have*

$$\frac{1}{2^k} \, d(W^x_k(y), W^{x'}_k(y)) \leq \varepsilon \quad ,$$

for an k such that $d(x, x') < 1/2^k$.

Theorem 10.8 *Let (X^ω, d, δ) be a dilation structure on (X^ω, d), where d is the standard distance on X^ω, such that $p_0 = 0$. Then there exists a smooth (according to definition 10.7) function*

$$W : \mathbb{N}^* \times X^\omega \to Isom(X^\omega) \quad , \quad W(n, x) = W_n^x$$

such that for any $q \in X^*$, $\alpha \in X$, $x, y \in X^\omega$ we have

$$\delta_2^{q\alpha x} q\bar{\alpha} y = q\alpha \bar{x}_1 W_{|q|+1}^{q\alpha x}(y) \quad . \tag{10.4.4}$$

Conversely, to any smooth function $W : \mathbb{N}^* \times X^\omega \to Isom(X^\omega)$ is associated a dilation structure (X^ω, d, δ), with $p_0 = 0$, induced by functions δ_2^x, defined by $\delta_2^x x = x$ and otherwise by relation (10.4.4).

Proof. Let (X^ω, d, δ) be a dilation structure on (X^ω, d), such that $p_0 = 0$. Any two different elements of X^ω can be written in the form $q\alpha x$ and $q\bar{\alpha} y$, with $q \in X^*$, $\alpha \in X$, $x, y \in X^\omega$. We also have $d(q\alpha x, q\bar{\alpha} y) = 2^{-|q|}$. From the following computation (using $p_0 = 0$ and axiom A1):

$$2^{-|q|-1} = \frac{1}{2} d(q\alpha x, q\bar{\alpha} y) = d(q\alpha x, \delta_2^{q\alpha x} q\bar{\alpha} y) \quad ,$$

we find that there exists $w_{|q|+1}^{q\alpha x}(y) \in X^\omega$ such that $\delta_2^{q\alpha x} q\bar{\alpha} y = q\alpha w_{|q|+1}^{q\alpha x}(y)$. Further on, we compute:

$$\frac{1}{2} d(q\bar{\alpha} x, q\bar{\alpha} y) = d(\delta_2^{q\alpha x} q\bar{\alpha} x, \delta_2^{q\alpha x} q\bar{\alpha} y) = d(q\alpha w_{|q|+1}^{q\alpha x}(x), q\alpha w_{|q|+1}^{q\alpha x}(y)) \quad .$$

From this equality we find that $1 > \frac{1}{2} d(x, y) = d(w_{|q|+1}^{q\alpha x}(x), w_{|q|+1}^{q\alpha x}(y))$, which means that the first letter of the word $w_{|q|+1}^{q\alpha x}(y)$ does not depend on y, and is equal to the first letter of the word $w_{|q|+1}^{q\alpha x}(x)$. Let us denote this letter by β (which depends only on q, α, x). Therefore we may write:

$$w_{|q|+1}^{q\alpha x}(y) = \beta W_{|q|+1}^{q\alpha x}(y) \quad ,$$

where the properties of the function $y \mapsto W_{|q|+1}^{q\alpha x}(y)$ remain to be determined later.

We go back to the first computation in this proof:

$$2^{-|q|-1} = d(q\alpha x, \delta_2^{q\alpha x} q\bar{\alpha} y) = d(q\alpha x, q\alpha\beta W_{|q|+1}^{q\alpha x}(y)) \quad .$$

This shows that $\bar{\beta}$ is the first letter of the word x. We proved the relation (10.4.4), excepting the fact that the function $y \mapsto W_{|q|+1}^{q\alpha x}(y)$ is an isometry. But this is true. Indeed, for any $u, v \in X^\omega$ we have

$$\frac{1}{2} d(q\bar{\alpha} u, q\bar{\alpha} v) = d(\delta_2^{q\alpha x} q\bar{\alpha} u, \delta_2^{q\alpha x} q\bar{\alpha} v) = d(q\alpha \bar{x}_1 W_{|q|+1}^{q\alpha x}(x), q\alpha \bar{x}_1 W_{|q|+1}^{q\alpha x}(y)) \quad .$$

This proves the isometry property.

The dilations of coefficient 2 induce all dilations (by axiom A2). In order to satisfy the continuity assumptions from axiom A1, the function $W : \mathbb{N}^* \times X^\omega \to Isom(X^\omega)$ has to be smooth in the sense of definition 10.7. Indeed, axiom A1 is equivalent to the fact that $\delta_2^{x'}(y')$ converges uniformly to $\delta_2^x(y)$, as $d(x, x'), d(y, y')$ go to zero. There are two cases to study.

Case 1: $d(x,x') \leq d(x,y)$, $d(y,y') \leq d(x,y)$. It means that $x = q\alpha q'\beta X$, $y = q\bar{\alpha}q"\gamma Y$, $x' = q\alpha q'\bar{\beta}X'$, $y' = q\bar{\alpha}q"\bar{\gamma}Y'$, with $d(x,y) = 1/2^k$, $k = \mid q \mid$.

Suppose that $q' \neq \emptyset$. We compute then:

$$\delta_2^x(y) = q\alpha \bar{q'_1} W_{k+1}^x(q"\gamma Y)$$

$$\delta_2^{x'}(y') = q\alpha \bar{q'_1} W_{k+1}^{x'}(q"\bar{\gamma}Y')$$

All the functions denoted by a capitalized "W" are isometries, therefore we get the estimation:

$$d(\delta_2^x(y), \delta_2^{x'}(y')) = \frac{1}{2^{k+2}}\, d(W_{k+1}^x(q"\gamma Y), W_{k+1}^{x'}(q"\bar{\gamma}Y')) \leq$$

$$\leq \frac{1}{2^{k+2}}\, d(q"\gamma Y, q"\bar{\gamma}Y') + \frac{1}{2^{k+2}}\, d(W_{k+1}^x(q"\gamma Y), W_{k+1}^{x'}(q"\gamma Y)) =$$

$$= \frac{1}{2} d(y,y') + \frac{1}{2^{k+2}}\, d(W_{k+1}^x(q"\gamma Y), W_{k+1}^{x'}(q"\gamma Y)) \quad .$$

We see that if W is smooth in the sense of definition 10.7 then the structure δ satisfies the uniform continuity assumptions for this case. Conversely, if δ satisfies A1 then W has to be smooth.

If $q' = \emptyset$ then a similar computation leads to the same conclusion.

Case 2: $d(x,x') > d(x,y) > d(y,y')$. It means that $x = q\alpha q'\beta X$, $x' = q\bar{\alpha}X'$, $y = q\alpha q'\bar{\beta}q"\bar{\gamma}Y$, $y' = q\alpha q'\bar{\beta}q"\gamma Y'$, with $d(x,x') = 1\ 2^k$, $k = \mid q \mid$.

We compute then: $\delta_2^x(y) = q\alpha q'\beta \bar{X}_1 W_{k+2+|q'|}^x(q"\bar{\gamma}Y)$,

$$\delta_2^{x'}(y') = q\bar{\alpha}\bar{X}'_1 W_{k+1}^{x'}(q'\bar{\beta}q"\gamma Y') \leq \frac{1}{2^k} = d(x,x') \quad .$$

Therefore in his case the continuity is satisfied, without any supplementary constraints on the function W.

The first part of the theorem is proven.

For the proof of the second part of the theorem we start from the function $W : \mathbb{N}^* \times X^\omega \rightarrow Isom(X^\omega)$. It is sufficient to prove for any $x, y, z \in X^\omega$ the equality

$$\frac{1}{2} d(y,z) = d(\delta_2^x y, \delta_2^x z) \quad .$$

Indeed, then we can construct the all dilations from the dilations of coefficient 2 (thus we satisfy A2). All axioms, excepting A1, are satisfied. But A1 is equivalent with the smoothness of the function W, as we proved earlier.

Let us prove now the before mentioned equality. If $y = z$ there is nothing to prove. Suppose that $y \neq z$. The distance d is ultrametric, therefore the proof splits in two cases.

Case 1: $d(x,y) = d(x,z) > d(y,z)$. This is equivalent to $x = q\bar{\alpha}x'$, $y = q\alpha q'\beta y'$, $z = q\alpha q'\bar{\beta}z'$, with $q, q' \in X^*$, $\alpha, \beta \in X$, $x', y', z' \in X^\omega$. We compute:

$$d(\delta_2^x y, \delta_2^x z) = d(\delta_2^{q\bar{\alpha}x'} q\alpha q'\beta y', \delta_2^{q\bar{\alpha}x'} q\alpha q'\bar{\beta}z') =$$

$$= d(q\bar{\alpha}\bar{x'_1} W_{|q|+1}^x(q'\beta y'), q\bar{\alpha}\bar{x'_1} W_{|q|+1}^x(q'\bar{\beta}z')) = 2^{-|q|-1} d(W_{|q|+1}^x(q'\beta y'), W_{|q|+1}^x(q'\bar{\beta}z')) =$$

$$= 2^{-|q|-1} d(q'\beta y', q'\bar{\beta}z')) = \frac{1}{2} d(q\alpha q'\beta y', q\alpha q'\bar{\beta}z') = \frac{1}{2} d(y,z) \quad .$$

Case 2: $d(x,y) = d(y,z) > d(x,z)$. If $x = z$ then we write $x = q\alpha u$, $y = q\bar{\alpha} v$ and we have

$$d(\delta_2^x y, \delta_2^x z) = d(q\alpha \bar{u}_1 W^x_{|q|+1}(v), q\alpha u) = 2^{-|q|+1} = \frac{1}{2} d(y,z) \quad .$$

If $x \neq z$ then we can write $z = q\bar{\alpha} z'$, $y = q\alpha q' \beta y'$, $x = q\alpha q' \bar{\beta} x'$, with $q, q' \in X^*$, $\alpha, \beta \in X$, $x', y', z' \in X^\omega$. We compute:

$$d(\delta_2^x y, \delta_2^x z) = d(\delta_2^{q\alpha q' \bar{\beta} x'} q\alpha q' \beta y', \delta_2^{q\alpha q' \bar{\beta} x'} q\bar{\alpha} z') =$$
$$= d(q\alpha q' \bar{\beta} \bar{x}_1' W^x_{|q|+|q'|+2}(y'), q\alpha\gamma W^x_{|q|+1}(z')) \quad ,$$

with $\gamma \in X$, $\bar{\gamma} = q_1'$ if $q' \neq \emptyset$, otherwise $\gamma = \beta$. In both situations we have $d(\delta_2^x y, \delta_2^x z) = 2^{-|q|-1} = \frac{1}{2} d(y,z)$. The proof is done. $\square$

10.5 Self-similar dilation structures

Let (X^ω, d, δ) be a dilation structure. There are induced dilations structures on $0X^\omega$ and $1X^\omega$.

Definition 10.9 *For any $\alpha \in X$ and $x, y \in X^\omega$ we define $\delta_2^{\alpha,x} y$ by the relation*

$$\delta_2^{\alpha x} \alpha y = \alpha \, \delta_2^{\alpha,x} y \quad .$$

The following proposition has a straightforward proof, therefore we skip it.

Proposition 10.10 *If (X^ω, d, δ) is a dilation structure and $\alpha \in X$ then $(X^\omega, d, \delta^\alpha)$ is a dilation structure.*

If (X^ω, d, δ') and $(X^\omega, d, \delta")$ are dilation structures then (X^ω, d, δ) is a dilation structure, where δ is uniquely defined by $\delta^0 = \delta'$, $\delta^1 = \delta"$.

Definition 10.11 *A dilation structure (X^ω, d, δ) is self-similar if for any $\alpha \in X$ and $x, y \in X^\omega$ we have*

$$\delta_2^{\alpha x} \, \alpha y = \alpha \, \delta_2^x y \quad .$$

Self-similarity is thus related to linearity. Indeed, let us compare self-similarity with the following definition of linearity.

Definition 10.12 *For a given dilation structure (X^ω, d, δ), a continuous transformation $A : X^\omega \to X^\omega$ is linear (with respect to the dilation structure) if for any $x, y \in X^\omega$ we have*

$$A \, \delta_2^x y \; = \; \delta_2^{Ax} A y$$

The previous definition provides a true generalization of linearity for dilation structures. This can be seen by comparison with the characterisation of linear (in fact affine) transformations in vector spaces from the proposition 5.6.

The definition of self-similarity 10.11 is related to linearity in the sense of definition 10.12. To see this, let us consider the functions $\hat{\alpha} : X^\omega \to X^\omega$, $\hat{\alpha} x = \alpha x$, for $\alpha \in X$. With this notations, the definition 10.11 simply states that a dilation structure is self-similar if these two functions, $\hat{0}$ and $\hat{1}$, are linear in the sense of definition 10.12.

The description of self-similar dilation structures on the boundary of the dyadic tree is given in the next theorem.

Theorem 10.13 *Let (X^ω, d, δ) be a self-similar dilation structure and $W : \mathbb{N}^* \times X^\omega \to Isom(X^\omega)$ the function associated to it, according to theorem 10.8. Then there exists a function $W : X^\omega \to Isom(X^\omega)$ such that:*

(a) for any $q \in X^*$ and any $x \in X^\omega$ we have $W^{qx}_{|q|+1} = W^x$,

(b) there exists $C > 0$ such that for any $x, x', y \in X^\omega$ and for any $\lambda > 0$, if $d(x, x') \leq \lambda$ then $d(W^x(y), W^{x'}(y)) \leq C\lambda$.

Proof. We define $W^x = W^x_1$ for any $x \in X^\omega$. We want to prove that this function satisfies (a), (b).

(a) Let $\beta \in X$ and any $x, y \in X^\omega$, $x = q\alpha u$, $y = q\bar{\alpha}v$. By self-similarity we obtain: $\beta q \alpha \bar{u}_1 W^{\beta x}_{|q|+2}(v) = \delta^{\beta x}_2 \beta y = \beta \delta^x_2 y = \beta q \alpha \bar{u}_1 W^x_{|q|+1}(v)$. We proved that

$$W^{\beta x}_{|q|+2}(v) = W^x_{|q|+1}(v)$$

for any $x, v \in X^\omega$ and $\beta \in X$ This implies (a).

(b) This is a consequence of smoothness, in the sense of definition 10.7, of the function $W : \mathbb{N}^* \times X^\omega \to Isom(X^\omega)$. Indeed, (X^ω, d, δ) is a dilation structure, therefore by theorem 10.8 the previous mentioned function is smooth.

By (a) the smoothness condition becomes: for any $\varepsilon > 0$ there is $\mu(\varepsilon) > 0$ such that for any $y \in X^\omega$, any $k \in \mathbb{N}$ and any $x, x' \in X^\omega$, if $d(x, x') \leq 2^k \mu(\varepsilon)$ then

$$d(W^x(y), W^{x'}(y)) \leq 2^k \varepsilon \quad .$$

Define then the modulus of continuity: for any $\varepsilon > 0$ let $\bar{\mu}(\varepsilon)$ be given by

$$\bar{\mu}(\varepsilon) = \sup \left\{ \mu \, : \, \forall x, x', y \in X^\omega \ d(x, x') \leq \mu \Longrightarrow d(W^x(y), W^{x'}(y)) \leq \varepsilon \right\} \quad .$$

We see that the modulus of continuity $\bar{\mu}$ has the property

$$\bar{\mu}(2^k \varepsilon) = 2^k \bar{\mu}(\varepsilon)$$

for any $k \in \mathbb{N}$. Therefore there exists $C > 0$ such that $\bar{\mu}(\varepsilon) = C^{-1}\varepsilon$ for any $\varepsilon = 1/2^p$, $p \in \mathbb{N}$. The point (b) follows immediately. $\square$

11 Sub-riemannian manifolds

Sub-riemannian geometry is the modern incarnation of non-holonomic spaces, discovered in 1926 by the romanian mathematician Gheorghe Vrănceanu [63], [64]. The sub-riemannian geometry is the study of non-holonomic spaces endowed with a Carnot-Carathéodory distance. Such spaces appear in applications to thermodynamics, to the mechanics of non-holonomic systems, in the study of hypo-elliptic operators cf. Hörmander [42], in harmonic analysis on homogeneous cones cf. Folland, Stein [28], and as boundaries of CR-manifolds.

The interest in these spaces comes from several intriguing features which they have: from the metric point of view they are fractals (the Hausdorff dimension with respect to the Carnot-Carathéodory distance is strictly bigger than the topological dimension, cf. Mitchell [51]); the metric tangent space to a point of a regular sub-riemannian manifold is a Carnot group (a simply connected nilpotent Lie group with a positive graduation), also known classicaly as a homogeneous cone; the asymptotic space (in the sense of Gromov-Hausdorff distance) of a finitely generated group with polynomial growth is also a Carnot group, by a famous theorem of Gromov [33] wich leads to an inverse to the Tits alternative; finally, on such spaces we have enough structure to develop a differential calculus resembling to the one proposed by Cheeger [23] and to prove theorems like Pansu' version of Rademacher theorem [53], leading to an ingenious proof of a Margulis rigidity result.

There are several fundamental papers dedicated to the establisment of the sub-riemannian geometry, among them Mitchell [51], Bellaïche [5], a substantial paper of Gromov asking for an intrinsic point of view for sub-riemannian geometry [39], Margulis, Mostow [48], [49], dedicated to Rademacher theorem for sub-riemannian manifolds and to the construction of a tangent bundle of such manifolds, and Vodopyanov [56] (among other papers), concerning the same subject.

There is a reason for the existence of so many papers, written by important mathematicians, on the same subject: the fundamental geometric properties of sub-riemannian manifolds are very difficult to prove. Maybe the most difficult problem is to provide a rigorous construction of the tangent bundle of such a manifold, starting from the properties of the Carnot-Carathéodory distance, and somehow allowing to generalize Pansu' differential calculus.

In several articles devoted to sub-riemannian geometry, these fundamental results were proved using differential geometry tools, which are not intrinsic to sub-riemannian geometry, therefore leading to very long proofs, sometimes with unclear parts, corrected or clarified in other papers dedicated to the same subject.

The fertile ideas of Gromov, Bellaïche and other founders of the field of analysis in sub-riemannian spaces are now developed into a hot research area. For the study of sub-riemannian geometry under weaker than usual regularity hypothesis see for example the string of papers by Vodopyanov, among them [56], [57]. In these papers Vodopyanov constructs a tangent bundle structure for a sub-riemannian manifold, under weak regularity hypothesis, by using notions as horizontal convergence.

Based on the notion of dilation structure [11], I tried to give a an intrinsic treatment to sub-riemannian geometry in the paper [15], after a series of articles [10], [17], [18] dedicated to the sub-riemannian geometry of Lie groups endowed with left invariant distributions.

In this article we show that normal frames are the central objects in the establishment of fundamental properties in sub-riemannian geometry, in the following precise sense. We prove that for regular sub-riemannian manifolds, the existence of normal frames (definition 11.7) implies that induced dilation structures exist (theorems 11.10, 11.11). The existence of normal frames has been proved by Bellaïche [5], starting with theorem 4.15 and ending in the first half of section 7.3 (page 62). From these facts all classical results concerning the structure of the tangent space to a point of a regular sub-riemannian manifold can be deduced as straightforward consequences of the structure theorems 13.6, 13.1, 13.5, 13.8 from the formalism of dilation structures.

In conclusion, our purpose is twofold: (a) we try to show that basic results in sub-riemannian geometry are particular cases of the abstract theory of dilation structures, and (b) we try to minimize the contribution of classical differential calculus in the proof of these basic results, by showing that in fact the differential calculus on the sub-riemannian manifold is needed only for proving that normal frames exist and after this stage an intrinsic way of reasoning is possible.

If we take the point of view of Gromov, that the only intrinsic object on a sub-riemannian manifold is the Carnot-Carathéodory distance, the underlying differential structure of the manifold is clearly not intrinsic. Nevertheless in all proofs that I know this differential structure is heavily used. Here we try to prove that in fact it is sufficient to take as intrinsic objects of sub-riemannian geometry the Carnot-Carathéodory distance and dilation structures compatible with it.

The closest results along these lines are maybe the ones of Vodopyanov. There is a clear correspondence between his way of defining the tangent bundle of a sub-riemannian manifold and the way of dilation structures. In both cases the tangent space to a point is defined only locally, as a neighbourhood of the point, in the manifold, endowed with a local group operation. Vodopyanov proves the existence of the (locally defined) operation under very weak regularity assumptions on the sub-riemannian manifold. The main tool of his proofs is nevertheless the differential structure of the underlying manifold. In distinction, we prove in [11], in an abstract setting, that the very existence of a dilation structure induces a locally defined operation. Here we show that the differential structure of the underlying manifold is important only in order to prove that dilation structures can indeed be constructed from normal frames.

Let M be a connected n dimensional real manifold. A distribution is a smooth subbundle D of M. To any point $x \in M$ there is associated the vector space $D_x \subset T_x M$. The dimension of the distribution D at point $x \in M$ is denoted by

$$m(x) = \, dim \, D_x$$

The distribution is smooth, therefore the function $x \in M \mapsto m(x)$ is locally constant. We suppose further that the dimension of the distribution is globally constant and we denote it by m (thus $m = m(x)$ for any $x \in M$). Clearly $m \leq n$; we are interested in the case $m < n$.

A horizontal curve $c : [a, b] \to M$ is a curve which is almost everywhere derivable and for almost any $t \in [a, b]$ we have $\dot{c}(t) \in D_{c(t)}$. The class of horizontal curves will be denoted by $Hor(M, D)$.

Further we shall use the following notion of non-integrability of the distribution D.

Definition 11.1 *The distribution D is completely non-integrable if M is locally connected by horizontal curves curves $c \in Hor(M, D)$.*

A sufficient condition for the distribution D to be completely non-integrable is given by Chow condition (C) [24].

Theorem 11.2 *(Chow) Let D be a distribution of dimension m in the manifold M. Suppose there is a positive integer number k (called the rank of the distribution D) such that for any $x \in X$ there is a topological open ball $U(x) \subset M$ with $x \in U(x)$ such that there are smooth vector fields $X_1, ..., X_m$ in $U(x)$ with the property:*

(C) the vector fields $X_1, ..., X_m$ span D_x and these vector fields together with their iterated brackets of order at most k span the tangent space $T_y M$ at every point $y \in U(x)$.

Then the distribution D is completely non-integrable in the sense of definition 11.1.

Definition 11.3 *A sub-riemannian (SR) manifold is a triple (M, D, g), where M is a connected manifold, D is a completely non-integrable distribution on M, and g is a metric (Euclidean inner-product) on the distribution (or horizontal bundle) D.*

11.1 The Carnot-Carathéodory distance

Given a distribution D which satisfies the hypothesis of Chow theorem 11.2, let us consider a point $x \in M$, its neighbourhood $U(x)$, and the vector fields $X_1, ..., X_m$ satisfying the condition (C).

One can define on $U(x)$ a filtration of bundles as follows. Define first the class of horizontal vector fields on U:

$$\mathcal{X}^1(U(x), D) \ = \ \{X \in \mathcal{X}^\infty(U) \,:\, \forall y \in U(x) \ , \ X(y) \in D_y\}$$

Next, define inductively for all positive integers j:

$$\mathcal{X}^{j+1}(U(x), D) \ = \ \mathcal{X}^j(U(x), D) \, + \, [\mathcal{X}^1(U(x), D), \mathcal{X}^j(U(x), D)]$$

Here $[\cdot, \cdot]$ denotes the bracket of vector fields. We obtain therefore a filtration $\mathcal{X}^j(U(x), D) \subset \mathcal{X}^{j+1}(U(x), D)$. Evaluate now this filtration at $y \in U(x)$:

$$V^j(y, U(x), D) \ = \ \left\{ X(y) \,:\, X \in \mathcal{X}^j(U(x), D) \right\}$$

According to Chow theorem there is a positive integer k such that for all $y \in U(x)$ we have

$$D_y = V^1(y, U(x), D) \subset V^2(y, U(x), D) \subset ... \subset V^k(y, U(x), D) = T_y M$$

Consequently, to the sub-riemannian manifold is associated the string of numbers:

$$\nu_1(y) = \dim V^1(y, U(x), D) < \nu_2(y) = \dim V^2(y, U(x), D) < ... < n = \dim M$$

Generally k, $\nu_j(y)$ may vary from a point to another.

The number k is called the step of the distribution at y.

Definition 11.4 *The distribution D is regular if $\nu_j(y)$ are constant on the manifold M. The sub-riemannian manifold $M, D, g)$ is regular if D is regular and for any $x \in M$ there is a topological ball $U(x) \subset M$ with $x \in U(M)$ and an orthonormal (with respect to the metric g) family of smooth vector fields $\{X_1, ..., X_m\}$ in $U(x)$ which satisfy the condition (C).*

The lenght of a horizontal curve is

$$l(c) \ = \ \int_a^b \left(g_{c(t)}(\dot{c}(t), \dot{c}(t)) \right)^{\frac{1}{2}} \ \mathrm{d}t$$

The length depends on the metric g.

Definition 11.5 *The Carnot-Carathéodory distance (or CC distance) associated to the sub-riemannian manifold is the distance induced by the length l of horizontal curves:*

$$d(x, y) \ = \ \inf \{ l(c) \,:\, c \in Hor(M, D) \ , \ c(a) = x \ , \ c(b) = y \}$$

The Chow theorem ensures the existence of a horizontal path linking any two sufficiently closed points, therefore the CC distance is locally finite. The distance depends only on the distribution D and metric g, and not on the choice of vector fields $X_1, ..., X_m$ satisfying the condition (C). The space (M, d) is locally compact and complete, and the topology induced by the distance d is the same as the topology of the manifold M. (These important details may be recovered from reading carefully the constructive proofs of Chow theorem given by Bellaïche [5] or Gromov [39].)

11.2 Normal frames

In the following we stay in a small open neighbourhood of an arbitrary, but fixed point $x_0 \in M$. All results are local in nature (that is they hold for some small open neighbourhood of an arbitrary, but fixed point of the manifold M). That is why we shall no longer mention the dependence of various objects on x_0, on the neighbourhood $U(x_0)$, or the distribution D.

We shall work further only with regular sub-riemannian manifolds, if not otherwise stated. The topological dimension of M is denoted by n, the step of the regular sub-riemannian manifold (M, D, g) is denoted by k, the dimension of the distribution is m, and there are numbers ν_j, $j = 1, ..., k$ such that for any $x \in M$ we have $dim\, V^j(x) = \nu_j$. The Carnot-Carathéodory distance is denoted by d.

Definition 11.6 *An adapted frame $\{X_1, ..., X_n\}$ is a collection of smooth vector fields which is obtained by the construction described below.*

We start with a collection $X_1, ..., X_m$ of vector fields which satisfy the condition (C). In particular for any point x the vectors $X_1(x), ..., X_m(x)$ form a basis for D_x. We further associate to any word $a_1....a_q$ with letters in the alphabet $1, ..., m$ the multi-bracket $[X_{a_1}, [..., X_{a_q}]...]$.

One can add, in the lexicographic order, $n - m$ elements to the set $\{X_1, ..., X_m\}$ until we get a collection $\{X_1, ..., X_n\}$ such that: for any $j = 1, ..., k$ and for any point x the set $\left\{X_1(x), ..., X_{\nu_j}(x)\right\}$ is a basis for $V^j(x)$.

Let $\{X_1, ..., X_n\}$ be an adapted frame. For any $j = 1, ..., n$ the degree $deg\, X_j$ of the vector field X_j is defined as the only positive integer p such that for any point x we have

$$X_j(x) \in V^p_x \setminus V^{p-1}(x)$$

Further we define normal frames. The name has been used by Vodopyanov [56], but for a slightly different object. The existence of normal frames in the sense of the following definition is the hardest technical problem in the classical establishment of sub-riemannian geometry. For the informed reader the referee pointed out that condition (a) Definition 11.7 is a part of the conclusion of Gromov approximation theorem, namely when one point coincides with the center of nilpotentization; also condition (b) is equivalent with a statement of Gromov concerning the convergence of rescaled vector fields to their nilpotentization (an informed reader must at least follow in all details the papers Bellaïche [5] and Gromov [39], where differential calculus in the classical sense is heavily used). Therefore the conditions of Definition 11.7 concentrate that part of the foundations of sub-riemannian geometry which makes use of classical differential calculus.

The key details in the Definition below are uniform convergence assumptions. This is in line with Gromov suggestions in the last section of Bellaïche [5].

Definition 11.7 *An adapted frame $\{X_1, ..., X_n\}$ is a normal frame if the following two conditions are satisfied:*

(a) we have the limit

$$\lim_{\varepsilon \to 0_+} \frac{1}{\varepsilon} d \left(\exp\left(\sum_1^n \varepsilon^{deg\, X_i} a_i X_i \right)(y), y \right) \; = \; A(y,a) \in (0, +\infty)$$

uniformly with respect to y in compact sets and $a = (a_1, ..., a_n) \in W$, with $W \subset \mathbb{R}^n$ compact neighbourhood of $0 \in \mathbb{R}^n$,

(b) for any compact set $K \subset M$ with diameter (with respect to the distance d) sufficiently small, and for any $i = 1, ..., n$ there are functions

$$P_i(\cdot, \cdot, \cdot) : U_K \times U_K \times K \to \mathbb{R}$$

with $U_K \subset \mathbb{R}^n$ a sufficiently small compact neighbourhood of $0 \in \mathbb{R}^n$ such that for any $x \in K$ and any $a, b \in U_K$ we have

$$\exp\left(\sum_1^n a_i X_i \right)(x) \; = \; \exp\left(\sum_1^n P_i(a,b,y) X_i \right) \circ \exp\left(\sum_1^n b_i X_i \right)(x)$$

and such that the following limit exists

$$\lim_{\varepsilon \to 0_+} \varepsilon^{-deg\, X_i} P_i(\varepsilon^{deg\, X_j} a_j, \varepsilon^{deg\, X_k} b_k, x) \in \mathbb{R}$$

and it is uniform with respect to $x \in K$ and $a, b \in U_K$.

The existence of normal frames is proven in Bellaïche [5], starting with theorem 4.15 and ending in the first half of section 7.3 (page 62).

In order to understand normal frames let us look to the case of a Lie group G endowed with a left invariant distribution. The distribution is completely non-integrable if it is generated by the left translation of a vector subspace D of the algebra $\mathfrak{g} = T_e G$ which bracket generates the whole algebra $\mathfrak{g}$. Take $\{X_1, ..., X_m\}$ a collection of $m = dim\, D$ left invariant independent vector fields and define with their help an adapted frame, as explained in definition 11.6. Then the adapted frame $\{X_1, ..., X_n\}$ is in fact normal.

11.3 Sub-riemannian dilation structures

To any normal frame of a regular sub-riemannian manifold we associate a dilation structure. (Technically this is a dilation structure defined only locally, as in the case of riemannian manifolds.)

Definition 11.8 *To any normal frame $\{X_1, ..., X_n\}$ of a regular sub-riemannian manifold (M, D, g) we associate the dilation structure (M, d, δ) defined by: d is the Carnot-Carathéodory distance, and for any point $x \in M$ and any $\varepsilon \in (0, +\infty)$ (sufficiently small if necessary), the dilation δ^x_ε is given by:*

$$\delta^x_\varepsilon \left(\exp\left(\sum_{i=1}^n a_i X_i \right)(x) \right) \; = \; \exp\left(\sum_{i=1}^n a_i \varepsilon^{deg X_i} X_i \right)(x)$$

We shall prove that (M, d, δ) is indeed a dilation structure. This allows us to get the main results concerning the infinitesimal geometry of a regular sub-riemannian manifold, as particular cases of theorems 13.6, 13.1, 13.5 and 13.8.

We only have to prove axioms A3 and A4 of dilation structures. We do this in the next two theorems. Before this let us decribe what we mean by "sufficiently closed".

Definition 11.9 *Further we shall say that a property $\mathcal{P}(x_1, x_2, x_3, ...)$ holds for $x_1, x_2, x_3, ...$ sufficiently closed if for any compact, non empty set $K \subset X$, there is a positive constant $C(K) > 0$ such that $\mathcal{P}(x_1, x_2, x_3, ...)$ is true for any $x_1, x_2, x_3, ... \in K$ with $d(x_i, x_j) \leq C(K)$.*

In the following we prove a result similar to Gromov local approximation theorem [39], p. 135, or to Bellaïche theorem 7.32 [5]. Note however that here we take as a hypothesis the existence of a normal frame.

Theorem 11.10 *Consider $X_1, ..., X_n$ a normal frame and the associated dilations provided by definition 11.8. Then axiom A3 of dilation structures is satisfied, that is the limit*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d\left(\delta^x_\varepsilon u, \delta^x_\varepsilon v\right) \ = \ d^x(u, v)$$

exists and it uniform with respect to x,u,v sufficiently closed.

Proof. Let $x, u, v \in M$ be sufficiently closed. We write

$$u \ = \ \exp\left(\sum_1^n u_i X_i\right)(x) \qquad , \quad v \ = \ \exp\left(\sum_1^n v_i X_i\right)(x)$$

we compute, using definition 11.8:

$$\frac{1}{\varepsilon} d\left(\delta^x_\varepsilon u, \delta^x_\varepsilon v\right) \ = \ \frac{1}{\varepsilon} d\left(\delta^x_\varepsilon \exp\left(\sum_1^n u_i X_i\right)(x), \delta^x_\varepsilon \exp\left(\sum_1^n v_i X_i\right)(x)\right) \ =$$

$$= \ \frac{1}{\varepsilon} d\left(\exp\left(\sum_1^n \varepsilon^{deg\, X_i} u_i X_i\right)(x), \exp\left(\sum_1^n \varepsilon^{deg\, X_i} v_i X_i\right)(x)\right) \ = \ A_\varepsilon$$

Let us denote by $u_\varepsilon = \exp\left(\sum_1^n \varepsilon^{deg\, X_i} u_i X_i\right)(x)$. Use the first part of the property (b), definition 11.7 of a normal system, to write further:

$$A_\varepsilon \ = \ \frac{1}{\varepsilon} d\left(u_\varepsilon, \exp\left(\sum_1^n P_i(\varepsilon^{deg\, X_j} v_j, \varepsilon^{deg\, X_k} u_k, x) X_i\right)(u_\varepsilon)\right) \ =$$

$$= \ \frac{1}{\varepsilon} d\left(u_\varepsilon, \exp\left(\sum_1^n \varepsilon^{deg\, X_i} \left(\varepsilon^{-deg\, X_i} P_i(\varepsilon^{deg\, X_j} v_j, \varepsilon^{deg\, X_k} u_k, x)\right) X_i\right)(u_\varepsilon)\right)$$

We make a final notation: for any $i = 1, ..., n$

$$a_i^\varepsilon \ = \ \varepsilon^{-deg\, X_i} P_i(\varepsilon^{deg\, X_j} v_j, \varepsilon^{deg\, X_k} u_k, x)$$

thus we have:

$$\frac{1}{\varepsilon}\, d\left(\delta^x_\varepsilon u, \delta^x_\varepsilon v\right) \; = \; \frac{1}{\varepsilon}\, d\left(u_\varepsilon, \exp\left(\sum_1^n \varepsilon^{deg\, X_i} a_i^\varepsilon X_i\right)(u_\varepsilon)\right)$$

By the second part of property (b), definition 11.7, the vector $a^\varepsilon \in \mathbb{R}^n$ converges to a finite value $a^0 \in \mathbb{R}^n$, as $\varepsilon \to 0$, uniformly with respect to x, u, v in compact set. In the same time u_ε converges to x, as $\varepsilon \to 0$. The proof ends by using property (a), definition 11.7. Indeed, we shall use the key assumption of uniform convergence. With the notations from definition 11.7, for fixed $\eta > 0$ the term

$$B(\eta, \varepsilon) = \frac{1}{\varepsilon}\, d\left(u_\eta, \exp\left(\sum_1^n \varepsilon^{deg\, X_i} a_i^\eta X_i\right)(u_\eta)\right)$$

converges to a real number $A(u_\eta, a_\eta)$ as $\varepsilon \to 0$, uniformly with respect to u_η and a_η. Since u_η converges to x and a_η converges to a^0 as $\eta \to 0$, by the uniform convergence assumption in (a), definition 11.7 we get that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\, d\left(\delta^x_\varepsilon u, \delta^x_\varepsilon v\right) \; = \; \lim_{\eta \to 0} A(u_\eta, a_\eta) \; = \; A(x, a^0)$$

The proof is done. □

In the next Theorem we prove that axiom A4 of dilation structures is satisfied. The referee informed us that Theorem 11.11 also follows from results of Vodopyanov and Karmanova [59], quoted in [57] p. 267; a complete version of this result will apear in a work by Karmanova and Vodopyanov "Geometry of Carnot-Carathéodory spaces, differentiability and coarea formula" in the book "Analysis and Mathematical Physics", Birchhäuser 2008.

Theorem 11.11 *Consider $X_1, ..., X_n$ a normal frame and the associated dilations provided by definition 11.8. Then axiom A4 of dilation structures is satisfied: as ε tends to 0 the quantity*

$$\Delta^x_\varepsilon(u, v) \; = \; \delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \circ \delta^x_\varepsilon(v)$$

converges, uniformly with respect to x, u, v sufficiently closed.

Proof. We shall use the notations from definition 11.6, 11.7, 11.8.

Let $x, u, v \in M$ be sufficiently closed. We write

$$u \; = \; \exp\left(\sum_1^n u_i X_i\right)(x) \qquad , \quad v \; = \; \exp\left(\sum_1^n v_i X_i\right)(x)$$

We compute now $\Delta^x_\varepsilon(u, v)$:

$$\Delta^x_\varepsilon(u, v) \; = \; \delta^{\exp\left(\sum_1^n \varepsilon^{deg\, X_i} u_i X_i\right)(x)}_{\varepsilon^{-1}} \exp\left(\sum_1^n \varepsilon^{deg\, X_i} v_i X_i\right)(x)$$

Let us denote by $u_\varepsilon = \delta^x_\varepsilon u$. Thus we have

$$\Delta^x_\varepsilon(u, v) \; = \; \delta^{u_\varepsilon}_{\varepsilon^{-1}} \exp\left(\sum_1^n \varepsilon^{deg\, X_i} v_i X_i\right)(x)$$

We use the first part of the property (b), definition 11.7, in order to write

$$\exp\left(\sum_{1}^{n} \varepsilon^{deg\, X_i} v_i X_i\right)(x) \ = \ \exp\left(\sum_{1}^{n} P_i(\varepsilon^{deg\, X_j} v_j, \varepsilon^{deg\, X_k} u_k, x) X_i\right)(u_\varepsilon)$$

We finish the computation:

$$\Delta_\varepsilon^x(u,v) \ = \ \exp\left(\sum_{1}^{n} \varepsilon^{-\, deg\, X_i}\, P_i(\varepsilon^{deg\, X_j} v_j, \varepsilon^{deg\, X_k} u_k, x) X_i\right)(u_\varepsilon)$$

As ε goes to 0 the point u_ε converges to x uniformly with respect to x, u sufficiently closed (as a corollary of the previous theorem, for example). The proof therefore ends by invoking the second part of the property (b), definition 11.7. $\square$

With the help of a normal frame we can prove the existence of strong dilation structures on regular sub-riemannian manifolds. The following is a consequence of theorems 6.3, 6.4 [14].

Theorem 11.12 *Let (M, D, g) be a regular sub-riemannian manifold, $U \subset M$ an open set which admits a normal frame. Define for any $x \in U$ and $\varepsilon > 0$ (sufficiently small if necessary), the dilation δ_ε^x given by:*

$$\delta_\varepsilon^x\left(\exp\left(\sum_{i=1}^{n} a_i X_i\right)(x)\right) \ = \ \exp\left(\sum_{i=1}^{n} a_i \varepsilon^{deg X_i} X_i\right)(x)$$

Then (U, d, δ) is a strong dilation structure.

12 The Kirchheim-Magnani counterexample to metric differentiability

In Kirchheim-Magnani [47] the authors construct a left invariant distance ρ on the Heisenberg group such that the identity map id is 1-Lipschitz but it is not metrically differentiable anywhere.

We shall give an interpretation of the Kirchheim-Magnani counterexample to metric differentiability. In fact we show that they construct something which fails shortly from being a dilation structure.

We shall construct a structure $(H(1), \rho, \bar{\delta})$ on $H(1)$ which satisfies all axioms of a dilation structure, excepting A3 and A4. We prove that for $(H(1), \rho, \bar{\delta})$ the axiom A4 implies A3. Finally we prove that A4 for $(H(1), \rho, \bar{\delta})$ is equivalent with id metrically differentiable from $(H(1), d)$ to $(H(1), \rho)$, where d is a left invariant CC distance.

For the elements of the Heisenberg group $H(1) = \mathbb{R}^2 \times \mathbb{R}$ we use the notation $\tilde{x} = (x, \bar{x})$, with $\tilde{x} \in H(1), x \in \mathbb{R}^2, \bar{x} \in \mathbb{R}$. In this subsection we shall use the following operation on $H(1)$:

$$\tilde{x}\tilde{y} = (x, \bar{x})(y, \bar{y}) = (x + y, \bar{x} + \bar{y} + 2\omega(x, y)),$$

where ω is the canonical symplectic form on $\mathbb{R}^2$. On $H(1)$ we consider the left invariant distance d uniquely determined by the formula:

$$d((0,0),(x,\bar{x})) = \max\left\{\|x\|, \sqrt{|\ \bar{x}\ |}\right\}.$$

The construction proposed by Kirchheim and Magnani is described further. Take an invertible, non decreasing function $g : [0, +\infty) \to [0, +\infty)$, continuous at 0, such that $g(0) = 0$.

For a function g which is well chosen, the function $\rho : H(1) \to [0, +\infty)$,

$$\rho(\tilde{x}) = \max\left\{\|x\|, g(|\ \bar{x}\ |)\right\}$$

induces a left invariant invariant distance on $H(1)$ (we use the same symbol)

$$\rho(\tilde{x}, \tilde{y}) = \rho(\tilde{x}^{-1}\tilde{y}).$$

In order to check this it is sufficient to prove that for any $\tilde{x}, \tilde{y} \in H(1)$ we have

$$\rho(\tilde{x}\tilde{y}) \leq \rho(\tilde{x}) + \rho(\tilde{y}),$$

and that $\rho(\tilde{x}) = 0$ if and only if $\tilde{x} = (0, 0)$. The following result is theorem 2.1 [47].

Theorem 12.1 *(Kirchheim-Magnani) If the function g has the expression*

$$g^{-1}(t) = k(t) + t^2$$

for any $t > 0$, where $k : [0, +\infty) \to [0, +\infty)$ is a convex function, strictly increasing, continuous at 0, and such that $k(0) = 0$, then the function ρ induces a left invariant distance (denoted also by ρ). Moreover, the identity function id is 1-Lipschitz from $(H(1), d)$ to $(H(1), \rho)$.

Further we shall work with a function g satisfying the hypothesis of theorem 12.1, and with the associated function ρ described in the previous subsection.

Definition 12.2 *Define for any $\varepsilon > 0$, the function*

$$\bar{\delta}_\varepsilon(x, \bar{x}) = \left(\varepsilon x, sgn(\bar{x}) g^{-1}\left(\varepsilon g(\mid \bar{x} \mid)\right)\right)$$

for any $\tilde{x} = (x, \bar{x}) \in H(1)$.

We define the following field of dilations $\bar{\delta}$ by: for any $\varepsilon > 0$ and $\tilde{x}, \tilde{y} \in H(1)$ let

$$\bar{\delta}^{\tilde{x}} \tilde{y} = \tilde{x} \bar{\delta}\left(\tilde{x}^{-1} \tilde{y}\right) \quad .$$

For any $\varepsilon > 0$ and $\tilde{x}, \tilde{y} \in H(1)$ we define

$$\bar{\beta}_\varepsilon(\tilde{x}, \tilde{y}) = \bar{\delta}_{\varepsilon^{-1}}\left(\bar{\delta}_\varepsilon(\tilde{x}) \bar{\delta}_\varepsilon(\tilde{y})\right) .$$

We want to know when $(H(1), \rho, \bar{\delta})$ is a dilation structure.

Proposition 12.3 *The structure $(H(1), \rho, \bar{\delta})$ satisfies the axioms A0, A1, A2. Moreover, A4 implies A3.*

Proof. It is easy to check that for any $\varepsilon, \mu \in (0, +\infty)$ we have

$$\bar{\delta}_\varepsilon \bar{\delta}_\mu = \bar{\delta}_{\varepsilon\mu}$$

and that $id = \delta_1$.

Moreover, from g non decreasing and continuous at 0 we deduce that

$$\lim_{\varepsilon \to 0} \bar{\delta}_\varepsilon \tilde{x} = (0, 0),$$

uniformly with respect to $\tilde{x}$ in compact sets.

Another computation shows that

$$\rho(\bar{\delta}_\varepsilon \tilde{x}) = \varepsilon \rho(\tilde{x})$$

for any $\tilde{x} \in H(1)$ and $\varepsilon > 0$. Otherwise stated, the function ρ is homogeneous with respect to $\bar{\delta}$.

All that is left to prove is that A4 implies A3. Remark that $\bar{\delta}$ is left invariant (in the sense of transport by left translations in $H(1)$) and the distance ρ is also left invariant. Then axiom A4 takes the form: there exists the limit

$$\lim_{\varepsilon \to 0} \bar{\beta}_\varepsilon(\tilde{x}, \tilde{y}) = \bar{\beta}(\tilde{x}, \tilde{y}) \in H(1) \tag{12.0.1}$$

uniform with respect to $\tilde{x}, \tilde{y} \in K$, K compact set.

From the homogeneity of the function ρ with respect to $\bar{\delta}$ we deduce that for any $\tilde{x}, \tilde{y} \in H(1)$ we have:

$$\frac{1}{\varepsilon} \rho\left(\bar{\delta}_\varepsilon(\tilde{x}), \bar{\delta}_\varepsilon(\tilde{y})\right) = \rho(\bar{\beta}_\varepsilon(\tilde{x}^{-1}, \tilde{y})).$$

From the left invariance of $\bar{\delta}$ and ρ it follows that A4 implies A3. $\square$

Theorem 12.4 *If the triple $(H(1), \rho, \bar{\delta})$ is a dilation structure then id is metrically differentiable from $(H(1), d)$ to $(H(1), \rho)$.*

Proof. We know that the triple $(H(1), \rho, \bar{\delta})$ is a dilation structure if and only if (12.0.1) is true. Taking (12.0.1) as hypothesis we deduce that the identity function is derivable from $(H(1), d, \delta)$ to $(H(1), \rho, \bar{\delta})$. Indeed, computation shows that id derivable is equivalent to the existence of the limit

$$\lim_{\varepsilon \to 0} \bar{\delta}_{\varepsilon^{-1}} \delta_\varepsilon \tilde{u} = (u, sgn(\bar{u}) g^{-1} \left(\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} g(\varepsilon^2 \mid \bar{u} \mid) \right))$$

uniform with respect to $\tilde{u}$ in compact set. Therefore the function id is derivable everywhere if and only if the uniform limit, with respect to $\bar{u}$ in compact set:

$$A(\bar{u}) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} g(\varepsilon^2 \mid \bar{u} \mid) \tag{12.0.2}$$

exists. We want to show that (12.0.1) implies the existence of this limit.

For this we shall use an equivalent (isomorphic) description of $(H(1), \rho, \bar{\delta})$. Consider the function $F : H(1) \to H(1)$, defined by

$$F(x, \bar{x}) = (x, sgn(\bar{x}) g(\mid \bar{x} \mid)).$$

The function F is invertible because g is invertible. For any $\varepsilon > 0$ let $\hat{\delta}_\varepsilon$ be the usual dilations:

$$\hat{\delta}_\varepsilon(x, \bar{x}) = (\varepsilon x, \varepsilon \bar{x}).$$

It is then straightforward that

$$\bar{\delta}_\varepsilon = F^{-1} \hat{\delta}_\varepsilon F,$$

for any $\varepsilon > 0$.

The function F can be made into a group isomorphism by re-defining the group operation on $H(1)$

$$\tilde{x} \cdot \tilde{y} = F((x, h(\bar{x}))(y, h(\bar{y})),$$

where h is the function

$$h(t) = sgn(t)(t^2 + k(\mid t \mid)).$$

Let μ be the transported left invariant distance on $H(1)$, defined by

$$\mu(F(\tilde{x}), F(\tilde{y})) = \rho(\tilde{x}, \tilde{y}).$$

Remark that μ has the simple expression

$$\mu((0,0), (x, \bar{x})) = \max \{\mid x \mid, \mid \bar{x} \mid\}.$$

Exactly as before we can construct the structure $\hat{\delta}$ by

$$\hat{\delta}^{\tilde{x}}_\varepsilon \tilde{y} = \tilde{x} \cdot \hat{\delta}_\varepsilon \left(\tilde{x}^{-1} \cdot \tilde{y} \right).$$

We get a dilation structure $(H(1), \mu, \hat{\delta})$ isomorphic with $(H(1), \rho, \bar{\delta})$.

The identity function id is derivable from $(H(1), d, \delta)$ to $(H(1), \rho, \bar{\delta})$ if and only if the function F is derivable from $(H(1), d, \delta)$ to $(H(1), \mu, \hat{\delta})$.

The axiom A4 for the dilation structure $(H(1), \mu, \hat{\delta})$ implies that for any $\tilde{x}, \tilde{y} \in H(1)$ the limit exists

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} g \left(\mid \varepsilon^2 \left(\frac{1}{2} \omega(x, y) + \mid \bar{x} \mid \bar{x} + \mid \bar{x} \mid \bar{x} \right) + sgn(\bar{x}) k(\varepsilon \mid \bar{x} \mid) + sgn(\bar{y}) k(\varepsilon \mid \bar{y} \mid) \mid \right),$$

uniform with respect to $\tilde{y}$ in compact set. Take in the previous limit $\bar{x} = \bar{y} = 0$ and denote $\bar{u} = \frac{1}{2}\omega(x, y)$. We get (12.0.2), therefore we proved that id is derivable from $(H(1), d, \delta)$ to $(H(1), \rho, \bar{\delta})$.

Finally, the derivability of id implies the metric differentiability. Indeed, we use (12.0.2) to compute ν, the metric differential of id. We obtain that

$$\nu_{\tilde{x}} = \mu((x, A(\bar{x}))) = \max\{| \, x \, |, A(\bar{u})\} \quad .$$

The proof is done. $\square$

In the counterexample of Kirchheim and Magnani the identity function id is not metric differentiable, therefore the corresponding triple $(H(1), \rho, \bar{\delta})$ is not a dilation structure.

13 Tangent bundle of a dilation structure

13.1 Metric profiles associated with dilation structures

In axiom A3 we take limits. In this subsection we shall look at dilation structures from the metric point of view, by using Gromov-Hausdorff distance and metric profiles.

We state the interpretation of the axiom A3 as a theorem. But before a definition: we denote by (δ, ε) the distance on

$$\bar{B}_{d^x}(x,1) \ = \ \{ y \in X \, : \, d^x(x,y) \leq 1 \}$$

given by

$$(\delta, \varepsilon)(u,v) \ = \ \frac{1}{\varepsilon} d(\delta^x_\varepsilon u, \delta^x_\varepsilon v) \ .$$

Theorem 13.1 *Let (X, d, δ) be a dilation structure. The following are consequences of axioms A0, ... , A3 only:*

(a) for all $u, v \in X$ such that $d(x,u) \leq 1$ and $d(x,v) \leq 1$ and all $\mu \in (0,A)$ we have:

$$d^x(u,v) \ = \ \frac{1}{\mu} d^x(\delta^x_\mu u, \delta^x_\mu v) \ .$$

We shall say that d^x has the cone property with respect to dilations.

(b) The curve $\varepsilon > 0 \mapsto \mathbb{P}^x(\varepsilon) \ = \ [\bar{B}_{d^x}(x,1), (\delta, \varepsilon), x]$ is a metric profile.

Proof. (a) Indeed, for $\varepsilon, \mu \in (0,1)$ we have:

$$\mid \frac{1}{\varepsilon\mu} d(\delta^x_\varepsilon \delta^x_\mu u, \delta^x_\varepsilon \delta^x_\mu v) \ - \ d^x(u,v) \mid \leq \mid \frac{1}{\varepsilon\mu} d(\delta^x_{\varepsilon\mu} u, \delta^x_\varepsilon \delta^x_\mu u) - \frac{1}{\varepsilon\mu} d(\delta^x_{\varepsilon\mu} v, \delta^x_\varepsilon \delta^x_\mu v) \mid +$$

$$+ \ \mid \frac{1}{\varepsilon\mu} d(\delta^x_{\varepsilon\mu} u, \delta^x_{\varepsilon\mu} v) \ - \ d^x(u,v) \mid \ .$$

Use now the axioms A2 and A3 and pass to the limit with $\varepsilon \to 0$. This gives the desired equality.

(b)We have to prove that $\mathbb{P}^x$ is a metric profile. For this we have to compare two pointed metric spaces:

$$\left((\delta^x, \varepsilon\mu), \bar{B}_{d^x}(x,1), x \right) \quad \text{and} \quad \left(\frac{1}{\mu}(\delta^x, \varepsilon), \bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,1), x \right) \ .$$

Let $u \in X$ such that

$$\frac{1}{\mu}(\delta^x, \varepsilon)(x,u) \leq 1 \ .$$

This means that:

$$\frac{1}{\varepsilon} d(\delta^x_\varepsilon x, \delta^x_\varepsilon u) \ \leq \ \mu \ .$$

Use further axioms A1, A2 and the cone property proved before:

$$\frac{1}{\varepsilon} d^x(\delta^x_\varepsilon x, \delta^x_\varepsilon u) \ \leq \ (\mathcal{O}(\varepsilon) + 1)\mu$$

therefore

$$d^x(x,u) \ \leq \ (\mathcal{O}(\varepsilon)+1)\mu \ .$$

It follows that for any $u \in \bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,1)$ we can choose $w(u) \in \bar{B}_{d^x}(x,1)$ such that

$$\frac{1}{\mu} d^x(u, \delta^x_\mu w(u)) \ = \ \mathcal{O}(\varepsilon) \ .$$

We want to prove that

$$\mid \frac{1}{\mu}(\delta^x,\varepsilon)(u_1,u_2) \ - \ (\delta^x, \varepsilon\mu)(w(u_1),w(u_2)) \mid \leq \ \mathcal{O}(\varepsilon\mu) + \frac{1}{\mu}\mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon) \ .$$

This goes as following:

$$\mid \frac{1}{\mu}(\delta^x,\varepsilon)(u_1,u_2) \ - \ (\delta^x, \varepsilon\mu)(w(u_1),w(u_2)) \mid$$

$$= \mid \frac{1}{\varepsilon\mu} d(\delta^x_\varepsilon u_1, \delta^x_\varepsilon u_2) \ - \ \frac{1}{\varepsilon\mu} d(\delta^x_{\varepsilon\mu} w(u_1), \delta^x_{\varepsilon\mu} w(u_2)) \mid \leq$$

$$\leq \ \mathcal{O}(\varepsilon\mu) + \ \mid \frac{1}{\varepsilon\mu} d(\delta^x_\varepsilon u_1, \delta^x_\varepsilon u_2) \ - \ \frac{1}{\varepsilon\mu} d(\delta^x_\varepsilon \delta^x_\mu w(u_1), \delta^x_\varepsilon \delta^x_\mu w(u_2)) \mid \leq$$

$$\leq \ \mathcal{O}(\varepsilon\mu) \ + \ \frac{1}{\mu}\mathcal{O}(\varepsilon) \ + \ \frac{1}{\mu} \mid d^x(u_1,u_2) \ - \ d^x(\delta^x_\mu w(u_1), \delta^x_\mu w(u_2)) \ .$$

In order to obtain the last estimate we used twice axiom A3. We continue:

$$\mathcal{O}(\varepsilon\mu) \ + \ \frac{1}{\mu}\mathcal{O}(\varepsilon) \ + \ \frac{1}{\mu} \mid d^x(u_1,u_2) \ - \ d^x(\delta^x_\mu w(u_1), \delta^x_\mu w(u_2)) \ \leq$$

$$\leq \mathcal{O}(\varepsilon\mu) \ + \ \frac{1}{\mu}\mathcal{O}(\varepsilon) \ + \ \frac{1}{\mu} d^x(u_1, \delta^x_\mu w(u_1)) \ + \ \frac{1}{\mu} d^x(u_1, \delta^x_\mu w(u_2)) \ \leq$$

$$\leq \ \mathcal{O}(\varepsilon\mu) + \frac{1}{\mu}\mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon) \ .$$

This shows that the property (b) of a metric profile is satisfied. The property (a) is proved in theorem 13.2. □

The following theorem is related to Mitchell [51] theorem 1, concerning sub-riemannian geometry.

Theorem 13.2 *In the hypothesis of theorem 13.1, we have the following limit:*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \sup \left\{ \mid d(u,v) - d^x(u,v) \mid \ : d(x,u) \leq \varepsilon \ , \ d(x,v) \leq \varepsilon \right\} \ = \ 0 \ .$$

Therefore if d^x is a true (i.e. nondegenerate) distance, then (X,d) admits a metric tangent space in x.

Moreover, the metric profile $[\bar{B}_{d^x}(x,1),(\delta,\varepsilon),x]$ is almost nice, in the following sense: let $c \in (0,1)$. Then we have the inclusion:

$$\delta^x_{\mu^{-1}}\left(\bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,c)\right) \ \subset \ \bar{B}_{d^x}(x,1) \quad .$$

Moreover, the following Gromov-Hausdorff distance is of order $\mathcal{O}(\varepsilon)$ for μ fixed (that is the modulus of convergence $\mathcal{O}(\varepsilon)$ does not depend on μ) :

$$\mu \ d_{GH}\left([\bar{B}_{d^x}(x,1),(\delta^x,\varepsilon),x], \ [\delta^x_{\mu^{-1}}\left(\bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,c)\right),(\delta^x,\varepsilon\mu),x]\right) = \mathcal{O}(\varepsilon) \ .$$

For another Gromov-Hausdorff distance we have the estimate:

$$d_{GH}\left([\bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,c),\frac{1}{\mu}(\delta^x,\varepsilon),x] \ , \ [\delta^x_{\mu^{-1}}\left(\bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,c)\right),(\delta^x,\varepsilon\mu),x]\right) = \mathcal{O}(\varepsilon\mu)$$

when $\varepsilon \in (0,\varepsilon(c))$.

Proof. We start from the axioms A0, A3 and we use the cone property. By A0, for $\varepsilon \in (0,1)$ and $u,v \in \bar{B}_d(x,\varepsilon)$ there exist $U,V \in \bar{B}_d(x,A)$ such that

$$u = \delta^x_\varepsilon U, v = \delta^x_\varepsilon V.$$

By the cone property we have

$$\frac{1}{\varepsilon} \mid d(u,v) - d^x(u,v) \mid = \mid \frac{1}{\varepsilon} d(\delta^x_\varepsilon U,\delta^x_\varepsilon V) - d^x(U,V) \mid \ .$$

By A2 we have

$$\mid \frac{1}{\varepsilon} d(\delta^x_\varepsilon U,\delta^x_\varepsilon V) - d^x(U,V) \mid \leq \mathcal{O}(\varepsilon).$$

This proves the first part of the theorem.

For the second part of the theorem take any $u \in \bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,c)$. We have then

$$d^x(x,u) \ \leq \ c\mu + \mathcal{O}(\varepsilon) \ .$$

Then there exists $\varepsilon(c) > 0$ such that for any $\varepsilon \in (0,\varepsilon(c))$ and u in the mentioned ball we have:

$$d^x(x,u) \leq \ \mu$$

In this case we can take directly $w(u) \ = \ \delta^x_{\mu^{-1}} u$ and simplify the string of inequalities from the proof of theorem 13.1, point (b), to get eventually the three points from the second part of the theorem. □

13.2 Infinitesimal translations

In this section we shall use the calculus with binary decorated trees introduced in section 4, for a space endowed with a dilation structure.

Theorem 13.3 *Let (X,d,δ) be a dilation structure. Then the "infinitesimal translations"*

$$L^x_u(v) = \lim_{\varepsilon\rightarrow 0} \Delta^x_\varepsilon(u,v)$$

are d^x isometries.

Proof. The first part of the conclusion of theorem 13.2 can be written as:

$$\sup \left\{ \frac{1}{\varepsilon} \mid d(u,v) \ - \ d^x(u,v) \mid \ : \ d(x,u) \leq \frac{3}{2}\varepsilon, \ d(x,v) \leq \frac{3}{2}\varepsilon \right\} \rightarrow 0 \tag{13.2.1}$$

as $\varepsilon \rightarrow 0$.

For $\varepsilon > 0$ sufficiently small the points $x, \delta^x_\varepsilon u, \delta^x_\varepsilon v, \delta^x_\varepsilon w$ are close one to another. Precisely, we have

$$d(\delta^\varepsilon_x u, \delta^\varepsilon_x v) = \varepsilon (d^x(u,v) + \mathcal{O}(\varepsilon)) \quad .$$

Therefore, if we choose u, v, w such that $d^x(u,v) < 1, d^x(u,w) < 1$, then there is $\eta > 0$ such that for all $\varepsilon \in (0,\eta)$ we have

$$d(\delta^\varepsilon_x u, \delta^\varepsilon_x v) \leq \frac{3}{2}\varepsilon \quad , \quad d(\delta^\varepsilon_x u, \delta^\varepsilon_x v) \leq \frac{3}{2}\varepsilon \quad .$$

We apply the estimate (13.2.1) for the basepoint $\delta^x_\varepsilon u$ to get:

$$\frac{1}{\varepsilon} \mid d(\delta^x_\varepsilon v, \delta^x_\varepsilon w) \ - \ d^{\delta^x_\varepsilon u}(\delta^x_\varepsilon v, \delta^x_\varepsilon w) \mid \rightarrow 0$$

when $\varepsilon \rightarrow 0$. This can be written, using the cone property of the distance $d^{\delta^x_\varepsilon u}$, like this:

$$\mid \frac{1}{\varepsilon} d(\delta^x_\varepsilon v, \delta^x_\varepsilon w) \ - \ d^{\delta^x_\varepsilon u}\left(\delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \delta^x_\varepsilon v, \delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \delta^x_\varepsilon w\right) \mid \rightarrow 0 \tag{13.2.2}$$

as $\varepsilon \rightarrow 0$. By the axioms A1, A3, the function

$$(x,u,v) \mapsto d^x(u,v)$$

is an uniform limit of continuous functions, therefore uniformly continuous on compact sets. We can pass to the limit in the left hand side of the estimate (13.2.2), using this uniform continuity and axioms A3, A4, to get the result. □

Let us define, in agreement with definition 7.4 (b):

$$\Sigma^x_\varepsilon(u,v) = \delta^x_{\varepsilon^{-1}} \delta^{\delta^x_\varepsilon u}_\varepsilon v.$$

Corollary 13.4 *If for any x the distance d^x is non degenerate then there exists $C > 0$ such that: for any x and u with $d(x,u) \leq C$ there exists a d^x isometry $\Sigma^x(u,\cdot)$ obtained as the limit:*

$$\lim_{\varepsilon \rightarrow 0} \Sigma^x_\varepsilon(u,v) = \Sigma^x(u,v)$$

uniformly with respect to x, u, v in compact set.

Proof. From theorem 13.3 we know that $\Delta^x(u,\cdot)$ is a d^x isometry. If d^x is non degenerate then $\Delta^x(u,\cdot)$ is invertible. Let $\Sigma^x(u,\cdot)$ be the inverse.

From proposition 7.5 we know that $\Sigma^x_\varepsilon(u,\cdot)$ is the inverse of $\Delta^x_\varepsilon(u,\cdot)$. Therefore

$$d^x(\Sigma^x_\varepsilon(u,w), \Sigma^x(u,w)) = d^x(\Delta^x(u, \Sigma^x_\varepsilon(u,w)), w) =$$

$$= d^x(\Delta^x(u, \Sigma^x_\varepsilon(u,w)), \Delta^x_\varepsilon(u, \Sigma^x_\varepsilon(u,w)).$$

From the uniformity of convergence in theorem 13.3 and the uniformity assumptions in axioms of dilation structures, the conclusion follows. □

The next theorem is the generalization of proposition 8.6.

Theorem 13.5 *Let (X, d, δ) be a dilation structure (which satisfies the strong form of the axiom A2), such that for any $x \in X$ the distance d^x is non degenerate. Then for any $x \in X$ $(U(x), \Sigma^x, \delta^x)$ is a conical group. Moreover, left translations of this group are d^x isometries.*

Proof. We start by proving that $(U(x), \Sigma^x)$ is a local uniform group. The uniformities are induced by the distance d.

We shall use the general relations written in terms of binary decorated trees. Indeed, according to proposition 7.5, we can pass to the limit with $\varepsilon \to 0$ and define:

$$inv^x(u) = \lim_{\varepsilon \to 0} \Delta^x_\varepsilon(u, x) = \Delta^x(u, x).$$

From relation (d) , proposition (7.5) we get (after passing to the limit with $\varepsilon \to 0$)

$$inv^x(inv^x(u)) = u.$$

We shall see that $inv^x(u)$ is the inverse of u. Relation (c), proposition (7.5) gives:

$$\Delta^x(u, v) = \Sigma^x(inv^x(u), v) \tag{13.2.3}$$

therefore relations (a), (b) from proposition 7.5 give

$$\Sigma^x(inv^x(u), \Sigma^x(u, v)) = v, \tag{13.2.4}$$

$$\Sigma^x(u, \Sigma^x(u, v)) = v. \tag{13.2.5}$$

Relation (e) from proposition 7.5 gives

$$\Sigma^x(u, \Sigma^x(v, w)) = \Sigma^x(\Sigma^x(u, v), w) \tag{13.2.6}$$

which shows that Σ^x is an associative operation. From (13.2.5), (13.2.4) we obtain that for any u, v

$$\Sigma^x(\Sigma^x(inv^x(u), u), v) = v, \tag{13.2.7}$$

$$\Sigma^x(\Sigma^x(u, inv^x(u)), v) = v. \tag{13.2.8}$$

Remark that for any x, v and $\varepsilon \in (0, 1)$ we have $\Sigma^x(x, v) = v$. Therefore x is a neutral element at left for the operation Σ^x. From the definition of inv^x, relation (13.2.3) and the fact that inv^x is equal to its inverse, we get that x is an inverse at right too: for any x, v we have

$$\Sigma^x(v, x) = v.$$

Replace now v by x in relations (13.2.7), (13.2.8) and prove that indeed $inv^x(u)$ is the inverse of u.

We still have to prove that $(U(x), \Sigma^x)$ admits δ^x as dilations.In this reasoning we need the axiom A2 in strong form.

Namely we have to prove that for any $\mu \in (0, 1)$ we have

$$\delta^x_\mu \Sigma^x(u, v) = \Sigma^x(\delta^x_\mu u, \delta^x_\mu v).$$

For this is sufficient to notice that

$$\delta^x_\mu \Delta^x_{\varepsilon\mu}(u, v) = \Delta^x_\varepsilon(\delta^x_\mu u, \delta^x_\mu v)$$

and pass to the limit as $\varepsilon \to 0$. Notice that here we used the fact that dilations δ_ε^x and δ_μ^x exactly commute (axiom A2 in strong form).

Finally, left translations L_u^x are d^x isometries. Indeed, this is a straightforward consequence of theorem 13.3 and corollary 13.4. $\square$

The conical group $(U(x), \Sigma^x, \delta^x)$ can be regarded as the tangent space of (X, δ, d) at x and denoted further by $T_x X$.

A reformulation of parts of theorems 6,7 [11] is the following.

Theorem 13.6 *A dilation structure (X, d, δ) has the following properties.*

(a) For all $x \in X$, $u, v \in X$ such that $d(x, u) \leq 1$ and $d(x, v) \leq 1$ and all $\mu \in (0, A)$ we have:

$$d^x(u, v) \ = \ \frac{1}{\mu} d^x(\delta_\mu^x u, \delta_\mu^x v) \ .$$

We shall say that d^x has the cone property with respect to dilations.

(b) The metric space (X, d) admits a metric tangent space at x, for any point $x \in X$. More precisely we have the following limit:

$$\lim_{\varepsilon \to 0} \ \frac{1}{\varepsilon} \sup \left\{ \mid d(u, v) - d^x(u, v) \mid \ : d(x, u) \leq \varepsilon \ , \ d(x, v) \leq \varepsilon \right\} \ = \ 0 \ .$$

For the next theorem (composite of results in theorems 8, 10 [11]) we need the previously introduced notion of a normed conical local group.

Theorem 13.7 *Let (X, d, δ) be a strong dilation structure. Then for any $x \in X$ the triple $(U(x), \Sigma^x, \delta^x)$ is a normed local conical group, with the norm induced by the distance d^x.*

The conical group $(U(x), \Sigma^x, \delta^x)$ can be regarded as the tangent space of (X, d, δ) at x. Further will be denoted by: $T_x X = (U(x), \Sigma^x, \delta^x)$.

The dilation structure on this conical group has dilations defined by

$$\bar{\delta}_\varepsilon^{x,u} y \ = \ \Sigma^x \left(u, \delta_\varepsilon^x \Delta^x(u, y) \right) \quad . \tag{13.2.9}$$

13.3 Topological considerations

In this subsection we compare various topologies and uniformities related to a dilation structure.

The axiom A3 implies that for any $x \in X$ the function d^x is continuous, therefore open sets with respect to d^x are open with respect to d.

If (X, d) is separable and d^x is non degenerate then $(U(x), d^x)$ is also separable and the topologies of d and d^x are the same. Therefore $(U(x), d^x)$ is also locally compact (and a set is compact with respect to d^x if and only if it is compact with respect to d).

If (X, d) is separable and d^x is non degenerate then the uniformities induced by d and d^x are the same. Indeed, let $\{u_n : n \in \mathbb{N}\}$ be a dense set in $U(x)$, with $x_0 = x$. We can embed $(U(x), (\delta^x, \varepsilon))$ (see definition 15.6) isometrically in the separable Banach space l^∞, for any $\varepsilon \in (0, 1)$, by the function

$$\phi_\varepsilon(u) = \left(\frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x x_n) - \frac{1}{\varepsilon} d(\delta_\varepsilon^x x, \delta_\varepsilon^x x_n) \right)_n \ .$$

A reformulation of point (a) in theorem 13.1 is that on compact sets ϕ_ε uniformly converges to the isometric embedding of $(U(x), d^x)$

$$\phi(u) = \left(d^x(u, x_n) - d^x(x, x_n)\right)_n .$$

Remark that the uniformity induced by (δ, ε) is the same as the uniformity induced by d, and that it is the same induced from the uniformity on l^∞ by the embedding ϕ_ε. We proved that the uniformities induced by d and d^x are the same.

From previous considerations we deduce the following characterization of tangent spaces associated to a dilation structure.

Corollary 13.8 *Let (X, d, δ) be a strong dilation structure with group $\Gamma = (0, +\infty)$. Then for any $x \in X$ the local group $(U(x), \Sigma^x)$ is locally a simply connected Lie group whose Lie algebra admits a positive graduation (a homogeneous group).*

Proof. Use the facts: $(U(x), \Sigma^x)$ is a locally compact group (from previous topological considerations) which admits δ^x as a contractive automorphism group (from theorem 13.5). Apply then Siebert proposition 8.12 (which is [54] proposition 5.4). $\square$

13.4 Differentiability with respect to dilation structures

We briefly explain the notion of differentiability associated to dilation structures (section 7.2 [11]). First we need the natural definition below.

Definition 13.9 *Let (N, δ) and $(M, \bar{\delta})$ be two conical groups. A function $f : N \to M$ is a conical group morphism if f is a group morphism and for any $\varepsilon > 0$ and $u \in N$ we have $f(\delta_\varepsilon u) = \bar{\delta}_\varepsilon f(u)$.*

The definition of the derivative with respect to dilations structures follows.

Definition 13.10 *Let (X, δ, d) and $(Y, \overline{\delta}, \overline{d})$ be two strong dilation structures and $f : X \to Y$ be a continuous function. The function f is differentiable in x if there exists a conical group morphism $Q^x : T_x X \to T_{f(x)} Y$, defined on a neighbourhood of x with values in a neighbourhood of $f(x)$ such that*

$$\lim_{\varepsilon \to 0} \sup \left\{ \frac{1}{\varepsilon} \overline{d} \left(f\left(\delta^x_\varepsilon u\right), \overline{\delta}^{f(x)}_\varepsilon Q^x(u) \right) \ : \ d(x, u) \leq \varepsilon \right\} = 0, \qquad (13.4.10)$$

The morphism Q^x is called the derivative of f at x and will be sometimes denoted by $Df(x)$.

The function f is uniformly differentiable if it is differentiable everywhere and the limit in (13.4.10) is uniform in x in compact sets.

14 Related constructions of tangent bundles

14.1 Mitchell's theorem 1. Bellaïche's construction

We collect here three key items in the edification of sub-Riemannian geometry. The first is Mitchell [51] theorem 1:

Theorem 14.1 *For a regular sub-Riemannian space (M, D, g), the tangent cone of (M, d_{CC}) at $x \in M$ exists and it is isometric to $(N(D), d_N)$, which is a Carnot group with a left invariant distribution and d_N is a induced Carnot-Carathéodory distance.*

We shall prove this theorem in the particular case of a Lie group with left-invariant distribution. For a detailed proof of the general case see Vodop'yanov & Greshnov [57] [58].

The Carnot group $N(D)$ is called the nilpotentisation of the regular distribution D and it can be constructed from D exclusively. However, the metric on $N(D)$ depends on the choice of metric g on the sub-Riemannian manifold.

Recall that the limit in the Gromov-Hausdorff sense is defined up to isometry. This means it this case that $N(D)$ is a model for the tangent space at x to (M, d_{CC}). In the Riemannian case $D = TM$ and $N(D) = R^n$, as a group with addition.

This theorem tells us nothing about the tangent bundle. There are however other ways to associate a tangent bundle to a metric measure space (Cheeger [23]) or to a regular sub-Riemannian manifold (Margulis & Mostow [48], [49]). These bundles differs. As Tyson (interpreting Cheeger) asserts (see Tyson paper in these proceedings (??)), Cheeger tangent bundle can be identified with the distribution D and Margulis-Mostow bundle is the same as the usual tangent bundle, but with the fiber isomorphic with $N(D)$, instead of R^n. We shall explain how the Margulis-Mostow tangent bundle is constructed a bit further (again in the particular case considered here).

Let us not, for the moment, be too ambitious and restrict to the question: is there a metric derivation of the group operation on $N(D)$? Bellaïche [5] writes that he asked Gromov this question, who pointed out that the key tool to construct the operation from metric is uniformity. Bellaïche proposed therefore the following construction, which starts from the proof of Mitchell theorem 1, where it can be seen that the Gromov-Hausdorff convergence to the tangent space *is uniform with respect to* $x \in M$. This means that for any $\varepsilon > 0$ there is $R(\varepsilon) > 0$ and map $\phi_{x,\varepsilon} : B_{CC}(x, R(\varepsilon)) \to N(D)$ such that

$$d_N(\phi_{x,\varepsilon}(y), \phi_{x,\varepsilon}(z)) = d_{CC}(y, z) + o(\varepsilon) \quad \forall y, z \in B_{CC}(x, R(\varepsilon))$$

Let us forget about ε (Bellaïche does not mention anything about it further) and take arbitrary $X, Y \in N(D)$. Pick then $y \in M$ such that $\phi_x(y) = X$. Denote $\phi_{xy} = \phi_y \phi_x^{-1}$. Then the operation in $N(D)$ is defined by:

$$X \overset{n}{\cdot} Y = \lim_{\lambda \to \infty} \phi_{xy}^{-1} \delta_\lambda^{-1} \phi_{xy} \delta_\lambda(Y)$$

It is easier to understand this in the Euclidean case, that is in R^n. We can take for example

$$\phi_x(y) = Q(x)(y - x) = X \quad , y = x + Q^{-1}X \quad , x \mapsto Q(x) \in SO(n) \text{ arbitrary}$$

(and we have no dependence on ε) Let us compute the operation. We get

$$\lim_{\lambda \to \infty} \phi_{xy}^{-1} \delta_\lambda^{-1} \phi_{xy} \delta_\lambda(Y) = \; X + Y$$

as expected. Notice that the arbitrary choice of the rotations $Q(x)$ does not influence the result. The tangent spaces at any point can rotate independently, which is a sign that this construction cannot lead to a tangent bundle.

There are several problems with Bellaïche's construction:

a) when $\lambda \rightarrow \infty$ the expression $\phi_{xy}\delta_\lambda(Y)$ might not make sense,

b) is not clear how ε and λ interact.

This is the reason why we introduced (first in [9], then here) the notion of uniform group, which encodes all that one really need to do the construction, again in the case of Lie groups with left invariant distributions.

Another way to transform Bellaïche's construction into an effective one (and more, to obtain a tangent bundle) is proposed by Margulis and Mostow. We shall explain further their construction. However, there are other problems emerging. as mentioned in the introduction.

14.2 Margulis & Mostow tangent bundle

In this section we shall apply Margulis & Mostow [49] construction of the tangent bundle to a SR manifold for the case of a group with left invariant distribution. It will turn that the tangent bundle does not have a group structure, due to the fact that, as previously, the non-smoothness of the right translations is not studied.

The main point in the construction of a tangent bundle is to have a functorial definition of the tangent space. This is achieved by Margulis & Mostow [49] in a very natural way. One of the geometrical definitions of a tangent vector v at a point x, to a manifold M, is the following one: identify v with the class of smooth curves which pass through x and have tangent v. If the manifold M is endowed with a distance then one can define the equivalence relation based in x by: $c_1 \equiv_x c_2$ if $c_1(0) = c_2(0) = x$ and the distance between $c_1(t)$ and $c_2(t)$ is negligible with respect to t for small t. The set of equivalence classes is the tangent space at x. One has to put then some structure on the tangent space (as, for example, the nilpotent multiplication).

To put is practice this idea is not so easy though. This is achieved by the following sequence of definitions and theorems. For commodity we shall explain this construction in the case $M = G$ connected Lie group, endowed with a left invariant distribution D. The general case is the one of a regular sub-Riemannian manifold. We shall denote by d_G the CC distance on G and we identify G with $\mathfrak{g}$, as previously. The CC distance induced by the distribution D^N, generated by left translations of G using nilpotent multiplication $\stackrel{\mathfrak{n}}{\cdot}$, will be denoted by d_N.

Definition 14.2 *A C^∞ curve in G with $x = c(0)$ is called rectifiable at $t = 0$ if $d_G(x, c(t)) \leq Ct$ as $t \rightarrow 0$.*

Two C^∞ curves $c', c"$ with $c'(0) = x = c"(0)$ are called equivalent at x if

$$t^{-1} d_G(c'(t), c"(t)) \rightarrow 0$$

as $t \rightarrow 0$.

The tangent cone to G as x, denoted by $C_x G$ is the set of equivalence classes of all C^∞ paths c with $c(0) = x$, rectifiable at $t = 0$.

Let $c : [-1, 1] \rightarrow G$ be a C^∞ rectifiable curve, $x = c(0)$ and

$$v \ = \ \lim_{t \rightarrow 0} \delta_t^{-1} \left(c(0)^{-1} \stackrel{\mathfrak{g}}{\cdot} c(t) \right) \tag{14.2.1}$$

The limit v exists because the curve is rectifiable.

Introduce the curve $c_0(t) \ = \ x \exp_G(\delta_t v)$. Then

$$d(x, c_0(t)) \ = \ d(e, x^{-1}c(t)) <\mid v \mid t$$

as $t \to 0$ (by the Ball-Box theorem) The curve c is equivalent with c_0. Indeed, we have (for $t > 0$):

$$\frac{1}{t} d_G(c(t), c_0(t)) \ = \ \frac{1}{t} d_G(c(t), x \stackrel{\mathfrak{g}}{\cdot} \delta_t v) \ = \ \frac{1}{t} d_G(\delta_t(v^{-1}) \stackrel{\mathfrak{g}}{\cdot} x^{-1} \stackrel{\mathfrak{g}}{\cdot} c(t), 0)$$

The latter expression is equivalent (by the Ball-Box Theorem) with

$$\frac{1}{t} d_N(\delta_t(v^{-1}) \stackrel{\mathfrak{g}}{\cdot} x^{-1} \stackrel{\mathfrak{g}}{\cdot} c(t), 0) \ = \ d_N \left(\delta_t^{-1} \left(\delta_t(v^{-1}) \stackrel{\mathfrak{g}}{\cdot} \delta_t \left(\delta_t^{-1} \left(x^{-1} \stackrel{\mathfrak{g}}{\cdot} c(t) \right) \right) \right) \right)$$

The right hand side (RHS) converges to $d_N(v^{-1} \stackrel{\mathfrak{n}}{\cdot} v, 0)$, as $t \to 0$, as a consequence of the definition of v.

Therefore we can identify $C_x G$ with the set of curves $t \mapsto x \exp_G(\delta_t v)$, for all $v \in \mathfrak{g}$. Remark that the equivalence relation between curves c_1, c_2, such that $c_1(0) \ = \ c_2(0) \ = \ x$ can be redefined as:

$$\lim_{t \to 0} \delta_t^{-1} \left(c_2(t)^{-1} \stackrel{\mathfrak{g}}{\cdot} c_1(t) \right) \ = \ 0 \tag{14.2.2}$$

In order to define the multiplication Margulis & Mostow introduce the families of segments rectifiable at t.

Definition 14.3 *A family of segments rectifiable at $t = 0$ is a C^∞ map*

$$\mathcal{F} : U \to G$$

where U is an open neighbourhood of $G \times 0$ in $G \times R$ satisfying

(a) $\mathcal{F}(\cdot, 0) \ = \ id$

(b) the curve $t \mapsto \mathcal{F}(x, t)$ is rectifiable at $t = 0$ uniformly for all $x \in G$, that is for every compact K in G there is a constant C_K and a compact neighbourhood I of 0 such that $d_G(y, \mathcal{F}(y, t)) < C_K t$ for all $(y, t) \in K \times I$.

Two families of segments rectifiable at $t = 0$ are called equivalent if

$$t^{-1} d_G(\mathcal{F}_1(x, t), \mathcal{F}_2(x, t)) \to 0$$

as $t \to 0$, uniformly on compact sets in the domain of definition.

Part (b) from the definition of a family of segments rectifiable can be restated as: there exists the limit

$$v(x) \ = \ \lim_{t \to 0} \delta_t^{-1} \left(x^{-1} \stackrel{\mathfrak{g}}{\cdot} \mathcal{F}(x, t) \right) \tag{14.2.3}$$

and the limit is uniform with respect to $x \in K$, K arbitrary compact set.

It follows then, as previously, that $\mathcal{F}$ is equivalent to $\mathcal{F}_0$, defined by:

$$\mathcal{F}_0(x, t) \ = \ x \stackrel{\mathfrak{g}}{\cdot} \delta_t v(x)$$

Also, the equivalence between families of segments rectifiable can be redefined as:

$$\lim_{t \to 0} \delta_t^{-1} \left(\mathcal{F}_2(x, t)^{-1} \stackrel{\mathfrak{g}}{\cdot} \mathcal{F}_1(x, t) \right) \ = \ 0 \tag{14.2.4}$$

uniformly with respect to $x \in K$, K arbitrary compact set.

Definition 14.4 *The product of two families $\mathcal{F}_1$, $\mathcal{F}_2$ of segments rectifiable at $t = 0$ is defined by*

$$(\mathcal{F}_1 \circ \mathcal{F}_2)(x,t) \ = \ \mathcal{F}_1(\mathcal{F}_2(x,t),t)$$

The product is well defined by Lemma 1.2 *op. cit.*. One of the main results is then the following theorem (5.5).

Theorem 14.5 *Let c_1, c_2 be C^∞ paths rectifiable at $t = 0$, such that $c_1(0) = x_0 = c_2(0)$. Let $\mathcal{F}_1$, $\mathcal{F}_2$ be two families of segments rectifiable at $t = 0$ with:*

$$\mathcal{F}_1(x_0,t) \ = \ c_1(t) \quad , \quad \mathcal{F}_2(x_0,t) \ = \ c_2(t)$$

Then the equivalence class of

$$t \mapsto \mathcal{F}_1 \circ \mathcal{F}_2(x_0,t)$$

depends only on the equivalence classes of c_1 and c_2. This defines the product of the elements of the tangent cone $C_{x_0}G$.

This theorem is the straightforward consequence of the following facts (5.1(5) and 5.2 in Margulis & Mostow [49]).

We shall denote by $\mathcal{F} \approx \mathcal{F}'$ the equivalence relation of families of segments rectifiable; the equivalence relation of rectifiable curves based at x will be denoted by $c \overset{x}{\approx} c'$.

Lemma 14.6 *(a) Let $\mathcal{F}_1 \approx \mathcal{F}_2$ and $\mathcal{G}_1 \approx \mathcal{G}_2$. Then $\mathcal{F}_1 \circ \mathcal{G}_1 \approx \mathcal{F}_2 \circ \mathcal{G}_2$.*

(b) The map $\mathcal{F} \mapsto \mathcal{F}_0$ is constant on equivalence classes of families of segments rectifiable.

Proof. Let

$$\mathcal{F}_0(x,t) \ = \ x \overset{\mathfrak{g}}{\cdot} \delta_t w_1(x) \quad , \quad \mathcal{G}_0(x,t) \ = \ x \overset{\mathfrak{g}}{\cdot} \delta_t w_2(x)$$

For the point (a) it is sufficient to prove that

$$\mathcal{F} \circ \mathcal{G} \approx \mathcal{F}_0 \circ \mathcal{G}_0$$

This is true by the following chain of estimates.

$$\frac{1}{t} d_G(\mathcal{F} \circ \mathcal{G}(x,t), \mathcal{F}_0 \circ \mathcal{G}_0(x,t)) \ =$$

$$= \ \frac{1}{t} d_G(\delta_t w_1(\mathcal{G}_0(x,t))^{-1} \overset{\mathfrak{g}}{\cdot} \delta_t w_2(x)^{-1} \overset{\mathfrak{g}}{\cdot} x^{-1} \overset{\mathfrak{g}}{\cdot} \mathcal{F}(\mathcal{G}(x,t),t), 0)$$

The RHS of this equality behaves like

$$d_N(\delta_t^{-1} \left(\delta_t w_1(\mathcal{G}_0(x,t))^{-1} \overset{\mathfrak{g}}{\cdot} \delta_t w_2(x)^{-1} \overset{\mathfrak{g}}{\cdot} \delta_t \left(\delta_t^{-1} \left(x^{-1} \overset{\mathfrak{g}}{\cdot} \mathcal{G}(x,t) \right) \right) \overset{\mathfrak{g}}{\cdot} \right.$$

$$\left. \overset{\mathfrak{g}}{\cdot} \delta_t \left(\delta_t^{-1} \left(\mathcal{G}(x,t)^{-1} \overset{\mathfrak{g}}{\cdot} \mathcal{F}(\mathcal{G}(x,t),t) \right) \right) \right), 0)$$

This quantity converges (uniformly with respect to $x \in K$, K an arbitrary compact) to

$$d_N(w_1(x)^{-1} \overset{\mathfrak{n}}{\cdot} w_2(x)^{-1} \overset{\mathfrak{n}}{\cdot} w_2(x) \overset{\mathfrak{n}}{\cdot} w_1(x), 0) \ = \ 0$$

The point (b) is easier: let $\mathcal{F} \approx \mathcal{G}$ and consider $\mathcal{F}_0$, $\mathcal{G}_0$, as above. We want to prove that $\mathcal{F}_0 \ = \ \mathcal{G}_0$, which is equivalent to $w_1 \ = \ w_2$.

Because $\approx$ is an equivalence relation all we have to prove is that if $\mathcal{F}_0 \approx \mathcal{G}_0$ then $w_1 = w_2$. We have:

$$\frac{1}{t} d_G(\mathcal{F}_0(x,t), \mathcal{G}_0(x,t)) = \frac{1}{t} d_G(x \overset{\mathfrak{g}}{\cdot} \delta_t w_1(x), x \overset{\mathfrak{g}}{\cdot} \delta_t w_2(x))$$

We use the $\overset{\mathfrak{g}}{\cdot}$ left invariance of d_G and the Ball-Box theorem to deduce that the RHS behaves like

$$d_N(\delta_t^{-1}\left(\delta_t w_2(x)^{-1} \overset{\mathfrak{g}}{\cdot} \delta_t w_1(x)^{-1}\right), 0)$$

which converges to $d_N(w_1(x), w_2(x))$ as t goes to 0. The two families are equivalent, therefore the limit equals 0, which implies that $w_1(x) = w_2(x)$ for all x. ■

We shall apply this theorem. Let $c_i(t) = x_0 \exp_G \delta_t v_i$, for $i = 1, 2$. It is easy to check that $\mathcal{F}_i(x,t) = x \exp_G(\delta_t v_i)$ are families of segments rectifiable at $t = 0$ which satisfy the hypothesis of the theorem. But then

$$(\mathcal{F}_1 \circ \mathcal{F}_2)(x,t) = x_0 \exp_G(\delta_t v_1) \exp_G(\delta_t v_2)$$

which is equivalent with

$$\mathcal{F} \exp_G\left(\delta_t(v_1 \overset{\mathfrak{n}}{\cdot} v_2)\right)$$

Therefore the tangent bundle defined by this procedure is the same as the virtual tangent bundle which we shall define soon, inspired from the construction proposed by Bellaïche.

Maybe I misunderstood the notations, but it seems to me that several times the authors claim that the exponential map which they construct is bi-Lipschitz (as in 5.1(4) and Corollary 4.5). This is false, as explained before. In Bellaïche [5], Theorem 7.32 and also at the beginning of section 7.6 we find that the exponential map is only $1/m$ Hölder continuous (where m is the step of the nilpotentization). However, the final results of Margulis & Mostow hold true, if not entirely proven facts, as statements at least.

14.3 Vodop'yanov & Greshnov definition of the derivability

We choose Vodop'yanov & Greshnov [57], [58] way of defining the derivability in order to explain Margulis & Mostow [48] Rademacher theorem 10.5 (or Vodop'yanov & Greshnov theorem 1). The definition (10.3.1) of derivability in the paper [48] is to be compared with the definition of $\mathcal{P}$ differentiability from [57], first page, which is in my opinion clearer. However, the reader entering for the first time in this subject might find hard to understand why such elementary notions as differentiability need so lengthy discussions. It is, I think, another sign of the fact that the foundations of non-Euclidean analysis are still in construction.

We stay, as previously, in the case of Lie groups with left invariant distributions. We put on such groups Lebesgue measures coming from arbitrary atlases.

Definition 14.7 *A mapping $f : G_1 \to G_2$ is said to be differentiable at $x \in G_1$ if the mapping $\exp_{G_2}^{-1} \circ L_{f(x)} \circ f \circ L_x \circ \exp_{G_1}$ is Pansu derivable at 0, when we identify the algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$ with the nilpotentisations of G_1, G_2 respectively.*

The following theorem then holds. The original (and stronger) versions of this theorem concern quasi-conformal mapping and can be found in Margulis & Mostow [48], Vodop'yanov & Greshnov [57] and the paper by Vodop'yanov in these proceedings.

Theorem 14.8 *Any Lipschitz map $f : E \subset G_1 \to G_2$, E measurable, is derivable almost everywhere.*

15 Infinitesimal affine geometry of dilation structures

15.1 Affine transformations

Definition 15.1 *Let (X, d, δ) be a dilation structure. A transformation $A : X \to X$ is affine if it is Lipschitz and it commutes with dilations in the following sense: for any $x \in X$, $u \in U(x)$ and $\varepsilon \in \Gamma$, $\nu(\varepsilon) < 1$, if $A(u) \in U(A(x))$ then*

$$A\delta^x_\varepsilon = \delta^{A(x)}_\varepsilon A(u) \quad .$$

The local group of affine transformations, denoted by $Aff(X, d, \delta)$ is formed by all invertible and bi-lipschitz affine transformations of X.

$Aff(X, d, \delta)$ is indeed a local group. In order to see this we start from the remark that if A is Lipschitz then there exists $C > 0$ such that for all $x \in X$ and $u \in B(x, C)$ we have $A(u) \in U(A(x))$. The inverse of $A \in Aff(X, d, \delta)$ is then affine. Same considerations apply for the composition of two affine, bi-lipschitz and invertible transformations.

In the particular case of X finite dimensional real, normed vector space, d the distance given by the norm, $\Gamma = (0, +\infty)$ and dilations $\delta^x_\varepsilon u = x + \varepsilon(u - x)$, an affine transformation in the sense of definition 15.1 is an affine transformation of the vector space X.

Proposition 15.2 *Let (X, d, δ) be a dilation structure and $A : X \to X$ an affine transformation. Then:*

(a) for all $x \in X$, $u, v \in U(x)$ sufficiently close to x, we have:

$$A\,\Sigma^x_\varepsilon(u, v) = \Sigma^{A(x)}_\varepsilon(A(u), A(v)) \quad .$$

(b) for all $x \in X$, $u \in U(x)$ sufficiently close to x, we have:

$$A\,inv^x(u) = inv^{A(x)} A(u) \quad .$$

Proposition 15.3 *Let (X, d, δ) be a strong dilation structure and $A : X \to X$ an affine transformation. Then A is uniformly differentiable and the derivative equals A.*

The proofs are straightforward, just use the commutation with dilations.

15.2 Infinitesimal linearity of dilation structures

We begin by an explanation of the term ”sufficiently closed“, which will be used repeatedly in the following.

We work in a dilation structure (X, d, δ). Let $K \subset X$ be a compact, non empty set. Then there is a constant $C(K) > 0$, depending on the set K such that for any $\varepsilon, \mu \in \Gamma$ with $\nu(\varepsilon), \nu(\mu) \in (0, 1]$ and any $x, y, z \in K$ with $d(x, y), d(x, z), d(y, z) \leq C(K)$ we have

$$\delta^y_\mu z \in V_\varepsilon(x) \quad , \quad \delta^x_\varepsilon z \in V_\mu(\delta^x_\varepsilon y) \quad .$$

Indeed, this is coming from the uniform (with respect to K) estimates:

$$d(\delta^x_\varepsilon y, \delta^x_\varepsilon z) \leq \varepsilon d^x(y, z) + \varepsilon \mathcal{O}(\varepsilon) \quad ,$$

$$d(x, \delta^y_\mu z) \leq d(x, y) + d(y, \delta^y_\mu z) \leq d(x, y) + \mu d^y(y, z) + \mu \mathcal{O}(\mu) \quad .$$

Definition 15.4 *A property $\mathcal{P}(x_1, x_2, x_3, ...)$ holds for $x_1, x_2, x_3, ...$ sufficiently closed if for any compact, non empty set $K \subset X$, there is a positive constant $C(K) > 0$ such that $\mathcal{P}(x_1, x_2, x_3, ...)$ is true for any $x_1, x_2, x_3, ... \in K$ with $d(x_i, x_j) \leq C(K)$.*

For example, we may say that the expressions

$$\delta_\varepsilon^x \delta_\mu^y z \quad , \quad \delta_\mu^{\delta_\varepsilon^x y} \delta_\varepsilon^x z$$

are well defined for any $x, y, z \in X$ sufficiently closed and for any $\varepsilon, \mu \in \Gamma$ with $\nu(\varepsilon), \nu(\mu) \in (0, 1]$.

Definition 15.5 *A dilation structure (X, d, δ) is linear if for any $\varepsilon, \mu \in \Gamma$ such that $\nu(\varepsilon), \nu(\mu) \in (0, 1]$, and for any $x, y, z \in X$ sufficiently closed we have*

$$\delta_\varepsilon^x \, \delta_\mu^y z \ = \ \delta_\mu^{\delta_\varepsilon^x y} \delta_\varepsilon^x z \quad .$$

This definition means simply that a linear dilation structure is a dilation structure with the property that dilations are affine in the sense of definition 15.1.

Let us look to a dilation structure in finer details. We do this by defining induced dilation structures from a given one.

Definition 15.6 *Let (X, δ, d) be a dilation structure and $x \in X$ a point. In a neighbourhood $U(x)$ of x, for any $\mu \in (0, 1)$ we define the distances:*

$$(\delta^x, \mu)(u, v) = \frac{1}{\mu} d(\delta_\mu^x u, \delta_\mu^x v).$$

The next theorem shows that on a dilation structure we almost have translations (the operators $\Sigma_\varepsilon^x(u, \cdot)$), which are almost isometries (that is, not with respect to the distance d, but with respect to distances of type (δ^x, μ)). It is almost as if we were working with a normed conical group, only that we have to use families of distances and to make small shifts in the tangent space, as it is done at the end of the proof of theorem 15.7.

Theorem 15.7 *Let (X, δ, d) be a (strong) dilation structure. For any $u \in U(x)$ and v close to u let us define*

$$\hat{\delta}_{\mu,\varepsilon}^{x,u} v = \Sigma_\mu^x(u, \delta_\varepsilon^{\delta_\mu^x u} \Delta_\mu^x(u, v)) = \delta_{\mu^{-1}}^x \delta_\varepsilon^{\delta_\mu^x u} \delta_\mu^x v \quad .$$

Then $(U(x), \hat{\delta}_\mu^x, (\delta^x, \mu))$ is a (strong) dilation structure.

The transformation $\Sigma_\mu^x(u, \cdot)$ is an isometry from $(\delta^{\delta_\mu^x u}, \mu)$ to (δ^x, μ). Moreover, we have $\Sigma_\mu^x(u, \delta_\mu^x u) = u$.

Proof. We have to check the axioms. The first part of axiom A0 is an easy consequence of theorem 13.1 for (X, δ, d). The second part of A0, A1 and A2 are true based on simple computations.

The first interesting fact is related to axiom A3. Let us compute, for $v, w \in U(x)$,

$$\frac{1}{\varepsilon}(\delta^x, \mu)(\hat{\delta}_{\mu,\varepsilon}^{x,u} v, \hat{\delta}_{\mu\varepsilon}^{x,u} w) = \frac{1}{\varepsilon\mu} d(\delta_\mu^x \hat{\delta}_{\mu\varepsilon}^{x,u} v, \delta_\mu^x \hat{\delta}_{\mu\varepsilon}^{x,u} w) =$$

$$= \frac{1}{\varepsilon\mu} d(\delta_{\varepsilon}^{\delta_{\mu}^{x} u} \delta_{\mu}^{x} v, \delta_{\varepsilon}^{\delta_{\mu}^{x} u} \delta_{\mu}^{x} w) = \frac{1}{\varepsilon\mu} d(\delta_{\varepsilon\mu}^{\delta_{\mu}^{x} u} \Delta_{\mu}^{x}(u,v), \delta_{\varepsilon\mu}^{\delta_{\mu}^{x} u} \Delta_{\mu}^{x}(u,w)) =$$

$$= (\delta^{\delta_{\mu}^{x} u}, \varepsilon\mu)(\Delta_{\mu}^{x}(u,v), \Delta_{\mu}^{x}(u,w)).$$

The axiom A3 is then a consequence of axiom A3 for (X, d, δ) and we have

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (\delta^{x}, \mu)(\hat{\delta}_{\mu\varepsilon}^{x,u} v, \hat{\delta}_{\mu\varepsilon}^{x,u} w) = d^{\delta_{\mu}^{x} u}(\Delta_{\mu}^{x}(u,v), \Delta_{\mu}^{x}(u,w)).$$

The axiom A4 is also a straightforward consequence of A4 for (X, d, δ). The second part of the theorem is a simple computation. □

The induced dilation structures $(U(x), \hat{\delta}_{\mu}^{x}, (\delta^{x}, \mu))$ should converge in some sense to the dilation structure on the tangent space at x, as $\nu(\mu)$ converges to zero. Remark that we have one easy convergence in strong dilation structures:

$$\lim_{\mu \to 0} \hat{\delta}_{\mu,\varepsilon}^{x,u} v \ = \ \bar{\delta}_{\varepsilon}^{x,u} v$$

where $\bar{\delta}^{x}$ are the dilations in the tangent space at x, cf. (13.2.9). Indeed, this comes from:

$$\hat{\delta}_{\mu,\varepsilon}^{x,u} v \ = \ \Sigma_{\mu}^{x}(u, \delta_{\varepsilon}^{\delta_{\mu}^{x} u} \Delta_{\mu}^{x}(u,v))$$

so, when $\nu(\mu)$ converges we get the mentioned limit.

The following proposition gives a more precise estimate: the order of approximation of the dilations δ by dilations $\hat{\delta}_{\varepsilon}^{x}$, in neighbourhoods of $\delta_{\varepsilon}^{x} y$ of order ε, as $\nu(\varepsilon)$ goes to zero.

Proposition 15.8 *Let (X, δ, d) be a dilation structure. With the notations of theorem 15.7 we introduce*

$$\hat{\delta}_{\varepsilon}^{x,u} v = \hat{\delta}_{\varepsilon,\varepsilon}^{x,u} v = \delta_{\varepsilon^{-1}}^{x} \delta_{\varepsilon}^{\delta_{\varepsilon}^{x} u} \delta_{\varepsilon}^{x} v \quad .$$

Then we have for any x, y, v sufficiently closed:

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \, (\delta^{x}, \varepsilon) \left(\delta_{\varepsilon}^{\delta_{\varepsilon}^{x} y} v \, , \, \hat{\delta}_{\varepsilon}^{x, \delta_{\varepsilon}^{x} y} v \right) \ = \ 0 \quad . \tag{15.2.1}$$

Proof. We start by a computation:

$$\frac{1}{\varepsilon} \, (\delta^{x}, \varepsilon) \left(\delta_{\varepsilon}^{\delta_{\varepsilon}^{x} y} v \, , \, \hat{\delta}_{\varepsilon}^{x, \delta_{\varepsilon}^{x} y} v \right) = \frac{1}{\varepsilon^{2}} \, d \, \left(\delta_{\varepsilon}^{x} \delta_{\varepsilon}^{\delta_{\varepsilon}^{x} y} v \, , \, \delta_{\varepsilon}^{x} \hat{\delta}_{\varepsilon}^{x, \delta_{\varepsilon}^{x} y} v \right) =$$

$$= \frac{1}{\varepsilon^{2}} \, d \, \left(\delta_{\varepsilon^{2}}^{x} \Sigma_{\varepsilon}^{x}(y, v) \, , \, \delta_{\varepsilon^{2}}^{x} \, \delta_{\varepsilon^{-2}}^{x} \delta_{\varepsilon^{2}}^{\delta_{\varepsilon^{2}}^{x} y} \, \Delta_{\varepsilon}^{x}(\delta_{\varepsilon}^{x} y, v) \right) =$$

$$= \frac{1}{\varepsilon^{2}} \, d \, \left(\delta_{\varepsilon^{2}}^{x} \Sigma_{\varepsilon}^{x}(y, v) \, , \, \delta_{\varepsilon^{2}}^{x} \Sigma_{\varepsilon^{2}}^{x} \, (y, \Delta_{\varepsilon}^{x}(\delta^{x} y, v)) \right) \quad .$$

This last expression converges as $\nu(\varepsilon)$ goes to 0 to

$$d^{x} \left(\Sigma^{x}(y, v), \Sigma^{x}(y, \Delta^{x}(x, v)) \right) = d^{x} \left(v, \Delta^{x}(x, v) \right) = 0$$

□

The result from this proposition indicates that strong dilation structures are infinitesimally linear. In order to make a precise statement we need a measure for nonlinearity of a dilation structure, given in the next definition. Then we have to repeat the computations from the proof of proposition 15.8 in a slightly different setting, related to this measure of nonlinearity.

Definition 15.9 *The following expression:*

$$Lin(x,y,z;\varepsilon,\mu) \ = \ d\left(\delta^x_\varepsilon\,\delta^y_\mu z\,,\,\delta_\mu^{\delta^x_\varepsilon y}\delta^x_\varepsilon z\right) \tag{15.2.2}$$

is a measure of lack of linearity, for a general dilation structure.

The next theorem shows that indeed, infinitesimally any strong dilation structure is linear.

Theorem 15.10 *Let (X,d,δ) be a strong dilation structure. Then for any $x,y,z \in X$ sufficiently close we have*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\,Lin(x,\delta^x_\varepsilon y,z;\varepsilon,\varepsilon) \ = \ 0 \quad . \tag{15.2.3}$$

Proof. From the hypothesis of the theorem we have:

$$\frac{1}{\varepsilon^2}\,Lin(x,\delta^x_\varepsilon y,\delta^x_\varepsilon z;\varepsilon,\varepsilon) \ = \ \frac{1}{\varepsilon^2}\,d\left(\delta^x_\varepsilon\,\delta_\varepsilon^{\delta^x_\varepsilon y}z\,,\,\delta_\varepsilon^{\delta^x_{\varepsilon^2}y}\delta^x_\varepsilon z\right) \ =$$

$$= \ \frac{1}{\varepsilon^2}\,d\left(\delta^x_{\varepsilon^2}\,\Sigma^x_\varepsilon(y,z)\,,\,\delta^x_{\varepsilon^2}\,\delta^x_{\varepsilon^{-2}}\,\delta_\varepsilon^{\delta^x_{\varepsilon^2}y}\delta^x_\varepsilon z\right) \ =$$

$$= \ \frac{1}{\varepsilon^2}\,d\left(\delta^x_{\varepsilon^2}\,\Sigma^x_\varepsilon(y,z)\,,\,\delta^x_{\varepsilon^2}\,\Sigma^x_{\varepsilon^2}(y\,,\,\Delta^x_\varepsilon(\delta^x_\varepsilon y,z))\right) \ =$$

$$= \ \mathcal{O}(\varepsilon^2) + d^x\left(\Sigma^x_\varepsilon(y,z)\,,\,\Sigma^x_{\varepsilon^2}(y\,,\,\Delta^x_\varepsilon(\delta^x_\varepsilon y,z))\right) \quad .$$

The dilation structure satisfies A4, therefore as ε goes to 0 we obtain:

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\,Lin(x,\delta^x_\varepsilon y,\delta^x_\varepsilon z;\varepsilon,\varepsilon) \ = \ d^x\left(\Sigma^x(y,z)\,,\,\Sigma^x(y\,,\,\Delta^x(x,z))\right) \ =$$

$$= \ d^x\left(\Sigma^x(y,z)\,,\,\Sigma^x(y,z)\right) \ = \ 0 \quad . \qquad \square$$

15.3 Linear strong dilation structures

Remark that for general dilation structures the "translations" $\Delta^x_\varepsilon(u,\cdot)$ are not affine. Nevertheless, they commute with dilation in a known way: for any u,v sufficiently close to x and $\mu\in\Gamma$, $\nu(\mu)<1$, we have:

$$\Delta^x_\varepsilon\left(\delta^x_\mu u,\delta^x_\mu v\right) = \delta_\mu^{\delta^x_{\epsilon\mu}u}\Delta^x_{\varepsilon\mu}(u,v) \quad .$$

This is important, because the transformations $\Sigma^x_\varepsilon(u,\cdot)$ really behave as translations. The reason for which such transformations are not affine is that dilations are generally not affine.

Linear dilation structures are very particular dilation structures. The next proposition gives a family of examples of linear dilation structures.

Proposition 15.11 *The dilation structure associated to a normed conical group is linear.*

Proof. Indeed, for the dilation structure associated to a normed conical group we have, with the notations from definition 15.5:

$$\delta_{\mu}^{\delta_{\varepsilon}^{x} y} \delta_{\varepsilon}^{x} z \;=\; \left(x\delta_{\varepsilon}(x^{-1}y)\right)\,\delta_{\mu}\left(\delta_{\varepsilon}(y^{-1}x)\,x^{-1}\,x\,\delta_{\varepsilon}(x^{-1}z)\right) \;=$$

$$=\; \left(x\delta_{\varepsilon}(x^{-1}y)\right)\,\delta_{\mu}\left(\delta_{\varepsilon}(y^{-1}x)\,\delta_{\varepsilon}(x^{-1}z)\right) \;=\; \left(x\delta_{\varepsilon}(x^{-1}y)\right)\,\delta_{\mu}\left(\delta_{\varepsilon}(y^{-1}z)\right) \;=$$

$$=\; x\left(\delta_{\varepsilon}(x^{-1}y)\,\delta_{\varepsilon}\,\delta_{\mu}(y^{-1}z)\right) \;=\; x\,\delta_{\varepsilon}\left(x^{-1}y\,\delta_{\mu}(y^{-1}z)\right) \;=\; \delta_{\varepsilon}^{x}\,\delta_{\mu}^{y} z \quad .$$

Therefore the dilation structure is linear. □

The affinity of translations Σ_{ε}^{x} is related to the linearity of the dilation structure, as described in the theorem below, point (a). As a consequence, we prove at point (b) that a linear and strong dilation structure comes from a conical group.

Theorem 15.12 *Let (X,d,δ) be a dilation structure.*

(a) If the dilation structure is linear then all transformations $\Delta_{\varepsilon}^{x}(u,\cdot)$ are affine for any $u \in X$.

(b) If the dilation structure is strong (satisfies A4) then it is linear if and only if the dilations come from the dilation structure of a conical group, precisely for any $x \in X$ there is an open neighbourhood $D \subset X$ of x such that $(\overline{D}, d^{x}, \delta)$ is the same dilation structure as the dilation structure of the tangent space of (X,d,δ) at x.

Proof. (a) If dilations are affine, then let $\varepsilon, \mu \in \Gamma$, $\nu(\varepsilon), \nu(\mu) \leq 1$, and $x, y, u, v \in X$ such that the following computations make sense. We have:

$$\Delta_{\varepsilon}^{x}(u, \delta_{\mu}^{y} v) = \delta_{\varepsilon^{-1}}^{\delta_{\varepsilon}^{x} u} \delta_{\varepsilon}^{x} \delta_{\mu}^{y} v \quad .$$

Let $A_{\varepsilon} = \delta_{\varepsilon^{-1}}^{\delta_{\varepsilon}^{x} u}$. We compute:

$$\delta_{\mu}^{\Delta_{\varepsilon}^{x}(u,y)} \Delta_{\varepsilon}^{x}(u,v) = \delta_{\mu}^{A_{\varepsilon}\delta_{\varepsilon}^{x} y} A_{\varepsilon} \delta_{\varepsilon}^{x} v \quad .$$

We use twice the affinity of dilations:

$$\delta_{\mu}^{\Delta_{\varepsilon}^{x}(u,y)} \Delta_{\varepsilon}^{x}(u,v) = A_{\varepsilon} \delta_{\mu}^{\delta_{\varepsilon}^{x} y} \delta_{\varepsilon}^{x} v = \delta_{\varepsilon^{-1}}^{\delta_{\varepsilon}^{x} u} \delta_{\varepsilon}^{x} \delta_{\mu}^{y} v \quad .$$

We proved that:

$$\Delta_{\varepsilon}^{x}(u, \delta_{\mu}^{y} v) = \delta_{\mu}^{\Delta_{\varepsilon}^{x}(u,y)} \Delta_{\varepsilon}^{x}(u,v) \quad ,$$

which is the conclusion of the part (a).

(b) Suppose that the dilation structure is strong. If dilations are affine, then by point (a) the transformations $\Delta_{\varepsilon}^{x}(u,\cdot)$ are affine as well for any $u \in X$. Then, with notations made before, for $y = u$ we get

$$\Delta_{\varepsilon}^{x}(u, \delta_{\mu}^{u} v) = \delta_{\mu}^{\delta_{\varepsilon}^{x} u} \Delta_{\varepsilon}^{x}(u,v) \quad ,$$

which implies

$$\delta_{\mu}^{u} v = \Sigma_{\varepsilon}^{x}(u, \delta_{\mu}^{x} \Delta_{\varepsilon}^{x}(u,v)) \quad .$$

We pass to the limit with $\varepsilon \to 0$ and we obtain:

$$\delta_{\mu}^{u} v = \Sigma^{x}(u, \delta_{\mu}^{x} \Delta^{x}(u,v)) \quad .$$

We recognize at the right hand side the dilations associated to the conical group $T_{x}X$.

By proposition 15.11 the opposite implication is straightforward, because the dilation structure of any conical group is linear. □

16 Noncommutative affine geometry

We propose here to call "noncommutative affine geometry" the generalization of affine geometry described in theorem 5.5, but without the restriction $\Gamma = (0, +\infty)$. For short, noncommutative affine geometry in the sense explained further is the study of the properties of linear strong dilation structures. Equally, by theorem 15.12, it is the study of normed conical groups.

As a motivation for this name, in the proposition below we give a relation, true for linear dilation structures, with an interesting interpretation. We shall explain what this relation means in the most trivial case: the dilation structure associated to a real normed affine space. In this case, for any points x, u, v, let us denote $w = \Sigma_\varepsilon^x(u, v)$. Then w equals (approximatively, due to the parameter ε) the sum $u +_x v$. Denote also $w' = \Delta_\varepsilon^u(x, v)$; then w' is (approximatively) equal to the difference between v and x based at u. In our space (a classical affine space over a vector space) we have $w = w'$. The next proposition shows that the same is true for any linear dilation structure.

Proposition 16.1 *For a linear dilation structure (X, δ, d), for any $x, u, v \in X$ sufficiently closed and for any $\varepsilon \in \Gamma$, $\nu(\varepsilon) \leq 1$, we have:*

$$\Sigma_\varepsilon^x(u, v) \ = \ \Delta_\varepsilon^u(x, v) \quad .$$

Proof. We have the following string of equalities, by using twice the linearity of the dilation structure:

$$\Sigma_\varepsilon^x(u, v) \ = \ \delta_{\varepsilon^{-1}}^x \delta_\varepsilon^{\delta_\varepsilon^x u} v \ = \ \delta_\varepsilon^u \, \delta_{\varepsilon^{-1}}^x v \ =$$
$$= \ \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^u x} \delta_\varepsilon^u v \ = \ \Delta_\varepsilon^u(x, v) \quad .$$

The proof is done. $\square$

16.1 Inverse semigroups and Menelaos theorem

Here we prove that for strong dilation structures linearity is equivalent to a generalization of the statement from corollary 5.8. The result is new for Carnot groups and the proof seems to be new even for vector spaces.

Definition 16.2 *A semigroup S is an inverse semigroup if for any $x \in S$ there is an unique element $x^{-1} \in S$ such that $x \, x^{-1} x = x$ and $x^{-1} x \, x^{-1} = x^{-1}$.*

An important example of an inverse semigroup is $I(X)$, the class of all bijective maps $\phi : dom\, \phi \to im\, \phi$, with $dom\, \phi, im\, \phi \subset X$. The semigroup operation is the composition of functions in the largest domain where this makes sense.

By the Vagner-Preston representation theorem [43] every inverse semigroup is isomorphic to a subsemigroup of $I(X)$, for some set X.

Definition 16.3 *A dilation structure (X, d, δ) has the Menelaos property if for any two sufficiently closed $x, y \in X$ and for any $\varepsilon, \mu \in \Gamma$ with $\nu(\varepsilon), \nu(\mu) \in (0, 1)$ we have*

$$\delta_\varepsilon^x \, \delta_\mu^y \ = \ \delta_{\varepsilon\mu}^w \quad ,$$

where $w \in X$ is the fixed point of the contraction $\delta_\varepsilon^x \delta_\mu^y$ (thus depending on x, y and ε, μ).

Theorem 16.4 *A linear dilation structure has the Menelaos property.*

Proof. Let $x, y \in X$ be sufficiently closed and $\varepsilon, \mu \in \Gamma$ with $\nu(\varepsilon), \nu(\mu) \in (0,1)$. We shall define two sequences $x_n, y_n \in X$, $n \in \mathbb{N}$.

We begin with $x_0 = x$, $y_0 = y$. Suppose further that x_n, y_n were defined and that they are sufficiently closed. Then we use twice the linearity of the dilation structure:

$$\delta_\varepsilon^{x_n} \delta_\mu^{y_n} \ = \ \delta_\mu^{\delta_\varepsilon^{x_n} y_n} \delta_\varepsilon^{x_n} \ = \ \delta_\varepsilon^{\delta_\mu^{\delta_\varepsilon^{x_n} y_n} x_n} \delta_\mu^{\delta_\varepsilon^{x_n} y_n} \quad .$$

We shall define then by induction

$$x_{n+1} \ = \ \delta_\mu^{\delta_\varepsilon^{x_n} y_n} x_n \quad , \quad y_{n+1} \ = \ \delta_\varepsilon^{x_n} y_n \quad . \tag{16.1.1}$$

Provided that we prove by induction that x_n, y_n are sufficiently closed, we arrive at the conclusion that for any $n \in \mathbb{N}$

$$\delta_\varepsilon^{x_n} \delta_\mu^{y_n} \ = \ \delta_\varepsilon^{x} \delta_\mu^{y} \quad . \tag{16.1.2}$$

The points x_0, y_0 are sufficiently closed by hypothesis. Suppose now that x_n, y_n are sufficiently closed. Due to the linearity of the dilation structure, we can write the first part of (16.1.1) as:

$$x_{n+1} \ = \delta_\varepsilon^{x_n} \delta_\mu^{y_n} x_n \quad .$$

Then we can estimate the distance between x_{n+1}, y_{n+1} like this:

$$d(x_{n+1}, y_{n+1}) \ = \ d(\delta_\varepsilon^{x_n} \delta_\mu^{y_n} x_n, \delta_\varepsilon^{x_n} y_n) \ = \ \nu(\varepsilon)\, d(\delta_\mu^{y_n} x_n, y_n) \ = \ \nu(\varepsilon\mu) d(x_n, y_n) \quad .$$

From $\nu(\varepsilon\mu) < 1$ it follows that x_{n+1}, y_{n+1} are sufficiently closed. By induction we deduce that for all $n \in \mathbb{N}$ the points x_{n+1}, y_{n+1} are sufficiently closed. We also find out that

$$\lim_{n \to \infty} d(x_n, y_n) \ = \ 0 \quad . \tag{16.1.3}$$

From relation (16.1.2) we deduce that the first part of (16.1.1) can be written as:

$$x_{n+1} \ = \delta_\varepsilon^{x_n} \delta_\mu^{y_n} x_n \ = \ \delta_\varepsilon^{x} \delta_\mu^{y} x_n \quad .$$

The transformation $\delta_\varepsilon^x \delta_\mu^y$ is a contraction of coefficient $\nu(\varepsilon\mu) < 1$, therefore we easily get:

$$\lim_{n \to \infty} x_n \ = \ w \quad , \tag{16.1.4}$$

where w is the unique fixed point of the contraction $\delta_\varepsilon^x \delta_\mu^y$.

We put together (16.1.3) and (16.1.4) and we get the limit:

$$\lim_{n \to \infty} y_n \ = \ w \quad , \tag{16.1.5}$$

Using relations (16.1.4), (16.1.5), we may pass to the limit with $n \to \infty$ in relation (16.1.2):

$$\delta_\varepsilon^x \delta_\mu^y \ = \ \lim_{n \to \infty} \delta_\varepsilon^{x_n} \delta_\mu^{y_n} \ = \ \delta_\varepsilon^w \delta_\mu^w \ = \ \delta_{\varepsilon\mu}^w \quad .$$

The proof is done. $\square$

Corollary 16.5 *Let (X, d, δ) be a strong linear dilation structure, with group Γ and the morphism ν injective. Then any element of the inverse subsemigroup of $I(X)$ generated by dilations is locally a dilation δ_ε^x or a left translation $\Sigma^x(y, \cdot)$.*

Proof. Let (X, d, δ) be a strong linear dilation structure. From the linearity and theorem 16.4 we deduce that we have to care only about the results of compositions of two dilations which are isometries.

The dilation structure is strong, therefore by theorem 15.12 the dilation structure is locally coming from a conical group.

Let us compute a composition of dilations $\delta^x_\varepsilon \delta^y_\mu$, with $\nu(\varepsilon\mu) = 1$. Because the morphism ν is injective, it follows that $\mu = \varepsilon^{-1}$. In a conical group we can make the following computation (here $\delta_\varepsilon = \delta^e_\varepsilon$ with e the neutral element of the conical group):

$$\delta^x_\varepsilon \delta^y_{\varepsilon^{-1}} z \ = \ x\delta_\varepsilon \left(x^{-1} y \delta_{\varepsilon^{-1}} \left(y^{-1} z\right)\right) \ = \ x\delta_\varepsilon \left(x^{-1} y\right) y^{-1} z \quad .$$

Therefore the composition of dilations $\delta^x_\varepsilon \delta^y_\mu$, with $\varepsilon\mu = 1$, is a left translation.

Another easy computation shows that composition of left translations with dilations are dilations. The proof end by remarking that all the statements are local. □

A counterexample. The Corollary 16.5 is not true without the injectivity assumption on ν. Indeed, consider the Carnot group $N = \mathbb{C} \times \mathbb{R}$ with the elements denoted by $X \in N$, $X = (x, x')$, with $x \in \mathbb{C}$, $x' \in \mathbb{R}$, and operation

$$X\,Y \ = \ (x, x')(y, y') \ = \ (x + y, x' + y' + \frac{1}{2} Im\, x\bar{y})$$

We take $\Gamma = \mathbb{C}^*$ and morphism $\nu : \Gamma \to (0, +\infty)$, $\nu(\varepsilon) =\mid \varepsilon \mid$. Dilations are defined as: for any $\varepsilon \in \mathbb{C}^*$ and $X = (x, x') \in N$:

$$\delta_\varepsilon X \ = \ (\varepsilon x, \mid \varepsilon \mid^2 x')$$

These dilations induce the field of dilations $\delta^X_\varepsilon Y \ = \ X\delta_\varepsilon(X^{-1}Y)$.

The morphism ν is not injective. Let now $\varepsilon, \mu \in \mathbb{C}^*$ with $\varepsilon\mu = -1$ and $\varepsilon \in (0, 1)$. An elementary (but a bit long) computation shows that for $X = (0, 0)$ and $Y = (y, y')$ with $y \neq 0$, $y' \neq 0$, the composition of dilations $\delta^X_\varepsilon \delta^Y_\mu$ is not a left translation in the group N, nor a dilation. □

Further we shall suppose that the morphism ν is always injective, if not explicitly stated otherwise. Therefore we shall consider $\Gamma \subset (0, +\infty)$ as a subgroup.

16.2 On the barycentric condition

The barycentric condition is (Af3): for any $\varepsilon \in (0, 1)$ $\delta^x_\varepsilon\, y \ = \ \delta^y_{1-\varepsilon}\, x$. In particular, the condition (Af3) tells that the transformation $y \mapsto \delta^y_\varepsilon x$ is also a dilation. Is this true for linear dilation structures? Theorem 5.5 indicates that (Af3) is true if and only if this is a dilation structure of a normed real affine space.

Proposition 16.6 *Let X be a normed conical group with neutral element e, dilations δ and distance d induced by the homogeneous norm $\| \cdot \|$, and $\varepsilon \in (0, 1) \cap \Gamma$. Then the function*

$$h_\varepsilon : X \to X \quad , \quad h_\varepsilon(x) = x\delta_\varepsilon(x^{-1}) = \delta^x_\varepsilon\, e$$

is invertible and the inverse g_ε has the expression

$$g_\varepsilon(y) \ = \ \prod_{k=0}^{\infty} \delta_{\varepsilon^k}(y) \ = \ \lim_{N\to\infty} \prod_{k=0}^{N} \delta_{\varepsilon^k}(y)$$

Remark 16.7 *As the choice of the neutral element is not important, the previous proposition says that for any $\varepsilon \in (0, 1)$ and any fixed $y \in X$ the function $x \mapsto \delta^x_\varepsilon y$ is invertible.*

Proof. Let $\varepsilon \in (0,1)$ be fixed. For any natural number N we define $g_N : X \to X$ by

$$g_N(y) \ = \ \prod_{k=0}^{N} \delta_{\varepsilon^k}(y)$$

For fixed $y \in X$ $(g_N(y))_N$ is a Cauchy sequence. Indeed, for any $N \in \mathbb{N}$ we have:

$$d(g_N(y), g_{N+1}(y)) \ = \ \|\delta_{\varepsilon^{N+1}}(y)\|$$

thus for any $N, M \in \mathbb{N}$, $M \geq 1$ we have

$$d(g_N(y), g_{N+M}(y)) \ \leq \ \left(\sum_{k=N+1}^{M} \varepsilon^k \right) \|y\| \ \leq \ \frac{\varepsilon^{N+1}}{1-\varepsilon} \|y\|$$

Let then $g_\varepsilon(y) \ = \ \lim_{N \to \infty} g_N(y)$. We prove that g_ε is the inverse of h_ε. We have, for any natural number N and $y \in X$

$$y \, \delta_\varepsilon g_N(y) \ = \ g_{N+1}(y)$$

By passing to the limit with N we get that $h_\varepsilon \circ g_\varepsilon(y) = y$ for any $y \in X$.

Let us now compute

$$g_N \circ h_\varepsilon(x) \ = \ \prod_{k=0}^{N} \delta_{\varepsilon^k}(x \delta_\varepsilon(x^{-1})) \ = \ \prod_{k=0}^{N} \delta_{\varepsilon^k}(x) \, \delta_{\varepsilon^{k+1}}(x^{-1}) \ =$$

$$= \ x \, \delta_{\varepsilon^{N+1}}(x^{-1})$$

therefore as N goes to infinity we get $g_\varepsilon \circ h_\varepsilon(x) \ = \ x$. $\square$

For any $\varepsilon \in (0,1)$ the functions $h_\varepsilon, g_\varepsilon$ are homogeneous, that is

$$h_\varepsilon(\delta_\mu x) \ = \ \delta_\mu \, h_\varepsilon(x) \quad , \quad g_\varepsilon(\delta_\mu y) \ = \ \delta_\mu \, g_\varepsilon(y)$$

for any $\mu > 0$ and $x, y \in X$.

In the presence of the barycentric condition we get the following:

Corollary 16.8 *Let (X, d, δ) be a strong dilation structure with group $\Gamma \subset (0, +\infty)$, which satisfies the barycentric condition (Af3). Then for any $u, v \in X$ and $\varepsilon \in (0,1) \cap \Gamma$ the points $inv^u(v)$, u and $\delta^u_\varepsilon v$ are collinear in the sense:*

$$d(inv^u(v), u) + d(u, \delta^u_\varepsilon v) \ = \ d(inv^u(v), \delta^u_\varepsilon v)$$

Proof. There is no restriction to work with the group operation with neutral element e and denote $\delta_\varepsilon := \delta^e_\varepsilon$. With the notation from the proof of the proposition 16.6, we use the expression of the function g_ε, we apply the homogeneous norm $\| \cdot \|$ and we obtain:

$$\|g_\varepsilon(y)\| \ \leq \ \left(\sum_{k=0}^{\infty} \varepsilon^k \right) \|x\| \ = \ \frac{1}{1-\varepsilon} \|y\|$$

with equality if and only if e, y and $y\delta_\varepsilon y$ are collinear in the sense $d(e, y) + d(y, y\delta_\varepsilon y) \ = \ d(e, y\delta_\varepsilon y)$. The barycentric condition can be written as: $h_\varepsilon(x) = \delta_{1-\varepsilon}(x)$. We have therefore:

$$\|x\| \ = \ \|g_\varepsilon \circ h_\varepsilon(x)\| \ \leq \ \frac{1}{1-\varepsilon} \|h_\varepsilon(x)\| \ = \ \frac{1-\varepsilon}{1-\varepsilon} \|x\| \ = \ \|x\|$$

therefore e, x and $x\delta_\varepsilon x$ are on a geodesic. This is true also for the choice: $e = inv^u(v)$, $x = u$, which gives the conclusion. $\square$

We can actually say more in the case $\Gamma = (0, +\infty)$.

Proposition 16.9 *Let (X, d, δ) be a strong dilation structure with group $\Gamma = (0, +\infty)$, which satisfies the barycentric condition (Af3). Then for any $x \in X$ the group operation Σ^x is abelian and moreover the graduation of X, as a homogeneous group with respect to the operation Σ^x has only one level.*

Proof. Let us denote the neutral element by e instead of x and denote $\delta_\varepsilon := \delta^e_\varepsilon$. According to corollary 13.8 X is a Lie homogeneous group. The barycentric condition implies: for any $x \in X$ and $\varepsilon \in (0, 1)$ we have $\delta_{1-\varepsilon} y \ = \ y \delta_\varepsilon y^{-1}$, which implies:

$$\delta_{1-\varepsilon}(y) \ \delta_\varepsilon(y) \ = \ y$$

for any y and for any $\varepsilon \in (0, 1)$. This fact implies that $\{\delta_\mu y : \mu \in (0, +\infty)\}$ is a one parameter semigroup. Moreover, let $f_y : \mathbb{R} \to X$, defined by: if $\varepsilon > 0$ then $f_y(\varepsilon) = \delta_\varepsilon y$, else $f_y(\varepsilon) = \delta_\varepsilon y^{-1}$. Then f_y is a group morphism from $\mathbb{R}$ to X, with $f_y(1) = y$. Therefore $f_y(\varepsilon) \ = \ \exp(\varepsilon y) \ = \ \varepsilon y$. According to definition 5.3 the group X is identified with its Lie algebra and any element y has a decomposition $y = y_1 + y_2 + ... + y_m$ and $\delta_\varepsilon y \ = \ \sum_{j=1}^{m} \varepsilon^j y_j$. We proved that $m = 1$, otherwise said that the graduation of the group has only one level, that is the group is abelian. $\square$

16.3 The ratio of three collinear points

In this section we prove that the noncommutative affine geometry is a geometry in the sense of the Erlangen program, because it can be described as the geometry of collinear triples (see definition 16.10). Collinear triples generalize the basic ratio invariant of classical affine geometry.

Indeed, theorem 16.4 provides us with a mean to introduce a version of the ratio of three collinear points in a strong linear dilation structure. We define here *collinear triples*, *the ratio function* and *the ratio norm*.

Definition 16.10 *Let (X, d, δ) be a strong linear dilation structure. Denote by $x^\alpha = (x, \alpha)$, for any $x \in X$ and $\alpha \in (0, +\infty)$. An ordered set $(x^\alpha, y^\beta, z^\gamma) \in (X \times (0, +\infty))^3$ is a collinear triple if:*

(a) $\alpha\beta\gamma = 1$ and all three numbers are different from 1,

(b) we have $\delta^x_\alpha \, \delta^y_\beta \, \delta^z_\gamma = id$.

The ratio norm $r(x^\alpha, y^\beta, z^\gamma)$ of the collinear triple $(x^\alpha, y^\beta, z^\gamma)$ is given by the expression:

$$r(x^\alpha, y^\beta, z^\gamma) \ = \ \frac{\alpha}{1 - \alpha\beta}$$

Let $(x^\alpha, y^\beta, z^\gamma)$ be a collinear triple. Then we have: $\delta^x_\alpha \, \delta^y_\beta \ = \ \delta^z_{\alpha\beta}$ with $\alpha, \beta, \alpha\beta$ not equal to 1. By theorem 16.4 the point z is uniquely determined by (x^α, y^β), therefore we can express it as a function $z = w(x, y, \alpha, \beta)$. The function w is called the ratio function.

In the next proposition we obtain a formula for $w(x, y, \alpha, \beta)$. Alternatively this can be seen as another proof of theorem 16.4.

Proposition 16.11 *In the hypothesis of proposition 16.6, for any $\varepsilon, \mu \in (0, 1)$ and $x, y \in X$ we have:*

$$w(x, y, \varepsilon, \mu) \ = \ g_{\varepsilon\mu}\left(h_\varepsilon(x) h_\mu(\delta_\varepsilon y)\right)$$

Proof. Any $z \in X$ with the property that for any $u \in X$ we have $\delta^x_\varepsilon \, \delta^y_\mu(u) \ = \ \delta^z_{\varepsilon\mu}(u)$ satisfies the equation:

$$x \, \delta_\varepsilon \left(x^{-1} y \delta_\mu(y^{-1})\right) \ = \ z \delta_{\varepsilon\mu}(z^{-1}) \qquad (16.3.6)$$

This equation can be put as:

$$h_\varepsilon(x) \, \delta_\varepsilon \left(h_\mu(y)\right) \ = \ h_{\varepsilon\mu}(z)$$

From proposition 16.6 we obtain that indeed exists and it is unique $z \in X$ solution of this equation. We use further homogeneity of h_μ and we get:

$$z \ = \ w(x, y, \varepsilon, \mu) \ = \ g_{\varepsilon\mu}\left(h_\varepsilon(x) h_\mu(\delta_\varepsilon y)\right) \quad \square$$

Remark that if $(x^\alpha, y^\beta, z^\gamma)$ is a collinear triple then any circular permutation of the triple is also a collinear triple. We can not deduce from here a collinearity notion for the triple of *points* $\{x, y, z\}$. Indeed, as the following example shows, even if $(x^\alpha, y^\beta, z^\gamma)$ is a collinear triple, it may happen that here are no numbers α', β', γ' such that $(y^{\beta'}, x^{\alpha'}, z^{\gamma'})$ is a collinear triple.

Collinear triples in the Heisenberg group. The Heisenberg group $H(n) = \mathbb{R}^{2n+1}$ is a 2-step Carnot group. For the points of $X \in H(n)$ we use the notation $X = (x, \bar{x})$, with $x \in \mathbb{R}^{2n}$ and $\bar{x} \in \mathbb{R}$. The group operation is :

$$X \, Y \ = \ (x, \bar{x})(y, \bar{y}) \ = \ (x + y, \bar{x} + \bar{y} + \frac{1}{2}\omega(x, y))$$

where ω is the standard symplectic form on $\mathbb{R}^{2n}$. We shall identify the Lie algebra with the Lie group. The bracket is

$$[(x, \bar{x}), (y, \bar{y})] = (0, \omega(x, y))$$

The Heisenberg algebra is generated by

$$V = \mathbb{R}^{2n} \times \{0\}$$

and we have the relations $V + [V, V] = H(n)$, $\{0\} \times \mathbb{R} \ = \ [V, V] \ = \ Z(H(n))$.

The dilations on $H(n)$ are

$$\delta_\varepsilon(x, \bar{x}) = (\varepsilon x, \varepsilon^2 \bar{x})$$

For $X = (x, \bar{x}), Y = (y, \bar{y}) \in H(n)$ and $\varepsilon, \mu \in (0, +\infty)$, $\varepsilon\mu \neq 1$, we compute $Z = (z, \bar{z}) = w(\tilde{x}, \tilde{y}, \varepsilon, \mu)$ with the help of equation (16.3.6). This equation writes:

$$((1 - \varepsilon)x, (1 - \varepsilon^2)\bar{x}) \left(\varepsilon(1 - \mu)y, \varepsilon^2(1 - \mu^2)\bar{y}\right) \ = \ ((1 - \varepsilon\mu)z, (1 - \varepsilon^2\mu^2)\bar{z})$$

After using the expression of the group operation we obtain:

$$Z = \left(\frac{1 - \varepsilon}{1 - \varepsilon\mu}x + \frac{\varepsilon(1 - \mu)}{1 - \varepsilon\mu}y, \frac{1 - \varepsilon^2}{1 - \varepsilon^2\mu^2}\bar{x} + \frac{\varepsilon^2(1 - \mu^2)}{1 - \varepsilon^2\mu^2}\bar{y} + \frac{\varepsilon(1 - \varepsilon)(1 - \mu)}{2(1 - \varepsilon^2\mu^2)}\omega(x, y)\right)$$

Suppose now that $(X^{\alpha}, Y^{\beta}, Z^{\gamma})$ and $(Y^{\beta'}, X^{\alpha'}, Z^{\gamma'})$ are collinear triples such that $X = (x,0)$, $Y = (y,0)$ and $\omega(x,y) \neq 0$. From the computation of the ratio function, we get that there exist numbers $k, k' \neq 0$ such that:

$$z \;=\; kx \,+\, (1-k)y \;=\; (1-k')x \,+\, k'y \quad ,$$

$$\bar{z} \;=\; \frac{k(1-k)}{2}\,\omega(x,y) \;=\; \frac{k'(1-k')}{2}\,\omega(y,x)$$

From the equalities concerning z we get that $k' = 1-k$. This lead us to contradiction in the equalities concerning $\bar{z}$. Therefore, in this case, if $(X^{\alpha}, Y^{\beta}, Z^{\gamma})$ is a collinear triple then there are no α', β', γ' such that $(Y^{\beta'}, X^{\alpha'}, Z^{\gamma'})$ is a collinear triple. $\square$

In a general linear dilation structure the relation (5.2.5) does not hold. Nevertheless, there is some content of this relation which survives in the general context.

Proposition 16.12 *For x, y sufficiently closed and for $\varepsilon, \mu \in \Gamma$ with $\nu(\varepsilon), \nu(\mu) \in (0,1)$, we have the distance estimates:*

$$d(x, w(x,y,\varepsilon,\mu)) \;\leq\; \frac{\nu(\varepsilon)}{1-\nu(\varepsilon\mu)} d(x, \delta^{y}_{\mu} x) \tag{16.3.7}$$

$$d(y, w(x,y,\varepsilon,\mu)) \;\leq\; \frac{1}{1-\nu(\varepsilon\mu)} d(y, \delta^{x}_{\varepsilon} y) \tag{16.3.8}$$

Proof. Further we shall use the notations from the proof of theorem 16.4, in particular $w = w(x,y,\varepsilon,\mu)$. We define by induction four sequences of points (the first two sequences are defined as in relation (16.1.1)):

$$x_{n+1} \;=\; \delta_{\mu}^{\delta_{\varepsilon}^{x_n} y_n} x_n \quad , \quad y_{n+1} \;=\; \delta_{\varepsilon}^{x_n} y_n$$

$$x'_{n+1} \;=\; \delta_{\mu}^{\delta_{\varepsilon}^{y'_n} x'_n} x_n \quad , \quad y'_{n+1} \;=\; \delta_{\varepsilon}^{x'_{n+1}} y'_n$$

with initial conditions $x_0 = x, y_0 = y, x'_0 = x, y'_0 = \delta^{x}_{\varepsilon} y$. The first two sequences are like in the proof of theorem 16.4, while for the third and fourth sequences we have the relations $x'_n = x_n$, $y'_n = y_{n+1}$. These last sequences come from the fact that they appear if we repeat the proof of theorem 16.4 starting from the relation:

$$\delta_{\mu}^{\delta_{\varepsilon}^{x} y} \delta_{\varepsilon}^{x} \;=\; \delta^{w}_{\varepsilon\mu}$$

We know that all these four sequences converge to w as n goes to ∞. Moreover, we know from the proof of theorem 16.4 that for all $n \in \mathbb{N}$ we have

$$d(x_n, x_{n+1}) \;=\; d(x, \delta^{x}_{\varepsilon} \delta^{y}_{\mu} x) \nu(\varepsilon\mu)^n$$

There is an equivalent relation in terms of the sequence y'_n, which is the following:

$$d(y'_n, y'_{n+1}) \;=\; d(\delta^{x}_{\varepsilon} y, \delta_{\mu}^{\delta_{\varepsilon}^{x} y} \delta^{x}_{\varepsilon} \delta^{x}_{\varepsilon} y) \nu(\varepsilon\mu)^n$$

This relation becomes: for any $n \in \mathbb{N}$, $n \geq 1$

$$d(y_n, y_{n+1}) \;=\; d(y, \delta^{x}_{\varepsilon} y) \nu(\varepsilon\mu)^{n+1}$$

For the first distance estimate we write:

$$d(x,w) \ \leq \ \sum_{n=0}^{\infty} d(x_n, x_{n+1}) \ = \ d(x, \delta^x_\varepsilon \delta^y_\mu x) \left(\sum_{n=0}^{\infty} \nu(\varepsilon\mu)^n \right) \ = \ \frac{\nu(\varepsilon)}{1 - \nu(\varepsilon\mu)} d(x, \delta^y_\mu x)$$

For the second distance estimate we write:

$$d(y,w) \ \leq \ d(y, y_1) + \sum_{n=1}^{\infty} d(y_n, y_{n+1}) \ = \ d(y, y_1) + \frac{\nu(\varepsilon\mu)}{1 - \nu(\varepsilon\mu)} d(y, \delta^x_\varepsilon y) \ =$$

$$= \ d(y, \delta^x_\varepsilon y) \left(1 + \frac{\nu(\varepsilon\mu)}{1 - \nu(\varepsilon\mu)} \right) \ = \ \frac{1}{1 - \nu(\varepsilon\mu)} d(y, \delta^x_\varepsilon y)$$

and the proof is done. $\square$

For a collinear triple $(x^\alpha, y^\beta, z^\gamma)$ in a general linear dilation structure we cannot say that x, y, z lie on the same geodesic. This is false, as shown by easy examples in the Heisenberg group, the simplest noncommutative Carnot group.

Nevertheless, theorem 16.4 allows to speak about collinearity in the sense of definition 16.10.

Affine geometry is the study of relations which are invariant with respect to the group of affine transformations. An invertible transformation is affine if and only if it preserves the ratio of any three collinear points. We are thus arriving to the following definition.

Definition 16.13 *Let (X, d, δ) be a linear dilation structure. A geometrically affine transformation $T : X \rightarrow X$ is a Lipschitz invertible transformation such that for any collinear triple $(x^\alpha, y^\beta, z^\gamma)$ the triple $((Tx)^\alpha, (Ty)^\beta, (Tz)^\gamma)$ is collinear.*

The group of geometric affine transformations defines a geometry in the sense of Erlangen program. The main invariants of such a geometry are collinear triples. There is no obvious connection between collinearity and geodesics of the space, as in classical affine geometry. (It is worthy to notice that in fact, there might be no geodesics in the metric space (X, d) of the linear dilation structure (X, d, δ). For example, there are linear dilation structures defined over the boundary of the dyadic tree [13], which is homeomorphic with the middle thirds Cantor set.)

The first result for such a geometry is the following.

Theorem 16.14 *Let (X, d, δ) be a strong linear dilation structure. Any Lipschitz, invertible, transformation $T : (X, d) \rightarrow (X, d)$ is affine in the sense of definition 15.1 if and only if it is geometrically affine in the sense of definition 16.13.*

Proof. The first implication, namely T affine in the sense of definition 15.1 implies T affine in the sense of definition 16.13, is straightforward: by hypothesis on T, for any collinear triple $(x^\alpha, y^\beta, z^\gamma)$ we have the relation

$$T \, \delta^x_\alpha \, \delta^y_\beta \, \delta^z_\gamma \, T^{-1} \ = \ \delta^{Tx}_\alpha \, \delta^{Ty}_\beta \, \delta^{Tz}_\gamma$$

Therefore, if $(x^\alpha, y^\beta, z^\gamma)$ is a collinear triple then the triple $((Tx)^\alpha, (Ty)^\beta, (Tz)^\gamma)$ is collinear.

In order to show the inverse implication we use the linearity of the dilation structure. Let $x, y \in X$ and $\varepsilon, \eta \in \Gamma$. Then

$$\delta^x_\varepsilon \, \delta^y_\eta \delta^x_{\varepsilon^{-1}} \ = \ \delta^{\delta^x_\varepsilon y}_\eta$$

This identity leads us to the description of $\delta_\varepsilon^x y$ in terms of the ratio function. Indeed, we have:

$$\delta_\varepsilon^x y \ = \ w(w(x, y, \varepsilon, \eta), \varepsilon\eta, \varepsilon^{-1})$$

If the transformation T is geometrically affine then we easily find that it is affine in the sense of definition 15.1:

$$T\left(\delta_\varepsilon^x y\right) \ = \ w(w(Tx, Ty, \varepsilon, \eta), \varepsilon\eta, \varepsilon^{-1}) \ = \ \delta_\varepsilon^{Tx} Ty$$

$\square$

As a conclusion for this section, theorem 16.14 shows that in a linear dilation structure we may take dilations as the basic affine invariants. It is surprising that in such a geometry there is no obvious notion of a line, due to the fact that not any permutation of a collinear triple is again a collinear triple.

17 The Radon-Nikodym property

Definition 17.1 *A dilation structure (X, d, δ) has the Radon-Nikodym property if any Lipschitz curve $c : [a, b] \rightarrow (X, d)$ is derivable almost everywhere.*

Example 17.1 For $(X, d) = (\mathbb{V}, d)$, a real, finite dimensional, normed vector space, with distance d induced by the norm, the (usual) dilations δ_ε^x are given by:

$$\delta_\varepsilon^x y \ = \ x + \varepsilon(y - x)$$

Dilations are defined everywhere. The group Γ is $(0, +\infty)$ and the function ν is the identity.

There are few things to check (see the appendix): axioms 0,1,2 are obviously true. For axiom A3, remark that for any $\varepsilon > 0$, $x, u, v \in X$ we have:

$$\frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) \ = \ d(u, v) \ ,$$

therefore for any $x \in X$ we have $d^x = d$.

Finally, let us check the axiom A4. For any $\varepsilon > 0$ and $x, u, v \in X$ we have

$$\delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v \ = \ x + \varepsilon(u - x) + \frac{1}{\varepsilon}\left(x + \varepsilon(v - x) - x - \varepsilon(u - x)\right) \ =$$

$$= \ x + \varepsilon(u - x) + v - u$$

therefore this quantity converges to

$$x + v - u \ = \ x + (v - x) - (u - x)$$

as $\varepsilon \rightarrow 0$. The axiom A4 is verified.

This dilation structure has the Radon-Nikodym property. □

Example 17.2 Because dilation structures are defined by local requirements, we can easily define dilation structures on riemannian manifolds, using particular atlases of the manifold and the riemannian distance (infimum of length of curves joining two points). Note that any finite dimensional manifold can be endowed with a riemannian metric. This class of examples covers all dilation structures used in differential geometry. The axiom A4 gives an operation of addition of vectors in the tangent space (compare with Bellaïche [5] last section). □

Example 17.3 Take $X = \mathbb{R}^2$ with the euclidean distance d. For any $z \in \mathbb{C}$ of the form $z = 1 + i\theta$ we define dilations

$$\delta_\varepsilon x = \varepsilon^z x \ .$$

It is easy to check that $(\mathbb{R}^2, d, \delta)$ is a dilation structure, with dilations

$$\delta_\varepsilon^x y = x + \delta_\varepsilon(y - x) \quad .$$

Two such dilation structures (constructed with the help of complex numbers $1 + i\theta$ and $1 + i\theta'$) are equivalent if and only if $\theta = \theta'$.

There are two other interesting properties of these dilation structures. The first is that if $\theta \neq 0$ then there are no non trivial Lipschitz curves in X which are differentiable almost everywhere. It means that such dilation structure does not have the Radon-Nikodym property.

The second property is that any holomorphic and Lipschitz function from X to X (holomorphic in the usual sense on $X = \mathbb{R}^2 = \mathbb{C}$) is differentiable almost everywhere, but there are Lipschitz functions from X to X which are not differentiable almost everywhere (suffices to take a $\mathcal{C}^\infty$ function from $\mathbb{R}^2$ to $\mathbb{R}^2$ which is not holomorphic). □

The Radon-Nikodym property can be stated in two equivalent ways.

Proposition 17.2 *Let (X,d,δ) be a dilation structure. Then the following are equivalent:*

(a) *(X,d,δ) has the Radon-Nikodym property;*

(b) *any Lipschitz curve $c' : [a',b'] \rightarrow (X,d)$ admits a reparametrization $c : [a,b] \rightarrow (X,d)$ such that for almost every $t \in [a,b]$ there is $\dot{c}(t) \in U(c(t))$ such that*

$$\frac{1}{\varepsilon} d(c(t+\varepsilon), \delta_{\varepsilon}^{c(t)} \dot{c}(t)) \rightarrow 0$$

$$\frac{1}{\varepsilon} d(c(t-\varepsilon), \delta_{\varepsilon}^{c(t)} inv^{c(t)}(\dot{c}(t))) \rightarrow 0 \quad ;$$

(c) *any Lipschitz curve $c' : [a',b'] \rightarrow (X,d)$ admits a reparametrization $c : [a,b] \rightarrow (X,d)$ such that for almost every $t \in [a,b]$ there is a conical group morphism*

$$\dot{c}(t) : \mathbb{R} \rightarrow T_{c(t)}X$$

such that for any $a \in \mathbb{R}$ we have

$$\frac{1}{\varepsilon} d(c(t+\varepsilon a), \delta_{\varepsilon}^{c(t)} \dot{c}(t)(a)) \rightarrow 0.$$

Proof. It is straightforward that a conical group morphism $f : \mathbb{R} \rightarrow (N, \delta)$ is defined by its value $f(1) \in N$. Indeed, for any $a > 0$ we have $f(a) = \delta_a f(1)$ and for any $a < 0$ we have $f(a) = \delta_a f(1)^{-1}$. From the morphism property we also deduce that

$$\delta v = \left\{ \delta_a v \, : \, a > 0, v = f(1) \text{ or } v = f(1)^{-1} \right\}$$

is a one parameter group and that for all $\alpha, \beta > 0$ we have

$$\delta_{\alpha+\beta} u = \delta_{\alpha} u \delta_{\beta} u \quad \square$$

Definition 17.3 *In a conical group N we shall denote by $D(N)$ the set of all $u \in N$ with the property that $\varepsilon \in ((0,\infty),+) \mapsto \delta_{\varepsilon} u \in N$ is a morphism of semigroups .*

$D(N)$ is always non empty, because it contains the neutral element of N. $D(N)$ is also a cone, with dilations δ_{ε}, and a closed set.

We shall always identify a conical group morphism $f : \mathbb{R} \rightarrow N$ with its value $f(1) \in D(N)$.

17.1 Length formula from Radon-Nikodym property

Theorem 17.4 *Let (X,d,δ) be a dilation structure with the Radon-Nikodym property, over a complete length metric space (X,d). Then for any Lipschitz curve $c : [a,b] \rightarrow X$ the length of $\gamma = c([a,b])$ is*

$$L(\gamma) = \int_a^b d^{c(t)}(c(t), \dot{c}(t)) \; dt.$$

Proof. The upper dilation of c in t is

$$Lip(c)(t) = \limsup_{\varepsilon \to 0} \sup \left\{ \frac{d(c(v), c(w))}{\mid v - w \mid} \ : \ v \neq w \ , \ \mid v - t \mid, \mid w - t \mid < \varepsilon \right\}.$$

From theorem 4.11 we deduce that for almost every $t \in (a,b)$ we have

$$Lip(c)(t) = \lim_{s \to t} \frac{d(c(s), c(t))}{\mid s - t \mid}.$$

If the dilation structure has the Radon-Nikodym property then for almost every $t \in [a,b]$ there is $\dot{c}(t) \in D(T_{c(t)}X)$ such that

$$\frac{1}{\varepsilon} d(c(t+\varepsilon), \delta_{\varepsilon}^{c(t)} \dot{c}(t)) \to 0.$$

Therefore for almost every $t \in [a,b]$ we have

$$Lip(c)(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d(c(t+\varepsilon), c(t)) = d^{c(t)}(c(t), \dot{c}(t)).$$

The formula for length follows from here. $\square$

17.2 Equivalent dilation structures and their distributions

Definition 17.5 Two strong dilation structures (X, δ, d) *and* $(X, \overline{\delta}, \overline{d})$ **are equivalent** *if*

(a) the identity map $id : (X, d) \to (X, \overline{d})$ *is bilipschitz and*

(b) for any $x \in X$ *there are functions* P^x, Q^x *(defined for* $u \in X$ *sufficiently close to* x*) such that*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \overline{d} \left(\delta_{\varepsilon}^x u, \overline{\delta}_{\varepsilon}^x Q^x(u) \right) = 0, \tag{17.2.1}$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d \left(\overline{\delta}_{\varepsilon}^x u, \delta_{\varepsilon}^x P^x(u) \right) = 0, \tag{17.2.2}$$

uniformly with respect to x, u *in compact sets.*

Proposition 17.6 (X, δ, d) *and* $(X, \overline{\delta}, \overline{d})$ *are equivalent if and only if*

(a) the identity map $id : (X, d) \to (X, \overline{d})$ *is bilipschitz,*

(b) for any $x \in X$ *there are conical group morphisms:*

$$P^x : T_x(X, \overline{\delta}, \overline{d}) \to T_x(X, \delta, d) \ \text{and} \ Q^x : T_x(X, \delta, d) \to T_x(X, \overline{\delta}, \overline{d})$$

such that the following limits exist

$$\lim_{\varepsilon \to 0} \left(\overline{\delta}_{\varepsilon}^x \right)^{-1} \delta_{\varepsilon}^x(u) = Q^x(u), \tag{17.2.3}$$

$$\lim_{\varepsilon \to 0} \left(\delta_{\varepsilon}^x \right)^{-1} \overline{\delta}_{\varepsilon}^x(u) = P^x(u), \tag{17.2.4}$$

and are uniform with respect to x, u *in compact sets.*

The next theorem shows a link between the tangent bundles of equivalent dilation structures.

Theorem 17.7 *Let (X, d, δ) and $(X, \overline{d}, \overline{\delta})$ be equivalent strong dilation structures. Then for any $x \in X$ and any $u, v \in X$ sufficiently close to x we have:*

$$\overline{\Sigma}^x(u, v) = Q^x\left(\Sigma^x\left(P^x(u), P^x(v)\right)\right). \tag{17.2.5}$$

The two tangent bundles are therefore isomorphic in a natural sense.

As a consequence, the following corollary is straightforward.

Corollary 17.8 *Let (X, d, δ) and $(X, \overline{d}, \overline{\delta})$ be equivalent strong dilation structures. Then for any $x \in X$ we have*

$$Q^x(D(T_x(X, \delta, d))) \;=\; D(T_x(X, \overline{\delta}, \overline{d}))$$

If (X, d, δ) has the Radon-Nikodym property , then $(X, \overline{d}, \overline{\delta})$ has the same property.

Suppose that (X, d, δ) and $(X, \overline{d}, \overline{\delta})$ are complete length spaces with the Radon-Nikodym property . If the functions P^x, Q^x from definition 17.5 (b) are isometries, then $d = \overline{d}$.

18 Tempered dilation structures

The notion of a tempered dilation structure is inspired by the results from Venturini [62] and Buttazzo, De Pascale and Fragala [34].

The examples of length dilation structures from this section are provided by the extension of some results from [34] (propositions 2.3, 2.6 and a part of theorem 3.1) to dilation structures.

The following definition gives a class of distances $\mathcal{D}(\Omega, \bar{d}, \bar{\delta})$, associated to a strong dilation structure $(\Omega, \bar{d}, \bar{\delta})$, which generalizes the class of distances $\mathcal{D}(\Omega)$ from [34], definition 2.1.

Definition 18.1 *For any strong dilation structure $(\Omega, \bar{d}, \bar{\delta})$ we define the class $\mathcal{D}(\Omega, \bar{d}, \bar{\delta})$ of all distance functions d on Ω such that*

(a) d is a length distance,

(b) for any $\varepsilon > 0$ and any x, u, v sufficiently close the are constants $0 < c < C$ such that:

$$c\,\bar{d}^x(u, v) \;\leq\; \frac{1}{\varepsilon}\, d(\bar{\delta}^x_\varepsilon u, \bar{\delta}^x_\varepsilon v) \;\leq\; C\,\bar{d}^x(u, v) \tag{18.0.1}$$

The dilation structure $(\Omega, \bar{d}, \bar{\delta})$ is **tempered** *if $\bar{d} \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$.*

On $\mathcal{D}(\Omega, \bar{d}, \bar{\delta})$ we put the topology of uniform convergence (induced by distance $\bar{d}$) on compact subsets of $\Omega \times \Omega$.

To any distance $d \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$ we associate the function:

$$\phi_d(x, u) \;=\; \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon}\, d(x, \delta^x_\varepsilon u)$$

defined for any $x, u \in \Omega$ sufficiently close. We have therefore

$$c\,\bar{d}^x(x, u) \;\leq\; \phi_d(x, u) \;\leq\; C\,\bar{d}^x(x, u) \tag{18.0.2}$$

Notice that if $d \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$ then for any x, u, v sufficiently close we have

$$-\bar{d}(x,u)\, O(\bar{d}(x,u)) \,+\, c\, \bar{d}^{x}(u,v) \,\leq$$

$$\leq\, d(u,v) \,\leq\, C\, \bar{d}^{x}(u,v) \,+\, \bar{d}(x,u)\, O(\bar{d}(x,u))$$

If $c : [0,1] \to \Omega$ is a d-Lipschitz curve and $d \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$ then we may decompose it in a finite family of curves $c_1, ..., c_n$ (with n depending on c) such that there are $x_1, ..., x_n \in \Omega$ with c_k is $\bar{d}^{x_k}$-Lipschitz. Indeed, the image of the curve $c([0,1])$ is compact, therefore we may cover it with a finite number of balls $B(c(t_k), \rho_k, \bar{d}^{c(t_k)})$ and apply (18.0.1). If moreover $(\Omega, \bar{d}, \bar{\delta})$ is tempered then it follows that $c : [0,1] \to \Omega$ d-Lipschitz curve is equivalent with c $\bar{d}$-Lipschitz curve.

By using the same arguments as in the proof of theorem 17.4, we get the following extension of proposition 2.4 [34].

Proposition 18.2 *If $(\Omega, \bar{d}, \bar{\delta})$ is tempered, with the Radon-Nikodym property, and $d \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$ then*

$$d(x,y) \;=\; \inf \left\{ \int_a^b \phi_d(c(t), \dot{c}(t)) \; dt \;:\; c : [a,b] \to X \;\bar{d}\text{-Lipschitz} \;, \right.$$

$$\left. c(a) = x, c(b) = y \right\}$$

The next theorem is a generalization of a part of theorem 3.1 [34].

Theorem 18.3 *Let $(\Omega, \bar{d}, \bar{\delta})$ be a strong dilation structure which is tempered, with the Radon-Nikodym property, and $d_n \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$ a sequence of distances converging to $d \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$. Denote by L_n, L the length functional induced by the distance d_n, respectively by d. Then L_n Γ-converges to L.*

Proof. This is the generalization of the implication (i) $\Rightarrow$ (iii), theorem 3.1 [34]. The proof (p. 252-253) is almost identical, we only need to replace everywhere expressions like $\mid x - y \mid$ by $\bar{d}(x,y)$ and use proposition 18.2, relations (18.0.2) and (18.0.1) instead of respectively proposition 2.4 and relations (2.6) and (2.3) [34]. $\square$

Using this result we obtain a large class of examples of length dilation structures.

Corollary 18.4 *If $(\Omega, \bar{d}, \bar{\delta})$ is strong dilation structure which is tempered and it has the Radon-Nikodym property then it is a length dilation structure.*

Proof. Indeed, from the hypothesis we deduce that $\bar{\delta}^{x}_{\varepsilon} \bar{d} \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$. For any sequence $\varepsilon_n \to 0$ we thus obtain a sequence of distances $d_n = \bar{\delta}^{x}_{\varepsilon_n} \bar{d}$ converging to $\bar{d}^{x}$. We apply now theorem 18.3 and we get the result. $\square$

19 Coherent projections

For a given dilation structure with the Radon-Nikodym property, we shall give a procedure to construct another dilation structure, such that the first one looks down to the the second one.

This will be done with the help of coherent projections.

Definition 19.1 *Let $(X, \bar{d}, \bar{\delta})$ be a strong dilation structure. A* **coherent projection** *of $(X, \bar{d}, \bar{\delta})$ is a function which associates to any $x \in X$ and $\varepsilon \in (0,1]$ a map $Q^x_\varepsilon : U(x) \to X$ such that:*

(I) $Q^x_\varepsilon : U(x) \to Q^x_\varepsilon(U(x))$ is invertible and the inverse will be denoted by $Q^x_{\varepsilon^{-1}}$; for any $\varepsilon, \mu > 0$ and any $x \in X$ we have

$$Q^x_\varepsilon \, \bar{\delta}^x_\mu \; = \; \bar{\delta}^x_\mu \, Q^x_\varepsilon$$

(II) the limit $\lim_{\varepsilon \to 0} Q^x_\varepsilon u \; = \; Q^x u$ is uniform with respect to x, u in compact sets.

(III) for any $\varepsilon, \mu > 0$ and any $x \in X$ we have $Q^x_\varepsilon \, Q^x_\mu \; = \; Q^x_{\varepsilon\mu}$. Also $Q^x_1 \; = \; id$ and $Q^x_\varepsilon x \; = \; x$.

(IV) define $\Theta^x_\varepsilon(u, v) \; = \; \bar{\delta}^x_{\varepsilon^{-1}} \, Q^{\bar{\delta}^x_\varepsilon Q^x_\varepsilon u}_{\varepsilon^{-1}} \, \bar{\delta}^x_\varepsilon Q^x_\varepsilon v$. Then the limit exists

$$\lim_{\varepsilon \to 0} \Theta^x_\varepsilon(u, v) \; = \; \Theta^x(u, v)$$

and it is uniform with respect to x, u, v in compact sets.

Remark 19.2 *Property (IV) is basically a smoothness condition on the coherent projection Q, relative to the strong dilation structure $(X, \bar{d}, \bar{\delta})$.*

Proposition 19.3 *Let $(X, \bar{d}, \bar{\delta})$ be a strong dilation structure and Q a coherent projection. We define $\delta^x_\varepsilon \; = \; \bar{\delta}^x_\varepsilon \, Q^x_\varepsilon$. Then:*

(a) for any $\varepsilon, \mu > 0$ and any $x \in X$ we have $\delta^x_\varepsilon \, \bar{\delta}^x_\mu \; = \; \bar{\delta}^x_\mu \, \delta^x_\varepsilon$.

(b) for any $x \in X$ we have $Q^x \, Q^x \; = \; Q^x$ (thus Q^x is a projection).

(c) δ satisfies the conditions A1, A2, A4 from definition 6.1.

Proof. (a) this is a consequence of the commutativity condition (I) (second part). Indeed, we have $\delta^x_\varepsilon \, \bar{\delta}^x_\mu \; = \; \bar{\delta}^x_\varepsilon \, Q^x_\varepsilon \, \bar{\delta}^x_\mu \; = \; \bar{\delta}^x_\varepsilon \, \bar{\delta}^x_\mu \, Q^x_\varepsilon \; = \; \bar{\delta}^x_\mu \, \bar{\delta}^x_\varepsilon \, Q^x_\varepsilon \; = \; \bar{\delta}^x_\mu \, \delta^x_\varepsilon$.

(b) we pass to the limit $\varepsilon \to 0$ in the equality $Q^x_{\varepsilon^2} \; = \; Q^x_\varepsilon \, Q^x_\varepsilon$ and we get, based on condition (II), that $Q^x \, Q^x \; = \; Q^x$.

(c) Axiom A1 for δ is equivalent with (III). Indeed, the equality $\delta^x_\varepsilon \, \delta^x_\mu \; = \; \delta^x_{\varepsilon\mu}$ is equivalent with: $\bar{\delta}^x_{\varepsilon\mu} \, Q^x_{\varepsilon\mu} \; = \; \bar{\delta}^x_{\varepsilon\mu} \, Q^x_\varepsilon \, Q^x_\mu$. This is true because $Q^x_\varepsilon \, Q^x_\mu \; = \; Q^x_{\varepsilon\mu}$. We also have $\delta^x_1 \; = \; \bar{\delta}^x_1 Q^x_1 \; = \; Q^x_1 \; = \; id$. Moreover $\delta^x_\varepsilon x \; = \; \bar{\delta}^x_\varepsilon \, Q^x_\varepsilon x \; = \; Q^x_\varepsilon \, \bar{\delta}^x_\varepsilon x \; = \; Q^x_\varepsilon x \; = \; x$. Let us compute now:

$$\Delta^x_\varepsilon(u, v) \; = \; \delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \, \delta^x_\varepsilon v \; = \; \bar{\delta}^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \, Q^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \, \delta^x_\varepsilon v \; =$$

$$= \; \bar{\delta}^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \, \bar{\delta}^x_\varepsilon \, \Theta^x_\varepsilon(u, v) \; = \; \bar{\Delta}^x_\varepsilon(Q^x_\varepsilon u, \Theta^x_\varepsilon(u, v))$$

We can pass to the limit in the last term of this string of equalities and we prove that the axiom A4 is satisfied by δ: there exists the limit

$$\Delta^x(u, v) \; = \; \lim_{\varepsilon \to 0} \Delta^x_\varepsilon(u, v) \tag{19.0.1}$$

which is uniform as written in A4, moreover we have the equality

$$\Theta^x_\varepsilon(u, v) \; = \; \bar{\Sigma}^x_\varepsilon(Q^x_\varepsilon u, \Delta^x_\varepsilon(u, v)) \tag{19.0.2}$$

□

We collect two useful relations in the next proposition.

Proposition 19.4 *Let $(X, \bar{d}, \bar{\delta})$ be a strong dilation structure and Q a coherent projection. We denote by δ the field of dilations induced by the coherent projection, as in the previous proposition, and by Δ^x is defined by (19.0.1). Then we have:*

$$\Delta^x(u,v) \, = \, \bar{\Delta}^x(Q^x u, \Theta^x(u,v)) \tag{19.0.3}$$

$$Q^x \Delta^x(u,v) \, = \, \bar{\Delta}^x(Q^x u, Q^x v) \tag{19.0.4}$$

Proof. After passing to the limit with $\varepsilon \to 0$ in the relation (19.0.2) we get the formula (19.0.3). In order to prove (19.0.4) we notice that:

$$Q_\varepsilon^{\delta_\varepsilon^x u} \, \Delta_\varepsilon^x(u,v) \, = \, Q_\varepsilon^{\delta_\varepsilon^x u} \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v \, =$$

$$= \, \bar{\delta}_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \bar{\delta}_\varepsilon^x Q_\varepsilon^x v \, = \, \bar{\Delta}_\varepsilon^x(Q_\varepsilon^x u, Q_\varepsilon^x v)$$

which gives(19.0.4) as we pass to the limit with $\varepsilon \to 0$ in this relation. □

Next is described the notion of Q-horizontal curve.

Definition 19.5 *Let $(X, \bar{d}, \bar{\delta})$ be a strong dilation structure and Q a coherent projection. A curve $c : [a,b] \to X$ is Q-**horizontal** if for almost any $t \in [a,b]$ the curve c is derivable and the derivative of c at t, denoted by $\dot{c}(t)$ has the property:*

$$Q^{c(t)} \dot{c}(t) \, = \, \dot{c}(t) \tag{19.0.5}$$

*A curve $c : [a,b] \to X$ is Q-**everywhere horizontal** if for all $t \in [a,b]$ the curve c is derivable and the derivative has the horizontality property (19.0.5).*

We shall look first at some induced dilation structures.

For any $x \in X$ and $\varepsilon \in (0,1)$ the dilation δ_ε^x can be seen as an isomorphism of strong dilation structures with coherent projections:

$$\delta_\varepsilon^x : (U(x), \delta_\varepsilon^x \bar{d}, \hat{\delta}_\varepsilon^x, \hat{Q}_\varepsilon^x) \to (\delta_\varepsilon^x U(x), \frac{1}{\varepsilon}\bar{d}, \bar{\delta}, Q)$$

which defines the dilations $\hat{\delta}_{\varepsilon,\cdot}^{x,\cdot}$ and coherent projection $\hat{Q}_\varepsilon^x$ by:

$$\hat{\delta}_{\varepsilon,\mu}^{x,u} \;\; = \;\; \delta_{\varepsilon^{-1}}^x \, \bar{\delta}_\mu^{\delta_\varepsilon^x u} \, \delta_\varepsilon^x$$

$$\hat{Q}_{\varepsilon,\mu}^{x,u} \;\; = \;\; \delta_{\varepsilon^{-1}}^x \, Q_\mu^{\delta_\varepsilon^x u} \, \delta_\varepsilon^x$$

Also the dilation $\bar{\delta}_\varepsilon^x$ is an isomorphism of strong dilation structures with coherent projections:

$$\bar{\delta}_\varepsilon^x : (U(x), \bar{\delta}_\varepsilon^x \bar{d}, \bar{\delta}_\varepsilon^x, \bar{Q}_\varepsilon^x) \to (\bar{\delta}_\varepsilon^x U(x), \frac{1}{\varepsilon}\bar{d}, \bar{\delta}, Q)$$

which defines the dilations $\bar{\delta}_{\varepsilon,\cdot}^{x,\cdot}$ and coherent projection $\bar{Q}_\varepsilon^x$ by:

$$\bar{\delta}_{\varepsilon,\mu}^{x,u} \;\; = \;\; \bar{\delta}_{\varepsilon^{-1}}^x \, \bar{\delta}_\mu^{\bar{\delta}_\varepsilon^x u} \, \bar{\delta}_\varepsilon^x$$

$$\bar{Q}_{\varepsilon,\mu}^{x,u} \;\; = \;\; \bar{\delta}_{\varepsilon^{-1}}^x \, Q_\mu^{\bar{\delta}_\varepsilon^x u} \, \bar{\delta}_\varepsilon^x$$

Because $\delta_\varepsilon^x \; = \; \bar{\delta}_\varepsilon^x \, Q_\varepsilon^x$ we get that

$$Q_\varepsilon^x : (U(x), \delta_\varepsilon^x \bar{d}, \hat{\delta}_\varepsilon^x, \hat{Q}_\varepsilon^x) \to (Q_\varepsilon^x U(x), \bar{\delta}_\varepsilon^x d, \bar{\delta}_\varepsilon^x, \bar{Q}_\varepsilon^x)$$

is an isomorphism of strong dilation structures with coherent projections.

Further is a useful description of the coherent projection $\hat{Q}_\varepsilon^x$.

Proposition 19.6 *With the notations previously made, for any $\varepsilon \in (0,1]$, $x, u, v \in X$ sufficiently close and $\mu > 0$ we have:*

(i) $\hat{Q}^{x,u}_{\varepsilon,\mu} v \ = \ \Sigma^x_\varepsilon(u, Q^{\delta^x_\varepsilon u}_\mu \Delta^x_\varepsilon(u,v))$,

(ii) $\hat{Q}^{x,u}_{\varepsilon} v \ = \ \Sigma^x_\varepsilon(u, Q^{\delta^x_\varepsilon u} \Delta^x_\varepsilon(u,v))$.

Proof. (i) implies (ii) when $\mu \to 0$, thus it is sufficient to prove only the first point. This is the result of a computation:

$$\hat{Q}^{x,u}_{\varepsilon,\mu} v \ = \ \delta^x_{\varepsilon^{-1}} \, Q^{\delta^x_\varepsilon u}_\mu \, \delta^x_\varepsilon \ =$$

$$= \ \delta^x_{\varepsilon^{-1}} \, \delta^{\delta^x_\varepsilon u}_\varepsilon \, Q^{\delta^x_\varepsilon u}_\mu \, \delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \, \delta^x_\varepsilon \ = \ \Sigma^x_\varepsilon(u, Q^{\delta^x_\varepsilon u}_\mu \Delta^x_\varepsilon(u,v))$$

□

Notation concerning derivatives. We shall denote the derivative of a curve with respect to the dilations $\hat{\delta}^x_\varepsilon$ by $\displaystyle \frac{\hat{d}^x_\varepsilon}{dt}$. Also, the derivative of the curve c with respect to $\bar{\delta}$ is denoted by $\displaystyle \frac{\bar{d}}{dt}$, and so on.

By computation we get: the curve c is $\hat{\delta}^x_\varepsilon$-derivable if and only if $\delta^x_\varepsilon c$ is $\bar{\delta}$-derivable and

$$\frac{\hat{d}^x_\varepsilon}{dt} c(t) \ = \ \delta^x_{\varepsilon^{-1}} \frac{\bar{d}}{dt} \left(\delta^x_\varepsilon c\right)(t)$$

With these notations we give a proposition which explains that the operator Θ^x_ε , from the definition of coherent projections, is a lifting operator.

Proposition 19.7 *If the curve $\delta^x_\varepsilon c$ is Q-horizontal then*

$$\frac{\bar{d}^x_\varepsilon}{dt} \left(Q^x_\varepsilon c\right)(t) \ = \ \Theta^x_\varepsilon(c(t), \frac{\hat{d}^x_\varepsilon}{dt} c(t))$$

Proof. If the curve $Q^x_\varepsilon c$ is $\bar{\delta}^x_\varepsilon$ derivable and $\bar{Q}^x_\varepsilon$ horizontal. We have therefore:

$$\frac{\bar{d}^x_\varepsilon}{dt} \left(Q^x_\varepsilon c\right)(t) \ = \ \bar{\delta}^x_{\varepsilon^{-1}} \, Q^{\delta^x_\varepsilon c(t)} \, \bar{\delta}^x_\varepsilon \, \frac{\bar{d}^x_\varepsilon}{dt} \left(Q^x_\varepsilon c\right)(t)$$

which implies:

$$\bar{\delta}^x_\varepsilon \, \frac{\bar{d}^x_\varepsilon}{dt} \left(Q^x_\varepsilon c\right)(t) \ = \ Q^{\delta^x_\varepsilon c(t)}_{\varepsilon^{-1}} \, \bar{\delta}^x_\varepsilon \, \frac{\bar{d}^x_\varepsilon}{dt} \left(Q^x_\varepsilon c\right)(t) \ = \ Q^{\delta^x_\varepsilon c(t)}_{\varepsilon^{-1}} \, \delta^x_\varepsilon \, \frac{\hat{d}^x_\varepsilon}{dt} c(t)$$

which is the formula we wanted to prove. □

19.1 Distributions in sub-riemannian spaces

The inspiration for the notion of coherent projection comes from sub-riemannian geometry.

Further we shall work locally, just as in the mentioned section. Same notations are used. Let $\{Y_1, ..., Y_n\}$ be a frame induced by a parameterization $\phi : O \subset \mathbb{R}^n \to U \subset M$ of a small open, connected set U in the manifold M. This parameterization induces a affine dilation structure on U, by

$$\tilde{\delta}_{\varepsilon}^{\phi(a)}\,\phi(b) \;=\; \phi\left(a+\varepsilon(-a+b)\right)$$

We take the distance $\tilde{d}(\phi(a),\phi(b)) \;=\; \|b-a\|$.

Let $\{X_1,...,X_n\}$ be a normal frame, cf. definition 11.7, d be the Carnot-Carathéodory distance and

$$\delta_{\varepsilon}^{x}\left(\exp\left(\sum_{i=1}^{n}a_iX_i\right)(x)\right) \;=\; \exp\left(\sum_{i=1}^{n}a_i\varepsilon^{degX_i}X_i\right)(x)$$

be the dilation structure associated.

We may take another dilation structure, constructed as follows: extend the metric g on the distribution D to a riemannian metric on M, denoted for convenience also by g. Let $\bar{d}$ be the riemannian distance induced by the riemannian metric g, and the dilations

$$\bar{\delta}_{\varepsilon}^{x}\left(\exp\left(\sum_{i=1}^{n}a_iX_i\right)(x)\right) \;=\; \exp\left(\sum_{i=1}^{n}a_i\varepsilon X_i\right)(x)$$

Then $(U,\bar{d},\bar{\delta})$ is a strong dilation structure which is equivalent with the dilation structure $(U,\tilde{d},\tilde{\delta})$.

From now we may define coherent projections associated either to the pair $(\tilde{\delta},\delta)$ or to the pair $(\bar{\delta},\delta)$. Because we put everything on the manifold (by the use of the chosen frames), we shall obtain different coherent projections, both inducing the same dilation structure (U,d,δ).

Let us define Q_{ε}^{x} by:

$$Q_{\varepsilon}^{x}\left(\exp\left(\sum_{i=1}^{n}a_iX_i\right)(x)\right) \;=\; \exp\left(\sum_{i=1}^{n}a_i\varepsilon^{degX_i-1}X_i\right)(x) \qquad (19.1.6)$$

Proposition 19.8 *Q is a coherent projection associated with the dilation structure $(U,\bar{d},\bar{\delta})$.*

Proof. (I) definition 19.1 is true, because $\delta_{\varepsilon}^{x}\,u \;=\; Q_{\varepsilon}^{x}\,\bar{\delta}_{\varepsilon}^{x}$ and $\delta_{\varepsilon}^{x}\,\bar{\delta}_{\varepsilon}^{x} \;=\; \bar{\delta}_{\varepsilon}^{x}\delta_{\varepsilon}^{x}$. (II), (III) and (IV) are consequences of these facts, with a proof similar to the one of proposition 19.3. □

Definition (19.1.6) of the coherent projection Q implies that:

$$Q^{x}\left(\exp\left(\sum_{i=1}^{n}a_iX_i\right)(x)\right) \;=\; \exp\left(\sum_{degX_i=1}a_iX_i\right)(x) \qquad (19.1.7)$$

Therefore Q^x can be seen as a projection onto the (classical differential) geometric distribution.

Remark 19.9 *The projection Q^x has one more interesting feature: for any x and*

$$u \;=\; \exp\left(\sum_{degX_i=1}a_iX_i\right)(x)$$

we have $Q^x u \;=\; u$ and the curve

$$s\in[0,1]\mapsto \delta_s^x\,u \;=\; \exp\left(s\sum_{degX_i=1}a_iX_i\right)(x)$$

is D-horizontal and joins x and u. This will be related to the supplementary condition (B) further.

We may equally define a coherent projection which induces the dilations δ from $\tilde{\delta}$. Also, if we change the chosen normal frame with another of the same kind, we shall pass to a dilation structure which is equivalent to (U, d, δ). In conclusion, coherent projections are not geometrical objects per se, but in a natural way one may define a notion of equivalent coherent projections such that the equivalence class is geometrical, i.e. independent of the choice of a pair of particular dilation structures, each in a given equivalence class. Another way of putting this is that a class of equivalent dilation structures may be seen as a category and a coherent projection is a functor between such categories. We shall not pursue this line here.

The bottom line is that $(U, \bar{d}, \bar{\delta})$ is a dilation structure which belongs to an equivalence class which is independent on the distribution D, and also independent on the choice of parameterization ϕ. It is associated to the manifold M only. On the other hand $(U, \bar{d}, \bar{\delta})$ belongs to an equivalence class which is depending only on the distribution D and metric g on D, thus intrinsic to the sub-riemannian manifold (M, D, g). The only advantage of choosing $\bar{\delta}, \delta$ related by the normal frame $\{X_1, ..., X_n\}$ is that they are associated with a coherent projection with a simple expression.

19.2 Length functionals associated to coherent projections

Definition 19.10 *Let $(X, \bar{d}, \bar{\delta})$ be a strong dilation structure with the Radon-Nikodym property and Q a coherent projection. We define the associated distance $d : X \times X \to [0, +\infty]$ by:*

$$d(x,y) \ = \ \inf \left\{ \int_a^b \bar{d}^{c(t)}(c(t), \dot{c}(t)) \ dt \ : \ c : [a,b] \to X \ \bar{d}\text{-Lipschitz} \ , \right.$$

$$\left. c(a) = x, c(b) = y, \ \text{and } \forall \text{a.e. } t \in [a,b] \quad Q^{c(t)} \dot{c}(t) \ = \ \dot{c}(t) \right\}$$

The relation $x \equiv y$ if $d(x,y) < +\infty$ is an equivalence relation. The space X decomposes into a reunion of equivalence classes, each equivalence class being connected by horizontal curves.

It is easy to see that d is a finite distance on each equivalence class. Indeed, from theorem 17.4 we deduce that for any $x, y \in X$ $d(x,y) \geq \bar{d}(x,y)$. Therefore $d(x,y) = 0$ implies $x = y$. The other properties of a distance are straightforward.

Later we shall give a sufficient condition (the generalized Chow condition (Cgen)) on the coherent projection Q for X to be (locally) connected by horizontal curves.

Proposition 19.11 *Suppose that X is connected by horizontal curves and (X, d) is complete. Then d is a length distance.*

Proof. Because (X, d) is complete, it is sufficient to check that d has the approximate middle property: for any $\varepsilon > 0$ and for any $x, y \in X$ there exists $z \in X$ such that

$$\max \{d(x,z), d(y,z)\} \leq \frac{1}{2} \, d(x,y) + \varepsilon$$

Given $\varepsilon > 0$, from the definition of d we deduce that there exists a horizontal curve $c : [a,b] \to X$ such that $c(a) = x$, $c(b) = y$ and $d(x,y) + 2\varepsilon \geq l(c)$ (where $l(c)$ is the length of c with respect to the distance $\bar{d}$). There exists then $\tau \in [a,b]$ such that

$$\int_a^\tau \bar{d}^{c(t)}(c(t),\dot{c}(t))\ \mathrm{d}t \ = \ \int_\tau^b \bar{d}^{c(t)}(c(t),\dot{c}(t))\ \mathrm{d}t \ = \ \frac{1}{2}\, l(c)$$

Let $z = c(\tau)$. We have then: $\max\{d(x,z), d(y,z)\} \ \leq \ \frac{1}{2}\, l(c) \ \leq \ \frac{1}{2}\, d(x,y) + \varepsilon$. Therefore d is a length distance. $\square$

Notations concerning length functionals. The length functional associated to the distance $\bar{d}$ is denoted by $\bar{l}$. In the same way the length functional associated with $\bar{\delta}^x_\varepsilon$ is denoted by $\bar{l}^x_\varepsilon$.

We introduce the space $\mathcal{L}_\varepsilon(X,d,\delta) \subset X \times Lip([0,1],X,d)$:

$$\mathcal{L}_\varepsilon(X,d,\delta) \ = \ \{(x,c) \in X \times \mathcal{C}([0,1],X) \, : \, c : [0,1] \in U(x) \ ,$$

$$\delta^x_\varepsilon c \text{ is } \bar{d} - Lip, \quad Q - \text{horizontal and } Lip(\delta^x_\varepsilon c) \leq 2\varepsilon l_d(\delta^x_\varepsilon c)\big\}$$

For any $\varepsilon \in (0,1)$ we define the length functional

$$l_\varepsilon : \mathcal{L}_\varepsilon(X,d,\delta) \to [0,+\infty] \quad , \quad l_\varepsilon(x,c) \ = \ l^x_\varepsilon(c) \ = \ \frac{1}{\varepsilon}\, \bar{l}(\delta^x_\varepsilon c)$$

By theorem 17.4 we have:

$$l^x_\varepsilon(c) \ = \ \int_0^1 \frac{1}{\varepsilon}\, \bar{d}^{\delta^x_\varepsilon c(t)} \left(\delta^x_\varepsilon c(t), \frac{\bar{d}}{dt}\left(\delta^x_\varepsilon c\right)(t)\right)\ \mathrm{d}t \ =$$

$$= \ \int_0^1 \frac{1}{\varepsilon}\, \bar{d}^{\delta^x_\varepsilon c(t)} \left(\delta^x_\varepsilon c(t), \delta^x_\varepsilon \frac{\hat{d}^x_\varepsilon}{dt}\, c(t)\right)\ \mathrm{d}t$$

Another description of the length functional l^x_ε is the following.

Proposition 19.12 *For any $(x,c) \in \mathcal{L}_\varepsilon(X,d,\delta)$ we have*

$$l^x_\varepsilon(c) \ = \ \bar{l}^x_\varepsilon\left(Q^x_\varepsilon c\right)$$

Proof. Indeed, we shall use an alternate definition of the length functional. Let c be a curve such that $\delta^x_\varepsilon c$ is $\bar{d}$-Lipschitz and Q-horizontal. Then:

$$l^x_\varepsilon(c) \ = \ \sup\left\{\sum_{i=1}^n \frac{1}{\varepsilon}\bar{d}\left(\delta^x_\varepsilon c(t_i), \delta^x_\varepsilon c(t_{i+1})\right) \, : \, 0 = t_1 < ... < t_{n+1} = 1\right\} \ =$$

$$= \ \sup\left\{\sum_{i=1}^n \frac{1}{\varepsilon}\bar{d}\left(\bar{\delta}^x_\varepsilon Q^x_\varepsilon c(t_i), \bar{\delta}^x_\varepsilon Q^x_\varepsilon c(t_{i+1})\right) \, : \, 0 = t_1 < ... < t_{n+1} = 1\right\} \ =$$

$$= \ \bar{l}^x_\varepsilon\left(Q^x_\varepsilon c\right)$$

$\square$

19.3 Supplementary hypotheses

Definition 19.13 *Let $(X, \bar{d}, \bar{\delta})$ be a strong dilation structure and Q a coherent projection. Further is a list of supplementary hypotheses on Q:*

(A) δ^x_ε is $\bar{d}$-bilipschitz in compact sets in the following sense: for any compact set $K \subset X$ and for any $\varepsilon \in (0,1]$ there is a number $L(K) > 0$ such that for any $x \in K$ and u, v sufficiently close to x we have:

$$\frac{1}{\varepsilon} \bar{d}\left(\delta^x_\varepsilon u, \delta^x_\varepsilon v\right) \leq L(K)\, \bar{d}(u,v)$$

(B) if $u = Q^x u$ then the curve $t \in [0,1] \mapsto Q^x \delta^x_t u = \bar{\delta}^x_t u = \delta^x_t u$ is Q-everywhere horizontal and for any $a \in [0,1]$ we have

$$\limsup_{a \to 0} \frac{\bar{l}\left(t \in [0,a] \mapsto \bar{\delta}^x_t u\right)}{\bar{d}(x, \bar{\delta}^x_a u)} = 1$$

uniformly with respect to x, u in compact set K.

Condition (A), as well as the property (IV) definition 19.1, is another smoothness condition on Q with respect to the strong dilation structure $(X, \bar{d}, \bar{\delta})$.

The condition (A) has several useful consequences, among them the fact that for any $\bar{d}$-Lipschitz curve c, the curve $\delta^x_\varepsilon c$ is also Lipschitz. Another consequence is that Q^x_ε is locally $\bar{d}$-Lipschitz. More precisely, for any compact set $K \subset X$ and for any $\varepsilon \in (0,1]$ there is a number $L(K) > 0$ such that for any $x \in K$ and u, v sufficiently close to x we have:

$$\left(\bar{\delta}^x_\varepsilon \bar{d}\right) \left(Q^x_\varepsilon u, Q^x_\varepsilon v\right) \leq L(K)\, \bar{d}(u,v) \tag{19.3.8}$$

with the notation

$$\left(\bar{\delta}^x_\varepsilon \bar{d}\right)(u,v) = \frac{1}{\varepsilon} \bar{d}\left(\bar{\delta}^x_\varepsilon u, \bar{\delta}^x_\varepsilon v\right)$$

Indeed, we have:

$$\left(\bar{\delta}^x_\varepsilon \bar{d}\right)\left(Q^x_\varepsilon u, Q^x_\varepsilon v\right) = \frac{1}{\varepsilon} \bar{d}\left(\delta^x_\varepsilon u, \delta^x_\varepsilon v\right) \leq L(K)\, \bar{d}(u,v)$$

See the remark 19.9 for the meaning of the condition B for the case sub-riemannian geometry, where it is explained why condition B is a generalization of the fact that the "distribution" $x \mapsto Q^x U(x)$ is generated by horizontal one parameter flows.

Condition (B) will be useful later, along with the generalized Chow condition (Cgen).

20 The generalized Chow condition

Notations about words. For any set A we denote by A^* the collection of finite words $q = a_1 ... a_p$, $p \in \mathbb{N}$, $p > 0$. The empty word is denoted by $\emptyset$. The length of the word $q = a_1 ... a_p$ is $\mid q \mid = p$; the length of the empty word is 0.

The collection of words infinite at right over the alphabet A is denoted by A^ω. For any word $w \in A^\omega \cup A^*$ and any $p \in \mathbb{N}$ we denote by $[w]_p$ the finite word obtained from the first p letters of w (if $p = 0$ then $[w]_0 = \emptyset$ (in the case of a finite word q, if $p > \mid q \mid$ then $[q]_p = q$).

For any non-empty $q_1, q_2 \in A^*$ and $w \in A^\omega$ the concatenation of q_1 and q_2 is the finite word $q_1 q_2 \in A^*$ and the concatenation of q_1 and w is the (infinite) word $q_1 w \in A^\omega$. The empty word $\emptyset$ is seen both as an infinite word or a finite word and for any $q \in A^*$ and $w \in A^\omega$ we have $q\emptyset = q$ (as concatenation of finite words) and $\emptyset w = w$ (as concatenation of a finite empty word and an infinite word).

20.1 Coherent projections as transformations of words

To any coherent projection Q in a strong dilation structure $(X, \bar{d}, \bar{\delta})$ we associate a family of transformations as follows.

Definition 20.1 *For any non-empty word $w \in (0,1]^{\omega}$ and any $\varepsilon \in (0,1]$ we define the transformation*

$$\Psi_{\varepsilon w} : X^*_{\varepsilon w} \subset X^* \setminus \{\emptyset\} \to X^*$$

given by: for any non-empty finite word $q = xx_1...x_p \in X^*_{\varepsilon w}$ we have

$$\Psi_{\varepsilon w}(xx_1...x_p) \ = \ \Psi^1_{\varepsilon w}(x)...\Psi^{k+1}_{\varepsilon w}(xx_1...x_k)...\Psi^{p+1}_{\varepsilon w}(xx_1...x_p)$$

The functions $\Psi^k_{\varepsilon w}$ are defined by: $\Psi^1_{\varepsilon w}(x) = x$, and for any $k \geq 1$ we have

$$\Psi^{k+1}_{\varepsilon w}([q]_{k+1}) \ = \ \delta^x_{\varepsilon^{-1}} \, Q^{\delta^x_\varepsilon \, \Psi^k_{\varepsilon w}([q]_k)}_{w_k} \, \delta^x_\varepsilon \, q_{k+1} \tag{20.1.1}$$

If $w = \emptyset$ then $\Psi^k_{\varepsilon\emptyset}$ is defined as previously $\Psi^1_{\varepsilon\emptyset}(x) = x$, with the only difference that for any $k \geq 1$ we have

$$\Psi^{k+1}_{\varepsilon\emptyset}([q]_{k+1}) \ = \ \delta^x_{\varepsilon^{-1}} \, Q^{\delta^x_\varepsilon \, \Psi^k_{\varepsilon w}([q]_k)} \, \delta^x_\varepsilon \, q_{k+1}$$

The domain $X^*_{\varepsilon w} \subset X^* \setminus \{\emptyset\}$ is such that the previous definition makes sense. By using the definition of a coherent projection, we may redefine $X^*_{\varepsilon w}$ as follows: for any compact set $K \subset X$ there is $\rho = \rho(K) > 0$ such that for any $x \in K$ the word $q = xx_1...x_p \in X^*_{\varepsilon w}$ if for any $k \geq 1$ we have

$$\bar{d}\left(x_{k+1}, \Psi^k_{\varepsilon w}([q]_k)\right) \ \leq \ \rho$$

We shall explain the meaning of these transformations for $\varepsilon = 1$.

Proposition 20.2 *Suppose that condition (B) holds for the coherent projection Q. If*

$$y \ = \ \Psi^{k+1}_{1\emptyset}(xx_1...x_k)$$

then there is a Q-horizontal curve joining x and y.

Proof. By definition 20.1 for $\varepsilon = 1$ we have:

$$\Psi^1_{1w}(x) = x \quad , \quad \Psi^2_{1w}(x, x_1) \ = \ Q^x_{w_1} \, x_1 \quad ,$$

$$\Psi^3_{1w}(x, x_1, x_2) \ = \ Q^{Q^x_{w_1} x_1}_{w_2} \, x_2 \quad ...$$

Suppose now that condition (B) holds for the coherent projection Q. Then the curve $t \in [0,1] \mapsto \bar{\delta}^x_t Q^x u$ is a Q-horizontal curve joining x with $Q^x u$. Therefore by applying inductively the condition (B) we get that there is a Q-horizontal curve between $\Psi^k_{1\emptyset}(xx_1...x_{k-1})$ and $\Psi^{k+1}_{1\emptyset}(xx_1...x_k)$ for any $k > 1$ and a Q-horizontal curve joining x and $\Psi^2_{1\emptyset}(xx_1)$. □

There are three more properties of the transformations $\Psi_{\varepsilon w}$.

Proposition 20.3 *With the notations from definition 20.1 we have:*

(a) $\Psi_{\varepsilon w} \, \Psi_{\varepsilon\emptyset} \ = \ \Psi_{\varepsilon\emptyset}$. *Therefore we have the equality of sets:*

$$\Psi_{\varepsilon\emptyset}\left(X^*_{\varepsilon\emptyset} \cap xX^*\right) \ = \ \Psi_{\varepsilon w}\left(\Psi_{\varepsilon\emptyset}\left(X^*_{\varepsilon\emptyset} \cap xX^*\right)\right)$$

(b) $\Psi^{k+1}_{\varepsilon\emptyset}(xq_1...q_k) \ = \ \delta^x_{\varepsilon^{-1}} \, \Psi^{k+1}_{1\emptyset}(x\delta^x_\varepsilon q_1...\delta^x_\varepsilon q_k)$

(c) $\lim_{\varepsilon \to 0} \delta^x_{\varepsilon^{-1}} \, \Psi^{k+1}_{1\emptyset}(x\delta^x_\varepsilon q_1...\delta^x_\varepsilon q_k) \ = \ \Psi^{k+1}_{0\emptyset}(xq_1...q_k)$ *uniformly with respect to $x, q_1, ..., q_k$ in compact set.*

Proof. (a) We use induction on k to prove that for any natural number k we have:

$$\Psi^{k+1}_{\varepsilon w}\left(\Psi^{1}_{\varepsilon\emptyset}(x)...\Psi^{k+1}_{\varepsilon\emptyset}(xq_1...q_k)\right) \;=\; \Psi^{k+1}_{\varepsilon\emptyset}(xq_1...q_k) \tag{20.1.2}$$

For $k = 0$ we have have to prove that $x = x$ which is trivial. For $k = 1$ we have to prove that

$$\Psi^{2}_{\varepsilon w}\left(\Psi^{1}_{\varepsilon\emptyset}(x)\,\Psi^{2}_{\varepsilon\emptyset}(xq_1)\right) \;=\; \Psi^{2}_{\varepsilon\emptyset}(xq_1)$$

This means:

$$\Psi^{2}_{\varepsilon w}\left(x\,\delta^{x}_{\varepsilon^{-1}}\,Q^{x}\delta^{x}_{\varepsilon}q_1\right) \;=\; \delta^{x}_{\varepsilon^{-1}}\,Q^{x}_{w_1}\,\delta^{x}_{\varepsilon}\,\delta^{x}_{\varepsilon^{-1}}\,Q^{x}\,\delta^{x}_{\varepsilon}x_1 \;=$$
$$=\; \delta^{x}_{\varepsilon^{-1}}\,Q^{x}\,\delta^{x}_{\varepsilon}x_1 \;=\; \Psi^{2}_{\varepsilon\emptyset}(xq_1)$$

Suppose now that $l \geq 2$ and for any $k \leq l$ the relations (20.1.2) are true. Then, as previously, it is easy to check (20.1.2) for $k = l + 1$.

(b) is true by direct computation. The point (c) is a straightforward consequence of (b) and definition of coherent projections. $\square$

Definition 20.4 *Let $N \in \mathbb{N}$ be a strictly positive natural number and $\varepsilon \in (0,1]$. We say that $x \in X$ is (ε, N, Q)*-**nested** *in a open neighbourhood $U \subset X$ if there is $\rho > 0$ such that for any finite word $q = x_1...x_N \in X^N$ with*

$$\bar{\delta}^{x}_{\varepsilon}\bar{d}\left(x_{k+1}, \Psi^{k}_{\varepsilon\emptyset}([xq]_k)\right) \;\leq\; \rho$$

for any $k = 1, ..., N$, we have $q \in U^N$.

If $x \in U$ is (ε, N, Q)-nested then denote by $U(x, \varepsilon, N, Q, \rho) \subset U^N$ the collection of words $q \in U^N$ such that $\bar{\delta}^{x}_{\varepsilon}\bar{d}\left(x_{k+1}, \Psi^{k}_{\varepsilon\emptyset}([xq]_k)\right) \;<\; \rho$ for any $k = 1, ..., N$.

Definition 20.5 *A coherent projection Q satisfies the* **generalized Chow condition** *if:*

(Cgen) for any compact set K there are $\rho = \rho(K) > 0$, $r = r(K) > 0$, a natural number $N = N(Q, K)$ and a function $F(\eta) = \mathcal{O}(\eta)$ such that for any $x \in K$ and $\varepsilon \in (0,1]$ there are neighbourhoods $U(x)$, $V(x)$ such that any $x \in K$ is (ε, N, Q)-nested in $U(x)$, $B(x, r, \bar{\delta}^{x}_{\varepsilon}\bar{d}) \subset V(x)$ and such that the mapping

$$x_1...x_N \in U(x, N, Q, \rho) \;\mapsto\; \Psi^{N+1}_{\varepsilon\emptyset}(xx_1...x_N)$$

is surjective from $U(x, \varepsilon, N, Q, \rho)$ to $V(x)$. Moreover for any $z \in V(x)$ there exist $y_1, ...y_N \in U(x, \varepsilon, N, Q, \rho)$ such that $z = \Psi^{N+1}_{\varepsilon\emptyset}(xy_1, ...y_N)$ and for any $k = 0, ..., N-1$ we have

$$\delta^{x}_{\varepsilon}\bar{d}\left(\Psi^{k+1}_{\varepsilon\emptyset}(xy_1...y_k), \Psi^{k+2}_{\varepsilon\emptyset}(xy_1...y_{k+1})\right) \;\leq\; F(\delta^{x}_{\varepsilon}\bar{d}(x,z))$$

Condition (Cgen) is inspired from lemma 1.40 Folland-Stein [28]. If the coherent projection Q satisfies also (A) and (B) then in the space $(U(x), \bar{\delta}^{x}_{\varepsilon})$, with coherent projection $\hat{Q}^{x,\cdot}_{\varepsilon,\cdot}$, we can join any two sufficiently close points by a sequence of at most N horizontal curves. Moreover there is a control on the length of these curves via condition (B) and condition (Cgen); in sub-riemannian geometry the function F is of the type $F(\eta) = \eta^{1/m}$ with m positive natural number.

Definition 20.6 *Suppose that the coherent projection Q satisfies conditions (A), (B) and (Cgen). Let us consider $\varepsilon \in (0,1]$ and $x, y \in K$, K compact in X. With the notations from* *definition 20.5, suppose that there are numbers $N = N(Q,K)$, $\rho = \rho(Q,K) > 0$ and words $x_1 ... x_N \in U(x, \varepsilon, N, Q, \rho)$ such that*

$$y = \Psi^{N+1}_{\varepsilon \emptyset}(x x_1 ... x_N)$$

To these data we associate a **short curve** *joining x and y, $c : [0,N] \rightarrow X$ defined by: for any $t \in [0,N]$ then let $k = [t]$, where $[b]$ is the integer part of the real number b. We define the short curve by*

$$c(t) \ = \ \bar{\delta}^{x, \Psi^{k+1}_{\varepsilon \emptyset}(x x_1 ... x_k)}_{\varepsilon, t+N-k} Q^{\Psi^{k+1}_{\varepsilon \emptyset}(x x_1 ... x_k)} x_{k+1}$$

Any short curve joining x and y is a increasing linear reparameterization of a curve c described previously.

20.2 The candidate tangent space

Let $(X, \bar{d}, \bar{\delta})$ be a strong dilation structure and Q a coherent projection. Then we have the induced dilations

$$\mathring{\delta}^{x,u}_{\mu} v \ = \ \Sigma^x(u, \delta^x_{\mu} \Delta^x(u,v))$$

and the induced projection

$$\mathring{Q}^{x,u}_{\mu} v \ = \ \Sigma^x(u, Q^x_{\mu} \Delta^x(u,v))$$

For any curve $c : [0,1] \rightarrow U(x)$ which is $\mathring{\delta}^x$-derivable and $\mathring{Q}^x$-horizontal almost everywhere:

$$\frac{\mathring{d}^x}{dt} c(t) \ = \ \mathring{Q}^{x,u} \frac{\mathring{d}^x}{dt} c(t)$$

we define the length

$$l^x(c) \ = \ \int_0^1 \bar{d}^x \left(x, \Delta^x(c(t), \frac{\mathring{d}^x}{dt} c(t)) \right) \ \mathrm{dt}$$

and the distance function:

$$\mathring{d}^x(u,v) \ = \ \inf \Big\{ l^x(c) \, : \, c : [0,1] \rightarrow U(x) \ \text{ is } \mathring{\delta}^x\text{-derivable,}$$

$$\text{and } \mathring{Q}^x\text{-horizontal a.e.}, \ c(0) = u, c(1) = v \Big\}$$

We want to prove that $(U(x), \mathring{d}^x, \mathring{\delta}^x)$ is a strong dilation structure and $\mathring{Q}^x$ is a coherent projection. For this we need first the following proposition.

Proposition 20.7 *The curve $c : [0,1] \rightarrow U(x)$ is $\mathring{\delta}^x$-derivable, $\mathring{Q}^x$-horizontal almost everywhere, and $l^x(c) < +\infty$ if and only if the curve $Q^x c$ is $\bar{\delta}^x$-derivable almost everywhere and $\bar{l}^x(Q^x c) < +\infty$. Moreover, we have*

$$\bar{l}^x(Q^x c) \ = \ l^x(c)$$

Proof. The curve c is $\mathring{Q}^x$-horizontal almost everywhere if and only if for almost any $t \in [0,1]$ we have

$$Q^x \, \Delta^x(c(t), \frac{\mathring{d}^x}{dt} c(t)) \;=\; \Delta^x(c(t), \frac{\mathring{d}^x}{dt} c(t))$$

We shall prove that c is $\mathring{Q}^x$-horizontal is equivalent with

$$\Theta^x(c(t), \frac{\mathring{d}^x}{dt} c(t)) \;=\; \frac{\bar{d}^x}{dt} \left(Q^x c \right)(t) \tag{20.2.3}$$

Indeed, (20.2.3) is equivalent with

$$\lim_{\varepsilon \to 0} \bar{\delta}^x_{\varepsilon^{-1}} \bar{\Delta}^x (Q^x c(t), Q^x c(t+\varepsilon)) \;=\; \bar{\Delta}^x(Q^x c(t), \Theta^x(c(t), \frac{\mathring{d}^x}{dt} c(t)))$$

which is equivalent with

$$\lim_{\varepsilon \to 0} \bar{\delta}^x_{\varepsilon^{-1}} \bar{\Delta}^x (Q^x c(t), Q^x c(t+\varepsilon)) \;=\; \Delta^x(c(t), \frac{\mathring{d}^x}{dt} c(t))$$

But this is equivalent with:

$$\lim_{\varepsilon \to 0} \bar{\delta}^x_{\varepsilon^{-1}} \bar{\Delta}^x (Q^x c(t), Q^x c(t+\varepsilon)) \;=\; \lim_{\varepsilon \to 0} \delta^x_{\varepsilon^{-1}} \Delta^x (c(t), c(t+\varepsilon)) \tag{20.2.4}$$

The horizontality condition for the curve c can be written as:

$$\lim_{\varepsilon \to 0} Q^x \delta^x_{\varepsilon^{-1}} \Delta^x (c(t), c(t+\varepsilon)) \;=\; \lim_{\varepsilon \to 0} \delta^x_{\varepsilon^{-1}} \Delta^x (c(t), c(t+\varepsilon))$$

We use now the properties of Q^x in the left hand side of the previous equality:

$$Q^x \delta^x_{\varepsilon^{-1}} \Delta^x (c(t), c(t+\varepsilon)) \;=\; \bar{\delta}^x_{\varepsilon^{-1}} Q^x \Delta^x (c(t), c(t+\varepsilon)) \;=$$
$$=\; \bar{\delta}^x_{\varepsilon^{-1}} \bar{\Delta}^x (Q^x c(t), Q^x c(t+\varepsilon))$$

thus after taking the limit as $\varepsilon \to 0$ we prove that the limit

$$\lim_{\varepsilon \to 0} \bar{\delta}^x_{\varepsilon^{-1}} \bar{\Delta}^x (Q^x c(t), Q^x c(t+\varepsilon))$$

exists and we obtain:

$$\lim_{\varepsilon \to 0} \delta^x_{\varepsilon^{-1}} \Delta^x (c(t), c(t+\varepsilon)) \;=\; \lim_{\varepsilon \to 0} \bar{\delta}^x_{\varepsilon^{-1}} \bar{\Delta}^x (Q^x c(t), Q^x c(t+\varepsilon))$$

This last equality is the same as (20.2.4), which is equivalent with (20.2.3).

As a consequence we obtain the following equality, for almost any $t \in [0,1]$:

$$\bar{d}^x \left(x, \Delta^x(c(t), \frac{\mathring{d}^x}{dt} c(t)) \right) \;=\; \bar{\Delta}^x(Q^x c(t), \frac{\bar{d}^x}{dt} \left(Q^x c \right)(t)) \tag{20.2.5}$$

This implies that $Q^x c$ is absolutely continuous and by theorem 4.11, as in the proof of theorem 17.4 (but without using the Radon-Nikodym property property, because we already know that $Q^x c$ is derivable a.e.), we obtain the following formula for the length of the curve $Q^x c$:

$$\bar{l}^x(Q^x c) \;=\; \int_0^1 \bar{d}^x \left(x, , \bar{\Delta}^x(Q^x c(t), \frac{\bar{d}^x}{dt} \left(Q^x c \right)(t)) \right) \, \mathrm{d}t$$

But we have also:

$$l^x(c) \;=\; \int_0^1 \bar{d}^x \left(x, \Delta^x(c(t), \frac{\mathring{d}^x}{dt} c(t)) \right) \, \mathrm{dt}$$

By (20.2.5) we obtain $\bar{l}^x(Q^x c) \;=\; l^x(c)$. □

Proposition 20.8 *If $(X, \bar{d}, \bar{\delta})$ is a strong dilation structure, Q is a coherent projection and $\mathring{d}^x$ is finite then the triple $(U(x), \Sigma^x, \delta^x)$ is a normed conical group, with the norm induced by the left-invariant distance $\mathring{d}^x$.*

Proof. The fact that $(U(x), \Sigma^x, \delta^x)$ is a conical group comes directly from the definition 19.1 of a coherent projection. Indeed, it is enough to use proposition 19.3 (c) and the formalism of binary decorated trees in [11] section 4 (or theorem 11 [11]), in order to reproduce the part of the proof of theorem 10 (p.87-88) in that paper, concerning the conical group structure. There is one small subtlety though. In the proof of theorem 13.5(a) the same modification of proof has been done starting from the axiom A4+, namely the existence of the uniform limit $\lim_{\varepsilon \to 0} \Sigma^x_\varepsilon(u,v) \;=\; \Sigma^x(u,v)$. Here we need first to prove this limit, in a similar way as in the corollary 9 [11]. We shall use for this the distance $\mathring{d}^x$ instead of the distance in the metric tangent space of (X, d) at x denoted by d^x (which is not yet proven to exist). The distance $\mathring{d}^x$ is supposed to be finite by hypothesis. Moreover, by its definition and proposition 20.7 we have

$$\mathring{d}^x(u,v) \;\geq\; \bar{d}^x(u,v)$$

therefore the distance $\mathring{d}^x$ is non degenerate. By construction this distance is also left invariant with respect to the group operation $\Sigma^x(\cdot, \cdot)$. Therefore we may repeat the proof of corollary 9 [11] and obtain the result that A4+ is true for (X, d, δ).

What we need to prove next is that $\mathring{d}^x$ induces a norm on the conical group $(U(x), \Sigma^x, \delta^x)$. For this it is enough to prove that

$$\mathring{d}^x(\mathring{\delta}^{x,u}_\mu v, \mathring{\delta}^{x,u}_\mu w) \;=\; \mu \, \mathring{d}^x(v,w) \tag{20.2.6}$$

for any $v, w \in U(x)$. This is a direct consequence of relation (20.2.5) from the proof of the proposition 20.7. Indeed, by direct computation we get that for any curve c which is $\mathring{Q}^x$-horizontal a.e. we have:

$$l^x(\mathring{\delta}^{x,u}_\mu c) \;=\; \int_0^1 \bar{d}^x \left(x, \Delta^x \left(\mathring{\delta}^{x,u}_\mu c(t), \frac{\mathring{d}^x}{dt} \left(\mathring{\delta}^{x,u}_\mu c \right)(t) \right) \right) \, \mathrm{dt} \;=$$

$$=\; \int_0^1 \bar{d}^x \left(x, \delta^x_\mu \Delta^x \left(c(t), \frac{\mathring{d}^x}{dt} c(t) \right) \right) \, \mathrm{dt}$$

But c is $\mathring{Q}^x$-horizontal a.e., which implies, via (20.2.5), that

$$\delta^x_\mu \Delta^x \left(c(t), \frac{\mathring{d}^x}{dt} c(t) \right) \;=\; \bar{\delta}^x_\mu \Delta^x \left(c(t), \frac{\mathring{d}^x}{dt} c(t) \right)$$

therefore we have

$$l^x(\mathring{\delta}^{x,u}_\mu c) \;=\; \int_0^1 \bar{d}^x \left(x, \bar{\delta}^x_\mu \Delta^x \left(c(t), \frac{\mathring{d}^x}{dt} c(t) \right) \right) \, \mathrm{dt} \;=\; \mu \, l^x(c)$$

This implies (20.2.6), therefore the proof is done. □

Theorem 20.9 *If the generalized Chow condition (Cgen) and condition (B) are true then* $(U(x), \Sigma^x, \delta^x)$ *is local conical group which is a neighbourhood of the neutral element of a Carnot group generated by* $Q^x U(x)$.

Proof. For any $\varepsilon \in (0,1]$, as a consequence of proposition 19.6 we can put the recurrence relations (20.1.1) in the form:

$$\Psi^{k+1}_{\varepsilon w}([q]_{k+1}) \;=\; \Sigma^x_\varepsilon \left(\Psi^k_{\varepsilon w}([q]_k), Q_{w_k}^{\delta^x_\varepsilon \Psi^k_{\varepsilon w}([q]_k)} \, \Delta^x_\varepsilon \left(\Psi^k_{\varepsilon w}([q]_k), q_{k+1} \right) \right) \tag{20.2.7}$$

This recurrence relation allows us to prove by induction that for any k the limit

$$\Psi^k_w([q]_k) \;=\; \lim_{\varepsilon \to 0} \Psi^k_{\varepsilon w}([q]_k)$$

exists and it satisfies the recurrence relation:

$$\Psi^{k+1}_{0w}([q]_{k+1}) \;=\; \Sigma^x \left(\Psi^k_{0w}([q]_k), Q^x_{w_k} \, \Delta^x \left(\Psi^k_{0w}([q]_k), q_{k+1} \right) \right) \tag{20.2.8}$$

and the initial condition $\Psi^1_{0w}(x) = x$. We pass to the limit in the generalized Chow condition (Cgen) and we thus obtain that a neighbourhood of the neutral element x is (algebraically) generated by $Q^x U(x)$. Then the distance $\mathring{d}^x$. Therefore by proposition 20.8 $(U(x), \Sigma^x, \delta^x)$ is a normed conical group generated by $Q^x U(x)$.

Let $c : [0,1] \to U(x)$ be the curve $c(t) = \delta^x_t u$, with $u \in Q^x U(x)$. Then we have $Q^x c(t) = c(t) = \bar{\delta}^x_t u$. From condition (B) we get that c is $\bar{\delta}$-derivable at $t = 0$. A short computation of this derivative shows that:

$$\frac{d\bar{\delta}}{dt} c(0) \;=\; u$$

Another easy computation shows that the curve c is $\bar{\delta}^x$-derivable if and only if the curve c is $\bar{\delta}$-derivable at $t = 0$, which is true, therefore c is $\bar{\delta}^x$-derivable, in particular at $t = 0$. Moreover, the expression of the $\bar{\delta}^x$-derivative of c shows that c is also Q^x-everywhere horizontal (compare with the remark 19.9). We use the proposition 20.7 and relation (20.2.3) from its proof to deduce that $c = Q^x c$ is $\mathring{\delta}^x$-derivable at $t = 0$, thus for any $u \in Q^x U(x)$ and small enough $t, \tau \in (0,1)$ we have

$$\mathring{\delta}^{x,x}_{t+\tau} u \;=\; \bar{\Sigma}^x(\bar{\delta}^x_t u, \bar{\delta}^x_\tau u) \tag{20.2.9}$$

By previous proposition 20.8 and corollary 6.3 [12] the normed conical group $(U(x), \Sigma^x, \delta^x)$ is in fact locally a homogeneous group, i.e. a simply connected Lie group which admits a positive graduation given by the eigenspaces of δ^x. Indeed, corollary 6.3 [11] is originally about strong dilation structures, but the generalized Chow condition implies that the distances d, $\bar{d}$ and $\mathring{d}^x$ induce the same uniformity, which, along with proposition 20.8, are the only things needed for the proof of this corollary. The conclusion of corollary 6.3 [12] therefore is true, that is $(U(x), \Sigma^x, \delta^x)$ is locally a homogeneous group. Moreover it is locally Carnot if and only if on the generating space $Q^x U(x)$ any dilation $\mathring{\delta}^{x,x}_\varepsilon u = \bar{\delta}^x_\varepsilon$ is linear in ε. But this is true, as shown by relation (20.2.9). This ends the proof. □

20.3 Coherent projections induce length dilation structures

Theorem 20.10 *If* $(X, \bar{d}, \bar{\delta})$ *is a tempered strong dilation structure, has the Radon-Nikodym property and* Q *is a coherent projection, which satisfies (A), (B), (Cgen) then* (X, d, δ) *is a length dilation structure.*

Proof. We shall prove that:

(a) for any function $\varepsilon \in (0,1) \mapsto (x_\varepsilon, c_\varepsilon) \in \mathcal{L}_\varepsilon(X,d,\delta)$ which converges to (x,c) as $\varepsilon \to 0$, with $c : [0,1] \to U(x)$ $\mathring{\delta}^x$-derivable and $\mathring{Q}^x$-horizontal almost everywhere, we have:

$$l^x(c) \ \leq \ \liminf_{\varepsilon \to 0} l^{x_\varepsilon}(c_\varepsilon)$$

(b) for any sequence $\varepsilon_n \to 0$ and any (x,c), with $c : [0,1] \to U(x)$ $\mathring{\delta}^x$-derivable and $\mathring{Q}^x$-horizontal almost everywhere, there is a recovery sequence $(x_n, c_n) \in \mathcal{L}_{\varepsilon_n}(X,d,\delta)$ such that

$$l^x(c) \ = \ \lim_{n \to \infty} l^{x_n}(c_n)$$

Proof of (a). This is a consequence of propositions 20.7, 19.12 and definition 19.1 of a coherent projection. With the notations from (a) we see that we have to prove

$$l^x(c) \ = \ \bar{l}^x(Q^x c) \ \leq \ \liminf_{\varepsilon \to 0} \bar{l}^{x_\varepsilon}(Q^{x_\varepsilon}_\varepsilon c_\varepsilon)$$

This is true because $(X, \bar{d}, \bar{\delta})$ is a tempered dilation structure and because of condition (A). Indeed from the fact that $(X, \bar{d}, \bar{\delta})$ is tempered and from (19.3.8) (which is a consequence of condition (A)) we deduce that Q_ε is uniformly continuous on compact sets in a uniform way: for any compact set $K \subset X$ there is are constants $L(K) > 0$ (from (A)) and $C > 0$ (from the tempered condition) such that for any $\varepsilon \in (0,1]$, any $x \in K$ and any u, v sufficiently close to x we have:

$$\bar{d}\,(Q^x_\varepsilon u, Q^x_\varepsilon v) \ \leq \ C\ \left(\bar{\delta}^x_\varepsilon \bar{d}\right)(Q^x_\varepsilon u, Q^x_\varepsilon v) \ \leq \ C\, L(K)\, \bar{d}(u,v)$$

The sequence Q^x_ε uniformly converges to Q^x as ε goes to 0, uniformly with respect to x in compact sets. Therefore if $(x_\varepsilon, c_\varepsilon) \in \mathcal{L}_\varepsilon(X,d,\delta)$ converges to (x,c) then $(x_\varepsilon, Q^{x_\varepsilon}_\varepsilon c_\varepsilon) \in \mathcal{L}_\varepsilon(X, \bar{d}, \bar{\delta})$ converges to $(x, Q^x c)$. Use now the fact that by corollary 18.4 $(X, \bar{d}, \bar{\delta})$ is a length dilation structure. The proof is done.

Proof of (b). We have to construct a recovery sequence. We are doing this by discretization of $c : [0,L] \to U(x)$. Recall that c is a curve which is $\mathring{\delta}^x$-derivable a.e. and $\mathring{Q}^x$-horizontal, that is for almost every $t \in [0,L]$ the limit

$$u(t) \ = \ \lim_{\mu \to 0} \delta^x_{\mu^{-1}}\, \Delta^x(c(t), c(t+\mu))$$

exists and $Q^x\, u(t) \ = \ u(t)$. Moreover we may suppose that for almost every t we have $\bar{d}^x(x, u(t)) \leq 1$ and $\bar{l}^x(c) \leq L$.

There are functions $\omega^1, \omega^2 : (0,+\infty) \to [0,+\infty)$ with $\lim_{\lambda \to 0} \omega^i(\lambda) = 0$, with the following property: for any $\lambda > 0$ sufficiently small there is a division $A_\lambda \ = \ \{0 < t_0 < ... < t_P < L\}$ such that

$$\frac{\lambda}{2} \ \leq \ \min \left\{ \frac{t_0}{t_1 - t_0}, \frac{L - t_P}{t_P - t_{P-1}}, t_k - t_{k-1} \,:\, k = 1, ..., P \right\} \tag{20.3.10}$$

$$\lambda \ \geq \ \max \left\{ \frac{t_0}{t_1 - t_0}, \frac{L - t_P}{t_P - t_{P-1}}, t_k - t_{k-1} \,:\, k = 1, ..., P \right\} \tag{20.3.11}$$

and such that $u(t_k)$ exists for any $k = 1, ..., P$ and

$$\mathring{d}^x(c(0), c(t_0)) \ \leq \ t_0 \ \leq \ \lambda^2 \tag{20.3.12}$$

$$\mathring{d}^x(c(L), c(t_P)) \leq L - t_P \leq \lambda^2 \tag{20.3.13}$$

$$\mathring{d}^x(u(t_{k-1}), \Delta^x(c(t_{k-1}), c(t_k)) \leq (t_k - t_{k-1})\, \omega^1(\lambda) \tag{20.3.14}$$

$$\mid \int_0^L \bar{d}^x(x, u(t))\, \mathrm{d}t - \sum_{k=0}^{P-1} (t_{k+1} - t_k)\, \bar{d}^x(x, u(t_k)) \mid \leq \omega^2(\lambda) \tag{20.3.15}$$

Indeed (20.3.12), (20.3.13) are a consequence of the fact that c is $\mathring{d}^x$-Lipschitz, (20.3.14) is a consequence of Egorov theorem applied to

$$f_\mu(t) = \delta^x_{\mu^{-1}}\, \Delta^x(c(t), c(t+\mu))$$

and (20.3.15) comes from the definition of the integral

$$l(c) = \int_0^L \bar{d}^x(x, u(t))\, \mathrm{d}t$$

For each λ we shall choose $\varepsilon = \varepsilon(\lambda)$ and we shall construct a curve c_λ with the properties:

(i) $(x, c_\lambda) \in \mathcal{L}_{\varepsilon(\lambda)}(X, d, \delta)$

(ii) $\lim_{\lambda \to 0} l^x_{\varepsilon(\lambda)}(c_\lambda) = l^x(c)$.

At almost every t the point $u(t)$ represents the velocity of the curve c seen as the the left translation of $\frac{\mathring{d}^x}{dt} c(t)$ by the group operation $\Sigma^x(\cdot, \cdot)$ to x (which is the neutral element for the mentioned operation). The derivative (with respect to $\mathring{\delta}^x$) of the curve c at t is

$$y(t) = \Sigma^x(c(t), u(t))$$

Let us take $\varepsilon > 0$, arbitrary for the moment. We shall use the points of the division A_λ and for any $k = 0, ..., P-1$ we shall define the point:

$$y^\varepsilon_k = \hat{Q}^{x, c(t_k)}_\varepsilon\, \Sigma^x_\varepsilon(c(t_k), u(t_k)) \tag{20.3.16}$$

Thus y^ε_k is obtained as the "projection" by $\hat{Q}^{x,c(t_k)}_\varepsilon$ of the "approximate left translation" $\Sigma^x_\varepsilon(c(t_k), \cdot)$ by $c(t_k)$ of the velocity $u(t_k)$. Define also the point:

$$y_k = \Sigma^x(c(t_k), u(t_k))$$

By construction we have:

$$y^\varepsilon_k = \hat{Q}^{x, c(t_k)}_\varepsilon\, y^\varepsilon_k \tag{20.3.17}$$

and by computation we see that y^ε_k can be expressed as:

$$y^\varepsilon_k = \delta^x_{\varepsilon^{-1}}\, Q^{\delta^x_\varepsilon c(t_k)}\, \delta^{\delta^x_\varepsilon c(t_k)}_\varepsilon\, u(t_k) = \tag{20.3.18}$$

$$= \Sigma^x_\varepsilon(c(t_k), Q^{\delta^x_\varepsilon c(t_k)}\, u(t_k)) = \delta^x_{\varepsilon^{-1}}\, \bar{\delta}^{\delta^x_\varepsilon c(t_k)}_\varepsilon\, Q^{\delta^x_\varepsilon c(t_k)}\, u(t_k)$$

Let us define the curve

$$c^\varepsilon_k(s) = \hat{\delta}^{x, c(t_k)}_{\varepsilon, s}\, y^\varepsilon_k \quad , \quad s \in [0, t_{k+1} - t_k] \tag{20.3.19}$$

which is a $\hat{Q}^x_\varepsilon$-horizontal curve (by supplementary hypothesis (B)) which joins $c(t_k)$ with the point

$$z^\varepsilon_k = \hat{\delta}^{x, c(t_k)}_{\varepsilon, t_{k+1} - t_k}\, y^\varepsilon_k \tag{20.3.20}$$

The point z_k^ε is an approximation of the point

$$z_k \,=\, \mathring{\delta}^{x,c(t_k)}_{t_{k+1}-t_k} y_k$$

We shall also consider the curve

$$c_k(s) \,=\, \mathring{\delta}^{x,c(t_k)}_{s}\, y_k \quad , \quad s \in [0, t_{k+1} - t_k] \tag{20.3.21}$$

There is a short curve g_k^ε which joins z_k^ε with $c(t_{k+1})$, according to condition (Cgen). Indeed, for ε sufficiently small the points $\delta^x_\varepsilon \, z_k^\varepsilon$ and $\delta^x_\varepsilon \, c(t_{k+1})$ are sufficiently close.

Finally, take g_0^ε and g_{P+1}^ε "short curves" which join $c(0)$ with $c(t_0)$ and $c(t_P)$ with $c(L)$ respectively.

Correspondingly, we can find short curves g_k (in the geometry of the dilation structure $(U(x), \mathring{d}^x, \mathring{\delta}^x, \mathring{Q}^x)$) joining z_k with $c(t_{k+1})$, which are the uniform limit of the short curves g_k^ε as $\varepsilon \to 0$. Moreover this convergence is uniform with respect to k (and λ). Indeed, these short curves are made by N curves of the type $s \mapsto \hat{\delta}^{x,u_\varepsilon}_{\varepsilon,s} v_\varepsilon$, with $\hat{Q}^{x,u_\varepsilon} v_\varepsilon \,=\, v_\varepsilon$. Also, the short curves g_k are made respectively by N curves of the type $s \mapsto \mathring{\delta}^{x,u}_s v$, with $\mathring{Q}^{x,u} v \,=\, v$. Therefore we have:

$$\bar{d}(\mathring{\delta}^{x,u}_s v, \hat{\delta}^{x,u_\varepsilon}_{\varepsilon,s} y_k^\varepsilon) \,=$$

$$=\, \bar{d}(\Sigma^x(u, \bar{\delta}^x_s \Delta^x(u,v)), \Sigma^x_\varepsilon(u_\varepsilon, \bar{\delta}_s^{\delta^x_\varepsilon u_\varepsilon} \Delta^x_\varepsilon(u_\varepsilon, v_\varepsilon)))$$

By an induction argument on the respective ends of segments forming the short curves, using the axioms of coherent projections, we get the result.

By concatenation of all these curves we get two new curves:

$$c_\lambda^\varepsilon \,=\, g_0^\varepsilon \left(\prod_{k=0}^{P-1} c_k^\varepsilon \, g_k^\varepsilon \right) g_{P+1}^\varepsilon$$

$$c_\lambda \,=\, g_0 \left(\prod_{k=0}^{P-1} c_k \, g_k \right) g_{P+1}$$

From the previous reasoning we get that as $\varepsilon \to 0$ the curve c_λ^ε uniformly converges to c_λ, uniformly with respect to λ.

By theorem 20.9, specifically from relation (20.2.9) and considerations below, we notice that for any $u \,=\, Q^x u$ the length of the curve $s \mapsto \delta^x_s u$ is:

$$l^x(s \in [0,a] \mapsto \delta^x_s u) \,=\, a \, \bar{d}^x(x,u)$$

From here and relations (20.3.12), (20.3.13), (20.3.14), (20.3.15) we get that

$$l^x(c) \,=\, \lim_{\lambda \to 0} l^x(c_\lambda) \tag{20.3.22}$$

Condition (B) and the fact that $(X, \bar{d}, \bar{\delta})$ is tempered imply that there is a positive function $\omega^3(\varepsilon) = \mathcal{O}(\varepsilon)$ such that

$$\mid l^x_\varepsilon(c_\lambda^\varepsilon) - l^x(c_\lambda) \mid \leq \, \frac{\omega^3(\varepsilon)}{\lambda} \tag{20.3.23}$$

This is true because if $v \, \hat{Q}^{x,u}_\varepsilon v$ then $\delta^x_\varepsilon v \,=\, Q^{\delta^x_\varepsilon u} \delta^x_\varepsilon v$, therefore by condition (B)

$$\frac{l^x_\varepsilon(s \in [0,a] \mapsto \hat{\delta}^{x,u}_{\varepsilon,s} v)}{\delta^x_\varepsilon \bar{d}(u,v)} \,=\, \frac{\bar{l}(s \in [0,a] \mapsto \bar{\delta}_s^{\delta^x_\varepsilon u} \delta^x_\varepsilon v)}{\bar{d}(\delta^x_\varepsilon u, \delta^x_\varepsilon v)} \,\leq\, \mathcal{O}(\varepsilon) + 1$$

Since each short curve is made by N segments and the division A_λ is made by $1/\lambda$ segments, the relation (20.3.23) follows.

We shall choose now $\varepsilon(\lambda)$ such that $\omega^3(\varepsilon(\lambda)) \leq \lambda^2$ and we define:

$$c_\lambda = c_\lambda^{\varepsilon(\lambda)}$$

These curves satisfy the properties (i), (ii). Indeed (i) is satisfied by construction and (ii) follows from the choice of $\varepsilon(\lambda)$, uniform convergence of c_λ^ε to c_λ, uniformly with respect to λ, and relations (20.3.23), and (20.3.22). □

21 Sub-riemannian symmetric spaces as braided dilation structures

Sub-riemannian symmetric spaces have been introduced in [55], section 9. We shall be interested in the description of sub-riemannian geometry by dilation structures, therefore we shall use the same notations as in the previous paper [14] (see also the relevant citations in that paper, as well as the long paper [19], where the study of sub-riemannian geometry as a length dilation structure is completed).

Definition 21.1 *(adaptation of [55] definition 8.1) Let (M, D, g) be a regular sub-riemannian manifold. We say that $\Psi : M \rightarrow M$ is an infinitesimal isometry if Ψ is $\mathcal{C}^1$ and $D\Psi$ preserves the metric g. An infinitesimal isometry is regular if for any $x \in M$ and any tangent vector $u \in T_x M$*

$$\Psi(exp_x(u)) \, = \, exp_{\Psi(x)}(D\Psi(x)u)$$

By [55] theorem 8.2., $\mathcal{C}^1$ isometries are regular infinitesimal isometries and, conversely, regular infinitesimal isometries are isometries.

An equivalent description of regular infinitesimal isometries is the following: they are $\mathcal{C}^1$ Pansu differentiable isometries.

Definition 21.2 *([55] definition 9.1) A sub-riemannian symmetric space is a regular sub-riemannian manifold $M, D, g)$ which has a transitive Lie group G of regular infinitesimal isometries acting differentiably on M such that:*

(i) there is a point $x \in X$ such that the isotropy subgroup K of x is compact,

(ii) K contains an element Ψ such that $D\Psi(x)_{|D_x} = -\, id$ and Ψ is involutive.

If G is a group for which (i), (ii) holds then we call G an admissible isometry group for M.

Theorem 21.3 *([55] theorem 9.2) If M is a sub-riemannian symmetric space and G is an admissible isometry group, then there exists an involution σ of G such that $\sigma(K) \subset K$ with the following properties (we write $\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^-$, where $\mathfrak{g}^+, \mathfrak{g}^-$ are the subspaces of $\mathfrak{g}$ on which $D\sigma$ acts as Id, $-Id$):*

(a) $\mathfrak{g}$ is generated as a Lie algebra by a subspace $\mathfrak{p}$ and the Lie algebra $\mathfrak{k}$ of K with $\mathfrak{p} \subset \mathfrak{g}^-$, $\mathfrak{k} \subset \mathfrak{g}^+$,

(b) there exists a positive definite quadratic form g on $\mathfrak{p}$ and $ad\, K$ maps $\mathfrak{p}$ to itself and preserves g. Furthermore, $\mathfrak{p}$ may be identified with D_x under the exponential map of the Lie algebra $\mathfrak{g}$, and g with the sub-riemannian metric on D_x.

Conversely, given a Lie group G and an involution σ such that (a) and (b) hold, then G/K forms a sub-riemannian symmetric space, where $D_{x_0} = \, exp\, \mathfrak{p}$ for the point x_0 identified with the coset K, and the sub-riemannian metric on D_{x_0} is given by g. The bundle D and its metric is then uniquely determined by the requirement that elements of G be infinitesimal isometries.

As a consequence of this theorem we see that we may endow a sub-riemannian symmetric space, with admissible isometry group G, with a (reflexive space) operation

$$(x,y) \in M^2 \mapsto \Psi(x,y) = \Psi^x y$$

such that Ψ is distributive, for any $x \in X$ the map Ψ^x satisfies (ii) definition 21.2, and for any $g \in G$ and any $x, y \in X$ we have

$$g\left(\Psi^x y\right) \, = \, \Psi^{g(x)} g(y)$$

We explained in [14] that we can construct a dilation structure over a regular sub-riemannian manifold by using adapted frames.

Let us consider now dilations structures with Γ isomorphic with $\mathbb{R} \times \mathbb{Z}_2$. That means Γ is the commutative group made by two copies of $(0,+\infty)$, generated by $(0,+\infty)$ and an element $\sigma \notin (0,+\infty)$, with the properties: for any $\varepsilon \in (0,+\infty)$ we have $\varepsilon\sigma = \sigma\varepsilon$ and $\sigma\sigma = 1$. The absolute we take has two elements, one corresponding to $\varepsilon \to 0$ (we denote it by "0") and the other one is the transport by σ of 0, denoted by "0σ". The morphism $|\cdot|$ is defined by

$$\mid \varepsilon \mid = \mid \sigma\varepsilon \mid = \varepsilon$$

Let (X, d, δ) be a dilation structure with respect to the group Γ , absolute $Abs(\Gamma)$ and morphism $|\cdot|$ described previously. Then for any $\varepsilon \in (0,+\infty)$ and any $x \in X$ we have the relations:

$$\delta^x_\sigma \, \delta^x_\varepsilon \, = \, \delta^x_\varepsilon \, \delta^x_\sigma \quad , \quad \delta^x_\sigma \, \delta^x_\sigma \, = \, id$$

Proposition 21.4 *Denote by $\sigma^x y = \delta^x_\sigma y$ and suppose that for any $x \in X$ the map σ^x is not the identity map. Then σ^x is involutive, a isometry of d^x and an isomorphism of the conical group $T^x X$.*

Proof. For any $x \in X$ clearly σ^x is involutive, commutes with dilations δ^x_ε and is an isometry of d^x. We need to show that it preserves the operation $+^x_\infty$. We shall work with the notations from dilation structures. We have then, for any $\varepsilon \in (0,+\infty)$:

$$\sigma^{\delta^x_{\varepsilon\sigma} u} \, \Delta^x_{\sigma\varepsilon}(u,v) \, = \, \delta^{\delta^x_{\sigma\varepsilon} u}_{\varepsilon^{-1}} \, \delta^x_{\sigma\varepsilon} v \, = \, \Delta^x_\varepsilon(\sigma^x u, \sigma^x v)$$

We pass to the limit with $\varepsilon \to 0$ and we get the relation:

$$\sigma^x \Delta^x(u,v) \, = \, \Delta^x(\sigma^x u, \sigma^x v)$$

which shows that σ^x is an isomorphism of $T^x X$. $\square$

This proposition motivates us to introduce braided $\mathbb{R} \times \mathbb{Z}_2$-dilation structures.

Definition 21.5 *Let (X, d, δ) be a dilation structure, with respect to the group Γ , absolute $Abs(\Gamma)$ and morphism $|\cdot|$ described previously, and such that for any $x \in X$ the map σ^x is not the identity map. This dilation structure is braided if the map*

$$(x,y) \in X^2 \mapsto (\sigma^x y, x)$$

is a braided map.

Theorem 21.6 *A sub-riemannian symmetric space M with admissible isometry group G can be endowed with a braided $\mathbb{R} \times \mathbb{Z}_2$-dilation structure which is G-invariant, that is for any $g \in G$, for any $x, y \in M$, and for any $\varepsilon \in \Gamma$ we have*

$$g\left(\delta^x_\varepsilon y\right) \, = \, \delta^{g(x)}_\varepsilon g(y)$$

Proof. In the particular case of a sub-riemannian symmetric space we may obviously take the adapted frames to be G-invariant, therefore we may construct a dilation structure (over the group $(0, +\infty)$ with multiplication) which is G-invariant. Because Ψ^x satisfies (ii) definition 21.2, it follows Ψ^x is differentiable in x in the sense of dilation structures. We extend the dilation structure to a braided one by defining for any $x \in X$

$$\sigma^x \, = \, T\Psi^x(x, \cdot)$$

By G-invariance of both the dilation structure and the operation Ψ it follows that

$$T\Psi^x(x, \cdot) \, = \, \Psi^x$$

therefore σ^x commutes with δ^x_ε, which ensures us that we well defined a braided dilation structure. $\square$

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