Uniform spaces, coarse spaces, dilation spaces

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Abstract

WARNING: WORKING VERSION, NOT FINAL, THANKS FOR COMMENTS.

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1 Preliminaries

Dilation structures have been introduced in [3] with the name "dilatation structures" (a latin friendly denomination). An alternative name for the same object is "metric spaces with dilations".

One of the axioms of dilation structures (axiom 0) specifies conditions on the domain of definition of dilations of coefficient $\varepsilon \in (0, 1]$. Such conditions seem convoluted at first sight, but they are needed for the other axioms, which are statements about functions which are constructed from dilations. We have to be sure that these functions make sense, i.e. they are at least well defined.

Therefore, at first view, the axiom 0 of dilation structures is just a collection of necessary, but boring facts. However, a first sign that this is not true can be inferred from the paper [4], about dilatation structures on ultrametric spaces. In this paper the axiom 0 has strong consequences concerning the form of dilation structures on such spaces.

... some connections between this axiom and the subject of approximate groups cf. Breuillard, Green and Tao [2] (and the references available therein).

2 A simple idea

In a normed real vector space $(V, \|\cdot\|)$ we may use dilations as replacements of metric balls. Here is the simple idea.

Let us define, for any $x \in V$, the domain of x as

$$U(x) = \{u \in V \,:\, \|x - u\| < 1\}$$

that is the metric ball centered at x, of radius one, with respect to the distance $d(x, y) = \|x - y\|$. In general, let

$$B(x, r) = \{u \in V \,:\, d(x, u) < 1\}$$

be the metric ball with respect to the distance induced by the norm.

Also, for any $x \in V$ and $\varepsilon \in (0, +\infty)$, the dilation based at x, with coefficient ε is the function

$$\delta^x_\varepsilon : V \to V \quad , \quad \delta^x_\varepsilon y = x + \varepsilon(-x+y)$$

(notice that I don't write the dilation as $\delta^x_\varepsilon = (1-\varepsilon)x + \varepsilon y$, because I don't want to use the commutativity of the operation "+", see further why).

I use these notations to write the closed ball of radius $r > 0$ as

$$B(x,r) = \delta^x_\varepsilon U(x)$$

and then ask what happens if we replace the structure of real vector space and the norm by the "field of dilations" $(x, \varepsilon) \mapsto (U(x), \delta^x_\varepsilon)$.

In fact, spaces with dilations, or dilatation structures, or dilation structures, are various names for spaces endowed with fields of dilations which satisfy certain axioms. Real normed vector spaces are examples of spaces with dilations, as a subclass of the larger class of conical groups with dilations, itself just a subclass of groups with dilations. Regular sub-riemannian manifolds are examples of spaces with dilations without a predefined group structure.

For any space with dilations, then, we may ask what can we get by forgetting the distance function, but keeping the dilation structure. From the beginning, the axioms of dilations structures have to be modified, because we have to know what is the correct translation, in terms of dilations fields of the following:

- the uniformity structure of the space with dilations expressed in terms of fields of dilations (or, when a field of dilations produces an uniformity),
- when the field of dilations is uniformly continuous with respect to the uniformity induced by it,
- express the various uniform convergence statements in the axioms of dilations structures in terms of the uniformity induced by the filed of dilations.

Another structure, as interesting as the uniformity structure, is the (bounded metric) coarse structure of the space, expressed again in terms of fields of dilations. As coarse structures and uniform structures are very much alike (only that one is interesting for the small scale, other for the large scale), is there a notion of dilation structure which is appropriate for coarse structures?

3 Fields of dilations

Let $\Gamma = (0, +\infty)$, seen as an ordered abelian topological group, with respect to the operation of multiplications of reals. I shall denote by $\bar{0} = (0, 1]$ and $\bar{\infty} = [1, +\infty)$ two of its subsemigroups.

Definition 3.1 *(X, δ) is a $\bar{0}$ field of dilations if it satisfies the following conditions marked with a "0". (X, δ) is a $\bar{\infty}$ field of dilations if it satisfies the following conditions marked with a "∞". For any $x \in X$ and*

(a0) for any $\varepsilon \in \bar{0}$ the function $U(x) \subset X \to U_\varepsilon(x) \subset X$ is a bijection.

(a∞) for any $\varepsilon \in \bar{\infty}$ the function $U(x) \subset X \to U_\varepsilon(x) \subset X$ is a bijection.

(b0) for any $\varepsilon \in \bar{0}$ we have $\delta^x_\varepsilon \delta^x_\mu = \delta^x_{\varepsilon\mu}$.

(b∞) for any $\varepsilon \in \bar{\infty}$ we have $\delta^x_\varepsilon \delta^x_\mu = \delta^x_{\varepsilon\mu}$.

(c)=(c0)=(c∞) $\delta^x_1 = id_{U(x)}$.

(d0) for any $\varepsilon \in \bar{\infty}$ we have $\delta^x_\varepsilon = (\delta^x_{\varepsilon^{-1}})^{-1}$.

(d∞) for any $\varepsilon \in \bar{0}$ we have $\delta^x_\varepsilon = (\delta^x_{\varepsilon^{-1}})^{-1}$.

(e)=(e0)=(e∞) for any $\varepsilon \in \Gamma$ we have $\delta^x_\varepsilon x = x$.

(f0) for any $\varepsilon \in \bar{0}$ $U_\varepsilon(x) \subset U(x)$ and $U_\varepsilon(x) \neq U(x)$.

(f∞) for any $\varepsilon \in \bar{\infty}$ $U(x) \subset U_\varepsilon(x)$ and $U(x) \neq U_\varepsilon(x)$.

Let (X, δ) be a $\bar{0}$ or a $\bar{\infty}$ field of dilations. Further are objects, or statements associated to fields of dilations, with the convention that the "$\bar{0}$" label indicates that the respective thing is associated to a $\bar{0}$ field of dilations and the and "$\bar{\infty}$" label indicates that the respective thing is associated to a $\bar{\infty}$ field of dilations.

Definition 3.2 $\bar{0}$ - *for any $\varepsilon \in \bar{0}$ and for any $x \in X$ the ball of center x and radius ε is $B(x, \varepsilon) = U_\varepsilon(x)$.*

$\bar{\infty}$ - *for any $\varepsilon \in \bar{\infty}$ and for any $x \in X$ the ball of center x and radius ε is $B(x, \varepsilon) = U_\varepsilon(x)$.*

$\bar{0}$ - *for any $\varepsilon \in \bar{0}$ the ε entourage is $U_\varepsilon = \{(x, \delta^x_\varepsilon u) \,:\, u \in U(x)\}$.*

$\bar{\infty}$ - *for any $\varepsilon \in \bar{\infty}$ the ε entourage is $U_\varepsilon = \{(x, \delta^x_\varepsilon u) \,:\, u \in U(x)\}$.*

$\bar{0}$ - *let Φ be the collection of all $A \subset X \times X$ such that there exists $\varepsilon \in \bar{0}$ with $U_\varepsilon \subset A$.*

$\bar{\infty}$ - *let Ξ be the collection of all $A \subset X \times X$ such that there exists $\varepsilon \in \bar{\infty}$ with $A \subset U_\varepsilon$.*

$\bar{0}$ - *for any $\varepsilon \in \bar{0}$ the ε-domain of the sum is the set $D^\Sigma_\varepsilon = \left\{(x, u, v) \in X^3 \,:\, u \in U(x)\,,\, v \in U(\delta^x_\varepsilon u)\right\}$.*

$\bar{\infty}$ - *for any $\varepsilon \in \bar{\infty}$ the ε-domain of the sum is the set $D^\Sigma_\varepsilon = \left\{(x, u, v) \in X^3 \,:\, u \in U(x)\,,\, v \in U(\delta^x_\varepsilon u)\right\}$.*

Lemma 3.3 *($\bar{0}$) If (X, δ) is a $\bar{0}$ field of dilations then Φ is a filter of $\Delta(X) = \{(x, x) \,:\, x \in X\}$, that is it satisfies the following:*

(a$\bar{0}$) for any $A \in \Phi$ $\Delta(X) \subset A$,

(b$\bar{0}$) $A, B \in \Phi$ implies $A \cap B \in \Phi$,

(c$\bar{0}$) $A \in \Phi$ and $A \subset B \subset X$ imply $B \in \Phi$.

($\bar{\infty}$) If (X, δ) is a $\bar{\infty}$ field of dilations then Ξ has the properties:

(a$\bar{\infty}$) $\Delta(X) \in \Xi$,

(b$\bar{\infty}$) $A, B \in \Xi$ implies $A \cup B \in \Xi$,

(c$\bar{\infty}$) $A \in \Xi$ and $B \subset A$ imply $B \in \Xi$.

Proof. In each case, the point (c) comes from the definition 3.2 of Φ, Ξ respectively. Property (e) from definition 3.1 implies the point (a) in each case. As for the point (b), in the case $\bar{0}$ it is a consequence of the relation $U_{\varepsilon\mu} \subset U_\varepsilon \cap U_\mu$ for any $\varepsilon, \mu \in \bar{0}$, coming from (b) and (f), definition 3.1; in the case $\bar{\infty}$, by the same reasoning we get the implication $U_{\varepsilon\mu} \subset U_\varepsilon \cup U_\mu$ for any $\varepsilon, \mu \in \bar{\infty}$. $\square$

For any $A, B \subset X \times X$ we denote by AB the set

$$AB = \{(x, z) : \exists y \in X \quad (x, y) \in A \quad \text{and} \quad (y, z) \in B\} \quad .$$

Lemma 3.4 *($\bar{0}$) Let (X, δ) be a $\bar{0}$ field of dilations. The set Φ satisfies*

(d$\bar{0}$) for any $A \in \Phi$ there is $B \in \Phi$ such that $BB \subset A$,

if and only if there is a non-decreasing function $\lambda_S : \bar{0} \to \bar{0}$, called the modulus of the sum, such that

(s$\bar{0}$) for any $\varepsilon \in \bar{0}$ we have $U_{\lambda_S(\varepsilon)} U_{\lambda_S(\varepsilon)} \subset U_\varepsilon$.

Moreover, the condition (s$\bar{0}$) is equivalent with the following:

(S$\bar{0}$) *for any $\varepsilon \in \bar{0}$ and for any $(x,u,v) \in D_{\lambda_S(\varepsilon)}$ (definition 3.2, "domain of the sum"), the "sum at scale ε" expression*

$$\Sigma^x_\varepsilon(\lambda_S; u, v) \; = \; \delta^x_{\varepsilon^{-1}} \, \delta^{\delta^x_{\lambda_S(\varepsilon)} u}_{\lambda_S(\varepsilon)} v$$

is well defined and

$$\Sigma^x_\varepsilon(\lambda_S; u, v) \; \in \; U(x) \quad .$$

($\bar{\infty}$) *Let (X,δ) be a $\bar{\infty}$ field of dilations. The set Ξ satisfies*

(d$\bar{\infty}$) *for any $A \in \Xi$ there is $B \in \Xi$ such that $AA \subset B$,*

if and only if there is a non-decreasing function $\lambda_S : \bar{\infty} \to \bar{\infty}$, called the modulus of the sum, such that

(s$\bar{\infty}$) *for any $\varepsilon \in \bar{\infty}$ we have $U_\varepsilon U_\varepsilon \subset U_{\lambda_S(\varepsilon)}$.*

Moreover, the condition (s$\bar{\infty}$) is equivalent with the following:

(S$\bar{\infty}$) *for any $\varepsilon \in \bar{\infty}$ and for any $(x,u,v) \in D_\varepsilon$ (definition 3.2, "domain of the sum"), the "sum at scale ε" expression*

$$\Sigma^x_\varepsilon(\lambda_S; u, v) \; = \; \delta^x_{\lambda_S(\varepsilon)^{-1}} \, \delta^{\delta^x_\varepsilon u}_\varepsilon v$$

is well defined and

$$\Sigma^x_\varepsilon(\lambda_S; u, v) \; \in \; U(x) \quad .$$

Proof. (d$\bar{0}$) $\Longleftrightarrow$ (s$\bar{0}$). By definition of Φ, (d$\bar{0}$) is equivalent with: for any $\varepsilon \in \bar{0}$ there is $\eta \in \bar{0}$ such that $U_\eta U_\eta \subset U_\varepsilon$. Using that for any $\eta, \mu \in \bar{0}$, $\mu \leq \eta$ implies $U_\mu \subset U_\eta$, we get that for any $\varepsilon \in \bar{0}$ there is a $\Lambda_s(\varepsilon) \in (0,1)$ such that

$$(0, \Lambda_s(\varepsilon)) \; \subset \; \{\eta \in \bar{0} \, : \, U_\eta U_\eta \subset U_\varepsilon\}$$

and moreover the function $\varepsilon \mapsto \Lambda_s(\varepsilon)$ is non-decreasing. Take then λ_S to be any positive, non-decreasing function smaller or equal to Λ_s.

(S$\bar{0}$) $\Longleftrightarrow$ (s$\bar{0}$). For $\eta \in \bar{0}$, let us compute $U_\eta U_\eta$. It gives:

$$U_\eta U_\eta \; = \; \left\{ (x, \delta^{\delta^x_\eta u}_\eta) v \, : \, u \in U(x) \, , \, v \in U(\delta^x_\eta u) \right\} \; = \; \left\{ (x, \delta^{\delta^x_\eta u}_\eta v) \, : \, (x,u,v) \in D_\eta \right\} \quad .$$

By taking $\eta = \lambda_S(\varepsilon)$, (s$\bar{0}$) is therefore equivalent with: for any $\varepsilon \in \bar{0}$ and for any $(x,u,v) \in D_{\lambda_S(\varepsilon)}$ there is $w \in U(x)$ such that

$$\delta^{\delta^x_{\lambda_S(\varepsilon)} u}_{\lambda_S(\varepsilon)} v \; = \; \delta^x_\varepsilon w \quad .$$

But this is equivalent with (S$\bar{0}$).

For the $\bar{\infty}$ part, the proof is similar. $\square$

For any $A \subset X \times X$ we denote by A^{-1} the set $A^{-1} = \{(y,x) \, : \, (x,y) \in A\}$.

Lemma 3.5 *($\bar{0}$) Let (X,δ) be a $\bar{0}$ field of dilations. The set Φ satisfies*

(e$\bar{0}$) *$A \in \Phi$ implies $A^{-1} \in \Phi$,*

if and only if there is a non-decreasing function $\lambda_I : \bar{0} \to \bar{0}$, called the modulus of inversion, such that

(i$\bar{0}$) *for any $\varepsilon \in \bar{0}$ we have $U_{\lambda_I(\varepsilon)} \subset (U_\varepsilon)^{-1}$.*

Moreover, the condition ($i\bar{0}$) is equivalent with the following:

($I\bar{0}$) *for any $\varepsilon \in \bar{0}$ and for any $(x,u) \in U_1$ the "inverse at scale ε" expression*

$$inv_{\varepsilon}^{x}(\lambda_I; u) \, = \, \delta_{\varepsilon^{-1}}^{\delta^x_{\lambda_I(\varepsilon)} u} x$$

is well defined and

$$inv_{\varepsilon}^{x}(\lambda_I; u) \, \in \, U(\delta^x_{\lambda_I(\varepsilon)} u) \quad .$$

($\bar{\infty}$) *Let (X, δ) be a $\bar{\infty}$ field of dilations. The set Ξ satisfies*

($e\bar{\infty}$) *$A \in \Xi$ implies $A^{-1} \in \Xi$,*

if and only if there is a non-decreasing function $\lambda_I : \bar{\infty} \to \bar{\infty}$, called the modulus of inversion, such that

($i\bar{\infty}$) *for any $\varepsilon \in \bar{\infty}$ we have $(U_\varepsilon)^{-1} \subset U_{\lambda_I(\varepsilon)}$.*

Moreover, the condition ($i\bar{\infty}$) is equivalent with the following:

($I\bar{\infty}$) *for any $\varepsilon \in \bar{\infty}$ and for any $(x,u) \in U_1$ the "inverse at scale ε" expression*

$$inv_{\varepsilon}^{x}(\lambda_I; u) \, = \, \delta_{\lambda_I(\varepsilon)^{-1}}^{\delta^x_{\varepsilon} u} x$$

is well defined and

$$inv_{\varepsilon}^{x}(\lambda_I; u) \, \in \, U(\delta^x_{\varepsilon} u) \quad .$$

Proof. Similar with the proof of lemma 3.4. $\square$

By putting together lemma 3.3, lemma 3.4, lemma 3.5, we obtain the following theorem.

Theorem 3.6 *($\bar{0}$) Let (X, δ) be a $\bar{0}$ field of dilations. The set Φ is an uniformity on X, i.e. it satisfies:*

($a\bar{0}$) *for any $A \in \Phi$ $\Delta(X) \subset A$,*

($b\bar{0}$) *$A, B \in \Phi$ implies $A \cap B \in \Phi$,*

($c\bar{0}$) *$A \in \Phi$ and $A \subset B \subset X$ imply $B \in \Phi$,*

($d\bar{0}$) *for any $A \in \Phi$ there is $B \in \Phi$ such that $BB \subset A$,*

($e\bar{0}$) *$A \in \Phi$ implies $A^{-1} \in \Phi$,*

if and only if the field of dilations satisfies:

($S\bar{0}$) *there is a non-decreasing function $\lambda_S : \bar{0} \to \bar{0}$, called the modulus of the sum, such that for any $\varepsilon \in \bar{0}$ and for any $(x,u,v) \in D_{\lambda_S(\varepsilon)}$ (definition 3.2, "domain of the sum"), the "sum at scale ε" expression*

$$\Sigma_{\varepsilon}^{x}(\lambda_S; u, v) \, = \, \delta^x_{\varepsilon^{-1}} \, \delta_{\lambda_S(\varepsilon)}^{\delta^x_{\lambda_S(\varepsilon)} u} v$$

is well defined and

$$\Sigma_{\varepsilon}^{x}(\lambda_S; u, v) \, \in \, U(x) \quad ,$$

($I\bar{0}$) *there is a non-decreasing function $\lambda_I : \bar{0} \to \bar{0}$, called the modulus of inversion, such that for any $\varepsilon \in \bar{0}$ and for any $(x,u) \in U_1$ the "inverse at scale ε" expression*

$$inv_{\varepsilon}^{x}(\lambda_I; u) \, = \, \delta_{\varepsilon^{-1}}^{\delta^x_{\lambda_I(\varepsilon)} u} x$$

is well defined and

$$inv_{\varepsilon}^{x}(\lambda_I; u) \, \in \, U(\delta^x_{\lambda_I(\varepsilon)} u) \quad .$$

($\bar{0}$) Let (X, δ) be a $\bar{\infty}$ field of dilations. The set Φ is a coarse structure on X, i.e. it satisfies:

(a$\bar{\infty}$) $\Delta(X) \in \Xi$,

(b$\bar{\infty}$) $A, B \in \Xi$ *implies* $A \cup B \in \Xi$,

(c$\bar{\infty}$) $A \in \Xi$ *and* $B \subset A$ *imply* $B \in \Xi$'

(d$\bar{\infty}$) for any $A \in \Xi$ *there is* $B \in \Xi$ *such that* $AA \subset B$,

(e$\bar{\infty}$) $A \in \Xi$ *implies* $A^{-1} \in \Xi$,

if and only if the field of dilations satisfies:

(S$\bar{\infty}$) there is a non-decreasing function $\lambda_S : \bar{\infty} \to \bar{\infty}$, called the modulus of the sum, such that for any $\varepsilon \in \bar{\infty}$ and for any $(x, u, v) \in D_\varepsilon$ (definition 3.2, "domain of the sum"), the "sum at scale ε" expression

$$\Sigma_\varepsilon^x(\lambda_S; u, v) \;=\; \delta^x_{\lambda_S(\varepsilon)^{-1}} \, \delta_\varepsilon^{\delta_\varepsilon^x u} v$$

is well defined and

$$\Sigma_\varepsilon^x(\lambda_S; u, v) \;\in\; U(x) \quad ,$$

(I$\bar{\infty}$) there is a non-decreasing function $\lambda_I : \bar{\infty} \to \bar{\infty}$, called the modulus of inversion, such that for any $\varepsilon \in \bar{\infty}$ and for any $(x, u) \in U_1$ the "inverse at scale ε" expression

$$inv_\varepsilon^x(\lambda_I; u) \;=\; \delta^{\delta_\varepsilon^x u}_{\lambda_I(\varepsilon)^{-1}} x$$

is well defined and

$$inv_\varepsilon^x(\lambda_I; u) \;\in\; U(\delta_\varepsilon^x u) \quad .$$

4 Approximate operations

We see the appearance of two expressions: the sum and the inverse at scale ε, which are slightly different, depending on the case $\bar{0}$ or $\bar{\infty}$. Using these two expressions, the difference at scale ε is introduced further.

Proposition 4.1 *($\bar{0}$) Let (X, δ) be a $\bar{0}$ field of dilations which satisfies (S$\bar{0}$) with modulus of the sum λ_S and (I$\bar{0}$) with modulus of inversion λ_I. Define the modulus of the difference to be the non-decreasing function $\lambda_D : \bar{0} \to \bar{0}$ with the expression $\lambda_D(\varepsilon) = \lambda_S(\varepsilon)\lambda_I(\lambda_S(\varepsilon))$ (alternatively we may define λ_D by the expression $\lambda_D(\varepsilon) = \min\{\lambda_I(\lambda_S(\varepsilon)), \lambda_S(\varepsilon)\}$).*

Then for any $\varepsilon \in \bar{0}$ we have $\left(U_{\lambda_D(\varepsilon)}\right)^{-1} U_{\lambda_D(\varepsilon)} \subset U_\varepsilon$.

Equivalently, for any $\varepsilon \in \bar{0}$ and for any $x \in X$ and $u, v \in U(x)$ the "difference at scale ε" expression

$$\Delta_\varepsilon^x(\lambda_D; u, v) \;=\; \delta_{\varepsilon^{-1}}^{\delta^x_{\lambda_D(\varepsilon)} u} \delta^x_{\lambda_D(\varepsilon)} v$$

is well defined and

$$\Delta_\varepsilon^x(\lambda_D; u, v) \;\in\; U(\delta^x_{\lambda_D(\varepsilon)} u) \quad .$$

($\bar{\infty}$) Let (X, δ) be a $\bar{\infty}$ field of dilations which satisfies (S$\bar{\infty}$) with modulus of the sum λ_S and (I$\bar{\infty}$) with modulus of inversion λ_I. Define the modulus of the difference to be the non-decreasing function $\lambda_D : \bar{\infty} \to \bar{\infty}$ with the expression $\lambda_D(\varepsilon) = \lambda_S(\varepsilon\lambda_I(\varepsilon))$ (alternatively we may define λ_D by the expression $\lambda_D(\varepsilon) = \lambda_S(\max\{\lambda_I(\varepsilon), \varepsilon\})$).

Then for any $\varepsilon \in \bar{\infty}$ we have $(U_\varepsilon)^{-1} U_\varepsilon \subset U_{\lambda_D(\varepsilon)}$.

Equivalently, for any $\varepsilon \in \bar{\infty}$ and for any $x \in X$ and $u, v \in U(x)$ the "difference at scale ε" expression

$$\Delta_\varepsilon^x(\lambda_D; u, v) \;=\; \delta^{\delta_\varepsilon^x u}_{\lambda_D(\varepsilon)^{-1}} \delta_\varepsilon^x v$$

is well defined and

$$\Delta_\varepsilon^x(\lambda_D; u, v) \;\in\; U(\delta_\varepsilon^x u) \quad .$$

Proof. ($\bar{0}$) By (I$\bar{0}$) we get that

$$\left(U_{\lambda_I(\lambda_S(\varepsilon))}\right)^{-1} \subset U_{\lambda_S(\varepsilon)}$$

therefore, by (b$\bar{0}$) definition 3.1 and because $\lambda_D(\varepsilon) \leq \lambda_I(\lambda_S(\varepsilon))$ and $\lambda_D(\varepsilon) \leq \lambda_S(\varepsilon)$, we obtain the following string of inclusions:

$$\left(U_{\lambda_D(\varepsilon)}\right)^{-1} U_{\lambda_D(\varepsilon)} \subset \left(U_{\lambda_I(\lambda_S(\varepsilon))}\right)^{-1} U_{\lambda_D(\varepsilon)} \subset U_{\lambda_S(\varepsilon)} U_{\lambda_D(\varepsilon)} \subset U_{\lambda_S(\varepsilon)} U_{\lambda_S(\varepsilon)} \quad .$$

By (S$\bar{0}$) we have

$$U_{\lambda_S(\varepsilon)} U_{\lambda_S(\varepsilon)} \subset U_\varepsilon$$

which proves this part of the proposition.

The proof of the equivalent formulation is similar to the one of lemma 3.4, part ($\bar{0}$).

($\bar{\infty}$) The proof goes along the same path. □

These three "approximate operations" satisfy interesting relations, which will be detailed further. (This was the first time explained in [3], section 4.2, as a consequence of a decorated binary trees formalism; later this was explored in [8] and in [9] the binary trees formalism was included into graphic lambda calculus).

Proposition 4.2 *Let (X, δ) be a $\bar{0}$ or $\bar{\infty}$ field of dilations which satisfies (S$\bar{0}$) and (I$\bar{0}$), or (S$\bar{\infty}$) and (I$\bar{\infty}$) respectively, with modulus of the sum λ_S and modulus of inversion λ_I. The inverse operation is then approximately an involution, in the following sense:*

- ($\bar{0}$) *the modulus of inversion satisfies $\lambda_I^2(\varepsilon) \leq \varepsilon$ for any $\varepsilon \in \bar{0}$ (here we denote by λ_I^2 the function $\lambda_I^2(\varepsilon) = \lambda_I(\lambda_I(\varepsilon))$). Moreover, for any $(u, v) \in U_1$ the terms of the following equality are well defined and*

$$inv_\varepsilon^{\delta^u_{\lambda_I^2(\varepsilon)} v} \left(\lambda_I ; \, inv^u_{\lambda_I(\varepsilon)} \left(\lambda_I ; v\right)\right) \, = \, \delta^u_{\varepsilon^{-1}\lambda_I^2(\varepsilon)} v \quad .$$

- ($\bar{\infty}$) *the modulus of inversion satisfies $\lambda_I^2(\varepsilon) \geq \varepsilon$ for any $\varepsilon \in \bar{\infty}$. Moreover, for any $(u, v) \in U_1$ the terms of the following equality are well defined and*

$$inv_{\lambda_I(\varepsilon)}^{\delta^u_\varepsilon v} \left(\lambda_I ; \, inv^u_\varepsilon \left(\lambda_I ; v\right)\right) \, = \, \delta^u_{\varepsilon(\lambda_I^2(\varepsilon))^{-1}} v \quad .$$

Proof. ($\bar{0}$): By (i$\bar{0}$), for any $\varepsilon \in \bar{0}$ we have $\left(U_{\lambda_I(\varepsilon)}\right)^{-1} \subset U_\varepsilon$. We repeat for $\lambda_I(\varepsilon)$ instead and we get that

$$U_{\lambda_I^2(\varepsilon)} \subset \left(U_{\lambda_I(\varepsilon)}\right)^{-1} \subset U_\varepsilon \tag{1}$$

which implies $\varepsilon^{-1}\lambda_I(\varepsilon) \in \bar{0}$. Now, the inclusion $\left(U_{\lambda_I(\varepsilon)}\right)^{-1} \subset U_\varepsilon$ defines a function:

$$(u, v) \in U_1 \, \mapsto \, Inv_\varepsilon(\lambda_I ; u, v) \in U_1 \quad ,$$

$$Inv_\varepsilon(\lambda_I ; u, v) \, = \, \left(\delta^u_{\lambda_I(\varepsilon)} v \, , \, inv^u_\varepsilon(\lambda_I ; v)\right)$$

with the property that for any $(u, v) \in U_1$ the pair $(x, y) \in U_1$ given by $(x, y) = Inv_\varepsilon(\lambda_I ; u, v)$ is the unique solution of the equation

$$(\delta^u_{\lambda_I(\varepsilon)} v \, u) \, = \, (x, \delta^x_\varepsilon y) \quad .$$

(This equation is the translation in terms of fields of dilations of (i$\bar{0}$).) The relation (1) implies that for any $(u, v) \in U_1$

$$Inv_\varepsilon(\lambda_I ; Inv_{\lambda_I(\varepsilon)}(\lambda_I ; u, v)) \, = \, (u, \delta^u_{\varepsilon^{-1}\lambda_I^2(\varepsilon)} v) \quad .$$

This is equivalent with the last part of the conclusion, by direct computation.

$(\bar{\infty})$: Similar proof, based on the double inequality

$$U_\varepsilon \subset \left(U_{\lambda_I(\varepsilon)}\right)^{-1} \subset U_{\lambda_I^2(\varepsilon)} \quad , \tag{2}$$

the inverse function

$$(u,v) \in U_1 \mapsto Inv_\varepsilon(\lambda_I; u, v) \in U_1 \quad ,$$
$$Inv_\varepsilon(\lambda_I; u, v) = (\delta^u_\varepsilon v \, , \, inv^u_\varepsilon(\lambda_I; v))$$

and the relation: for any $(u,v) \in U_1$

$$Inv_{\lambda_I(\varepsilon)}(\lambda_I; Inv_\varepsilon(\lambda_I; u, v)) = (u, \delta^u_{\varepsilon(\lambda_I^2(\varepsilon))^{-1}} v) \quad ,$$

which translates (2) in terms of fields of dilations. $\square$

Proposition 4.3 *Let (X,δ) be a $\bar{0}$ or $\bar{\infty}$ field of dilations which satisfies (S$\bar{0}$) and (I$\bar{0}$), or (S$\bar{\infty}$) and (I$\bar{\infty}$) respectively, with modulus of the sum λ_S and modulus of inversion λ_I. The sum operation is then approximately associative, in the following sense:*

($\bar{0}$) *let us make the notations $\lambda_S^2(\varepsilon) = \lambda_S(\lambda_S(\varepsilon))$ and $\lambda'_S(\varepsilon) = \lambda^2_S(\varepsilon)\lambda_S(\varepsilon)^{-1}$. For any $\varepsilon \in \bar{0}$ we have the inclusions:*

$$\left(U_{\lambda^2_S(\varepsilon)} U_{\lambda^2_S(\varepsilon)}\right) U_{\lambda_S(\varepsilon)} \subset U_\varepsilon \quad , \quad U_{\lambda_S(\varepsilon)} \left(U_{\lambda^2_S(\varepsilon)} U_{\lambda^2_S(\varepsilon)}\right) \subset U_\varepsilon \quad .$$

Moreover, we may choose λ_S such that $\lambda_S(\varepsilon) \leq \varepsilon$ for any $\varepsilon \in \bar{0}$. With this choice, for any $\varepsilon \in \bar{0}$ and for any $(x,u,v,w) \in X^4$ such that $u \in U(x)$, $v \in U(\delta^x_{\lambda^2_S(\varepsilon)} u)$, $w \in U_{\lambda'_S(\varepsilon)}(\delta^{\delta^x_{\lambda^2_S(\varepsilon)} u}_{\lambda^2_S(\varepsilon)} v)$, the terms of the following equality are well defined and

$$\Sigma^x_\varepsilon \left(\lambda_S; \Sigma^x_{\lambda_S(\varepsilon)}\left(\lambda_S; u, v\right), w\right) = \Sigma^x_\varepsilon \left(\lambda_S; \delta^x_{\lambda'_S(\varepsilon)} u, \Sigma^{\delta^x_{\lambda^2_S(\varepsilon)} u}_{\lambda_S(\varepsilon)} \left(\lambda_S; v, \delta^{\delta^{\delta^x_{\lambda^2_S(\varepsilon)} u}_{\lambda^2_S(\varepsilon)} v}_{\lambda'_S(\varepsilon)^{-1}} w\right)\right) \quad .$$

($\bar{\infty}$) *For any $\varepsilon \in \bar{\infty}$ we have the inclusions:*

$$\left(U_\varepsilon U_\varepsilon\right) U_{\lambda_S(\varepsilon)} \subset U_{\lambda^2_S(\varepsilon)} \quad , \quad U_{\lambda_S(\varepsilon)} \left(U_\varepsilon U_\varepsilon\right) \subset U_{\lambda^2_S(\varepsilon)} \quad .$$

Moreover, we may choose λ_S such that $\lambda_S(\varepsilon) \geq \varepsilon$ for any $\varepsilon \in \bar{\infty}$. With this choice, for any $\varepsilon \in \bar{\infty}$ and for any $(x,u,v,w) \in X^4$ such that $u \in U(x)$, $v \in U(\delta^x_\varepsilon u)$, $w \in U(\delta^{\delta^x_\varepsilon u}_\varepsilon v)$, the terms of the following equality are well defined and

$$\Sigma^x_{\lambda_S(\varepsilon)} \left(\lambda_S; \Sigma^x_\varepsilon\left(\lambda_S; u, v\right), \delta^{\delta^{\delta^x_\varepsilon u}_\varepsilon v}_{\varepsilon\lambda_S(\varepsilon)^{-1}} w\right) = \Sigma^x_{\lambda_S(\varepsilon)} \left(\lambda_S; \delta^x_{\varepsilon\lambda_S(\varepsilon)^{-1}} u, \Sigma^{\delta^x_\varepsilon u}_\varepsilon \left(\lambda_S; v, w\right)\right) \quad .$$

Proof. ($\bar{0}$): For the first part we apply twice (s$\bar{0}$), in two ways, namely:

$$U_{\lambda^2_S(\varepsilon)} U_{\lambda^2_S(\varepsilon)} \subset U_{\lambda_S(\varepsilon)} \quad , \quad U_{\lambda_S(\varepsilon)} U_{\lambda_S(\varepsilon)} \subset U_\varepsilon$$

imply that $\left(U_{\lambda^2_S(\varepsilon)} U_{\lambda^2_S(\varepsilon)}\right) U_{\lambda_S(\varepsilon)} \subset U_{\lambda_S(\varepsilon)} U_{\lambda_S(\varepsilon)} \subset U_\varepsilon$ and also the other inclusion, by the same type of reasoning.

For the second part we start by noticing that the proof of lemma 3.4 shows that if λ_S is a modulus of the sum then $\varepsilon \in \bar{0} \mapsto \min\{\lambda_S(\varepsilon), \varepsilon\}$ is another modulus of the sum. Therefore we may indeed choose λ_S such that $\lambda_S(\varepsilon) \leq \varepsilon$ for any $\varepsilon \in \bar{0}$. For this modulus we have $U_{\lambda^2_S(\varepsilon)} \subset U_{\lambda_S(\varepsilon)}$, which implies the inclusions

$$U_{\lambda^2_S(\varepsilon)} U_{\lambda^2_S(\varepsilon)} U_{\lambda^2_S(\varepsilon)} \subset \left(U_{\lambda^2_S(\varepsilon)} U_{\lambda^2_S(\varepsilon)}\right) U_{\lambda_S(\varepsilon)} \cap U_{\lambda_S(\varepsilon)} \left(U_{\lambda^2_S(\varepsilon)} U_{\lambda^2_S(\varepsilon)}\right) \subset U_\varepsilon \quad .$$

We try to identify in two ways the elements of the set $U_{\lambda_S^2(\varepsilon)} U_{\lambda_S^2(\varepsilon)} U_{\lambda_S^2(\varepsilon)}$, first as elements of the set $\left(U_{\lambda_S^2(\varepsilon)} U_{\lambda_S^2(\varepsilon)}\right) U_{\lambda_S(\varepsilon)}$, then as elements of the set $U_{\lambda_S(\varepsilon)} \left(U_{\lambda_S^2(\varepsilon)} U_{\lambda_S^2(\varepsilon)}\right)$.

For this, recall that the approximate sum function $(x,u,v) \mapsto \Sigma^x_\varepsilon(\lambda_S; u, v)$ is defined on triples $(x,u,v) \in D_{\lambda_S(\varepsilon)}$, with the property that if $(x,y) \in U_{\lambda_S(\varepsilon)}$ and $(y,z) \in U_{\lambda_S(\varepsilon)}$ are parametrized by

$$(x,y) = (x, \delta^x_{\lambda_S(\varepsilon)} u) \quad , \quad (y,z) = (\delta^x_{\lambda_S(\varepsilon)} u, \delta^{\delta^x_{\lambda_S(\varepsilon)} u}_{\lambda_S(\varepsilon)} v)$$

then $(x,z) \in U_\varepsilon$ is uniquely given by the parametrization

$$(x,z) = (x, \delta^x_\varepsilon \Sigma^x_\varepsilon(\lambda_S; u, v)) \quad .$$

We look at the inclusion of $U_{\lambda_S^2(\varepsilon)} U_{\lambda_S^2(\varepsilon)} U_{\lambda_S^2(\varepsilon)}$ into $\left(U_{\lambda_S^2(\varepsilon)} U_{\lambda_S^2(\varepsilon)}\right) U_{\lambda_S(\varepsilon)}$. We take three pairs: $(x,y), (y,z) \in U_{\lambda_S^2(\varepsilon)}$ and $(z,q) \in U_{\lambda_S^2(\varepsilon)} \subset U_{\lambda_S(\varepsilon)}$, such that $(x,q) \in U_\varepsilon$. These pairs are parametrized by $(x,u,v,w) \in X^4$ such that $u \in U(x)$, $v \in U(\delta^x_{\lambda_S^2(\varepsilon)} u)$, $w \in U_{\lambda'_S(\varepsilon)}(\delta^{\delta^x_{\lambda_S^2(\varepsilon)} u}_{\lambda_S^2(\varepsilon)} v)$ and

$$(x,y) = (x, \delta^x_{\lambda_S^2(\varepsilon)} u) \quad , \quad (y,z) = (\delta^x_{\lambda_S^2(\varepsilon)} u, \delta^{\delta^x_{\lambda_S^2(\varepsilon)} u}_{\lambda_S^2(\varepsilon)} v) \quad , \quad (z,q) = (\delta^{\delta^x_{\lambda_S^2(\varepsilon)} u}_{\lambda_S^2(\varepsilon)} v, \delta^{\delta^{\delta^x_{\lambda_S^2(\varepsilon)} u}_{\lambda_S^2(\varepsilon)} v}_{\lambda_S(\varepsilon)} w) \quad .$$

Because of the choice of w we have $(z,q) \in U_{\lambda_S^2(\varepsilon)}$ therefore $(x,q) \in U_{\lambda_S^2(\varepsilon)} U_{\lambda_S^2(\varepsilon)} U_{\lambda_S^2(\varepsilon)}$. We notice that we may write:

$$(x,z) = (x, \delta^x_{\lambda_S(\varepsilon)} \Sigma^x_{\lambda_S(\varepsilon)}(\lambda_S; u, v)) \quad , \quad (z,q) = (\delta^x_{\lambda_S(\varepsilon)} \Sigma^x_{\lambda_S(\varepsilon)}(\lambda_S; u, v), \delta^z_{\lambda_S(\varepsilon)} w)$$

It follows that $q = \delta^x_\varepsilon A$ with $A \in U(x)$ being the expression:

$$A = \Sigma^x_\varepsilon \left(\lambda_S; \Sigma^x_{\lambda_S(\varepsilon)} (\lambda_S; u, v), w\right) \quad .$$

We look now at the inclusion of $U_{\lambda_S^2(\varepsilon)} U_{\lambda_S^2(\varepsilon)} U_{\lambda_S^2(\varepsilon)}$ into $U_{\lambda_S(\varepsilon)} \left(U_{\lambda_S^2(\varepsilon)} U_{\lambda_S^2(\varepsilon)}\right)$. This time we may realize the three pairs (x,y), (y,z), (z,q) by the following parametrization: $(x,u',v',w') \in X^4$, satisfy $u' \in U(x)$, $v' \in U(\delta^x_{\lambda_S(\varepsilon)} u')$, $w' \in U(\delta^{\delta^x_{\lambda_S(\varepsilon)} u'}_{\lambda_S^2(\varepsilon)} v')$ and

$$(x,y) = (x, \delta^x_{\lambda_S(\varepsilon)} u') \quad , \quad (y,z) = (\delta^x_{\lambda_S(\varepsilon)} u', \delta^{\delta^x_{\lambda_S(\varepsilon)} u'}_{\lambda_S^2(\varepsilon)} v') \quad , \quad (z,q) = (\delta^{\delta^x_{\lambda_S(\varepsilon)} u'}_{\lambda_S^2(\varepsilon)} v', \delta^{\delta^{\delta^x_{\lambda_S(\varepsilon)} u'}_{\lambda_S^2(\varepsilon)} v'}_{\lambda_S^2(\varepsilon)} w') \quad .$$

We get that $u' = \delta^x_{\lambda'_S(\varepsilon)} u$, $v = v'$ and moreover

$$w' = \delta^{\delta^{\delta^x_{\lambda_S^2(\varepsilon)} u}_{\lambda_S^2(\varepsilon)} v}_{\lambda'_S(\varepsilon)^{-1}} w \quad ,$$

which is possible because the choice of $w \in U_{\lambda'_S(\varepsilon)}(\delta^{\delta^x_{\lambda_S^2(\varepsilon)} u}_{\lambda_S^2(\varepsilon)} v)$. We notice that

$$(x,y) = (x, \delta^x_{\lambda_S(\varepsilon)} u') \quad , \quad (y,q) = (\delta^x_{\lambda_S(\varepsilon)} u', \delta^{\delta^x_{\lambda_S(\varepsilon)} u'}_{\lambda_S(\varepsilon)} \Sigma^{\delta^x_{\lambda_S(\varepsilon)} u'}_{\lambda_S(\varepsilon)} (\lambda_S; v', w'))$$

therefore $q = \delta^x_\varepsilon B$ with $B \in U(x)$ being the expression:

$$B = \Sigma^x_\varepsilon \left(\lambda_S; u', \Sigma^{\delta^x_{\lambda_S(\varepsilon)} u'}_{\lambda_S(\varepsilon)} (\lambda_S; v', w')\right) \quad .$$

In conclusion $A = B$, which gives the desired equality after replacing u', v', w' by their expressions written with u, v, w.

$(\bar{\infty})$ has a similar proof. $\square$

5 Topological or coarse notions and properties

In this section are explained various notions or properties of topological or coarse nature, in the particular case of fields of dilations. Every such notion or property will be expressed in terms of relations involving the dilation field and various moduli.

6 Dilation structures

Before giving the definition of a dilation structure, let me explain what "sufficiently close" means further.

Definition 6.1 *Let (X, d) be a metric space. We say that a property*

$$\mathcal{P}(x_1, x_2, x_3, ...)$$

is true for $x_1, x_2, x_3, ...$ **sufficiently close** *if for any compact, non empty set $K \subset X$, there is a positive constant $C(K) > 0$ such that $\mathcal{P}(x_1, x_2, x_3, ...)$ is true for any $x_1, x_2, x_3, ... \in K$ with $d(x_i, x_j) \leq C(K)$.*

I shall give a detailed definition of what a dilation structure is. (In this definition I shall use the name "dilation structure" for what I previously called "strong dilation structure". In older papers I called "dilation structure" one which satisfies axioms A0-A3 and "strong dilation structure" one which satisfies A0-A4. Here I shall use the name "dilation structure" for any structure which satisfies A0-A4.)

Definition 6.2 *A* **dilation structure** *(X, d, δ) is a triple made by a complete metric space (X, d), such that that for any $x \in X$ the closed ball $\bar{B}(x, 3)$ is compact, and a* **dilation field**

$$\delta : dom\, \delta \subset X \times X \times [0, +\infty) \rightarrow X \quad , \quad \delta(x, y, \varepsilon) \;=\; \delta_{\varepsilon}^{x} y$$

such that the axioms listed further are satisfied.

A0. *For any $x \in X$ and for any $\varepsilon \in (0, 1]$ the dilation $\delta_{\varepsilon}^{x}(\cdot) = \delta(x, \cdot, \varepsilon)$ is a homeomorphism (w.r.t. the topology induced by the distance d)*

$$\delta_{\varepsilon}^{x} : U(x) \rightarrow V_{\varepsilon}(x) = \delta_{\varepsilon}^{x} U(x)$$

For any $x \in X$ the the set $U(x) \subset X$ is an open neighbourhood of x. Moreover, for any compact set $K \subset X$ there are numbers $B > A > 1$ such that for any $x \in K$ we have

$$\bar{B}(x, A) \subset U(x) \subset B(x, B)$$

For any $x \in X$ and for any $\varepsilon > 1$ the dilation δ_{ε}^{x} is defined as

$$\delta_{\varepsilon}^{x} : V_{\varepsilon^{-1}}(x) U(x) \rightarrow U(x)$$

the inverse of the dilation $\delta_{\varepsilon^{-1}}^{x}$.

For any $x \in X$ and $u \in U(x)$ we have $\delta_{0}^{x} u = x$. Thus the domain $dom\, \delta$ of the dilation field is

$$dom\, \delta = \{(x, y, \varepsilon) \in X \times X \times [0, +\infty) \,:\, if\ \varepsilon \leq 1\ then\ y \in U(x)\ ,$$
$$else\ y \in V_{\varepsilon^{-1}}(x)\}$$

On this set we put the topology inherited from $X \times X \times [0, +\infty)$ with the product topology.

We suppose that δ is continuous.

Finally, let us introduce for any $\varepsilon \in (0, 1)$ the following set:

$$P(\varepsilon) \;=\; \left\{ (x, u, v) \,:\, u, v \in U(x)\ and\ \delta^{x} v \in \delta_{\varepsilon}^{\delta_{\varepsilon}^{x} u} U(\delta_{\varepsilon}^{x} u) \right\}$$

Then for any compact set $K \subset X$ there are $\alpha, \gamma \in (0, 1)$ such that for any $\varepsilon \in (0, \gamma]$ and for any $x \in K$ we have

$$\{x\} \times \delta_{\alpha}^{x} U(x) \times \delta_{\alpha}^{x} U(x) \subset P(\varepsilon)$$

A1. *We have* $\delta^x_\varepsilon x = x$ *for any point* x. *We also have* $\delta^x_1 = id$ *for any* $x \in X$.

A2. *For any* $x, \in X$, $\varepsilon, \mu \in (0, +\infty)$ *and* $u \in U(x)$ *we have the equality:*

$$\delta^x_\varepsilon \delta^x_\mu u = \delta^x_{\varepsilon\mu} u$$

whenever one of the sides are well defined.

A3. *For any* x *there is a distance function* $(u, v) \mapsto d^x(u, v)$, *defined for any* u, v *in the closed ball (in distance* d*)* $\bar{B}(x, A)$, *such that*

$$\lim_{\varepsilon \to 0} \quad \sup \left\{ \mid \frac{1}{\varepsilon} d(\delta^x_\varepsilon u, \delta^x_\varepsilon v) \ - \ d^x(u, v) \mid \ : \ u, v \in \bar{B}_d(x, A) \right\} \ = \ 0$$

uniformly with respect to x *in compact sets.*

A4. *Let us define for any* $\varepsilon \in (0, 1)$ *and for any* $(x, u, v) \in P(\varepsilon)$ *the function* $\Delta^x_\varepsilon(u, v) = \delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \delta^x_\varepsilon v$. *Then we have the limit*

$$\lim_{\varepsilon \to 0} \Delta^x_\varepsilon(u, v) = \Delta^x(u, v)$$

uniformly with respect to x, u, v *in compact sets.*

Definition 6.3 *A dilation structure* (X, d, δ) *is* **domain symmetric** *if for any* $x, u \in X$ *and* $\varepsilon \in (0, 1]$

$$u \in U(x) \text{ is equivalent with } inv^x_\varepsilon u \in U(\delta^x_\varepsilon u). \tag{3}$$

A dilation structure (X, d, δ) *is* **symmetric** *if it is domain symmetric and for any* $x, u \in X$ *and* $\varepsilon \in (0, 1]$

$$d(inv^x_\varepsilon u, \delta^x_\varepsilon u) = d(x, u) \tag{4}$$

Normed groups with dilations are examples of symmetric dilation structures. Indeed, if $(G, \| \mid \|, \delta)$ is such a group then it has a distance defined by $d(x, y) = \|x^{-1} y\|$ and a dilation structure associated to it, with dilations defined by:

$$\delta^x_\varepsilon : xU \to x\delta_\varepsilon U \quad , \quad \delta^x_\varepsilon u = x\delta_\varepsilon(x^{-1}u)$$

for any $\varepsilon \in (0, 1]$. The set U is a neighbourhood of the neutral element $e \in G$ such that it contains the ball of radius 1 and it is contained in a ball of radius $B > 1$. Moreover, we suppose that for any $u \in U$ we have $u^{-1} \in U$ (i.e. U is *g*roup symmetric).

It is then easy to see that (G, d, δ) is a symmetric dilation structure. We start with a computation:

$$inv^x_\varepsilon u \ = \ \delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} x \ = \ (\delta^x_\varepsilon u) \, u^{-1} x$$

By definition we have $U(x) = xU$, therefore $u \in U(x)$ is equivalent with $x^{-1}u \in U$. But U is group symmetric, therefore $u^{-1}x \in U$. From here we use the left invariance of the dilation structure and the previous computation to get: $u \in U(x)$ is equivalent with $inv^x_\varepsilon u \in (\delta^x_\varepsilon u) \, U = U \, (\delta^x_\varepsilon u)$. We proved that the dilation structure is domain symmetric. Again by using the previous computation we have:

$$d(inv^x_\varepsilon u, \delta^x_\varepsilon u) \ = \ d((\delta^x_\varepsilon u) \, u^{-1} x, \delta^x_\varepsilon u) \ = \ d(u^{-1} x, e) \ = \ d(x, u)$$

therefore the dilation structure is symmetric.

More general, emergent symmetric spaces as defined in [8] give examples of symmetric dilation structures. Another class of examples comes from taking on riemannian manifolds the dilation structure associated to the geodesic exponential. Then, locally, such a dilation structure is symmetric.

The following property is resembling with the doubling condition for metric spaces.

Definition 6.4 *A dilation structure (X, d, δ) is* **doubling** *if for any $c > 1$ and any compact set $K \subset X$ there is a natural number $N = N(c, K)$ and $a = a(c, K) \in (0, 1]$ such that for any $x \in K$ and any $\varepsilon \in (0, a]$ there are $u_1^\varepsilon, ..., u_N^\varepsilon \in X$ such that*

$$\delta^x_{c\varepsilon} U(x) \subset \bigcup_{i=1}^{N} \delta_\varepsilon^{u_i^\varepsilon} U(u_i^\varepsilon) \tag{5}$$

We obviously have: if the metric space (X, d) is metrically doubling then (X, d, δ) is doubling.

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