Inverse topological pressure with applications to holomorphic dynamics of several complex variables

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Abstract

In this paper we introduce a few notions of inverse topological pressure ($\tilde{P}^-, P^-, P_-$), defined in terms of backward orbits (prehistories) instead of forward orbits. This inverse topological pressure has some properties similar to the regular (forward) pressure but, in general if the map is not a homeomorphism, they do not coincide. In fact, there are several ways to define inverse topological pressure; for instance we show that the Bowen type definition coincides with the one using spanning sets.

Then we consider the case of a holomorphic map $f : \mathbb{P}^2\mathbb{C} \to \mathbb{P}^2\mathbb{C}$ which is Axiom A and such that its critical set does not intersect a particular basic set of saddle type Λ.

We will prove that, under a technical condition, the Hausdorff dimension of the intersection between the local stable manifold and the basic set is equal to t^s, i.e $HD(W^s_\delta(x) \cap \Lambda) = t^s$, for all points x belonging to Λ. Here t^s represents the unique zero of the function $t \to P^-(t\phi^s)$, with P^- denoting the inverse topological pressure and $\phi^s(y) := \log |Df|_{E^s_y}|, y \in \Lambda$. In general $HD(W^s_\delta(x) \cap \Lambda)$ will be estimated above by t^s and below by t^s_-, where t^s_- is the unique zero of the map $t \to P_-(t\phi^s)$.

As a corollary we obtain that, if the stable dimension is non-zero, then Λ must be a non-Jordan curve, and also, if $f|_\Lambda$ happens to be a homeomorphism (like in the examples from [13]), then the stable dimension cannot be zero.

Keywords: Inverse topological pressure, preimage entropy, stable Hausdorff dimension, holomorphic Axiom A maps

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1 Introduction

The notions of entropy and topological pressure have found many interesting applications in dynamical systems. In particular Ruelle ([17]) proved (see [2] for the first result of this type) that, for a hyperbolic rational map f on $\mathbb{P}^1\mathbb{C}$, the Hausdorff dimension of its Julia set is equal to the unique zero of the pressure function of $\phi(x) := -\log|Df(x)|$. This can be used further to give estimates on the Hausdorff dimension of the Julia set for some rational maps.

Notation: In the sequel we will denote the Hausdorff dimension of a set A by $HD(A)$; also, $\mathbb{P}^k$ will denote the k-dimensional complex projective space.

Relations between Lyapunov exponents, Hausdorff dimension and entropy were also given by Manning in [9]. Equalities between stable/unstable Hausdorff dimension and the zero of the contraction/expansion in the stable/unstable direction have been given for complex Henon mappings by Verjovsky-Wu [19]). They proved the following theorem (shown in the case of plane horseshoes by Manning and McCluskey in [10]):

Theorem (Verjovsky-Wu). *For any hyperbolic Henon map g on $\mathbb{C}^2$ and $x \in J$, (with J denoting the Julia set of g), the Hausdorff dimension t^s of $W^s_\delta(x) \cap J$ is given by Bowen's formula:*

$$P_{g|J}(t\Phi^s) = 0,$$

where $0 < t^s < 2$ is independent of $x \in J$, and $\Phi^s(x) := \log|Dg|_{E^s_x}|$, with E^s_x the stable tangent space at x.

A similar equality is true for the unstable dimension $HD(W^u_\delta(x) \cap J)$.

In [11] we noticed that a similar equality is not valid for the stable dimension of a holomorphic endomorphism of $\mathbb{P}^2$ which is hyperbolic on its nonwandering set $\Omega(f)$. However we still have one inequality:

Theorem ([11]). *If f is an Axiom A, holomorphic map on $\mathbb{P}^2$, and Λ is a basic set of saddle type of $\Omega(f)$, then $HD(W^s_\delta(x) \cap \Lambda) \leq t^s_0$, with t^s_0 the unique zero of the pressure function $t \to P(t\phi^s)$; in general the inequality is strict.*

Counterexamples to the equality are given in [11]. In [11] we gave also a similar estimate for the Hausdorff dimension of the set K^- of points with "bounded inverse iterates"; K^- represents actually the union of unstable sets of points from the saddle part S_1 of the nonwandering set.

The problem of estimating $HD(W^s_\delta(x) \cap \Lambda)$ for a hyperbolic holomorphic map f on $\mathbb{P}^2$ was studied in [12] as well. In that paper we obtained an upper bound using the number of preimages that a point from Λ can have in Λ.

Theorem (Mihailescu-Urbanski). *Assume f is an Axiom A, holomorphic map of degree $d \geq 2$ on $\mathbb{P}^2$, and Λ is one of the basic sets with unstable index equal to 1. Suppose $C_f \cap \Lambda = \emptyset$ (C_f= critical set of f) and also that $f|_\Lambda$: $\Lambda \to \Lambda$ has the property that each point $x \in \Lambda$ has at least*

$d' \leq d$ *preimages in* Λ. *Then* $HD(W^s_\delta(x) \cap \Lambda) \leq \frac{\log d' - h_{top}(f|_\Lambda)}{\log \alpha + \log \gamma}$, *where* $\alpha := \frac{\sup_{x\in\Lambda} |Df|_{E^s_x}|}{\inf_{y\in\Lambda} |Df|_{E^s_y}|}$ *and* $\gamma := \sup_{x\in\Lambda} |Df|_{E^s_x}|$, *as long as* $\alpha \leq \gamma^{-1}$.

The number of preimages of a point, belonging to Λ, is not always constant and is not stable under perturbation. In [13] we gave a large class of perturbations of the map $f(z,w) = (z^2 + c, w^2), |c|$ small, which are homeomorphisms on their respective basic sets close to the basic set $\{p_0(c)\} \times S^1$ of the initial map, where $p_0(c)$ is a fixed attracting point of $z^2 + c$.

We would like now to introduce a notion of inverse topological pressure which is better suited to the stable Hausdorff dimension problem. This inverse topological pressure has some properties similar to the regular (forward) pressure, but in general they do not coincide if the map is not a homeomorphism. (compare also to the notions of inverse entropy studied in [6], [14], etc.).

In the sequel, let us introduce several notions that will be used throughout the paper. We will start with the topological entropy defined in the usual manner. The general setting is that of (X, d), a compact metric space, and $f\colon X \to X$ a continuous map. For n, a positive integer, $d_n(x,y) := \max\{d(f^i x, f^i y),\ i = 0, ..., n-1\}$ is a metric on X inducing the same topology as the metric d.

Definition 1. A subset $E \subset X$ is called (n,ε)-separated (for some $\varepsilon > 0$) if for all $x, y \in E$, $x \neq y$, we have $d_n(x,y) \geq \varepsilon$.

Definition 2. The *topological pressure* of f is the functional $P_f\colon \mathcal{C}(X,\mathbb{R}) \to \overline{\mathbb{R}}$ defined on the space of continuous functions by:

$$P_f(\varphi) = \lim_{\varepsilon\to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \log \sup \left\{ \sum_{x\in E} \exp\left(\sum_{i=0}^{n-1} \varphi(f^i x)\right), E \subset X, (n,\varepsilon)\text{-separated set.} \right\}.$$

Definition 3. When considering $\varphi \equiv 0$ in Definition 2, we obtain the notion of *topological entropy* of f.

There exists an interesting relationship between Borel invariant measures and P_f, contained in the following:

Theorem (Variational Principle). *In the above setting,* $P_f(\varphi) = \sup_\mu \{h_\mu(f) + \int \varphi \, d\mu\}$, *where the supremum is taken over all* f*-invariant Borel probability measures* μ, *and* $h_\mu(f) =$ *measure-theoretic entropy of* μ.

For the definition of $h_\mu(f)$ and proofs of all these facts, as we mentioned, a good reference is [20].

The topological pressure has several useful properties:

Theorem (Properties of Pressure). *If* $f\colon X \to X$ *is a continuous transformation, and* $\varphi, \psi \in \mathcal{C}(X,\mathbb{R})$, *then:*

1) $\varphi \leq \psi \Rightarrow P_f(\varphi) \leq P_f(\psi)$

2) $P_f(\cdot)$ *is either finitely valued or constantly* ∞

3) P_f *is convex*

4) for a strictly negative function φ*, the mapping* $t \to P_f(t\varphi)$ *is strictly decreasing if* $P(0) < \infty$.

5) P_f *is a topological conjugacy invariant.*

The need appears however for a notion of topological pressure on non-compact sets. This was done beautifully in a paper by Pesin-Pitskel [15]. However, in the case of defining the inverse topological pressure, there is no apriori backward sum similar to the forward sum $\phi(x) + \phi(fx) + \phi(f^2x) + ... + \phi(f^n x)$ that was taken in the definition of $P(\phi)$ in the usual case.

One can define a notion of inverse pressure using the supremum over all prehistories, or the infimum, or by restricting to a certain set of points and prehistories.

We will give in the following section a definition for the inverse pressure, P^-, which is good from the point of view of its similarity to the Pesin-Pitskel notion and, more importantly, since it will give an upper estimate for the stable dimension.

Let us now also give the definition of inverse entropy, studied by Hurley [6], Nitecki-Przytycki [14], etc. Although the notion we will introduce will be in general different from this one, parallels between the two shall prove interesting. We will call a *branch* of length ℓ (or *prehistory* of length ℓ) in X, a sequence of preimages, $\beta = (z_0, z_{-1}, \ldots, z_{-\ell})$, with $z_i \in X$, $-\ell \le i \le 0$, such that $f(z_{i-1}) = z_i$, $-\ell + 1 \le i \le 0$. For another branch $\beta' = (z'_0, \ldots, z'_{-\ell})$ of same length, define their *branch distance* to be $d^b(\beta, \beta') = \max_{j=\overline{0,\ell}} d(z_{-j}, z'_{-j})$. If two branches have different lengths, we define their distance to be the diameter of X. The reader can notice the similarity between the branch distance and $d_n(\cdot,\cdot)$ introduced earlier. Like $d_n(\cdot,\cdot)$ for forward iterates, d^b_n measures the growth of inverse iterates. Using this, we now define a *branch metric* on X:

$$d^b_n(x, x') < \varepsilon,$$

if for every branch β of length ℓ $(0 \le \ell \le n)$ with $z_0 = x$, there exists a branch β' with $z'_0 = x'$ such that $d^b(\beta, \beta') < \varepsilon$, and vice versa. We say that a subset A of X is ε-*spanning* for X in the d^b_n metric if, for any $x \in X$, there exists a point $y \in A$ with $d^b_n(x, y) < \varepsilon$. For any positive integer n, denote by $N_{\text{span}}(\varepsilon, d^b_n, X)$ the smallest cardinality of an ε-spanning set for X in the d^b_n metric. We will also say that a subset A of X is ε-*separated* in the d^b_n-metric if, for any two different point $x, y \in A$, we have $d^b_n(x, y) > \varepsilon$. For any positive integer n, let $N_{\text{sep}}(\varepsilon, d^b_n, X)$ denote the largest cardinality of an ε-separated set for X in the d^b_n metric. Like in the case of usual(forward) entropy, the above spanning and separated sets can each be used to define the preimage branch entropy:

Proposition ([14]). *For* $f\colon X \to X$ *continuous,* (X, d) *compact metric space, we have*

$$\lim_{\varepsilon\to 0} \overline{lim}_{n\to\infty} \frac{1}{n} \log N_{sep}(\varepsilon, d^b_n, X) = \lim_{\varepsilon\to 0} \overline{lim}_{n\to\infty} \frac{1}{n} \log N_{\text{span}}(\varepsilon, d^b_n, X)$$

and the common value is called the preimage (branch) entropy*, denoted by* $h_i(f)$. ☐

Proposition. *In the same setting as above, if f is a homeomorphism, then $h_i(f) = h(f)$.*

So, in the particular case of homeomorphisms, the two notions coincide; the proof is immediate. Let us recall now two cases when $h_i(f) = 0$.

a. Forward-expansive coverings

If X is a metric space, then a continuous map $f\colon\ X \to X$ is called *forward expansive* if there exists $\varepsilon_0 > 0$ such that for all $x, y \in X$ with $x \neq y$ there exists $m \geq 0$ with

$$d(f^m x, f^m y) \geq \varepsilon_0 > 0.$$

For example, f is forward-expansive on any invariant subset of a Riemannian manifold on which Df is expanding by a constant factor $\lambda > 1$. Recall that $f\colon\ X \to X$ is a *covering map* if for all $x \in X$ there exists a neighborhood U_x of x, such that $f^{-1}(U_x) = \bigcup\limits_i V^i$ with $\{V^i\}$ open disjoint sets, and such that $f\colon\ V^i \to U_x$ is homeomorphism.

Proposition ([14]). *If $f\colon\ X \to X$ is a* forward expansive covering *map, then $h_i(f) = 0$.* □

b. Graph maps.

A *finite graph* is a compact metric space K with a distinguished finite set of points called *vertices*, whose complement has finitely many connected components, *edges*, homeomorphic to the open interval (0,1). We fix the metric on K by assigning length 1 to each edge and the distance between two points in K is the length of the shortest path connecting them.

Theorem (Nitecki-Przytycki, [14]). *Let K be a finite graph and $f\colon\ K \to K$ a continuous map. Then $h_i(f) = 0$.* □

Corollary. *For any continuous self-map f of a closed interval $[a, b]$, or of the circle S^1, we have $h_i(f) = 0$.* □

We end this section with a short discussion of possible definitions for inverse topological pressure and their advantages or disadvantages.

First, one may try to generalize the definition of preimage branch entropy $h_i(f)$ and obtain a notion of inverse pressure using the spanning sets in the metric d_n^b and then taking supremum over all the sums of the test function ϕ along prehistories of points in the spanning set. This could be done with a definition modeled after the one for pressure on a non-compact set from [15] (see also [1]). Indeed we can define $\tilde{P}_i(\phi, \varepsilon) := \inf\{\lambda, \lim\limits_{N\to\infty} M_i(\lambda, \phi, N, \varepsilon) = 0\}$, with $M_i(\lambda, \phi, N, \varepsilon) := \inf\{\sum\limits_{x\in F} \exp(-\lambda n_x + S_{i,n_x}\phi(x)), F \subset X, \bigcup\limits_{x\in F} B^i_{n_x}(x, \varepsilon) = X$, and $n_x \geq N, \forall x \in F\}$, where $B^i_n(x, \varepsilon)$ is the ball of radius ε centered at x in the metric d_n^b, and $S_{i,n}\phi(x) := \sup\{\phi(x_{-n+1}) + ... + \phi(x), (x, x_{-1}, ..., x_{-n+1})$ is a prehistory of x in $X\}$. Finally let $\tilde{P}_i(\phi) := \lim\limits_{\varepsilon\to 0} \tilde{P}_i(\phi, \varepsilon)$.

However in this definition, if we concatenate prehistories of length $n_1, n_2, ..., n_m$, then it is not possible to obtain small sets in the $d^b_{n_1+n_2+...+n_m}$ metric. This happens because the chosen prehistories form just a strict subset in the set of all $(n_1 + ... + n_m)$-prehistories of points in the tail (see the proof of Theorem 1 for this concatenation procedure given in a situation where it *does* work).

By a similar method to the one used in Section 2 for P^-, one could define now another notion of inverse pressure, namely P_i. For an arbitrary continuous real function ϕ, number $\varepsilon > 0$ and integer $m > 0$, let $P_{i,m}(\phi, \varepsilon) := \inf\{ \sum_{x \in F} \exp(S_{i,m}\phi(x)), F \subset X$ is a $(m, \varepsilon)-$spanning set for X in the d^b_m metric$\}$. Then define $P_i(\phi, \varepsilon) := \overline{\lim_{m\to\infty}} \frac{1}{m} \log P_{i,m}(\phi, \varepsilon)$. Similarly as in Section 2 one may show that the limit $\lim_{\varepsilon \to 0} P_i(\phi, \varepsilon)$ does exist and denote this limit by $P_i(\phi)$. So this definition for P_i uses at step m only m- prehistories (instead of n-prehistories with $n \geq m$, as for $\tilde{P}_i$).

However the fact that prehistories do not concatenate with respect to d^b_m makes the equality between P_i and the notion of $\tilde{P}_i$ defined above, to break down. Therefore we do not have an analogue of Theorem 1 for the pressures $\tilde{P}_i$ and P_i. On the other hand, we would need the outer measure construction of $\tilde{P}_i$ since it is better suited for the different diameters of the sets appearing in the definition of Hausdorff dimension. (P_i has however some advantages over P^- due to its definition involving the metric d^b_m; these advantages, as well as more properties of P_i are going to appear in a future paper).

In the next section we will address these questions and will introduce two notions of inverse topological pressure, $\tilde{P}^-$ and P^- which, in the case of f surjective, coincide; (for them, the concatenation procedure mentioned above will work). The common value P^- will prove to be good from the point of view of stable dimension. By *stable dimension at a point* x we will understand the Hausdorff dimension of the intersection between the local stable manifold at x and the respective basic set Λ. We will define $\tilde{P}^-$ with a construction similar to [15] where instead of working with the metric d^b_n, we will say that two points are (n, ε)-close if they have some n-prehistories which are ε close at each level. For a fixed $\varepsilon > 0$, and given a prehistory C of x, we will define the set $X(C)$ of points y which have a prehistory ε-shadowed by C; in the definition of $\tilde{P}^-$ (and P^-) we cover X with such sets $X(C)$.

We shall define also another notion of inverse pressure, called P_- which uses even bigger sets for the cover, namely for a point x and a given $\varepsilon > 0$ we define $B^-_n(x, \varepsilon) := \cup X(C)$, where the union is taken over all n-prehistories C of x in X. The stable dimension will be contained between the unique zeros of the functions $t \to P_-(t\phi^s)$ and $t \to P^-(t\phi^s)$.

2 Two definitions for inverse topological pressure

We shall start with the definition of $\tilde{P}^-$ and P^-. Let us fix $\varepsilon > 0$ small enough.

Let X be a compact metric space, $Y \subset X$, and $f : X \to X$ a continuous map.

Denote by $\mathcal{C}_m(\varepsilon)$ the set of chains (collections) C of length m of balls of radius ε centered at points of prehistories, i.e $C = \{U_0 = B(x_0, \varepsilon), ..., U_{m-1} = B(x_{-m+1}, \varepsilon)\}$, where $f(x_{-1}) = x_0, ..., f(x_{-m+1}) = x_{-m+2}$ and of collections C of length k, $k < m$ which are *terminal*, i.e $C = \{U_0 = B(x_0, \varepsilon), ..., U_{k-1} = B(x_{-k+1}, \varepsilon)\}$, with $f(x_{-i}) = x_{-i+1}, i = 1, ..., k-1$ and $f^{-1}(x_{-k+1}) = \emptyset$. We denote by $n(C)$ the number of elements of C. It is clear that terminal branches with $k < m$

can be taken only if the map f is not surjective. In most of our applications the map f will be surjective on X, however. Now let $C = \{U_0, ..., U_{k-1}\} \in \mathcal{C}_m(\varepsilon), k = n(C)$, and define

$$X(C) := \{y \in U_0, \exists y_{-1} \in f^{-1}(y) \cap U_1, \exists y_{-2} \in f^{-1}(y_{-1}) \cap U_2, ...\exists y_{-k+1} \in f^{-1}(y_{-k+2}) \cap U_{k-1}\}$$

For a real continuous function ϕ, on X, define also $S_k^- \phi(C) := sup\{\phi(y) + ... + \phi(y_{-k+1}), y \in X(C)$ and the prehistory $y, ...y_{-k+1}$ is as in the definition of $X(C)\}$, where $k = n(C)$. If y is a point in $X(C)$, we will say that a prehistory $(y, y_{-1}, ..., y_{-n(C)+1})$ is **attached** to $X(C)$ (or to C) if it satisfies the conditions in the definition of $X(C)$.

Remark: Let us denote by $\delta_\phi(\varepsilon)$ the maximum oscillation of the function ϕ on a ball of radius ε in X, i.e $\delta_\phi(\varepsilon) := \sup\{|\phi(x) - \phi(y)|\}$, where the supremum is taken over all pairs $x, y \in X$ for which there exists $z \in X$ such that $x, y \in B(z, \varepsilon)$. Then, for any $y, y' \in X(C), |\phi(y) + \phi(y_{-1}) + ... + \phi(y_{-n(C)+1}) - \phi(y') - \phi(y'_{-1}) - ... - \phi(y'_{-n(C)+1})| \leq n(C)\delta_\phi(\varepsilon)$. So, up to a difference of at most $n(C)\delta_\phi(\varepsilon)$, it does not matter which point in $X(C)$ we take to calculate $S_{n(C)}^- \phi(C)$. □

Let now $\mathcal{C}(\varepsilon) := \bigcup_{m=1}^\infty \mathcal{C}_m(\varepsilon)$.

For an arbitrary function $\phi \in \mathcal{C}(X, \mathbb{R})$, a positive integer N, and a real number λ let:

$M(\lambda, \phi, Y, N, \varepsilon) := \inf\{\sum_{C \in \Gamma} \exp(-\lambda n(C) + S_{n(C)}^- \phi(C)), \Gamma \subset \mathcal{C}(\varepsilon),\ Y \subset \bigcup_{C \in \Gamma} X(C)$, and for all $C \in \Gamma$ we have either $n(C) \geq N$, or $n(C) < N$ and C is terminal$\}$.

When N increases, the pool of possible candidates Γ appearing in the definition of $M(\lambda, \phi, Y, N, \varepsilon)$ decreases, implying that $M(\lambda, \phi, Y, N, \varepsilon)$ increases with N.

Hence, there exists the limit $\lim_{N \to \infty} M(\lambda, \phi, Y, N, \varepsilon) =: m(\lambda, \phi, Y, \varepsilon)$. The notation $m(\lambda, \phi, Y, \varepsilon)$ emphasizes the nature of the construction in the spirit of Hausdorff outer measure. Now let $\tilde{P}^-(\phi, Y, \varepsilon) := \inf\{\lambda, m(\lambda, \phi, Y, \varepsilon) = 0\}$.

Proposition 1. *Given a continuous function $f : X \to X$ as above, and $Y \subset X$, the limit $\lim\limits_{\varepsilon \to 0} \tilde{P}^-(\phi, Y, \varepsilon)$ exists and is called the* ***inverse topological pressure*** *of ϕ on Y, and denoted by $\tilde{P}^-(\phi, Y)$.*

Proof. Assume $0 < \varepsilon' < \varepsilon$ and take a set of chains $\Gamma' \subset \mathcal{C}(\varepsilon')$ covering Y, i.e such that $Y \subset \bigcup_{C' \in \Gamma'} X(C')$. Then taking the balls of same centers as the ones in $C' \in \Gamma'$, but radius ε, we will obtain another cover of Y, this time from $\mathcal{C}(\varepsilon)$ and whose elements are denoted by C. If $\delta_\phi(\varepsilon)$ denotes the maximum oscillation of ϕ on a ball of radius ε in X, we get $S_{n(C)}^- \phi(C) \leq S_{n(C)}^- \phi(C') + n(C)\delta_\phi(\varepsilon)$.

Therefore, $\tilde{P}^-(\phi, Y, \varepsilon) - \delta_\phi(\varepsilon) \leq \liminf\limits_{\varepsilon' \to 0} \tilde{P}^-(\phi, Y, \varepsilon')$. This shows that the limit in the proposition does exist. □

Notation: When we want to emphasize the dependence of the inverse topological pressure on f we will write $\tilde{P}_f^-(\phi, Y)$. We will also denote by $\tilde{P}^-(\phi)$, $M(\lambda, \phi, N, \varepsilon)$, and $m(\lambda, \phi, \varepsilon)$, respectively, the quantities $\tilde{P}^-(\phi, X)$, $M(\lambda, \phi, X, N, \varepsilon)$ and $m(\lambda, \phi, X, \varepsilon)$, when no confusion arises.

A few properties of $\tilde{P}^-$ are given in the following

Proposition 2. *(1) If $Y_1 \subset Y_2 \subset X$, then $\tilde{P}^-(\phi, Y_1) \leq \tilde{P}^-(\phi, Y_2)$.*

(2) If $Y = \bigcup\limits_m Y_m$ is a finite or countable union, then $\tilde{P}^-(\phi, Y) = \sup\limits_m \tilde{P}^-(\phi, Y_m)$.

(3) $\tilde{P}^-(\phi, Y)$ is invariant to topological conjugacy.

Proof. Although the proofs of the above properties are relatively easy, we will outline them since they will be useful later on. Properties (1) and (3) follow from the definitions, and from the classical (forward) case ([20]).

Proof of (2): From (1) it follows that $\tilde{P}^-(\phi, Y) \geq \sup_m \tilde{P}^-(\phi, Y_m)$. We will show now also the opposite inequality. For this purpose, let $t > \sup_m \tilde{P}^-(\phi, Y_m)$; in this case there exists some small $\alpha > 0$ such that $t - \alpha > \sup_m \tilde{P}^-(\phi, Y_m)$. Now using the definition of $\tilde{P}^-$ and Proposition 1 it follows that for any $\varepsilon > 0$ small and any m, $\tilde{P}^-(\phi, Y_m, \varepsilon) - \delta_\phi(\varepsilon) \leq \tilde{P}^-(\phi, Y_m)$. So we have $t - \alpha > \sup_m \tilde{P}^-(\phi, Y_m) \geq \tilde{P}^-(\phi, Y_m, \varepsilon) - \delta_\phi(\varepsilon), \forall \varepsilon > 0, \forall m$. Therefore it is enough to take $\varepsilon > 0$ so small that $\delta_\phi(\varepsilon) < \frac{\alpha}{2}$; in this case, we obtain $t - \frac{\alpha}{2} > \tilde{P}^-(\phi, Y_m, \varepsilon), \forall m$. Thus $m(t - \frac{\alpha}{2}, \phi, Y_m, \varepsilon) = 0, \forall m$. But we have noticed that in general, $m(\lambda, \phi, Y, \varepsilon)$ is given as the limit of an increasing sequence $(M(\lambda, \phi, Y, N, \varepsilon))_N$. Therefore $M(t - \frac{\alpha}{2}, \phi, Y_m, N, \varepsilon) = 0, \forall m, N$. Fix N and let for each integer m, a set $\Gamma_m \subset \mathcal{C}(\varepsilon)$ such that $Y_m \subset \underset{C \in \Gamma_m}{\cup} X(C)$ and for all $C \in \Gamma_m$, either $n(C) \geq N$, or $n(C) < N$ when C is terminal, and we have also:

$$\sum_{C \in \Gamma_m} \exp(-(t - \frac{\alpha}{2})n(C) + S^-_{n(C)}\phi(C)) \leq \frac{1}{2^m}$$

Let now the collection $\Gamma := \underset{m}{\cup} \Gamma_m$. Then $Y = \underset{m}{\cup} Y_m \subset \underset{C \in \Gamma}{\cup} X(C)$ and for all $C \in \Gamma$ we have either $n(C) \geq N$ or $n(C) < N$ in the case when C is terminal. By summing the above inequalities we get

$$M(t - \frac{\alpha}{2}, \phi, Y, N, \varepsilon) \leq \sum_{C \in \Gamma} \exp(-(t - \frac{\alpha}{2})n(C) + S^-_{n(C)}\phi(C)) \leq 1,$$

which implies $M(t, \phi, Y, N, \varepsilon) \leq e^{-\alpha N/2}$. So $m(t, \phi, Y, \varepsilon) = 0$ and $t \geq \tilde{P}^-(\phi, Y, \varepsilon)$. So, if we let $\varepsilon \to 0$, we get $t \geq \tilde{P}^-(\phi, Y)$. But t was taken arbitrarily larger than $\sup_m \tilde{P}^-(\phi, Y_m)$ so we obtain the required equality, $\sup_m \tilde{P}^-(\phi, Y_m) = \tilde{P}^-(\phi, Y)$.

□

We can also define another notion of inverse pressure using only chains C of the same length or which are terminal. This time we define it directly on the whole space X. Take $P^-_m(\phi, \varepsilon) := \inf\{\sum_{C \in \Gamma} exp(S^-_{n(C)}\phi(C)), \Gamma \subset \mathcal{C}_m(\varepsilon), X = \underset{C \in \Gamma}{\cup} X(C)\}$. Then, set $P^-(\phi, \varepsilon) := \overline{\lim}_{m \to \infty} \frac{1}{m} \log P^-_m(\phi, \varepsilon)$. Similarly as for $\tilde{P}^-$, we can prove that $P^-(\phi) := \lim_{\varepsilon \to 0} P^-(\phi, \varepsilon)$ does exist. Again we write $P^-(\phi)$ when the mapping f is fixed and there can be no confusion; if we want to record also the dependence on f we will use the notation $P^-_f(\phi)$.

Theorem 1. *If $f : X \to X$ is surjective, then $P^-(\phi) = \tilde{P}^-(\phi)$, for any continuous function $\phi \in \mathcal{C}(X, \mathbb{R})$.*

Proof. Let us note in the begining of our proof that, since f is surjective, we do not need to worry about terminal branches in $\mathcal{C}_m(\varepsilon)$ of length $n(C) < m$.

First we show that $\tilde{P}^-(\phi) \leq P^-(\phi)$. Let us take $\lambda > P^-(\phi, \varepsilon)$. For every $0 < \eta < \lambda - P^-(\phi, \varepsilon)$ and all N large enough, we have

$$m(\lambda, \phi, \varepsilon) \leq M(\lambda, \phi, N, \varepsilon) + \eta. \tag{1}$$

Since there exists $m > N$ such that $|P^-(\phi, \varepsilon) - \frac{1}{m} \log P_m^-(\phi, \varepsilon)| \leq \eta$, we obtain

$$\begin{aligned} M(\lambda, \phi, N, \varepsilon) &\leq \inf\{\sum_{C \in \Gamma} \exp(-\lambda m + S_m^- \phi(C)), \Gamma \in \mathcal{C}_m(\varepsilon)\ X = \bigcup_{C \in \Gamma} X(C)\} \\ &\leq e^{-\lambda m} P_m^-(\phi, \varepsilon) \leq \exp((-\lambda + P^-(\phi, \varepsilon) + \eta)m) \end{aligned}$$

as long as $m \geq N$. Combining this and (1) we get

$$m(\lambda, \phi, \varepsilon) \leq \exp((-\lambda + P^-(\phi, \varepsilon) + \eta)m) + \eta$$

and, since $-\lambda + P^-(\phi, \varepsilon) + \eta < 0$, letting $m \to \infty$, we get $m(\lambda, \phi, \varepsilon) \leq \eta$. Letting in turn $\eta \to 0$, we get $m(\lambda, \phi, \varepsilon) \leq 0$ which implies that $\tilde{P}^-(\phi, \varepsilon) \leq \lambda$. Since λ was chosen arbitrarily larger than $P^-(\phi, \varepsilon)$, the required inequality thus follows.

Let us show now the opposite inequality: $\tilde{P}^-(\phi) \geq P^-(\phi)$. Firstly, we study the concatenation of two prehistories. This is the main advantage of this definition for inverse pressure, i.e the possibility of joining two different prehistories to form another one, of length equal to the sum of lengths of its components.

Let $\Gamma_m \subset \mathcal{C}_m(\varepsilon), \Gamma_n \subset \mathcal{C}_n(\varepsilon)$, each covering X. Since X is compact, we may assume that both Γ_m and Γ_n have finite number of chains. Take now $C \in \Gamma_m, C' \in \Gamma_n$. Assume that $C = \{U_0, ..., U_{m-1}\}, C' = \{U'_0, ..., U'_{n-1}\}$. For a point x from $X(C)$, let us denote by $(x, x_{-1}, ..., x_{-m+1})$ a prehistory of x satisfying the conditions in the definition of $X(C)$. We call any such prehistory of x, a prehistory *attached* to C. Let us define $X(CC') := \{x \in X(C) : x_{-m+1} \in X(C')$ for some prehistory of x attached to C$\}$. If $X(CC') \neq \emptyset$, then if y, z are points in $X(CC')$, consider prehistories $(y, y_{-1}, ..., y_{-m+1})$ and $(z, z_{-1}, ...z_{-m+1})$ which are attached to C, such that $y_{-m+1}, z_{-m+1} \in X(C')$. So z_{-i} has a preimage $z_{-i-1} \in U_{i+1}$, and y_{-i} has a preimage $y_{-i-1} \in U_{i+1}$, hence $d(y_{-i-1}, z_{-i-1}) < 2\varepsilon$, for $0 \leq i \leq m-1$. Also, y_{-m+1} has a preimage y_{-m} belonging to U'_1; so does z_{-m+1}, hence $d(y_{-m}, z_{-m}) < 2\varepsilon$. Repeating the reasoning, we get that, if $C'' := \{B(y_0, 2\varepsilon), ..., B(y_{-m-n+1}, 2\varepsilon)\}$, then $C'' \in \mathcal{C}_{m+n}(2\varepsilon)$, and $X(CC') \subset X(C'')$. Therefore

$$S_{m+n}^- \phi(C'') \leq S_m^-(C) + S_n^-(C') + (m+n)\delta_\phi(2\varepsilon). \tag{2}$$

We now return to the problem of showing the inequality $\tilde{P}^-(\phi) \geq P^-(\phi)$. In order to prove it fix $\lambda > \tilde{P}^-(\phi)$. Then $m(\lambda, \phi, \varepsilon) = 0$ and therefore $\lim\limits_{N \to \infty} M(\lambda, \phi, N, \varepsilon) = 0$. If N is large enough, then $M(\lambda, \phi, N, \varepsilon) < \frac{1}{2}$. Thus, there exists a covering of X with sets $X(C), C \in \Gamma \subset \mathcal{C}(\varepsilon)$, such that

$$\sum_{C \in \Gamma} \exp(-\lambda n(C) + S_{n(C)}^- \phi(C)) < \frac{1}{2}$$

Since $X = \bigcup_{C \in \Gamma} X(C)$ and X is compact, we can assume that Γ is finite. So we can write $\Gamma = \{C_1, ..., C_q\}$, and let $N_0 := \max\limits_{1 \leq i \leq q} n(C_i)$. For $1 \leq i \leq q$, denote also $n(C_i)$ by n_i. By raising the last displayed sum to power s and then summing over s, we obtain that

$$\sum_{s\geq 0}\sum_{1\leq j_1,..,j_s\leq q}\exp(-\lambda(n(C_{j_1})+...+n(C_{j_s}))+S^-_{n(C_{j_1})}\phi(C_{j_1})+..+S^-_{n(C_{j_s})}\phi(C_{j_s}))<$$
$$<\sum_{s=0}^{\infty}\left(\frac{1}{2}\right)^s=2<\infty. \tag{3}$$

Similarly as above, for any ordered s-tuple $(j_1,...j_s)$, with $s\geq 0$ and $j_1,...,j_s\in\{1,...,q\}$, we can associate a chain denoted by $C_{j_1}...C_{j_s}$, obtained by concatenation. By the same argument as in (2), we obtain the following:

$$S^-_{n_{j_1}+...+n_{j_s}}\phi(C_{j_1}..C_{j_s})\leq S^-_{n_{j_1}}\phi(C_{j_1})+...+S^-_{n_{j_s}}\phi(C_{j_s})+(n_{j_1}+...+n_{j_s})\cdot\delta_\phi(2\varepsilon) \tag{4}$$

For every positive integer n with $n>2N_0$, denote by Γ_n the collection of chains $C_{j_1}...C_{j_s}, s\geq 0$, for which $n\leq n_{j_1}+...+n_{j_s}<n+N_0$. Note that $\Gamma_n\subset\mathcal{C}(2\varepsilon)$ and for every chain $C_{j_1}...C_{j_s}$ from Γ_n, we have
$$S^-_n\phi(C_{j_1}..C_{j_s})\leq S^-_{n_{j_1}+...+n_{j_s}}\phi(C_{j_1}..C_{j_s})+N_0||\phi||_\infty.$$
Recall now that f is surjective on X and the sets $X(C_j), 1\leq j\leq q$, cover X. Hence, if y is an arbitrary point of X, and n is a positive integer $(n>2N_0)$, there exists a chain $C_{j_1}...C_{j_{s-1}}$ (for some integer $s\geq 0$ and some coefficients $1\leq j_1,...,j_{s-1}\leq q$) such that $y\in X(C_{j_1}...C_{j_{s-1}})$ and $n_{j_1}+...+n_{j_{s-1}}<n$. Denote by s' the largest of the integers s satisfying the above property; s' is finite since $n(C)\geq 1,\forall C\in\Gamma$. Also s' depends on y and n. Let us also take a prehistory of y, $(y,y_{-1},...,y_{-n_{j_1}-..-n_{j_{s'-1}}})$ attached to the chain $C_{j_1}..C_{j_{s'-1}}$. But, since the sets $X(C), C\in\Gamma$, cover X, we know that there must exist some integer $j_{s'}\in\{1,...,q\}$ with $y_{-n_{j_1}-..-n_{j_{s'-1}}}\in X(C_{j_{s'}})$. Therefore y has a prehistory $(y,y_{-1},...y_{-n_{j_1}-..-n_{j_{s'}}})$ attached to the chain $C_{j_1}...C_{j_{s'}}$ which has length equal to $n_{j_1}+...+n_{j_{s'}}$. But N_0 was defined as $\max\limits_{1\leq i\leq q}n(C_i)$, hence using the maximality property of s' and the fact that $y\in X(C_{j_1}...C_{j_{s'}})$, we obtain that $n\leq n_{j_1}+...+n_{j_{s'}}<n+N_0$. Thus $y\in X(C_{j_1}...C_{j_{s'}})$ for some chain $C_{j_1}..C_{j_{s'}}\in\Gamma_n$, and remember that y was taken arbitrarily in X. In conclusion we proved that $X=\bigcup\limits_{C\in\Gamma_n}X(C)$. Hence, applying (3) and (4), we get

$$\sum_{C\in\Gamma_n}\exp(-\lambda n+S^-_n\phi(C_{j_1}..C_{j_s})-n\delta_\phi(2\varepsilon))\leq$$
$$\leq\exp(N_0||\phi||_\infty)\sum_{C\in\Gamma_n}\exp(-\lambda n+S^-_{n_{j_1}+...+n_{j_s}}\phi(C_{j_1}..C_{j_s})-n\delta_\phi(2\varepsilon))$$
$$\leq\exp(N_0(||\phi||_\infty+|\lambda|))\sum_{C\in\Gamma_n}\exp(-\lambda(n_{j_1}+..+n_{j_s})+S^-_{n_{j_1}}\phi(C_{j_1})+...+S^-_{n_{j_s}}\phi(C_{j_s})+$$
$$(n_{j_1}+..n_{j_s})\delta_\phi(2\varepsilon)-n\delta_\phi(2\varepsilon))$$
$$\leq 2\exp(N_0(||\phi||_\infty+|\lambda|+\delta_\phi(2\varepsilon)))<\infty$$

This proves that

$$\inf_{\Gamma'}\{\sum_{C'\in\Gamma'}\exp(S^-_n\phi(C')),\Gamma'\subset\mathcal{C}_n(2\varepsilon),X=\bigcup_{C'\in\Gamma'}X(C')\}\leq 2\exp[N_0(||\phi||_\infty+\delta_\phi(2\varepsilon)+|\lambda|)+(\lambda+\delta_\phi(2\varepsilon))n]$$

In conclusion $\lambda + \delta_\phi(2\varepsilon) \geq P^-(\phi, 2\varepsilon)$. But ε can be taken arbitrarily small and λ was taken arbitrarily larger than $\tilde{P}^-(\phi)$. Hence $\tilde{P}^-(\phi) \geq \lim\limits_{\varepsilon\to 0} P^-(\phi, 2\varepsilon) = P^-(\phi)$. This finishes the proof of the required equality $P^-(\phi) = \tilde{P}^-(\phi)$. □

Corollary 1. *If f is a homeomorphism of X, then $\tilde{P}^-(\phi, X) = P(\phi)$, i.e the inverse pressure of a function ϕ coincides with the usual (forward) one in the case of homeomorphisms.*

Proof. Since f is surjective on X, we can apply the previous theorem. Thus we obtain that $\tilde{P}^-(\phi, X)$ is equal to $P^-(\phi)$. Let us take a collection $\Gamma \subset \mathcal{C}_m(\varepsilon)$ as in the definition of P^- and suppose that each C from Γ is written as a chain $C = \{B(x(C), \varepsilon), ..., B(x(C)_{-m+1}, \varepsilon)\}$. Then $X = \bigcup\limits_{C\in\Gamma} X(C)$ if and only if the set $\{x(C)_{-m+1}, C \in \Gamma\}$ is (m, ε)-spanning in the usual (forward) sense. Also $S_m^-\phi(C) = \phi(x(C)_{-m+1}) + ... + \phi(\phi^{m-1}(x(C)_{-m+1})), \forall C \in \Gamma$. This proves that $P^-(\phi) = P(\phi)$. □

Remark: If f is not surjective, it is not true in general in the proof of Theorem 1 that, if C' is terminal with $n(C') < n$, then CC' gives also a terminal branch C'' with $n(C'') < m + n$.

However, even if f is not surjective the proof of Theorem 1 still gives the following:

Proposition 3. *For any continuous map $f : X \to X$, $\tilde{P}^-(\phi) \leq P^-(\phi), \forall\phi \in \mathcal{C}(X, \mathbb{R})$.*

The following properties of inverse pressure are similar to those of the usual (forward) topological pressure; we denote by P^- the functional P_f^- in the following:

Proposition 4. *If $f : X \to X$ is a continuous map of a compact metric space, and if $\phi, \psi \in \mathcal{C}(X, \mathbb{R})$, we have:*

(a) $\tilde{P}^-(\phi + \alpha) = \tilde{P}^-(\phi) + \alpha$, for a real constant α.

(b) if $\phi \leq \psi$, then $\tilde{P}^-(\phi) \leq \tilde{P}^-(\psi)$. Hence if we denote by $\tilde{h}^-(f) := \tilde{P}_f^-(0)$, then $\tilde{h}^-(f) + inf\, \phi \leq \tilde{P}^-(\phi) \leq \tilde{h}^-(f) + sup\ \phi$.

(c) $\tilde{P}^-(\cdot)$ is either finitely valued or constantly ∞.

(d) $|\tilde{P}^-(\phi) - \tilde{P}^-(\psi)| \leq ||\phi - \psi||$ as long as $\tilde{P}^-(\cdot)$ is finite.

(e) $\tilde{P}^-(\phi + \psi \circ f - \psi) = \tilde{P}^-(\phi)$.

(f) for a strictly negative function ϕ, the mapping $t \to \tilde{P}^-(t\phi)$ is strictly decreasing if $\tilde{P}^-(0) < \infty$.

Proof. The items (a) and (d) follow immediately from the definition. The first part of (b) is clear from the definition of inverse pressure. The second part follows from the first part combined with (a).

In order to prove (c) notice that from (b) we have

$$\tilde{h}^-(f) + \inf\phi \leq \tilde{P}^-(\phi) \leq \tilde{h}^-(f) + \sup\phi.$$

Hence, if there exists $\phi \in \mathcal{C}(X, \mathbb{R})$, such that $\tilde{P}^-(\phi) = \infty$, then, since ϕ is bounded on X, it follows that $\tilde{h}^-(f) = \infty$, hence $\tilde{P}^-(\psi)$ is infinite for any $\psi \in \mathcal{C}(X, \mathbb{R})$. The item (e) follows from the fact that

$$\exp(S_n^-(\phi + \psi \circ f - \psi)(\beta)) = \exp(S_n^-(\phi)(\beta)) \cdot \exp(\psi(fx) - \psi(x_{-n+1})),$$

where $\beta = (x_{-n+1}, ..., x_0 = x)$ is a prehistory of x and in general we define $S_n^- \phi(\beta) := \phi(x_{-n+1}) + ... + \phi(x)$. Now we use the fact that, for two prehistories β, γ corresponding to two points x, y in a set of the form $X(C)$, (i.e assume β, γ follow C), $|S_n^- \phi(\beta) - S_n^- \phi(\gamma)| \le n\delta_\phi(\varepsilon)$, with $\delta_\phi(\varepsilon)$ the maximum oscillation of ϕ on a set of radius ε in X. Then by taking the limit over n approaching infinity, one obtains the equality in the statement.

Item (f) follows easily from (a) and (b). □

The notions of inverse pressure introduced will give in particular two inverse entropies, $\tilde{h}^- := \tilde{P}^-(0), h^- := P^-(0)$ and from Proposition 3, $\tilde{h}^- \le h^-$.

Proposition 5. $0 \le \tilde{h}^- \le h^- \le h_i$.

Proof. Let us denote the ball in the d_n^b metric, centered at a point x from X and of radius ε, by $B(x, \varepsilon, d_n^b)$. Then $B(x, \varepsilon, d_n^b)$ is in fact the intersection of sets $X(C)$, over all collections $C \in \mathcal{C}_n(\varepsilon), C = \{B(x, \varepsilon), ..., B(x_{-n+1}, \varepsilon)\}$ centered at points of various prehistories $(x, ..., x_{-n+1})$ of x with respect to f. This follows immediately from the definitions. So, any cover of X with balls in the d_n^b metric gives a cover of X with sets of type $X(C)$. Therefore $h^- \le h_i$. □

We shall give now an example showing that in general $h^- \ne h_i$.

Example with $h^- \ne h_i$. The example below is derived from the case of a continuous map with infinite h_i given in [14]. We will need a compact space X and a continous function $f : X \to X$ with the following properties :

(i) $X := K \cup S(1)$, where for $r > 0$, $S(r) := \{z \in \mathbb{C}, |z| = r\}$ and K is some compact set; in fact $K = B(\frac{1}{2})$ in the notation from [14]; (in order to clarify a notational aspect, remark that $B(\frac{1}{2})$ is *not* the ball of radius $\frac{1}{2}$ centered at 0).

(ii) $f|_{S(1)}$ is the map $z \to z^2$ and also $f(K) \subset S(1)$;

(iii) $K \subset S(\frac{1}{2})$;

(iv) the metric on X is the one induced from the real plane.

(v) $h_i(f) = \infty$. For the convenience of the reader and to illustrate the concept of preimage entropy we give next the idea of the proof of (v):

For every positive integer n, there exists a partition $\mathcal{P}_n$ of a subset $J \subset S(1)$ so that $\mathcal{P}_n$ has 2^{α_n} atoms, where the integers α_n satisfy $\frac{\alpha_n}{n} \to \infty$ as $n \to \infty$. Moreover if x, y are points belonging to different atoms of $\mathcal{P}_n$, then one of them (for example x) has a terminal $(k+1)$-prehistory $(x, x_{-1}, ...x_{-k})$, for some $k < n$, with $x_{-k} \in K \cap f^{-k}(x)$ (recall that $f^{-1}(K) = \emptyset$), while for any prehistory of y, $(y, y_{-1}, ..., y_{-k})$ we have $y_{-k} \in S(1)$ ($f^{-k}(y) \subset S(1)$ from properties of atoms of $\mathcal{P}_n$). Then, since the distance between K and $S(1)$ is $\frac{1}{2}$, we see that these two k-prehistories of x and y will be $\frac{1}{2}$-separated. Hence there exists a $\frac{1}{2}$-separated set for the metric d_n^b, which contains at least 2^{α_n} points. Thus $h_i(f) \ge \overline{\lim_{n\to\infty}}(\frac{\alpha_n}{n}\log 2) = \infty$. We gave this brief idea of the calculation of $h_i(f)$ since it is typical of the way in which one has to guarantee that *all* k-prehistories($k \le n$) of two points x, y have to be ε-close, in order for $d_n^b(x, y)$ to be smaller than ε.

We now calculate also $h^-(f)$ for this example; the difference between the definitions of h_i and h^- will become visible. First, since $f(X) \subset S(1)$ and $K \subset S(\frac{1}{2})$, we notice that any point in K does not have any preimages in X, hence the only prehistories of points in K are the 0-prehistories (i.e prehistories consisting of only one point). Therefore if $C \in \mathcal{C}_m(\varepsilon), m \geq 0$, is a chain attached to a prehistory of a point $y \in K$, then $C = \{B(y,\varepsilon)\}$ and $X(C) = B(y,\varepsilon)$. Denote by $N(\varepsilon)$ the minimal number of balls of radius ε $(0 \leq \varepsilon \leq \frac{1}{2})$, needed to cover X. So we can cover the set K with a number $N(\varepsilon)$ of sets $X(C), C \in \mathcal{C}_n(\varepsilon)$ and notice also that the number $N(\varepsilon)$ of these sets is independent of n. Also, if C is an element of $\mathcal{C}_n(\varepsilon)$ attached to some prehistory from $S(1)$ of a point $z \in S(1)$, we see that $X(C) = B(z,\varepsilon)$, since f is expanding on $S(1)$. So the minimal number of sets $X(C), C \in \mathcal{C}_n(\varepsilon)$ needed to cover $S(1)$ is again independent of n. Thus the minimal number of sets $X(C), C \in \mathcal{C}_n(\varepsilon)$ needed to cover the entire space X, is equal to $N(\varepsilon)$ and is independent of n; from definition of h^-, we see that $h^-(f) = 0$.

In conclusion we proved that $h_i(f) = \infty$ and $h^-(f) = 0$, therefore $h_i(f) \neq h^-(f)$ in this case. □

Let us introduce now another notion of inverse topological pressure, this time using inverse spanning sets.

We will start with a continuous *surjective* map $f : X \to X$, where X is a compact metric space. The model we have in mind is that of a holomorphic map of algebraic degree $d \geq 2$ on $\mathbb{P}^2$ which is Axiom A and of a basic set Λ of unstable index 1, from the nonwandering set of f. Since $f : X \to X$ is a surjective map, any point $x \in X$ will have n-prehistories for any positive integer n. Given an n-prehistory $\beta := (x_0, ..., x_{-n+1})$ of x and a positive number ε, we say that C is a chain (branch) ε-**modeled** after β if $C = \{B(x_0,\varepsilon), ..., B(x_{-n+1},\varepsilon)\}$.

Definition 4. We shall call (n,ε)-inverse ball centered at x, the set $\underset{C}{\cup} X(C)$, where C ranges over all chains modeled after the n-prehistories of x. It will be denoted by $B_n^-(x,\varepsilon)$.

Obviously if $x \in B_n^-(y,\varepsilon)$, then $y \in B_n^-(x,\varepsilon)$ as well.

Similar to the first definition of inverse topological pressure, and given $f : X \to X$ surjective, we introduce for an arbitrary function $\phi \in \mathcal{C}(X,\mathbb{R})$, a real number λ, a positive integer N, and a subset Y of X, the following quantity:

$$M_-(\lambda,\phi,Y,N,\varepsilon) := \inf\{\sum_{x\in F} \exp(-\lambda n_x + S_{n_x,-}\phi(x)), \text{where } F \subset X, Y \subset \underset{x\in F}{\cup} B_{n_x}^-(x,\varepsilon), n_x \geq N, \forall x \in F\},$$

where for every $n \geq 0$ we define $S_{n,-}\phi(x) := \inf\{\phi(x)+\phi(x_{-1})+...+\phi(x_{-n+1}),\ (x, x_{-1}, ..., x_{-n+1})$ is an n-prehistory of $x\}$. If N increases, we have less sets in the infimum above, therefore the limit $\lim_{N\to\infty} M_-(\lambda,\phi,Y,N,\varepsilon)$ exists. We shall denote this limit by $m_-(\lambda,\phi,Y,\varepsilon)$ in order to keep a similar notation as for P^-. We define also $P_-(\phi,Y,\varepsilon) := \inf\{\lambda, m_-(\lambda,\phi,Y,\varepsilon) = 0\}$. Similarly as for P^- one can prove that:

Proposition 6. *The limit* $\lim_{\varepsilon\to 0} P_-(\phi,Y,\varepsilon)$ *exists; it will be denoted by* $P_-(\phi,Y)$ *and will be called the inverse lower topological pressure of* ϕ *on* Y *relative to the map* f.

Remark: The inverse lower pressure P_- depends on f, although we did not record this dependence in order not to burden notation. When it will be necessary to record the dependence on f we will write $P_{-,f}$. □

The following properties of P_- are similar to those of P^- from Proposition 2. Notice also that since f was assumed surjective on X, P^- and $\tilde{P}^-$ coincide.

Proposition 7. *If $f : X \to X$ is a continuous map of a compact metric space, and if $\phi, \psi \in \mathcal{C}(X, \mathbb{R})$, then*

(a) $P_-(\phi + \alpha) = P_-(\phi) + \alpha$ for a constant α.

(b) if $\phi \le \psi$, then $P_-(\phi) \le P_-(\psi)$. If $h_-(f) := P_-(0, f)$, then $h_-(f) + \inf \phi \le P_-(\phi) \le h_-(f) + \sup \phi$.

(c) $P_-(\cdot)$ is either finitely valued or constantly ∞.

(d) $|P_-(\phi) - P_-(\psi)| \le ||\phi - \psi||$ when $P_-(\cdot)$ is finite.

(e) $P_-(\phi + \psi \circ f - \psi) = P_-(\phi)$.

(f) if $\phi < 0$ on X and $h_-(f) < \infty$, then the map $t \to P_-(t\phi)$ is strictly decreasing and has a unique zero.

(g) if f is a homeomorphism on X, then $P^-(\phi) = P_-(\phi)$, $\forall \phi \in \mathcal{C}(X, \mathbb{R})$. Hence P_-, P^-, P coincide for homeomorphisms.

The decreasing part in consequence (f) above follows from (a) and (b); the uniqueness of the zero follows from the fact that $P_-(0) = h_-(f) \ge 0$ and $P_-(t\phi) < 0$ if t is large enough.

The name "inverse lower pressure" is justified by the following Proposition:

Proposition 8. *For a continuous surjective map $f : X \to X$, and a continuous arbitrary function $\phi \in \mathcal{C}(X, \mathbb{R})$, we have $P^-(\phi) \ge P_-(\phi)$.*

Proof. Let us take a covering of X with sets $X(C)$, where C belongs to a finite set Γ. For each $C \in \Gamma$, assume that C corresponds to an $n(C)$-prehistory of a point $x(C)$. Let us denote the set of all points $x(C)$ obtained in this fashion (when $C \in \Gamma$), by F. We could have several C's from Γ corresponding to the same $x \in F$. If this happens then we take $n(x)$ to be the smallest $n(C)$ among all the C's giving x. Then $B^-_{n(x)}(x, \varepsilon)$ contains all the sets of the form $X(C)$ for all the C's in Γ with $x(C) = x$. This implies that $X = \underset{x \in F}{\cup} B^-_{n(x)}(x, \varepsilon)$. Also, it is clear that, if $n(C) \ge N$ for all $C \in \Gamma$, then also $n(x) \ge N$, for all $x \in F$. On the other hand, if for a branch C it happens that $x(C) = x$ and $n(C) = n(x)$ (obviously such a branch must exist for every $x \in F$), then from definitions it follows that $S^-_{n(C)}\phi(C) \ge S_{n(x),-}\phi(x)$. If, for another branch C, it happens that $x(C) = x$, but $n(C) > n(x)$, then we do not even consider the corresponding term in the sum from the definition of $M_-(\lambda, \phi, X, N, \varepsilon)$. Hence in the sum from the definition of $M_-(\lambda, \phi, X, N, \varepsilon)$ we have less sets than in $M(\lambda, \phi, X, N, \varepsilon)$, and for the chains that appear in both sums, we have $S^-_n\phi(C) \ge S_{n,-}\phi(x(C))$. Therefore $M_-(\lambda, \phi, X, N, \varepsilon) \le M(\lambda, \phi, X, N, \varepsilon), \forall \lambda, N, \varepsilon$. In conclusion we get $P_-(\phi) \le P^-(\phi)$. $\square$

3 Stable dimension

This section will present the main application of the previously introduced notions of inverse topological pressure. We will study the availability of a Bowen type relation for the Hausdorff dimension of the intersection between a local stable manifold and a given basic set for a holomorphic Axiom

A map f of $\mathbb{P}^2$. We shall call this Hausdorff dimension *the stable dimension* for short. Precise definitions will be given below.

It was observed in [11] that the stable dimension is in general less than or equal to than the zero t_0^s of the function $t \to P(t\phi^s)$, where $P(\cdot)$ denotes the usual (forward) pressure. In that article there are also given examples with strict inequality. This is due to the fact that $P(\cdot)$ takes into consideration only forward orbits and hence we cannot estimate from below the diameters of the sets in a covering by using derivatives of the form $||Df^n|_{E^s}||$.

In [12] we also showed that the gap between t_0^s and the stable dimension can be explained partially by the number of preimages that a point in Λ has in Λ.

Here we will prove some estimates between the unique zero of the inverse pressure P^-, t^s, the unique zero of the inverse lower pressure P_-, t_-^s, and the stable dimension. This will imply that if the stable dimension is non-zero, then the basic set cannot be a Jordan curve. Let us first introduce some notation.

Our setting throughout this section is that of a holomorphic mapping on the complex projective plane $f : \mathbb{P}^2 \to \mathbb{P}^2$.

Set

$$\Omega(f) := \{x \in \mathbb{P}^2, \forall (r > 0) \exists (n \geq 1) \ s.t \ f^n(B(x,r)) \cap B(x,r) \neq \emptyset\},$$

the non-wandering set of f. The space of prehistories in Ω is denoted by $\hat{\Omega} := \{\hat{x} := (x_n)_{n \leq 0}, f(x_{n-1}) = x_n, x_n \in \Omega, \forall n \leq 0\}$.

From now on f is assumed to be hyperbolic. This means that there exists a continuous splitting of the tangent bundle over $\hat{\Omega}$ into subspaces which are invariant under Df, $T_{\hat{\Omega}}(\hat{x}) = E_{x_0}^s \oplus E_{\hat{x}}^u$ and constants $c \geq 0$ and $\lambda > 1$ such that

$$||Df^n(v)|| \leq c\lambda^{-n} \cdot ||v||, v \in E_{x_0}^s, \text{ and}$$

$$||Df^n(v')|| \geq c^{-1}\lambda^n ||v'||, v' \in E_{\hat{x}}^u, \forall n \geq 0.$$

Up to a change of metric it can be proved that in the above inequalities one can take $c = 1$. It can be shown that the above splitting will give birth to local stable and unstable manifolds respectively denoted by

$$W_\delta^s(x) := \{y \in \mathbb{P}^2, d(f^n x, f^n y) \leq \delta, \forall n \geq 0\} \text{ and}$$

$$W_\delta^u(\hat{x}) := \{y \in \mathbb{P}^2, y \text{ has a prehistory } \hat{y} = (y_n)_{n \leq 0}, d(x_n, y_n) \leq \delta, \forall n \leq 0\}.$$

$W_\delta^s(x)$ and $W_\delta^u(\hat{x})$ are complex disks in our case. More information about this subject and proofs can be found in [5], [7], [18].

We now assume that f is **Axiom A** on $\mathbb{P}^2$, meaning that there exists a hyperbolic splitting of the tangent bundle over $\hat{\Omega}$ as above, and that periodic points are dense in Ω.

In this case The Spectral Decomposition Theorem ([18]) gives a partition of $\Omega(f)$ into finitely many closed f-invariant subsets $\Omega_i, i = 1, ..., p$, so that f is topologically transitive on each Ω_i (i.e there exists a point y_i with dense orbit in $\Omega_i, i = 1, .., p$); the sets $\Omega_i, i = 1, .., p$ are uniquely determined up to order and they are called **the basic sets** of f. A good general reference for

Axiom A in the case of endomorphisms is for example Ruelle's book [18]. For Axiom A maps it also makes sense to define the **No-Cycle Property** which says that there can be no cycles $\Omega_{i_1} \succ ... \succ \Omega_{i_k} \succ \Omega_{i_1}$ among the basic sets for the ordering: $\Omega_i \succ \Omega_j$ iff $W^u(\Omega_i) \cap W^s(\Omega_j) \neq \emptyset$ (where W^u, W^s are the global unstable/stable sets as defined for example in [5]).

In the sequel we will consider only basic sets $\Lambda \subset \mathbb{P}^2$ of *saddle type* (called also basic sets of unstable index 1), i.e sets which have both stable and unstable directions (so the dimensions of E^s_x and $E^u_{\hat{x}}$ as complex spaces are both equal to 1, $\forall \hat{x} \in \hat{\Lambda}$). By **stable dimension** at the point x from Λ, we will understand $HD(W^s_\delta(x) \cap \Lambda)$, for some fixed small number δ, $0 < \delta < \text{diam}\Lambda/4$.

It follows from the definition of local manifolds that $W^s_\delta(x)$ depends only on its base point x, whereas $W^u_\delta(\hat{x})$ can depend on the whole prehistory $\hat{x} \in \hat{\Omega}$.

Denote also by C_f the critical set of f. This is an analytic variety in $\mathbb{P}^2$. We fix a basic set of saddle type, Λ, and will assume in the sequel that $C_f \cap \Lambda = \emptyset$.

Any holomorphic map on $\mathbb{P}^2$ is of the form $[z : w : t] \to [P_1(z, w, t) : P_2(z, w, t) : P_3(z, w, t)]$, with P_i homogeneous polynomials in (z, w, t) of the same degree. This common degree is called the *algebraic degree* of f. We will assume that this algebraic degree is larger or equal than 2.

Proposition 9. *If f is an Axiom A holomorphic map on $\mathbb{P}^2$ of algebraic degree $d \geq 2$, and Λ is a basic set of saddle type, then $\tilde{P}^-_{f|_\Lambda}(\phi) = P^-_{f|_\Lambda}(\phi)$, for any $\phi \in \mathcal{C}(\Lambda, \mathbb{R})$.*

Proof. Since f is surjective on Λ, we can apply Theorem 1 on the compact space $X = \Lambda$ and we are done. □

If f is an Axiom A map with the No Cycle Property, and Λ is a basic set of f, then one can apply the Stability Theorem ([16]) for a close perturbation g of f, in order to obtain that:

i) g is also Axiom A and there exists a basic set Λ_g of g, such that Λ_g is close to Λ and

ii) there exists a homeomorphism $h : \widehat{\Lambda} \to \widehat{\Lambda_g}$ commuting with the liftings $\hat{f}, \hat{g}$, i.e $h \circ \hat{f} = \hat{g} \circ h$, or equivalently $\hat{g}^{-1} \circ h = h \circ \hat{f}^{-1}$.

The homeomorphism h is uniquely defined with the above commuting property. Let us notice also that h depends continuously on g; when we will want to emphasize this dependence on g, we shall write h_g. Hence from the continuity of the homeomorphism h_g with respect to g, it follows that $\pi \circ h_g$ converges towards the canonical projection $\pi : \widehat{\Lambda} \to \Lambda$, when $g \to f$.

Let us say next that an Axiom A map f on $\mathbb{P}^2$ is **strongly hyperbolic** on one of its basic sets of saddle type Λ, if it satisfies the following conditions:

i) $f|_\Lambda : \Lambda \to \Lambda$ is open with respect to the induced topology from $\mathbb{P}^2$;

ii) $\Lambda \cap \mathcal{C}_f = \emptyset$, where $\mathcal{C}_f$ is the critical set of f;

A similar, stronger notion has been introduced in [5]. Notice that, if there exists a neighbourhood U of Λ with $f^{-1}(\Lambda) \cap U = \Lambda$ (like in the definition of s-hyperbolicity from [5]), then automatically the map $f|_\Lambda$ is open.

Examples of strongly hyperbolic maps are numerous: there are for instance the examples of product maps, Ueda maps or perturbations of solenoids from [5] and the examples of maps g obtained as perturbations in [12] and which have basic sets Λ_g so that $g|_{\Lambda_g}$ is a homeomorphism.

Proposition 10. *(a) Consider f an Axiom A, holomorphic map on $\mathbb{P}^2$ which is strongly hyperbolic on a basic set of saddle type Λ and which satisfies the No Cycle Property. Assume also that Λ can be written as a union of finitely (countably) many compact, simply connected subsets $(V_i)_i$. Perturb the map f to a holomorphic map g on $\mathbb{P}^2$, such that the corresponding basic set of g close to Λ is Λ_g. Then Λ_g can also be written as a (possibly uncountable) union of compact, simply connected subsets which are homeomorphic images of the sets $V_i, i > 0$.*

(b) Let f be a holomorphic, Axiom A map on $\mathbb{P}^2$ satisfying the No Cycle Property and let Λ be a basic set of saddle type of f such that $\mathcal{C}_f \cap \Lambda = \emptyset$ and such that $f|_\Lambda$ is a homeomorphism. Then, if the holomorphic map g is a close perturbation of f, it follows that g has a basic set Λ_g close to Λ, so that $g|_{\Lambda_g}$ is also a homeomorphism.

Proof. (a) Let us take V a subset of Λ which is compact and simply connected. Now, since V is simply connected and Λ does not intersect the critical set $\mathcal{C}_f$ of f, we can define a branch of f^{-1} on V, call it f_*^{-1}, which takes its values in Λ; this can be done since $f|_\Lambda$ is open. But in this case, $f_*^{-1}(V)$ will be also simply connected, and contained in the set Λ which does not intersect the critical set $\mathcal{C}_f$, hence we can define again a branch of f^{-1}, this time on $f_*^{-1}(V)$; in this way, we can define on V a sequence of inverse branches $f_*^{-n}, n > 0$. If x is a point in Λ, let us define $\sigma(x) := (x, x^*_{-1}, x^*_{-2}, ..., x^*_{-n}, ...)$, where in general $x^*_{-n} := f_*^{-n}(x)$, for all integers $n > 0$. So σ is a section over V of the canonical projection $\pi : \widehat{\Lambda} \to \Lambda$. Let $\tilde{V}$ denote the set $\sigma(V)$, which is contained in $\hat{\Lambda}$. From construction, $\sigma : V \to \tilde{V}$ is a homeomorphism. Let also $V_g := \pi(h(\tilde{V}))$, where $h : \hat{\Lambda} \to \hat{\Lambda}_g$ is the homeomorphism given by the Stability Theorem and $\pi : \hat{\Lambda} \to \Lambda$ is the canonical projection. We shall prove that $\pi \circ h \circ \sigma : V \to V_g$ is a homeomorphism, which will imply the conclusion of part (a) of the theorem. Let us assume that there exist different points x, y in V such that $\pi(h(\hat{x})) = \pi(h(\hat{y}))$, where $\hat{x} = \sigma(x), \hat{y} = \sigma(y)$. In this case we can write:

$$h(\hat{x}) = (z_0, z_{-1}, ..., z_{-k+1}, z_{-k}, z_{-k-1}, ...),$$

$$h(\hat{y}) = (z_0, z_{-1}, ..., z_{-k+1}, z'_{-k}, z'_{-k-1}, ...)$$

where k is the first positive integer such that the k-preimages of z_0, denoted above by z_{-k} and z'_{-k} are different; so we have $z_{-k} \neq z'_{-k}$ and $z_{-j} = z'_{-j}, 0 \le j < k$. Also, from the definition of h, we have z_{-i}, z'_{-i} in Λ_g for all i.

Notice that from the construction of h, we get that $\hat{g}^{-1} \circ h = h \circ \hat{f}^{-1}$, and by induction, $\hat{g}^{-n} \circ h = h \circ \hat{f}^{-n}$, for all $n \ge 0$. But $\hat{g}^{-k} \circ h(\hat{x}) = (z_{-k}, z_{-k-1}, ...)$ and $\hat{g}^{-k} \circ h(\hat{y}) = (z'_{-k}, z'_{-k-1}, ...)$.

On the other hand,

$$\hat{f}^{-k}(\hat{x}) = (x_{-k}, x_{-k-1}, ...), \text{ and} \hat{f}^{-k}(\hat{y}) = (y_{-k}, y_{-k-1}, ...),$$

where x_{-k}, y_{-k} are the images of x, y by the k-th inverse iterate f_*^{-k} on V.

Since $\Lambda \cap \mathcal{C}_f = \emptyset$, it follows that, if the perturbation g is close enough to f, there exists a positive constant δ_0 depending only on f such that

$$d(\xi, \xi') > \delta_0,$$

for any $\xi, \xi' \in \Lambda_g, \xi \neq \xi'$ with $g(\xi) = g(\xi')$. This is true since Λ_g is close to Λ_f and $\mathcal{C}_g$ is close to $\mathcal{C}_f$. Fix now $\eta > 0$ sufficiently small such that $\eta < \frac{\delta_0}{3}$ and

$$d(\xi, \xi') < 2\eta \text{ implies } d(f_*^{-1}(\xi), f_*^{-1}(\xi')) < \frac{\delta_0}{3}, \forall \xi, \xi' \in V$$

Consider now g so close to f such that $d(\pi \circ h_g(\hat{\xi}), \xi_0) < \eta$, for all $\hat{\xi} \in \widehat{\Lambda}$.

But in our case, $\hat{g}^{-j} \circ h_g(\hat{x}) = (z_{-j}, ...) = h_g \circ \hat{f}^{-j}(\hat{x}), \forall j > 0$. Hence from above, $d(x_{-j}, z_{-j}) < \eta, \forall j > 0$. Similarly, $d(y_{-j}, z'_{-j}) < \eta, \forall j > 0$.

This implies that x_{-k+1} is η-close to z_{-k+1} and y_{-k+1} is η-close to $z'_{-k+1} = z_{-k+1}$.

But $x_{-k} = f_*^{-1}(x_{-k+1}), y_{-k} = f_*^{-1}(y_{-k+1})$ and from above, $d(x_{-k+1}, y_{-k+1}) < 2\eta$. So, from the way η was chosen, we obtain

$$d(x_{-k}, y_{-k}) < \frac{\delta_0}{3}$$

But, from the way g was taken close to f,

$$d(\pi \circ h_g(\hat{f}^{-k}\hat{x}),\ x_{-k}) < \eta, \text{ and } d(\pi \circ h_g(\hat{f}^{-k}\hat{y}), y_{-k}) < \eta$$

So from the preceding two inequalities, we get

$$d(\pi \circ h_g(\hat{f}^{-k}\hat{x}), \pi \circ h_g(\hat{f}^{-k}\hat{y})) < \delta_0$$

But $\pi \circ h_g(\hat{f}^{-k}\hat{x}) = z_{-k}, \pi \circ h_g(\hat{f}^{-k}\hat{y}) = z'_{-k}$, so the last inequality gives $d(z_{-k}, z'_{-k}) < \delta_0$. But, on the other hand z_{-k}, z'_{-k} are different g-preimages in Λ_g of the same point z_{-k+1}, hence from the definition of δ_0, we should have $d(z_{-k}, z'_{-k}) > \delta_0$. We have thus obtained a contradiction.

Therefore the map $\pi \circ h_g \circ \sigma : V \to V_g$ is a homeomorphism, so V_g has all the topological properties of V, in particular it is compact and simply connected.

Now, σ was determined by a fixed prehistory of an arbitrary point x from V. By taking all the possible prehistories of x in Λ, and the corresponding sections σ given by them as above, we will obtain, for each such σ, a homeomorphic image in Λ_g. In conclusion if we take all such sections σ for all the sets V_i which cover Λ, we will obtain a cover of Λ_g.

(b) For the proof of (b), we already have the (unique) sequence of inverse iterates $((f|_\Lambda)^{-k})_{k \geq 0}$ defined on the whole Λ, since $f|_\Lambda$ is a homeomorphism. So we do not need anymore that Λ is written as a union of simply connected subsets. Thus we have just one section σ as in (a) and thus Λ_g is homeomorphic to Λ. $\square$

Example:

(a) As an example of a map f and a basic set Λ satisfying the conditions of Proposition 10 (a), we consider $f(z, w) := (z^2 + c, w^2 + d)$, with $0 \neq |c|$ small and $|d|$ small; let $\Lambda := \{p_0(c)\} \times J_d$, where $p_0(c)$ is the attracting fixed point of $P(z) := z^2 + c$ and J_d is the Julia set of the rational map $Q(w) := w^2 + d$. If $|c|$ and $|d|$ are small enough, then P and Q are hyperbolic on their respective Julia sets ([4]), hence f is Axiom A.

From the one complex variable theory ([4]) we know also that J_d is completely invariant with respect to Q (i.e $Q(J_d) = J_d = Q^{-1}(J_d)$); hence using also the fact that Q is open on $\mathbb{C}$, it follows that f is open on Λ. Moreover, if $|d|$ is small enough, it follows that J_d is a quasicircle ([4]). This implies then that Λ can be written as the union of two subsets which are homeomorphic to open intervals.

It can also be seen easily that f has only two basic sets of saddle type in $\mathbb{C}^2$, i.e Λ and $\Lambda' := J_c \times \{p_0(d)\}$, and one basic set of saddle type Λ_0 contained in the line at infinity ($t = 0$) (where we extended the coordinates (z, w) from $\mathbb{C}^2$ to homogeneous coordinates $[z : w : t]$ in the projective space $\mathbb{P}^2$). It is then immediate to prove that f does not have any cycles among $\Lambda, \Lambda', \Lambda_0$. In conclusion f and Λ satisfy all the conditions of the statement and we can apply Proposition 10.

(b) The example at the very end of this paper shows that there exist maps f_ε and basic sets Λ_ε satisfying the conditions of Proposition 10 (b), i.e such that $f_\varepsilon|_{\Lambda_\varepsilon}$ is a homeomorphism. Hence we can apply the result of part (b) for close perturbations g of f_ε, with $\varepsilon = \varepsilon_0$ being fixed. $\square$

As was already said, in this section we assume $C_f \cap \Lambda = \emptyset$. Therefore $|Df|E^s_x| \neq 0, \forall x \in \Lambda$. Hence, it makes sense to consider the function ϕ^s, from $\mathcal{C}(\Lambda, \mathbb{R})$, defined by $\phi^s(x) := \log |Df|_{E^s_x}|, x \in \Lambda$. We now use the fact that, since ϕ^s is strictly negative on the space $X = \Lambda$, the mapping $t \to P^-(t\phi^s)$ is strictly decreasing (Proposition 3 and 4 (f)). We know from the definition of P^- that $P^-(0) = h^-(f|_\Lambda) \geq 0$. We assume also that $h^-(f|_\Lambda) < \infty$. Now, since $\phi^s < 0$, it is not difficult to see that $P^-(t\phi^s) < 0$ for t large enough. Hence this implies that there exists a unique $t^s \geq 0$ such that $P^-(t^s\phi^s) = 0$.

The same argument can be used to find a unique zero of $t \to P_-(t\phi^s)$, in case $h_-(f|_\Lambda) < \infty$.

Definition 5. In the above setting, the unique $t \geq 0$ such that $P^-(t\phi^s) = 0$ will be denoted by t^s and will be called the *zero of* $P^-(t\phi^s)$. The unique zero of $t \to P_-(t\phi^s)$ will be denoted by t^s_- and will be called the *zero of* $P_-(t\phi^s)$.

We are now ready to prove that, under a certain technical condition, t^s is equal to the Hausdorff dimension of the intersection between any local stable manifold and Λ, where Λ is a basic set with both stable and unstable directions (such basic sets are called *of saddle type*), and that in general, without the technical condition, $t^s_- \leq HD(W^s_\delta(x) \cap \Lambda) \leq t^s$. Let us denote by $\delta^s(x) := HD(W^s_\delta(x) \cap \Lambda), x \in \Lambda$.

Theorem 2. *(a) The inequality $t^s \geq \delta^s(x)$ holds under the general assumptions: f holomorphic Axiom A map on $\mathbb{P}^2$ and Λ a basic set of saddle type of f, with $\Lambda \cap \mathcal{C}_f = \emptyset$ and $h^-(f|_\Lambda) < \infty$.*

(b) Let now f be a holomorphic map on $\mathbb{P}^2$ which is strongly hyperbolic on a basic set of saddle type Λ. Assume that $h^-(f|_\Lambda) < \infty$ and that Λ can be written as a union of countably many compact, simply connected subsets. Then $t^s = \delta^s(x)$, for any point $x \in \Lambda$. In particular, $HD(W^s_\delta(x) \cap \Lambda)$ does not depend on $x \in \Lambda$ in this case.

(c) Let f and Λ satisfy the conditions from (a). Assume also that the map f is strongly hyperbolic on Λ. (here Λ is not necessarily the union of countably many compact, simply connected subsets). Then $t^s \geq \delta^s(x) \geq \sup\limits_V t^s(V)$, where the supremum in the last inequality is taken over all compact, simply connected subsets V of Λ, and where $t^s(V)$ denotes the unique zero of the map $t \to P^-(t\phi^s, V)$.

Proof. In this proof we take $X = \Lambda$, $f : \Lambda \to \Lambda$ and x an arbitrary point in Λ. All the inverse pressures considered are relative to f as a surjective mapping on X. Therefore from Theorem 1, we know that $\tilde{P}^-(\cdot) = P^-(\cdot)$. Therefore we will use interchangeably these two definitions of P^-.

Proof of (a):

First let us show that $t^s \geq \delta^s(x)$, where we remind the notation $HD(W^s_\delta(x) \cap \Lambda) =: \delta^s(x)$. So, take an arbitrary $t > t^s$. Then, $P^-(t\phi^s) < \beta$ for some $\beta < 0$. Hence $\lim\limits_{n\to\infty} \frac{1}{n} \log P_n^-(\phi, \varepsilon') < \beta < 0$, for $\varepsilon' > 0$ small enough. So, there exists a collection $\Gamma \subset \mathcal{C}_n(\varepsilon')$ such that $\sum\limits_{C\in\Gamma} \exp(S_n^-\phi(C)) < e^{n\beta}$, for n large enough. But since Γ covers Λ, i.e. since $\Lambda = \bigcup\limits_{C\in\Gamma} \Lambda(C)$, we then have $\Lambda(C) \cap W^s_\delta(x) \subset f^{n-1}(W^s_\delta(z_{-n+1}))$, where $z \in \Lambda$ and C is associated to an n-prehistory of z in Λ, $(z_0, z_{-1}, ..., z_{-n+1})$, (with $z_0 = z$).

Denote $W := W^s_\delta(x) \cap \Lambda$. Now by using the Distortion Lemma from [11], and the fact that f is conformal on its stable sets, we obtain that $\mathrm{diam}(\Lambda(C) \cap W) \leq A\delta |Df^n|_{E^s_{z_{-n+1}}}|$, where A is a fixed constant independent of n. This implies that:

$$\sum_{C\in\Gamma} (\mathrm{diam}\Lambda(C) \cap W)^t \leq A^t \delta^t \sum_{C\in\Gamma} \exp(S_n^-(t\phi^s)(C)) \leq e^{n\beta}\delta^t$$

But in the last inequality, $\beta < 0$ is independent of δ, which implies that for n large, $\sum\limits_{C\in\Gamma} (\mathrm{diam}\Lambda(C))^t < 1$. Since $\mathrm{diam}(\Lambda(C)) \to 0$ when $n \to \infty$, we obtain $\delta^s(x) \leq t$. But since t was taken arbitrarily larger than t^s, we conclude that $t^s \geq \delta^s(x)$.

Proof of (b):

We now proceed to the proof of the other inequality, i.e $\delta^s(x) \geq t^s$. Take an arbitrary $t > \delta^s(x)$. We shall show that $t \geq t^s$, i.e. that $P^-(t\phi^s) \leq 0$.

Fix $\varepsilon' > 0$ and $\eta > 0$. Since Λ was assumed to be a basic set, the map $f|_\Lambda$ is transitive. Using this and the local product structure on the hyperbolic set Λ, it follows as in the proof of Theorem 2 of [12], that there exists a positive integer m such that $f^{-m}(W^s_\delta(x) \cap \Lambda) \cap \Lambda$ is $\varepsilon'/2$-dense in Λ and that any local unstable manifold, $W^u_{\varepsilon'/2}(\hat{z})$ intersects it, for all $\hat{z} \in \hat{\Lambda}$. Since f is a local homeomorphism near Λ (due to $C_f \cap \Lambda = \emptyset$), we have also that

$$HD(f^{-m}(W^s_\delta(x) \cap \Lambda) \cap \Lambda) = HD(W^s_\delta(x) \cap \Lambda)$$

We can use now the fact that Λ is written as the union of countably many compact, simply connected subsets $V_i, i > 0$ integer.

From Proposition 2 we know that $P^-(\phi) = \sup_i P^-(\phi, V_i)$. If we denote the zero of the map $t \to P^-(t\phi^s, V_i)$ by $t^s(V_i)$, it is easy to see that $t^s = \sup_i t^s(V_i)$.

Therefore, if we prove that for all integers $i > 0$, $\delta^s(x) \geq t^s(V_i)$, then the inequality $\delta^s(x) \geq t^s$ will hold as well. So, in the sequel we restrict attention to such a simply connected subset V_i, which will be denoted for brevity by V.

As in the proof of Proposition 10, one can define on V a sequence of inverse iterates $f_*^{-n}, n > 0$. Therefore we have on V a well-defined distribution of unstable local manifolds given by the prehistories obtained from the inverse iterates f_*^{-n}.

For a point y from V denote by $\psi(y)$ the local unstable manifold $W^u_{\frac{\varepsilon'}{2}}(\hat{y})$, where $\hat{y}$ is the prehistory defined by taking $y_{-j} = f_*^{-j}(y), j > 0, y_0 = y$.

Consider now a point y from V and z belonging to $\psi(y)$. Then there exists a prehistory $(z_{-k})_{k\geq 0}$ of z such that $d(z_{-k}, y_{-k}) < \frac{\varepsilon'}{2}, \forall k \geq 0$.

We claim that there is only one prehistory of z with this property. Indeed, if $(z'_{-k})_k$ were another such prehistory, we would have $d(z_{-k}, z'_{-k}) < \varepsilon', \forall k > 0$. Assuming that N is an integer such that $z_{-k} = z'_{-k}, 0 < k < N$ and $z_{-N} \neq z'_{-N}$, it follows that z_{-N} and z'_{-N} are two different preimages of the same point z_{-N+1}, and since $\mathcal{C}_f \cap \Lambda = \emptyset$, we cannot have $d(z_{-N}, z'_{-N}) < \varepsilon'$ if ε' has been chosen small enough. Therefore the prehistory $(z_{-k})_k$ is uniquely defined by being the prehistory of z which follows the prehistory of y corresponding to $\psi(y)$.

Hence we can extend the inverse iterates $f_*^{-k}, k > 0$ also to the unstable manifolds $\psi(y), y \in V$.

Since $t > \delta^s(x)$, there exists $\mathcal{U} = (U_i)_{i\in I}$, a cover of $f^{-m}(W^s_\delta(x) \cap \Lambda) \cap \Lambda$ by open balls with $mesh\ (\mathcal{U}) < \eta/2,\ \eta << \varepsilon'$ and such that

$$\sum_i (\text{diam}(U_i))^t \leq 1.$$

In the sequel we shall denote $|Df^k|_{E^s_y}|$ by $|Df^k_s(y)|, \forall y \in \Lambda$.

For every set U_i let us take $\tilde{U}_i := \underset{y\in V, \psi(y)\cap U_i \neq \emptyset}{\cup} \psi(y), i > 0$. Obviously $\underset{i}{\cup} \tilde{U}_i$ covers the entire set V. From the discussion above, the inverse iterates f_*^{-n} are well defined on the sets $\tilde{U}_i$.

For each i, take n_i the first positive integer such that $\text{diam} f_*^{-n}(\tilde{U}_i) \leq \varepsilon', n < n_i$ and $\text{diam} f_*^{-n_i}(\tilde{U}_i) > \varepsilon'$. If $z \in \psi(y), y \in V$, then $f_*^{-n}(z)$ becomes closer and closer to $f_*^{-n}(y)$ when n increases, in any case closer than $\frac{\varepsilon'}{2}$. Hence $d(f_*^{-n}(\tilde{U}_i), f_*^{-n}(U_i)) < \frac{\varepsilon'}{2}, \forall n > 0$.

Notice however that the set $U_i \cap \tilde{U}_i$ is contained in a small open analytic disk inside $f^{-m}(W^s_\delta(x))$. Also, the stable tangent space makes sense at all points in $f^{-m}(W^s_\delta(x))$, not only at points from Λ.

Let us denote $U_i^* := U_i \cap \tilde{U}_i$. By inflating slightly U_i^* to an open set D_i contained in $f^{-m}(W^s_\delta(x))$, we can still define the inverse iterates $f_*^{-k}, k \leq n_i$ on D_i. This is possible due to the fact that, first, $\text{diam} f_*^{-1}(U_i^*) < \varepsilon'$ and the distance between different preimages of an open analytic disk neighbourhood of U_i inside $f^{-m}(W^s_\delta(x))$, is bounded below by a positive constant α independent of i (since $\mathcal{C}_f \cap \Lambda = \emptyset$ and $\tilde{U}_i$ is ε'-close to Λ). One can arrange for ε' to be less than $\frac{\alpha}{4}$. Hence $f_*^{-1}(U_i^*)$ is contained in just one analytic disk D_i^{-1}, preimage of the analytic disk D_i containing U_i^* (since the distance between different components of $f^{-1}(D_i)$ would be, as we saw, larger than $\alpha > 2\text{diam} f_*^{-1}(U_i^*)$).

Afterwards, we apply the same argument again to preimages of D_i^{-1}. Inductively we obtain that $f^{-n_i}(U_i^*)$ is contained in an analytic disk, denoted by $D_i^{-n_i}$, and $D_i^{-n_i} \subset f^{-m-n_i}(W^s_\delta(x))$. The tangent space to this analytic disk is in the stable direction, induced from $W^s_\delta(x)$, i.e $Df^{m+n_i}(E^s_z) = E^s_{f^{m+n_i}z}, z \in D_i^{-n_i}$. So one can define $Df_s^{n_i}$ also on $D_i^{-n_i}$.

Now, the Mean Value Inequality on $D_i^{-n_i}$ and the definition of n_i imply that:

— there exists a positive constant $\tilde{c}$ and a point $\xi_i \in D_i^{-n_i}$ (ξ_i is actually in the convex cover of $f_*^{-n_i}(U_i^*)$ inside the analytic disk $D_i^{-n_i}$), such that $d(f^k(\xi_i), f_*^{k-n_i}(U_i^*)) < \tilde{c}\varepsilon', 0 \leq k \leq n_i$ and

$$\varepsilon' < \mathrm{diam} f_*^{-n_i}(U_i^*) \leq \mathrm{diam} U_i \cdot |Df_s^{n_i}(\xi_i)|^{-1}$$

But $d(f^k(\xi_i), f_*^{k-n_i}(U_i^*)) < \tilde{c}\varepsilon', k \leq n_i$, and for every point $y_i \in f_*^{-n_i}(V \cap \tilde{U}_i)$ we know that $d(f^k(y_i), f_*^{k-n_i}(U_i^*)) < \frac{\varepsilon'}{2}$. Therefore $d(f^k(\xi_i), f^k(y_i)) < c\varepsilon', 0 \leq k \leq n_i$, for some positive constant $c > 2$. Hence the above inequality becomes:

$$(\varepsilon')^t < (\mathrm{diam} f_*^{-n_i}(U_i^*))^t \leq (\mathrm{diam} U_i)^t \cdot |Df_s^{n_i}(y_i)|)^{-t} \exp(tn_i \cdot \delta_{\phi^s}(c\varepsilon')),$$

where $\delta_{\phi^s}(c\varepsilon')$ is the maximum oscillation of ϕ^s on a ball of radius $c\varepsilon'$. Clearly we can do this entire procedure for all $i \in I$.

We may in addition require $mesh(\mathcal{U})$ to be so small that $n_i \geq N, \forall i \in I$, with N arbitrarily large. We shall now denote by C_i, $i \in I$, the ordered collection of balls

$\{B(f^{n_i}(y_i), c\varepsilon'), B(f^{n_i-1}(y_i), c\varepsilon'), ..., B(f(y_i), c\varepsilon')\}$, with $y_i, i \in I$, found above. We can then estimate as follows:

$$1 \geq \sum_{i \in I} (\mathrm{diam} U_i)^t \geq (\varepsilon')^t \sum_{i \in I} |Df_s^{n_i}(y_i)|^t \exp(-n_i t \delta_{\phi^s}(c\varepsilon')) \geq (\varepsilon')^t \sum_{i \in I} \exp(S_{n_i}^-(t\phi^s)(C_i) - n_i \delta_{\phi^s}(c\varepsilon')t) \tag{5}$$

We shall show now that $V \subset \bigcup_i \Lambda(C_i)$. This follows immediately from the fact that $V = \bigcup_i \tilde{U}_i$ and since $\tilde{U}_i \subset \Lambda(C_i)$. (since $\tilde{U}_i$ is just a union of local unstable manifolds of the form $\psi(y)$).

Using the definition of $M(\cdot,\cdot,\cdot,\cdot)$, and inequality (5), we therefore get that for any $\lambda > 2\delta_{\phi^s}(c\varepsilon')$

$$\begin{aligned} M(\lambda, t\phi^s, N, \varepsilon') &\leq \sum_{i \in I} \exp(-\lambda n(C_i) + S_{n_i}^-(t\phi^s)(C_i)) \leq \\ &\leq e^{-\delta_{\phi^s}(c\varepsilon')N} \sum_{i \in I} \exp(S_{n_i}^-(t\phi^s)(C_i) - n_i \delta_{\phi^s}(c\varepsilon')) \leq (\varepsilon')^{-t} e^{-\delta_{\phi^s}(c\varepsilon')N} \end{aligned}$$

Hence, taking the limit if $N \to \infty$, we obtain $m(\lambda, t\phi^s, \varepsilon') = 0$, hence $\tilde{P}^-(t\phi^s, V, \varepsilon') \leq \lambda$. But λ was taken arbitrarily larger than $2\delta_{\phi^s}(c\varepsilon')$, so we obtain $\tilde{P}^-(t\phi^s, V, \varepsilon') \leq 2\delta_{\phi^s}(c\varepsilon')$. Then, by taking ε' approaching zero, it follows that $\tilde{P}^-(t\phi^s, V) \leq 0$ and therefore $t \geq t^s(V)$. Consequently $\delta^s(x) \geq \sup_i t^s(V_i)$, hence $\delta^s(x) \geq t^s$. In conclusion the proof of (a) is finished.

Proof of (c):

In this case the proof follows in the same way as for (b). The inequality $t^s \geq \delta^s(x)$ does not use the fact that Λ can be written as a countable union of simply connected sets V_i, hence it still holds in case (c). In the end of the proof of (b), since the simply connected set V is arbitrary, we get that $\delta^s(x) \geq \sup_V t^s(V)$, which represents the lower estimate from the statement. □

Another lower estimate for the stable dimension can be given using the inverse lower pressure P_- and the unique zero t_-^s of the map $t \to P_-(t\phi^s)$:

Theorem 3. *Let f be a holomorphic Axiom A map of $\mathbb{P}^2$, and Λ a basic set of saddle type for f such that $\mathcal{C}_f \cap \Lambda = \emptyset$ and $h_-(f|_\Lambda) < \infty$. Then $HD(W_\delta^s(x) \cap \Lambda) \geq t_-^s$, for any point x in Λ.*

Proof. The proof here will follow similarly as in the proof of Theorem 2. We denote $\delta^s(x) := HD(W^s_\delta(x) \cap \Lambda)$ and then take $t > \delta^s(x)$ arbitrary. Therefore there will exist $\mathcal{U} = (U_i)_{i \in I}$, a cover of $f^{-m}(W^s_\delta(x) \cap \Lambda) \cap \Lambda$ with $mesh(\mathcal{U}) < \eta << \varepsilon'$, such that $\sum_i (\text{diam}(U_i))^t \le 1$; m and ε' have been taken as in the proof of part (b) of Theorem 2.

Now, for each U_i let us take n_i the largest integer such that all components of $f^{-k}(U_i)$ which intersect Λ have diameter less than ε' small, and $f^{-n_i}(U_i)$ has a component $U_i^{n_i}$ of diameter larger or equal than ε' and such that $U_i^{n_i} \cap \Lambda \ne \emptyset$. But then, from the Mean Value Inequality and the Laminated Distortion Lemma of [11], it follows that there exists a positive constant A and a point ξ_i in $U_i^{n_i} \cap \Lambda$ such that

$$\text{diam}(U_i) \ge A\varepsilon' |Df_s^{n_i}(\xi_i)| \ge A\varepsilon' \exp(S_{n_i,-}\phi^s(f^{n_i}(\xi_i))), i \in I.$$

The point $f^{n_i}(\xi_i)$ belongs to $U_i \cap \Lambda$, for all $i \in I$. Since all unstable manifolds of points in Λ, of size $\frac{\varepsilon'}{2}$ intersect the set $f^{-m}(W^s_\delta(x) \cap \Lambda) \cap \Lambda$, it follows that Λ is the union of the inverse balls $B^-_{n_i}(f^{n_i}(\xi_i), 2\varepsilon'), i \in I$. For each $i \in I$ we have only one inverse ball $B^-_{n_i}(f^{n_i}(\xi_i), 2\varepsilon')$ and $(\text{diam}(U_i))^t \ge A^t(\varepsilon')^t \exp(S_{n_i,-}(t\phi^s)(f^{n_i}(\xi_i)))$, so

$$1 \ge \sum_i (\text{diam}(U_i))^t \ge C \sum_i \exp(S_{n_i,-}(t\phi^s)(f^{n_i}(\xi_i)))$$

But then, using also the fact that the sets $B^-_{n_i}(f^{n_i}(\xi_i), 2\varepsilon'), i \in I$, cover Λ, it follows that $P_-(t\phi^s) \le 0$, hence $t \ge t^s_-$. Since t was chosen arbitrarily bigger than $\delta^s(x)$, we get that $\delta^s(x) \ge t^s_-$, for all x in Λ. $\square$

Corollary 2. *In the above setting, let f be an Axiom A holomorphic map on $\mathbb{P}^2$ and Λ a basic set of saddle type, on which f is strongly hyperbolic. Then, if there exists a point $x \in \Lambda$ such that $HD(W^s_\delta(x) \cap \Lambda) \ne 0$, it follows that Λ cannot be a finite graph (hence in particular Λ cannot be a Jordan curve).*

Proof. Let us assume on the contrary, that there exists a basic set Λ such that $HD(W) \ne 0$ and Λ is a finite graph, where again we denoted $W := W^s_\delta(x) \cap \Lambda$. But we proved in Proposition 5 that $0 \le h^- \le h_i$. Now we use the Theorem of Nitecki-Przytycki from Section 1, saying that, if f is a continuous map of a finite graph Λ, then $h_i(f|_\Lambda) = 0$. (as notations, $h^- = h^-(f), h_i(f) = h_i$ in the current situation). Therefore, $h^- = P^-(0) = 0$, implying that $t^s = 0$; contradiction with the fact that $t^s \ge HD(W^s_\delta(x) \cap \Lambda) > 0$ (see Theorem 2). $\square$

For some examples of maps f and basic sets Λ, it may happen to be easy to calculate (or estimate) the stable dimension at some point $x \in \Lambda$. The previous corollary says that, in case there exists one such point where the stable dimension is non-zero, then Λ has a complicated shape, since it is not Jordan. Equivalently, if Λ happens to be a Jordan curve, then the stable dimension at all points of Λ must be zero. So, if Λ is a Jordan curve, then the basic set Λ is "mostly oriented in the unstable direction".

This situation is illustrated by the case of the map $f(z, w) = (z^2 + c, w^2), |c|$ small, and $\Lambda = \{p_0(c)\} \times S^1$, where $p_0(c)$ is a fixed attracting point for $z \to z^2 + c$. In this example, for any $x \in \Lambda$

and $\delta > 0$ small, we have $W^s_\delta(x) \cap \Lambda = \{x\}$, so the stable dimension at every x is zero; notice also that Λ is a Jordan curve and $\Lambda \subset E^u_{\hat{x}}$. This suggests that, in general, if Λ is a Jordan curve, then f must have an "almost-expanding" character on Λ.

Corollary 3. *In the above setting, if $h(f|_\Lambda) \neq 0$ and $f|_\Lambda$ is a homeomorphism, then $HD(W^s_\delta(x) \cap \Lambda) \neq 0, \forall x \in \Lambda$.*

Proof. Indeed, if $f|_\Lambda$ is a homeomorphism, then, from the properties of lower inverse pressure (Proposition 7 g)), we have that $h(f|_\Lambda) = h^-(f|_\Lambda) = h_-(f|_\Lambda) \neq 0$. This implies that $P_-(0) \neq 0$, hence $t^s_- \neq 0$, so Theorem 3 now shows that $HD(W^s_\delta(x) \cap \Lambda) \neq 0$.

□

Examples:

In [13] we gave a large class of examples of perturbations of the map $(z, w) \to (z^2 + c, w^2)$, on $\mathbb{P}^2$, with basic sets on which they are homeomorphisms. These are maps of the form $f_\varepsilon(z, w) = (z^2 + a\varepsilon z + b\varepsilon w + c + d\varepsilon zw + e\varepsilon w^2, w^2)$, where $|c| \neq 0$, small, and $|c| < c(a, b, d, e), b \neq 0, \varepsilon < \varepsilon(a, b, c, d, e)$. Λ_ε denotes the basic set of f_ε close to the circle $\{p_0(c)\} \times S^1$, where $p_0(c)$ is a fixed attracting point of the map $z \to z^2 + c$. It is shown in [13] that $f_\varepsilon|_{\Lambda_\varepsilon}$ is a homeomorphism.

Being perturbations of $f(z, w) = (z^2 + c, w^2)$ and since their considered basic set is close to $p_0(c) \times S^1$, it will follow from the Stability Theorem ([16]) that there exists a homeomorphism $h : \hat{\Lambda} \to \hat{\Lambda}_\varepsilon$ commuting with $\hat{f}$ and $\hat{f}_\varepsilon$. Hence $\log 2 = h(f|_\Lambda) = h(\hat{f}|_{\hat{\Lambda}}) = h(\hat{f}_\varepsilon|_{\hat{\Lambda}_\varepsilon}) = h(f_\varepsilon|_{\Lambda_\varepsilon}) = h_-(f_\varepsilon|_{\Lambda_\varepsilon})$. Therefore, for this type of maps, Corollary 3 implies that the stable dimension is non-zero.

Also, let us notice that, according to Proposition 10 b), if g is itself a small perturbation of f_ε, (for an $\varepsilon > 0$ fixed, small) then $g|_{\Lambda_g} : \Lambda_g \to \Lambda_g$ is a homeomorphism too, so we can apply again Corollary 3 to get that the stable dimension on Λ_g is non-zero and that Λ_g is not a finite graph.

References

[1] R. Bowen, Topological entropy for non-compact sets, Trans. Amer. Math. Soc. **184** (1973), 125–136.

[2] R. Bowen, Hausdorff dimension of quasi-circles, Publ. Math. IHES, 50 (1980), 11-25.

[3] M. Brin and A. Katok, On local entropy, in Geometric Dynamics, 30-38, Lecture Notes in Mathematics 1007, Springer Verlag, New York, 1983.

[4] L. Carleson and T. Gamelin, Complex Dynamics, Springer-Verlag, Berlin-New York, 1993.

[5] J.E. Fornaess and N. Sibony, Hyperbolic maps on $\mathbb{P}^2$, Math. Ann. **311** (1998), 305–333.

[6] M. Hurley, On topological entropy of maps, Ergodic Theory and Dynamical Systems **15** (1995), 557–568.

[7] A. Katok and B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Cambridge Univ. Press, London-New York, 1995.

[8] R. Langevin and F. Przytycki, Entropie de l'image inverse d' une application, Bull Soc. Math. France **120** (1992), 237–250.

[9] A. Manning, A relation between Lyapunov exponents, Hausdorff dimension and entropy, Ergod. Th. and Dyn. Syst. **1** (1981), 451–459.

[10] A. Manning, H. McCluskey, Hausdorff dimension for horseshoes, Ergod. Th. and Dynam. Sys. 3 (1983), 251-260.

[11] E. Mihailescu, Applications of thermodynamic formalism in complex dynamics on $\mathbb{P}^2$, Discrete and Cont. Dyn. Syst. vol 7, no 4, (2001), 821-836.

[12] E. Mihailescu and M. Urbanski, Estimates for the stable dimension for holomorphic maps, accepted for publication Houston J. Math, 2003.

[13] E. Mihailescu and M. Urbanski, Maps for which the unstable manifolds depend on the prehistories, Discrete and Cont. Dyn. Syst., vol.9, no. 2, March 2003.

[14] Z. Nitecki and F. Przytycki, Preimage entropy for mappings, Inter. J. Bifur. and Chaos, **9**, (1999), no. 9, 1815–1843.

[15] Y. Pesin and B. Pitskel, Topological pressure and the variational principle for non-compact sets, Functional Analysis and Appl., 18, 1984, 307-318.

[16] F. Przytycki, Ω-stability and structural stability of endomorphisms, Studia Mathematica **60** (1977), 61-77.

[17] D. Ruelle, Repellers for real analytic maps, Ergod. Theory and Dyn. Syst. 2 (1982), 99-107.

[18] D. Ruelle, Elements of differentiable dynamics and bifurcation theory, Academic Press, New York, 1989.

[19] A. Verjovsky and H. Wu, Hausdorff dimension of Julia sets of complex Hénon mappings, Ergod. Th. & Dyn. Syst. **16** (1996), 849–861.

[20] P. Walters, An introduction to ergodic theory, Springer-Verlag, New York,-Berlin, 1982.

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