# Essential and Approximate operator amenability

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# History of essential and approximate amenability of Banach algebras and operator space.

The notion of essential and approximate amenability of Banach algebras was introduced by F. Ghahramani and R. J. Loy [3].
Also the notion of operator amenability is defined by J. Ruan [9]. He introduced a variant of Johnson's definition of amenability for completely contractive Banach algebras called operator amenability.

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**Motivition:** Can we define essential and approximate amenability of operator Banach algebra ?

# Preliminary Matters

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- Let $\mathcal{A}$ be a Banach algebra and $E$ be an $\mathcal{A}$-bimodule Banach algebra. A bounded linear map $d : \mathcal{A} \longrightarrow E$ is called a derivation if for every $a, b \in \mathcal{A}$: $d(ab) = d(a)b + ad(b)$.
  Let $\mathcal{A}$ be a Banach algebra and $E$ be an $\mathcal{A}$-module Banach algebra, $E$ is called neo-unital if
  $E = \mathcal{A}.E.\mathcal{A} = \{a.x.b : a, b \in \mathcal{A}, x \in E\}$.

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**Definition (Ghahramani and Loy [3])**

A Banach algebra $\mathcal{A}$ is called essentially amenable if for any neo-unital $\mathcal{A}$-module $E$, every continuous derivation $d : \mathcal{A} \longrightarrow E$ is inner.

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- A derivation $d : \mathcal{A} \longrightarrow E$ is called approximately inner if there is a net $(e_\alpha) \subset E$ such that $d(a) = lim_\alpha(a.e_\alpha - e_\alpha.a)$, for every $a \in \mathcal{A}$.

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**Definition (Ghahramani and Loy [3])**

$\mathcal{A}$ is approximately amenable if for any Banach $\mathcal{A}$-bimodule $E$, every continuous derivation $d : \mathcal{A} \longrightarrow E^*$ is approximately inner.

# Preliminary Matters

## Definition ( Ruan )

An operator space is a linear space $\mathcal{A}$, equipped with a family $\| \ \|_n$ of Banach space norm on $M_n(\mathcal{A})$, such that the following properties hold

$$(i)- \left\| \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right\|_{m+n} = \max\{\| A \|_m, \| B \|_n\}$$

for all $A \in M_m(\mathcal{A}), B \in M_n(\mathcal{A})$, and

$$(ii) \ \| [a_{i,j}]A[b_{i,j}] \|_n \leq \| [a_{i,j}] \| \| A \|_n \| [b_{i,j}] \|$$

where $[a_{i,j}]_n$ and $[b_{i,j}]_n$ are scalar matrices.

# Preliminary Matters

Let $\mathcal{A}$ and $\mathcal{B}$ be two operator spaces, and $U : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear map. For any $n \geq 1$ let $M_n(\mathcal{A}) = \{(a_{i,j}) : i,j \leq n \,|a_{i,j} \in \mathcal{A}\}$, be the space of $n \times n$ matrices with entries in $\mathcal{A}$. For any $n \geq 1$ consider the map $U_n : M_n(\mathcal{A}) \longrightarrow M_n(\mathcal{B})$ with $U_n([a_{i,j}]) = [U(a_{i,j})]$, where $[a_{i,j}] \in M_n(\mathcal{A})$.
The map $U$ is completely bounded if $\|U\|_{cb} = \sup_{n\geq 1} \|U_n\| < \infty$. Also, $U$ is completely contractive if $\|U\|_{cb} \leq 1$. If $n = 1$ then $M_1(\mathcal{A}) = \mathcal{A}$ and $U_1 = U$. We note that in general $\|U\| \leq \|U\|_{cb}$.

# Preliminary Matters-Amenability of operator Banach algebra.

**Definition ( Ruan )**

Let $\mathcal{A}$ be an operator space, if the product map $(a, b) \to ab$ from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$ determines a completely contractive linear mapping, from $\mathcal{A}\hat{\otimes}\mathcal{A}$ to $\mathcal{A}$, then $\mathcal{A}$ is called an operator Banach algebra.

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Let $\mathcal{A}$ be an operator Banach algebra, the operator space $E$ is called a left (resp. right) operator $\mathcal{A}$-module if the bilinear map $(a, x) \longrightarrow ax$ from $\mathcal{A} \times E \longrightarrow E$ ($(x, a) \longrightarrow xa$ from $E \times \mathcal{A} \longrightarrow E$) is completely contractive with respect to the operator projective tensor product, that is to say, the module map $\pi : A \widehat{\otimes} E \longrightarrow E$ is completely contractive. If $\mathcal{A}$ is a left and right operator $\mathcal{A}$-module then $\mathcal{A}$ is called an operator $\mathcal{A}$-bimodule.

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## Definition ( Ruan )

The operator Banach algebra $\mathcal{A}$ is called amenable if for every operator $\mathcal{A}$-bimodule $E$, every completely bounded derivation $D : \mathcal{A} \to E^*$ is inner.

# Essential amenability of operator Banach algebra.

## Definition

let $\mathcal{A}$ be an operator Banach algebra and $E$ be an operator $\mathcal{A}$-bimodule, $E$ is called neo-unital operator $\mathcal{A}$-bimodule if $E = \mathcal{A}E\mathcal{A} = \{a.x.b : a, b \in \mathcal{A}, x \in E\}$.

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The operator Banach algebra $\mathcal{A}$ is called essentially operator amenable if for every neo-unital operator $\mathcal{A}$-bimodule $E$, every completely bounded derivation $D : \mathcal{A} \to E^*$ is inner.

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The operator Banach algebra $\mathcal{A}$ is called essentially operator amenable if for every neo-unital operator $\mathcal{A}$-bimodule $E$, every completely bounded derivation $D : \mathcal{A} \to E^*$ is inner.

- Remark: *Clearly every neo-unital operator $\mathcal{A}$-bimodule is operator $\mathcal{A}$-bimodule. Therefore every operator amenable Banach algebra is essential operator amenable.*

# Essential operator amenability-Results

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- **Lemma**: Let $\mathcal{A}$ be an operator Banach algebra with bounded left approximate identity and $E$ be an operator $\mathcal{A}$-bimodule such that $E\mathcal{A} = 0$. Then every completely bounded derivation $D : \mathcal{A} \to E^*$ is inner.

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- **Lemma**: Let $\mathcal{A}$ be an operator Banach algebra with bounded left approximate identity and $E$ be an operator $\mathcal{A}$-bimodule such that $E\mathcal{A} = 0$. Then every completely bounded derivation $D : \mathcal{A} \to E^*$ is inner.

- **Proposition:** Let $\mathcal{A}$ be an operator Banach algebra with a bounded approximate identity, then the following assertions are equivalent:
  *(i)*- $\mathcal{A}$ is operator amenable Banach algebra.
  *(ii)*- $\mathcal{A}$ is essentially operator amenable Banach algebra.

# An example

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*Let $X$ be a non zero operator Banach space and $\mathcal{A} = X^*$ with action $ab = 0$ for all $a, b \in \mathcal{A}$. If $E$ is a neo-unital operator $\mathcal{A}$-bimodule then*

$$E = \mathcal{A}.E.\mathcal{A} = \mathcal{A}.(\mathcal{A}.E.\mathcal{A}).\mathcal{A} = (\mathcal{A}.\mathcal{A}).E.(\mathcal{A}.\mathcal{A}) = 0.$$

*Therefore $\mathcal{A}$ is essentially operator amenable. It is obvious that $\mathcal{A}$ is not operator amenable, indeed $\mathcal{A}$ has not a approximate identity.*

# Essential operator amenability-Results

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- **Remark:** *Let $\mathcal{A}$ and $\mathcal{B}$ be operator Banach algebras and $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a completely bounded epimorphism. If E is a neo-unital operator $\mathcal{B}$-bimodule then E is a neo-unital operator $\mathcal{A}$-bimodule under the operator $\mathcal{A}$-module action $a.e = \phi(a)e$ and $e.a = e.\phi(a)$. where $e \in E$ and $a \in \mathcal{A}$*

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- **Proposition:** Let $\mathcal{A}$ and $\mathcal{B}$ be operator Banach algebras and $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a completely bounded epimorphism, then if $\mathcal{A}$ is essentially operator amenable then $B$ is too.

# Approximate operator amenability.

**Definition**

*Suppose $\mathcal{A}$ is an operator Banach algebra and E be an operator $\mathcal{A}$-bimodule. A completely bounded derivation $D : \mathcal{A} \longrightarrow E$ is called approximately inner if there is a net $\{e_\alpha\}$ in E such that $D(a) = \lim_\alpha (a.e_\alpha - e_\alpha.a)$, for all $a \in \mathcal{A}$.*

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**Definition**

*An operator Banach algebra $\mathcal{A}$ is called approximately operator amenable if for every operator $\mathcal{A}$-bimodule $E$, every completely bounded derivation $D : \mathcal{A} \longrightarrow E^*$ is approximately inner.*

Recall that a point derivation *d* at a character $\phi$ of an algebra $\mathcal{A}$ is a linear functional satisfying
$d(xy) = d(x)\varphi(y) + \phi(x)d(y), \ \ (x, y \in \mathcal{A})$ That is, *d* is a derivation into the bimodule $\mathcal{C}$; where the latter has operations
$x.\lambda = \lambda.x = \lambda\phi(x), \ \ x \in \mathcal{A} and \lambda \in \mathbb{C}$
Proposition: Suppose that $\mathcal{A}$ admits a non-zero continuous point derivation. Then $\mathcal{A}$ is not approximately weakly amenable.([3]proposition 2.1)

# Approximate operator amenability-Results

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- **Remark:** *For an operator Banach algebra $\mathcal{A}$, Since $\mathcal{A}^*$ is an operator $\mathcal{A}$-bimodule then, the completely bounded right(resp. left) $\mathcal{A}$-module operation on $\mathcal{A}^*$ induces a completely bounded left(resp. right) $\mathcal{A}$-module operation on $\mathcal{A}^{**}$.*

# Approximate operator amenability-Results

- **Proposition:** Every approximately operator amenable Banach algebra has left (right) approximate identity.

# Approximate operator amenability-Results

In following Proposition we discuss on hereditary properties of approximate operator amenability.

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**Proposition**: Let $\mathcal{A}$ and $\mathcal{B}$ be two operator Banach algebra and $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ is a completely bounded epimorphism. If $\mathcal{A}$ is approximately operator amenable then, $\mathcal{B}$ is approximately operator amenable.

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**Proposition**: Let $\mathcal{A}$ and $\mathcal{B}$ be two operator Banach algebra and $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ is a completely bounded epimorphism. If $\mathcal{A}$ is approximately operator amenable then, $\mathcal{B}$ is approximately operator amenable.

**Corollary:** Let the operator Banach algebra $\mathcal{A}$ be a approximately operator amenable and *I* be a two-sided ideal of $\mathcal{A}$, then $\mathcal{A}/I$ is approximately operator amenable.

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2- Let $\mathcal{A}$ and $\mathcal{B}$ be two essentially (approximately) operator amenable Banach algebras. When is $\mathcal{A}\hat{\otimes}\mathcal{B}$ an essential (approximate) amenable operator Banach algebra? Or in the particular case when $\mathcal{B} = \mathcal{A}^{op}$.

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1-Does an essentially (approximately) operator amenable Banach algebra have an approximate identity?

2- Let $\mathcal{A}$ and $\mathcal{B}$ be two essentially (approximately) operator amenable Banach algebras. When is $\mathcal{A}\hat{\otimes}\mathcal{B}$ an essential (approximate) amenable operator Banach algebra? Or in the particular case when $\mathcal{B} = \mathcal{A}^{op}$.

3- Let $\mathcal{A}$ be a essential operator amenable Banach algebra, $E$ be an operator Banach $\mathcal{A}-$bimodule, and $D : \mathcal{A} \longrightarrow E^*$ be a derivation. Is there some $\varphi \in E^*$ such that $\langle axb, D(c)\rangle = \langle axb, c\varphi - \varphi c\rangle$.

- E. G. Effros and Z.-J. Ruan, *Operator Spaces*. Oxford University Press, 2000.
- H. G. Dales, *Banach algebras and automatic continuity*, Clarendon Press,Oxford, (2000).
- F. Ghahramani and R. J. Loy, *Generalized notions of amenability*, Journal of Functional Analysis. **208** (2004), 229 - 260.
- E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis I*, Springer Verlag, Berlin, 1963.
- B. E. Johnson, R. V. Kadison, and J. Ringrose, *Cohomology of operator algebras*, III. Bull. Soc. Math. France. **100** (1972),73-79.

M. Lashkarizade and H. Samea, *Amenability and essential amenability of certain Banach algebras*. Studia Scientiarum Mathematicarum Hungarica, To appear.

G. Pisier, *introduction to operator space theory*. Cambridge University Press, Cambridge, 2003.

Z. -J. Ruan and G. Xu, *Splitting properties of operator bimodules and operator amenability of Kac algebra, in A. Ghenondea, R. N. Gologan and D. Timotin, Operator Theory, Operator Algebras and Related Topics*, PP. 193-216. The Theta Foundation, (1997).

Z. J. Ruan, *The operator amenability of* $A(G)$, Amer. J. Math. **117** (1995), 1449-1474.

V. Runde, *Lecture on amenability*. Springer-Verlag, Berlin, 2002.

***Thank you very much for your attention***