Injective envelopes and local multiplier algebras of some spatial continuous trace C*-algebras

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Local Multipliers

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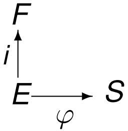

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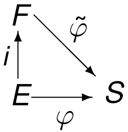

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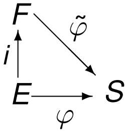

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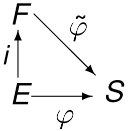

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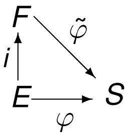

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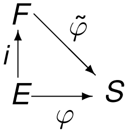

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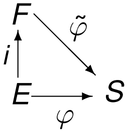

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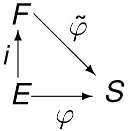

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- Type I AW*-algebras (Hamana, 1981)

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Choi & Effros (1977): Each injective operator system I is completely order isomorphic to a C^*-algebra.

Therefore, the injective envelope $I(E)$ of E is unique, as a C^*-algebra, up to isomorphism. Henceforth, we consider $I(E)$ as a C^*-algebra.

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Examples

- $K(H)$ is an essential ideal of $B(H)$.
- If Y is locally compact and Hausdorff, then K is an essential ideal of $C_0(Y)$ if and only if
$$K = C_0(X),$$
for some open, dense subset $X \subseteq Y$.
- If A is a type I AW*-algebra, then the ideal K generated by the abelian projections of A is an essential ideal of A.

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Plus more applications (cfr. Ara & Mathieu)

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For $A = C[0,1] \otimes K(H)$, $M_{\text{loc}}(A) \neq I(A)$ (more on this soon).

Fact of Life: $M_{\text{loc}}(A)$ is difficult to determine explicitly.
$M(A)$ "lives" naturally in A'' (as the idealizer of A in A'')
Where does $M_{\text{loc}}(A)$ live ?

3. Iterates of $M_{\text{loc}}(\cdot)$

Theorem

(Frank & Paulsen, 2002)

$$A \subseteq M_{\text{loc}}(A) \subseteq M_{\text{loc}}(M_{\text{loc}}(A)) \subseteq I(A),$$

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If $M_{\mathrm{loc}}(A)$ is an AW*-algebra, then

$$(\dagger) \qquad M_{\mathrm{loc}}(A) = M_{\mathrm{loc}}[M_{\mathrm{loc}}(A)] .$$

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(Ara & Mathieu, 2006) There is a separable, AFD, prime, antiliminal C^*-algebra A such that

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Such an A is a particular example of a **Fell Algebra**.

4. Bundles

Definition

A continuous Hilbert bundle is a triple $(T, \{H_t\}_{t\in T}, \Omega)$, *where* Ω *is a set of vector fields on* T *with fibres* H_t *such that:*

(I) Ω *is a* $C(T)$*-module with the action* $(f \cdot \omega)(t) = f(t)\omega(t)$;

(II) *for each* $t_0 \in T$, $\{\omega(t_0) : \ \omega \in \Omega\} = H_{t_0}$;

(III) *the map* $t \mapsto \|\omega(t)\|$ *is continuous, for all* $\omega \in \Omega$;

(IV) Ω *is closed under local uniform approximation.*

Bundles (continued)

Definition

A vector field $\nu : T \to \bigsqcup_t H_t$ *is said to be weakly continuous with respect to* $(T, \{H_t\}_{t\in T}, \Omega)$ *if the function*

$$t \longmapsto \langle \nu(t), \omega(t) \rangle$$

is continuous for all $\omega \in \Omega$*. The set of all bounded weakly continuous vector fields with respect to a given* Ω *will be denoted by* Ω_{wk}*, that is*

$$\Omega_{\mathrm{wk}} = \{\nu : T \to \bigsqcup_t H_t : \sup_t \|\nu(t)\| < \infty \text{ and } \nu \text{ is weakly continuous}\}.$$

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But it is lower-semicontinuous, and with Δ stonean, it differs from a continuous function off a meagre set.

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But it is lower-semicontinuous, and with Δ stonean, it differs from a continuous function off a meagre set.
Thus, one can canonically define $\langle \nu, \xi \rangle \in C(\Delta)$ for $\nu, \xi \in \Omega_{\mathrm{wk}}$.

Going Stonean

Assumption: T Stonean, further denoted by Δ.
Why?

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Theorem (A, Farenick, Massey 2009)

Ω_{wk} is a Kaplansky–Hilbert module over $C(\Delta)$. Moreover, Ω is a C^*-submodule of Ω_{wk} and $\Omega^\perp = 0$.

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(i) if $e_i \cdot \nu = 0$ for some family $\{e_i\}_i \subset C(\Delta)$ of pairwise-orthogonal projections and $\nu \in E$, then also $e \cdot \nu = 0$, where $e = \sup_i e_i$;

(ii) if $\{e_i\}_i \subset C(\Delta)$ is a family of pairwise-orthogonal projections such that $1 = \sup_i e_i$, and if $\{\nu_i\}_i \subset E$ is a bounded family, then there is a $\nu \in E$ such that $e_i \cdot \nu = e_i \cdot \nu_i$ for all i;

(iii) if $\nu \in E$, then $g \cdot \nu = 0$ for all $g \in C(\Delta)$ only if $\nu = 0$.

Definition

An operator field $a : \Delta \to \bigsqcup_{s\in\Delta} K(H_s)$ *is:*

1. *almost finite-dimensional if for each* $s_0 \in \Delta$ *and* $\varepsilon > 0$ *there exist an open set* $U \subset \Delta \ni s_0$ *and* $\omega_1, \ldots, \omega_n \in \Omega$ *such that*
   (a) $\omega_1(s), \ldots, \omega_n(s)$ *are linearly independent for every* $s \in U$*, and*
   (b) $\|p_s a(s) p_s - a(s)\| < \varepsilon$ *for all* $s \in U$*, where* $p_s = [\mathrm{Span}\,\{\omega_j(s) : 1 \le j \le n\}]$*;*
2. *weakly continuous if* $s \longmapsto \langle a(s)\omega_1(s), \omega_2(s)\rangle$ *is continuous for every* $\omega_1, \omega_2 \in \Omega$*.*

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Let Γ be the set of all weakly continuous, almost finite-dimensional operator fields $a : \Delta \to \coprod_{s\in\Delta} K(H_s)$ for which $s \mapsto \|a(s)\|$ is C_0,

Theorem (Fell 1961)

$(\Delta, \{K(H_s)\}_{s\in\Delta}, \Gamma)$ *is a continuous* C^**-bundle and the* C^**-algebra* A *of this bundle is a continuous trace* C^**-algebra with spectrum* $\hat{A} \simeq \Delta$.

5. Some results

Define

$$\Theta_{\nu_1,\nu_2}(\nu) \;=\; \langle \nu, \nu_1 \rangle \cdot \nu_2 \,, \quad \nu \in \Omega_{\mathrm{wk}} \,.$$

("rank-one operators").

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$$\begin{aligned} B(\Omega_{\mathrm{wk}}) &= \{\text{adjointable } C(\Delta) - \text{endomorphisms of } \Omega_{\mathrm{wk}}\} , \\ K(\Omega_{\mathrm{wk}}) &= \overline{\mathrm{Span}_{\mathbb{C}} \{\Theta_{\nu_1,\nu_2} \,:\, \nu_1, \nu_2 \in \Omega_{\mathrm{wk}}\}}^{\|\,\|} \subseteq B(\Omega_{\mathrm{wk}}) , \\ K(\Omega) &= \overline{\mathrm{Span}_{\mathbb{C}} \{\Theta_{\omega_1,\omega_2} \,:\, \omega_1, \omega_2 \in \Omega\}}^{\|\,\|} \subseteq K(\Omega_{\mathrm{wk}}) . \end{aligned}$$

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Theorem (A, Farenick, Massey 2009)

There exists a sequence of C^*-algebra embeddings such that

$$K(\Omega) \subseteq A \subseteq B(\Omega) \subseteq B(\Omega_{\mathrm{wk}}) = I(A)\,. \tag{1}$$

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Theorem (A, Farenick, Massey 2009)

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For $K(\Omega_{\text{wk}})$, the situation is radically different:

$$M(K(\Omega_{\text{wk}})) = B(\Omega_{\text{wk}}) \quad \text{(Kasparov)},$$

so

$$M_{\text{loc}}(K(\Omega_{\text{wk}})) = M_{\text{loc}}(M_{\text{loc}}(K(\Omega_{\text{wk}}))) = B(\Omega_{\text{wk}})$$

regardless of the choice of Ω.

Further Results

Detailed structure of the product in $B(\Omega_{\mathrm{wk}})$, extending Hamana's work on the product structure of $C(\Delta)\overline{\otimes}B(H)$.

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Thank you!