Commutativity of projections and characterization of tracial functionals on von Neumann algebras

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Definitions

Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{A}^+$ be the set of positive elements in $\mathcal{A}$. The modulus $(X^*X)^{1/2}$ of $X \in \mathcal{A}$ is written as $|X|$.

Definition. A *weight* on C^*-algebra $\mathcal{A}$ is a function $\varphi : \mathcal{A}^+ \to [0, +\infty]$ such that

- $\varphi(0) = 0$,
- $\varphi(\lambda X) = \lambda\varphi(X)$ for $X \in \mathcal{A}^+$ and $\lambda > 0$,
- $\varphi(X + Y) = \varphi(X) + \varphi(Y)$ for all $X, Y \in \mathcal{A}^+$.

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Definition. A *trace* on C^*-algebra $\mathcal{A}$ is a weight τ on $\mathcal{A}$ satisfying $\tau(X^*X) = \tau(XX^*)$ for all $X \in \mathcal{A}$.

Information

If τ is a trace on $\mathcal{A}$ and U is a unitary in $\tilde{\mathcal{A}}$, then $\tau(U^*XU) = \tau(X)$ for every $X \in \mathcal{A}^+$ (τ is unitarily invariant). An interesting question is whether the converse holds: is a unitarily invariant weight necessarily a trace? The answer is "no"in general.

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If τ is a trace on $\mathcal{A}$ and U is a unitary in $\tilde{\mathcal{A}}$, then $\tau(U^* X U) = \tau(X)$ for every $X \in \mathcal{A}^+$ (τ is unitarily invariant). An interesting question is whether the converse holds: is a unitarily invariant weight necessarily a trace? The answer is "no"in general.

A positive linear functional φ on a von Neumann algebra $\mathcal{M}$ is said to be *normal* if $\varphi(\sup A_i) = \sup \varphi(A_i)$ for every bounded increasing net $\{A_i\}$ of positive operators in $\mathcal{M}$.
A linear functional φ on $\mathcal{M}$ is said to be *tracial* if $\varphi(AB) = \varphi(BA)$ for all A, B in $\mathcal{M}$.

Problems

1. Characterization of traces among arbitrary weights on C^*-algebras.

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2. Characterization of the tracial functionals among all positive functionals on a C^*-algebras.

Trace characterization - History

1. L.T. Gardner (1979)
2. H. Upmeier (1981)
3. G. Pedersen, E. Stormer (1982)
4. Sh.A. Aupov (1986)
5. D. Petz, J. Zemanek (1988)
6. A.N. Stolyarov, O.E. Tikhonov, A.N. Sherstnev (2002)
7. A.M. Bikchentaev, A.S. Rusakov, O.E. Tikhonov (2004)
8. O.E. Tikhonov (2005)
9. A.M. Bikchentaev, O.E. Tikhonov (2005)
10. T. Sano, T. Yatsu (2006)
11. A.M. Bikchentaev, O.E. Tikhonov (2007)
12. Dinh Trung Hoa, O.E. Tikhonov (2007)
13. K. Cho, T. Sano (2009)
14. A.M. Bikchentaev (2009)
15. Dinh Trung Hoa, O.E. Tikhonov (2010)
16. A.M. Bikchentaev (2010)

L.T. Gardner's characterization of the trace (1979)

Theorem 1.
The finite traces on a C^*-algebra $\mathcal{A}$ are precisely those (positive) linear functionals φ on $\mathcal{A}$ which satisfy

$$|\varphi(X)| \leq \varphi(|X|) \text{ for all } X \in \mathcal{A}.$$

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Theorem 2.
Let $\mathcal{A}$ be a W^*-algebra, and φ a normal, strongly semifinite weight on $\mathcal{A}$ satisfying the condition

$$\begin{array}{c}\text{For every } \varphi\text{-finite projection } P \in \mathcal{A} \\ |\varphi(X)| \leq \varphi(|X|) \text{ for all } X \in P\mathcal{A}P.\end{array}$$

Then φ is trace on $\mathcal{A}$.

D. Petz – J. Zemanek's characterization of the trace (1988)

Theorem 3.
Let φ be a linear functional on a C^*-algebra $\mathcal{A}$. The following are equivalent:

(i) φ is positive and tracial;
(ii) for every positive integer k and every X in $\mathcal{A}$ we have $|\varphi(X^k)| \leq \varphi(|X|^k)$;
(iii) there exists a positive integer k such that $|\varphi(X^k)| \leq \varphi(|X|^k)$ for all X in $\mathcal{A}$.

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Theorem 4.
A positive linear functional φ on a von Neumann algebra is tracial if and only if it is subadditive ($\varphi(P \vee Q) \leq \varphi(P) + \varphi(Q)$) on the lattice of projections in $\mathcal{M}$.

Corollary 1. A von Neumann algebra is commutative if and only if every state is subadditive on the lattice of projections.

Characterization of the trace by monotonocity inequalities

Theorem 5. (A.M. Bikchentaev, O.E. Tikhonov (2004; 2007))
Let $1 < p < \infty$ and φ be a positive linear functional on M_n, such that

$$\varphi(A^p) \leq \varphi(B^p)$$

whenever $0 \leq A \leq B$. Then φ is a nonnegative scalar multiple of the trace.

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Corollary 2. Let φ be a positive linear functional on M_n, such that for any pair $A, B \in M_n^h$ with $A \leq B$ the inequality

$$\varphi(e^A) \leq \varphi(e^B)$$

holds. Then φ is a nonnegative scalar multiple of the trace.

A.M. Bikchentaev, O.E. Tikhonov (2004; 2007)

Corollary 3. Let $1 < p < \infty$, $0 < \lambda < \infty$, and φ be a positive linear functional on M_n, such that

$$\varphi((A+\lambda)^p) \leq \varphi((B+\lambda)^p),$$

whenever $0 \leq A \leq B$. Then φ is a nonnegative scalar multiple of the trace.

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whenever $0 \leq A \leq B$. Then φ is a nonnegative scalar multiple of the trace.

Problem. Let f be a nondecreasing function defined on an interval S, which is not matrix monotone of order 2. Let φ be a positive linear functional on M_n, such that

$$\varphi(f(A)) \leq \varphi(f(B))$$

whenever $f(A)$, $f(B)$ are well-defined and $A \leq B$. Does it follow that φ is a nonnegative scalar multiple of the trace?

Characterization of the trace by Young's inequality

Theorem 6. (A.M. Bikchentaev, O.E. Tikhonov (2005))
Let φ be a positive linear functional on M_n and p, q be positive numbers such that $1/p + 1/q = 1$. If for any pair $A, B \in M_n^+$ the inequality

$$\varphi(|AB|) \leq \frac{\varphi(A^p)}{p} + \frac{\varphi(B^q)}{q}$$

holds, then φ is a nonnegative scalar multiple of the trace.

Young's inequality and trace

Theorem 7. (K. Cho, T. Sano (2009))
Let φ be a positive linear functional on M_n and f, g mutually conjugate in the sense of Young. Then the inequality

$$|\varphi(A^*B)| \leq \varphi(f(|A|)) + \varphi(g(|B|)) \quad (A, B \in M_n)$$

holds if and only if one of the following conditions is satisfied:
(i) the function $f(x)$ is a positive scalar multiple of the quadratic function x^2;
(ii) the functional φ is a positive scalar multiple of the trace.

Young's inequality and trace

Theorem 8. (K. Cho, T. Sano (2009))
Let φ be a positive linear functional on M_n and f, g mutually conjugate in the sense of Young. Then

$$\varphi(|A^*B|) \leq \varphi(f(|A|)) + \varphi(g(|B|))$$

for all matrices $A, B \in M_n$ if and only if φ is a positive scalar multiple of the trace tr.

The Peierls-Bogoliubov inequality

It is an important issue in statistical mechanics to calculate the value of the so-called *partition function* $\mathrm{tr}(e^{\widehat{H}})$, where the Hermitian matrix $\widehat{H}$ is the Hamiltonian of a physical system. Since that computation is often difficult, it is simpler to compute the related quantity $\mathrm{tr}(e^{H})$, where H is a convenient approximation of the Hamiltonian $\widehat{H}$. Indeed, let $\widehat{H} = H + K$. The *Peierls-Bogoliubov inequality* provides useful information on $\mathrm{tr}(e^{H+K})$ from $\mathrm{tr}(e^{H})$. This inequlity states that, for two Hermitian operators H and K

$$\mathrm{tr}(e^{H}) \exp \frac{\mathrm{tr}(e^{H}K)}{\mathrm{tr}(e^{H})} \leq \mathrm{tr}(e^{H+K}).$$

Characterization of the trace by Peierls-Bogoliubov inequality

Theorem 9. (A.M. Bikchentaev (2010))
A positive functional φ on C^*-algebra $\mathcal{A}$ is tracial if and only if

$$\varphi(e^H) \exp \frac{\varphi(e^{H/2} K e^{H/2})}{\varphi(e^H)} \leq \varphi(e^{H+K})$$

for all positive operators H, K in $\mathcal{A}$.

The Araki–Lieb–Thirring inequality

In [Lett. Math. Phys. (1990)] Araki proved the following inequality:

$$\mathrm{tr}((A^{1/2}BA^{1/2})^{rp}) \leq \mathrm{tr}((A^{r/2}B^rA^{r/2})^p), \quad r \geq 1, p > 0.$$

Here, A, B are positive operators on a Hilbert space. This inequality is a generalization of the one due to Lieb and Thirring, and closely related to the Golden-Thompson inequality.

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Here, A, B are positive operators on a Hilbert space. This inequality is a generalization of the one due to Lieb and Thirring, and closely related to the Golden-Thompson inequality.

Theorem 10. (A.M. Bikchentaev (2010))
A positive functional φ on C^*-algebra $\mathcal{A}$ is tracial if and only if

$$\varphi((A^{1/2}BA^{1/2})^{rp}) \leq \varphi((A^{r/2}B^rA^{r/2})^p)$$

for some $r > 1$, $p > 0$ and for all positive operators A, B in $\mathcal{A}$.

We also give the affirmative answer to the question of J. Zemánek (a review Zbl 0942.15015 Zentralblatt MATH).

We prove that either of inequalities Hölder, Cauchy-Schwarz-Bunyakovskii, Golden-Tompson, etc., which holds only for projections characterizes the tracial functionals among all positive normal functionals on a von Neumann algebra. Most of these inequalities imply commutativity of projections. We characterize traces among arbitrary weights on a von Neumann algebra in terms of the commutation of products of projections under the weight sign.

Commutativity of projections

Theorem 11. (A.M. Bikchentaev (2009, 2010))
For $P, Q \in \mathcal{B}(\mathcal{H})^{pr}$ the following conditions are equivalent:

(i) $PQ = QP$;

(ii) $PQP = QPQ$;

(iii) $s_r(PQP) = s_r(QPQ)$ ($s_r(A)$ denote the support projection of $A \in \mathcal{B}(\mathcal{H})^+$);

(iv) $PQP \leq Q$;

(v) $e^{PQP} \leq e^Q$;

(vi) $|P - Q| \leq P + Q$ (D. Topping (1965)).

Theorem 12. (A.M. Bikchentaev (2009, 2010))
For a normal positive functional φ on a von Neumann algebra $\mathcal{M}$ the following conditions are equivalent:

(i) φ is tracial;

(ii) $\varphi(PQP) = \varphi(QPQ)$ for all $P, Q \in \mathcal{M}^{pr}$;

(iii) $\varphi(s_r(PQP)) = \varphi(s_r(QPQ))$ for all $P, Q \in \mathcal{M}^{pr}$;

(iv) $\varphi(PQP) \leq \varphi(Q)$ for all $P, Q \in \mathcal{M}^{pr}$;

(v) $\varphi(e^{PQP}) \leq \varphi(e^Q)$ for all $P, Q \in \mathcal{M}^{pr}$;

(vi) $\varphi(|P - Q|) \leq \varphi(P + Q)$ for all $P, Q \in \mathcal{M}^{pr}$.

Characterization of traces among weights

Theorem 13. (A.M. Bikchentaev (1998, 2009))
Let a weight φ on a von Neumann algebra $\mathcal{M}$ satisfy the condition

$$X_n X = X X_n,\ \|X - X_n\| \to 0,\ X_n \nearrow X \Rightarrow \varphi(X) = \lim_{n\to\infty} \varphi(X_n),\ X_n, X \in \mathcal{M}^+. \quad (1)$$

Then φ is a trace if and only if $\varphi(PQP) = \varphi(QPQ)$ for all $P, Q \in \mathcal{M}^{pr}$.

Characterization of traces among weights

Theorem 13. (A.M. Bikchentaev (1998, 2009))
Let a weight φ on a von Neumann algebra $\mathcal{M}$ satisfy the condition

$$X_nX = XX_n,\ \|X - X_n\| \to 0,\ X_n \nearrow X \Rightarrow \varphi(X) = \lim_{n\to\infty} \varphi(X_n),\ X_n, X \in \mathcal{M}^+. \tag{1}$$

Then φ is a trace if and only if $\varphi(PQP) = \varphi(QPQ)$ for all $P, Q \in \mathcal{M}^{pr}$.

Theorem 13 has applications in the theory of splitting subspaces (A.N. Sherstnev, E.A. Turilova (1999)). The norm lower semicontinuous weights satisfy condition (1); so are, for example, normal or finite weights.

Theorem 14 and Conjecture

Theorem 14. (A.M. Bikchentaev (2009))
A weight φ on a von Neumann algebra $\mathcal{M}$ is a trace if and only if $\varphi((PQR)^*(PQR)) = \varphi((PQR)(PQR)^*)$ for all $P, Q, R \in \mathcal{M}^{pr}$.

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Conjecture. (A.M. Bikchentaev (1998, 2010))
A weight φ on a von Neumann algebra $\mathcal{M}$ is a trace if and only if $\varphi(PQP) = \varphi(QPQ)$ for all $P, Q \in \mathcal{M}^{pr}$.

Thank you!