# Fibered products of continuous fields of $C^*$-algebras

É. Blanchard (CNRS)

## Continuous fields of C*-algebras

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(c) $\forall\ (a_x)_{x \in X} \in A, \quad \mathbf{x} \mapsto \|\mathbf{a_x}\|_{\mathbf{A_x}}$ is **continuous.**

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**Definition 1.3**
$A$ is a **continuous field of $C^*$-algebras over $X$ with fibres $A_x$**

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$\forall\, a \in A$, the function $x \mapsto \|a_x\|$ is $\boxed{\text{continuous}}$.

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(1) For all compact Hausdorff space $Y$ and all cont. $C(Y)$-algebra $B$, $A \overset{M}{\otimes} B$ is a continuous $C(X \times Y)$-algebra with fibres $(A \overset{M}{\otimes} B)_{(x,y)} \cong A_x \overset{M}{\otimes} B_y$

# Tensor products of $C(X)$-algebras

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# Nuclearity and Exactness

A C*-algebra $A$ is **nuclear** iff for any C*-algebra $B$, there is a unique C*-norm on $A \underset{\mathbb{C}}{\odot} B$.

A C*-algebra $A$ is **exact** iff for any closed two sided ideal $J$ in a C*-algebra $B$, the sequence

$$0 \to A \overset{m}{\underset{\mathbb{C}}{\otimes}} J \to A \overset{m}{\underset{\mathbb{C}}{\otimes}} B \to A \overset{m}{\underset{\mathbb{C}}{\otimes}} B/J \to 0$$

is exact.

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There are a $\boxed{\text{minimal}}$ Hausdorff completion $A \overset{m}{\underset{C(X)}{\otimes}} B$ of $A \underset{C(X)}{\odot} B$

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N.B. Wrong if $X = \{x\}$

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**Answer.** For all continuous $C(Y)$-algebra $B$,
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**b)** If $B$ is a C*-algebra, $\mathcal{B} := C(X; B)$ and $D := A \overset{m}{\underset{C(X)}{\otimes}} \mathcal{B}$, then

$$\begin{array}{ccccccccc} 0 \to & C_x(X)\, D & \to & D & \to & D_x & \to 0 & & \boxed{\text{exact}} \\ & \| & & \| & & \| & & & \\ & C_x(X) A \overset{m}{\otimes} B & & A \overset{m}{\otimes} B & & A_x \overset{m}{\otimes} \mathcal{B}_x & = A_x \overset{m}{\otimes} B & & \end{array}$$

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**c)** If $C$ is a continuous $C(X)$-algebra and $\boxed{x \in X}$,

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# Proof of (3) $\Rightarrow$ (4) in Proposition 2.4

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Hence $d \in A \overset{m}{\otimes} K$. $\qquad \Rightarrow A$ $\boxed{\text{exact}}$

**Corollary 2.5** *Let $X$ be a* $\boxed{\textit{perfect}}$ *metric compact space and $A$ a separable* $\boxed{\textit{continuous}}$ *$C(X)$-algebra.*

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(2) for all exact sequence of continuous $C(X)$-algebras

$$0 \to J \to B \to D \to 0$$

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# Proof of Corollary 2.5

(2)⇒(1) If $A$ is a unital continuous $C(X)$-algebra satisfying (2)

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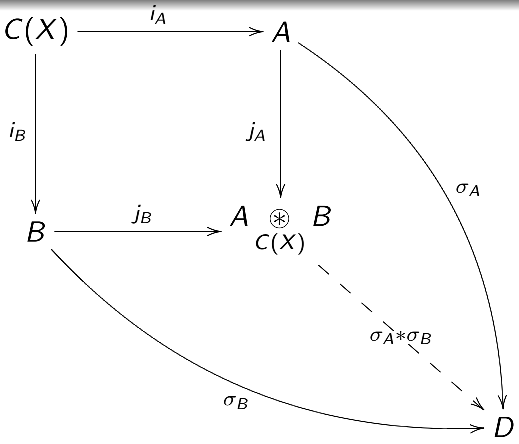
And $A \underset{C(X)}{\otimes} B = A \overset{m}{\otimes} B_0 \ldots \qquad \rightsquigarrow A$ exact.

(1)⇒(2) If $A$ is a unital exact continuous $C(X)$-algebra,

$$\begin{array}{ccccc} C_\Delta(X \times X) A \overset{m}{\otimes} J & \to & C_\Delta(X \times X) A \overset{m}{\otimes} B & \to & C_\Delta(X \times X) A \overset{m}{\otimes} D \\ \downarrow & & \downarrow & & \downarrow \\ A \overset{m}{\otimes} J & \to & A \overset{m}{\otimes} B & \to & A \overset{m}{\otimes} D \\ \downarrow & & \downarrow & & \downarrow \\ A \underset{C(X)}{\overset{m}{\otimes}} J & \dashrightarrow & A \underset{C(X)}{\overset{m}{\otimes}} B & \dashrightarrow & A \underset{C(X)}{\overset{m}{\otimes}} D \end{array}$$

# Amalgamated free product over $C(X)$



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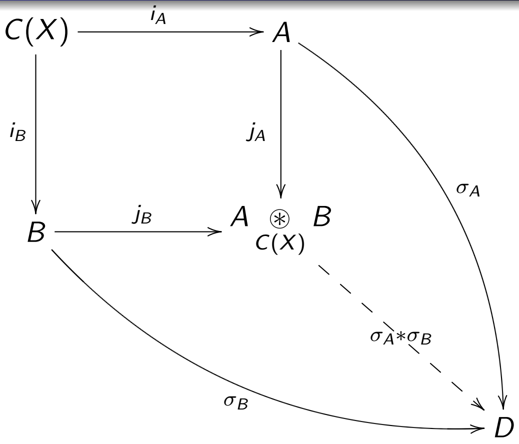


where $A \underset{C(X)}{\circledast} B := A \underset{\mathbb{C}}{\circledast} B \Big/ < f1_A - f1_B \,;\, f \in C(X) >$

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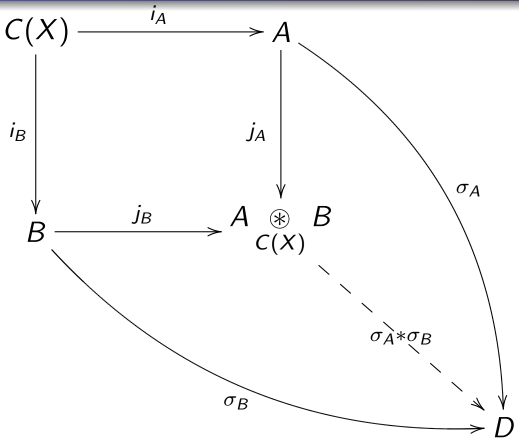


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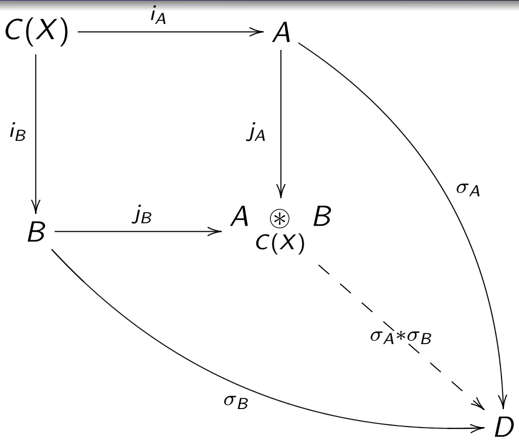


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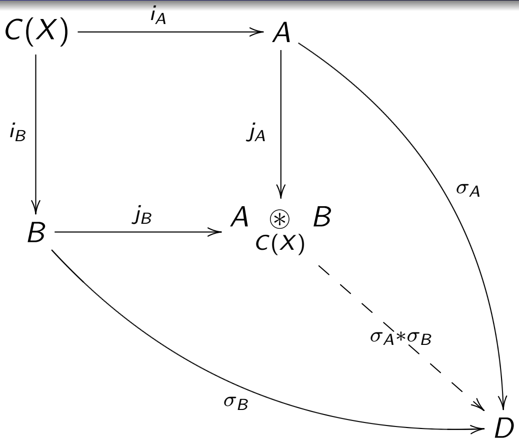


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# Amalgamated free product over $C(X)$

Assume that the unital $C(X)$-algebras $A$ and $B$ are continuous.

**Question 2** Is the $C(X)$-algebra $A \underset{C(X)}{\overset{\alpha}{*}} B$ continuous?

# Reduced amalgamated free product over $C(X)$

Assume that $\phi : A \to C(X)$ and $\psi : B \to C(X)$ are continuous fields of faithful states on the unital $C(X)$-algebras $A$ and $B$.

**Definition 3.2** (Voiculescu)
$(C, \phi * \psi) = (A, \phi) \underset{C(X)}{*} (B, \psi)$ is the **only** unital $C(X)$-algebra endowed with a cont. field of states $\phi * \psi : C \to C(X)$ such that:

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– $A$ and $B$ are free in $(C, \phi * \psi)$.

# Reduced amalgamated free product over $C(X)$

**Proposition 3.3** (B.) If $X$ is $\boxed{\text{perfect}}$ and $\phi$ is a cont. field of faithful states on the unital $C(X)$-algebra $A$, TFAE

(1) The continuous $C(X)$-algebra $A$ is an $\boxed{\text{exact}}$ C*-alg.

(2) The $C(X)$-alg. $(C, \phi * \psi) := (A, \phi) \underset{C(X)}{\overset{r}{*}} (B, \psi)$ is $\boxed{\text{continuous}}$

for :

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(1) $D$ is a continuous $C(X)$-algebra.
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If $D$ is a unital C*-algebra and $F$ is a Hilbert $D$-bimodule,

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– $\ell(\xi) \in \mathcal{L}_D(\mathscr{F}_D(F))$ $\boxed{\text{creation operator}}$ given by

$\ell(\xi).d = \xi d$ and $\ell(\xi)(\zeta_1 \otimes \ldots \otimes \zeta_k) = \xi \otimes \zeta_1 \otimes \ldots \otimes \zeta_k$

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then $D$ is a continuous $C(X)$-algebra if and only if $\mathcal{T}_D(F)$ is a continuous $C(X)$-algebra with fibres isomorphic to $\mathcal{T}_{D_x}(F_x)$.

**Proposition 3.6** (Pedersen)

If the separable unital continuous $C(X)$-algebras $A$ and $B$ are $\boxed{\text{nuclear}}$, then

$$A \underset{C(X)}{\overset{f}{*}} B \subset C(X;\mathcal{O}_2) \underset{C(X)}{\overset{f}{*}} C(X;\mathcal{O}_2)$$

# Full amalgamated free product over $C(X)$

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**Proposition 3.7** (B.)

The $C(X)$-algebra $A \underset{C(X)}{\overset{f}{*}} B$ is **always** continuous.

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**Sketch of proof.**

If $d \in A \underset{C(X)}{\odot} B$,

$$\|d_x\|_h = \inf\left\{ \|\sum_i a_i a_i^*\|^{\frac{1}{2}} . \|\sum_i b_i^* b_i\|^{\frac{1}{2}} \; ; \; d_x = \sum_i a_i \otimes b_i \right\}$$

$$= \sup\left\{ \left| \langle \xi, \sum_i \pi(a_i).\sigma(b_i)\eta \rangle \right| \; ; \; \pi \, , \, \sigma \; * -\text{rep. unifères} \right\}$$