bose–einstein condensation on non homogeneous networks: mathematical aspects and physical applications

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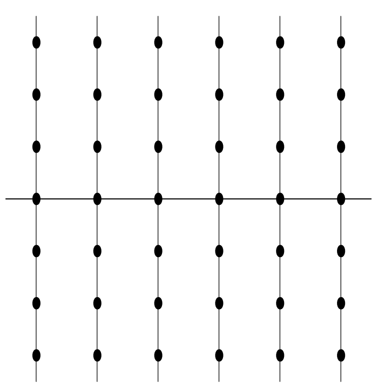
abstract

We study the Bose Einstein Condensation (BEC for short) for the pure hopping model describing a sea of Bardeen–Cooper pairs in arrays of Josephson junctions on non homogeneous networks. The graphs under investigation are obtained by adding density zero perturbations to homogeneous Cayley Trees. The resulting topological model is described by the Hamiltonian which is, up to an additive constant, the opposite of the adjacency operator on the graph. It is enough to perturb in a negligible way the original graph in order to obtain a new network whose mathematical and the deeply connected physical properties dramatically change. In fact, in the regime condensation, particles condensate even in the configuration space due to non homogeneity. We show, as in the amenable case (i.e. density zero perturbations of $\mathbb{Z}^d$ lattices), that it is

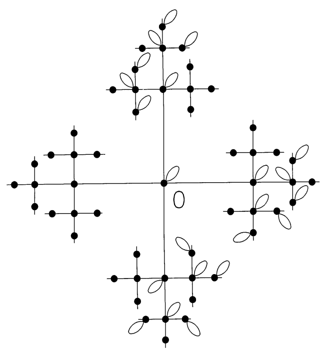
enough to add a finite perturbation to the exponentially growing networks under consideration (i.e. density zero perturbations of homogeneous Cayley Trees), to obtain a spatial distribution of the condensate around the perturbed zone of the network. The mathematical aspects of these phenomena are deeply connected to the investigation of the spectral property of the adjacency matrix A of the graph, for λ near $\|A\|$. Among them we list the following ones. The behavior of the *density of the state* distribution for $\lambda \approx \|A\|$, which is connected with the computation of the critical density of the model. The explicit description of the *Perron Frobenius eigenvector*, which is nothing but the wave function of the ground state describing the spatial distribution of the condensate in the condensation regime. Finally, the transience of the adjacency matrix connected with the possibility to exhibit locally normal states describing the BEC, that is states for which the density of the particles is locally finite. The key point to obtain these results is the explicit description of the adjacency operator for the perturbed networks under consideration.

the model

The framework is a sea of *Bardeen–Cooper pairs* in arrays of *Josephson junctions* on a network G: particles are located on vertices VG, and edges EG describe the presence of a Josephson junction. The networks consist of *density zero* additive perturbations of homogeneous ones



The comb graph $G_1 = \mathbb{Z}^1 \dashv \mathbb{Z}$



$\mathbb{G}^{4,3}$: The Cayley Tree $\mathbb{G}^4$ perturbed along $\mathbb{G}^3$

The Hamiltonian of the system is the *Bose Hubbard Hamiltonian*

$$H_{BH} = m\sum_i n_i + \sum_{i,j} A_{i,j}\Big(Vn_in_j - J_0 a_i^\dagger a_j\Big)\,. \quad (1)$$

Here, $a_i^\dagger$ is the Bosonic creator, and $n_i = a_i^\dagger a_i$ the number operator on the site i. Finally, A is the adjacency operator whose matrix element $A_{i,j}$ in the place ij is the number of the edges connecting the site i with the site j. When m and V are negligible with respect to J_0, the hopping term dominates the physics of the system. Thus, under this approximation, (1) becomes the quadratic *pure hopping*

Hamiltonian given by

$$H_{PH} = -J \sum_{i,j} A_{i,j} a_i^\dagger a_j \,, \tag{2}$$

where the constant $J > 0$ is a mean field coupling constant which might be different from the J_0 appearing in the more realistic Hamiltonian (1).

mathematical aspects

The previously described model is a free theory (the Hamiltonian (2) is quadratic). Then it is enough to study the selfadjoint operator $-A$ on the one–particle space $\ell^2(VG)$. We put $J_0 = 1$ in (2), and normalize such that the bottom of the spectrum of the energy is zero. The resulting Hamiltonian for the purely topological model under consideration is

$$H = \|A\|\mathbf{1} - A \,, \tag{3}$$

where A is the adjacency of the fixed graph G, acting on the Hilbert space $\ell^2(VG)$.

The appearance of the BEC is connected with the asymptotic close to zero, of the spectrum of the Hamiltonian. For free Bosonic models, mathematically described by the Canonical Commutation Relations, most of the physical relevant quantities are computed by using the functional calculus of suitable functions of the Hamiltonian. The critical density (cf. (4)) is one of them. But, the asymptotic behavior of the Hamiltonian (3) near zero corresponds to the asymptotic of the spectrum of A close to $\|A\|$. Indeed, by using the Taylor expansion, we heuristically get for the function appearing in the Bose Gibbs occupation number for the chemical potential $\mu < 0$ at small energies,

$$\begin{aligned}&(e^{H-\mu\mathbf{I}}-1)^{-1}\approx(H-\mu\mathbf{I})^{-1}\\=&((\|A\|-\mu)\mathbf{I}-A)^{-1}\equiv R_A(\|A\|-\mu)\,.\end{aligned}$$

Then the mathematics of the BEC is reduced to the investigation of the spectral properties of the more familiar object for mathematicians, the resolvent $R_A(\lambda)$ for $\lambda \approx \|A\|$.

By following the lines of the previous paper (Fidaleo F., Guido D., Isola T.: Bose Einstein condensation on inhomogeneous amenable graphs, preprint 2008), the non homogeneous graphs we deal with are density zero additive perturbations of homogeneous Cayley trees. The emerging results are quite surprising even if the graphs under consideration are exponentially growing and even in the case of finite additive perturbation.

hidden spectrum

The appearance of the hidden spectrum is the combination of two opposite phenomena arising from the perturbation. If the perturbation is sufficiently big (in many cases it is enough a finite one), the norm $\|A_p\|$ of the adjacency of the perturbed graph becomes bigger than the analogous one $\|A\|$ of the unperturbed adjacency. On the other hand, as the perturbation is sufficiently small (i.e. density zero), the part of the spectrum $\sigma(A_p)$ in the segment $(\|A\|, \|A_p\|]$ does not contribute to the density of the states. This allows us to compute any function of the perturbed adjacency by using the integrated density of the states F of the unperturbed one. For example, we get for the critical density $\rho_c(\beta)$ at the inverse temperature β for the perturbed model,

$$\rho_c(\beta) = \int \frac{\mathrm{d}F_X(x)}{e^{\beta(x+(\|A_p\|-\|A\|))} - 1}\,. \qquad (4)$$

The resulting effect on the critical density of the perturbed model exhibiting the hidden spectrum (i.e. when $\|A_p\| - \|A\| > 0$) is that it is always finite. This is because

$$F_Y(x) = F_X(x+\delta)\,, \qquad (5)$$

being F_X, F_Y be the integrated density of the states of the adjacency and the perturbed adjacency, respectively, and $\delta := \|A^X\| - \|A^Y\| < 0$.*
Notice that in presence of the hidden spectrum the critical density of the model is always finite independently on the geometrical dimension of the network.†

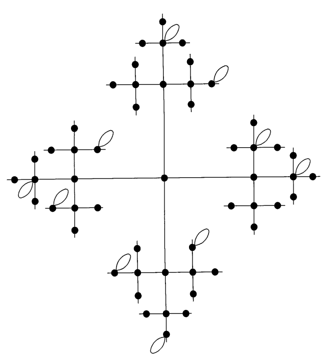
for density zero additive perturbation, in order to check if the adjacency exhibits hidden spectrum it is enough to find out out if whether $\|A_p\| > \|A\|$

*δ has the meaning of a chemical potential, see (4).

†The linearized part of (1) has the form $-\Delta + V(x)$, where Δ is the discrete Laplacian given by $\Delta = A - D$, with $D_{i,j} := (\deg i)\delta_{i,j}$. Notice that $-\Delta$ is positive but not positive preserving, whereas A is not positive but positive preserving. They differ by a diagonal term which is constant for homogeneous graphs. Thus, when the graph is not homogeneous they are completely different operators. The possibility to have BEC might depend on V. For example, if $V = 0$ we can show (cf. FGI) that it is impossible to have hidden spectrum.

This can be done by solving the *secular equation* (see FGI).

To simplify the computations, we deal with perturbations by self loops. On the other hand, it is expected (cf. FGI) that our simplified model captures all the qualitative phenomena appearing in more complicated examples relative to general additive negligible perturbations.


(finite) additive perturbations by self loops

In our situation, secular equation is written as

$$\|P_{\ell^2(S)} R_{A_{\mathbb{G}Q}}(\lambda) P_{\ell^2(S)}\| = 1\,, \qquad (6)$$

where $S \in \mathbb{G}^Q$ is the density zero set of vertices where are localized the self loops. This means also

$$S(\lambda) := P_{\ell^2(S)} R_{A_{\mathbb{G}^Q}}(\lambda) P_{\ell^2(S)}$$

is analytic for $\lambda > \lambda_*$ where λ_* is the at most unique solution of (6) and allows us to write the explicit formula for $R_{A_p}(\lambda)$.[‡]

transience character

By using the explicit formula for the resolvent of the perturbed adjacency, we are able to investigate the *transience character* of the adjacency, that is when

$$\lim_{\lambda \downarrow \|A\|} \langle R_A(\lambda)\delta_x, \delta_x \rangle < +\infty \,,$$

[‡]Such a formula can be analytically extended on the complex plane for $\|\lambda\| > \lambda_*$.

which does not depend on the point $x \in VG$.[§] In our situation, the matter is reduced to the investigation of the limit when $\lambda \downarrow \lambda_*$ of

$$\left\langle R_{A_p}(\lambda)\delta_0, \delta_0 \right\rangle = \left\langle S(\lambda)\Big(\mathbf{I}_{\mathbb{Z}} - S(\lambda)\Big)^{-1}\delta_0, \delta_0 \right\rangle .$$

The transience character is connected with the possibility to exhibit *locally normal* states enjoying BEC. A locally normal state ω describes a situation for which the local density of the particles

$$\rho_\Lambda(\omega) := \frac{1}{|\Lambda|}\sum_{j\in\Lambda}\omega(a^\dagger(\delta_j)a(\delta_j))$$

is finite. If the adjacency is recurrent it is expected that, for each choice of a sequence of chemical potentials $\mu_{\Lambda_n} \uparrow 0$ for the finite volume *Gibbs grand canonical ensamble* state

[§]If the generator of the process is the Laplacian, the transience character is connected with probabilistic properties of the *random walk* on the graph under consideration.

ω_{Λ_n}, $\Lambda_n \uparrow G$, we get that the two–point function diverges:

$$\lim_n \omega_{\Lambda_n}(a^\dagger(\delta_j)a(\delta_j)) = +\infty\,.$$

Namely, **it is impossible to construct any locally normal state exhibiting BEC if the adjacency is recurrent**. Conversely, in the transient case we are able to construct locally normal states describing BEC.

the Perron Frobenious weight

Let B be a bounded matrix with positive entries acting on $\ell^2(VX)$. Such an operator is called *positive preserving* as it preserves the elements of $\ell^2(VX)$ with positive entries. A sequence $\{v(x)\}_{x\in VX}$ is called a *(generalized) Perron Frobenius eigenvector* (or equivalently Perron Frobenius weight) if it has positive entries and

$$\sum_{y\in VX} B_{xy}v(y) = \|B\|v(x)\,, \quad x\in VX\,.$$

For finite additive perturbations the (ℓ^2) Perron Frobenius (normalized at 1 on a fixed root) can be explicitly written (cf. FGI). If $S \in \mathbb{G}^Q$ is a finite connected set supporting the perturbation by self loops, we get

$$v_S(x) = a(\lambda_S)^{d(x,S)} w_S(y(x)) \,.$$

Here,

$$a(\lambda) := \frac{1 - \sqrt{1 - \frac{4(Q-1)}{\lambda^2}}}{\frac{2(Q-1)}{\lambda}} \,, \tag{7}$$

λ_S is the unique solution of the secular equation, w_S is the unique Perron Frobenius eigenvector for the convolution operator T_a^S on S by the function $f_a := a^{d(\,\cdot\,,0)}$ (0 is a fixed root on S), and finally $y(x)$ is the unique nearest element of S to x.[¶]

Consider an infinite, connected and density zero set $S \in \mathbb{G}^Q$, together with the elements $S_n :=$

[¶]Notice that $S(\lambda)$ appearing in the secular equation is expressed in terms of such a convolution operator.

$S \bigcap B_n$ of S in the ball of the radius n centered in a fixed root 0. As $a(\lambda_{S_n}) \to a(\lambda_S)$, we can prove that

$$v_{S_n}(x) \to a(\lambda_S)^{d(x,S)} w_S(y(x)) \,,$$

provided the sequence $\{w_{S_n}\}$ of the Perron Frobenius eigenvectors of the convolution operator $T_a^{S_n}$ converges to a Perron Frobenius weight of T_a^S.[‖] We show that this is the case for all the situation under consideration.

We can prove that the finite volume sequence of the *Perron–Frobenius* eigenvectors, normalized to 1 in a "root", converges pointwise to a (generalized) PF eigenvector for the adjacency.[**] The surprising fact is that it decays exponentially far away from the perturbed zone of the graph.

[‖]Notice that such a Perron Frobenius weight is unique if T_a^S is recurrent.

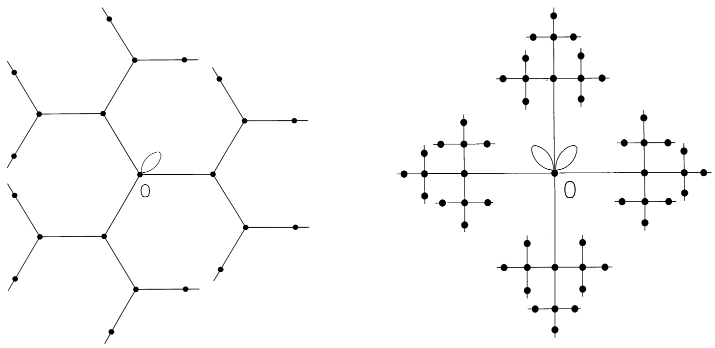
[**]When the graph is transient the set of Perron–Frobenius eigenvectors might be not unique.

The PF (generalized) eigenvector is nothing but the (generalized) wave function of the physical ground state.[††]. Then it describes the distribution of the condensate in the configuration space (due to nonhomogeneity, particle condensate on the network as well). As it exponentially decreases far away to the perturbation (even for the exponentially growing networks under consideration here), the condensate distribution is concentrated close to the base space S supporting the perturbation. Such results are in accordance with the other amenable models previously considered in FGI.

the graphs under consideration

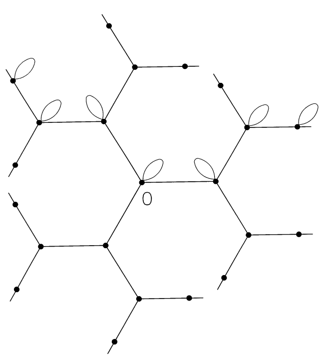
We briefly describe the networks under considerations.

[††]Here "generalized" stands for non normalizable.



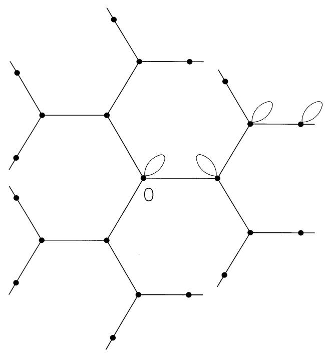
the perturbation of $\mathbb{G}^3$ and $\mathbb{G}^4$ by self loops in a fixed root

When $Q = 3$ it is enough only one self loop, for $Q = 4$ we need at least two. Such graphs are recurrent and the Perron Frobenius weight is normalizable and unique.



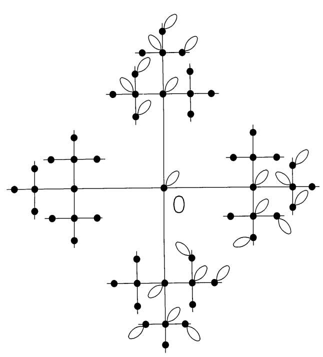
$\mathbb{G}^{3,2}$: the perturbation of $\mathbb{G}^3$ along $S \sim \mathbb{Z}$

Such a graph is recurrent, then the Perron Frobenius weight is unique.



$\mathbb{H}^3$: the perturbation of $\mathbb{G}^3$ along $S \sim \mathbb{N}$

Such a graph is transient. For the case $\mathbb{G}^{Q,2}$ and $\mathbb{H}^Q$, we get hidden spectrum for $2 < Q < 8$.



$\mathbb{G}^{4,3}$: the perturbation of $\mathbb{G}^4$ along $S \sim \mathbb{G}^4$

For such graphs, after fixing $q \geq 2$ we can compute $Q(q) \geq q$ such that $\mathbb{G}^{Q,q}$ exhibits hidden spectrum. Such graphs are transient.

Relatively to the amenable graphs previously treated (FGI, preprint; FGI2, in preparation) we list the following cases.

$\mathbb{N}$: the critical density is infinite (no hidden spectrum, direct computation), it is transient (so it is possible to exhibit locally normal states enjoyng BEC), $3 = d_{PF} > d_G = 1$ (this means we have to construct states exhibiting BEC by fixing a sequence $\{\rho_n\}$ of finite volume densities such that $\rho_n \to +\infty$).[‡‡] The resulting

[‡‡]The Perron–Frobenius and the geometrical dimension d_{PF}, d_G are defined as follows. Consider the ball $\Lambda_n \uparrow G$ of radius n centered in any fixed root of the graph. Consider the Perron–Frobenius eigenvector v, previously described. The *geometrical dimension* d_G of G is defined to be a if $|\Lambda_n| \sim n^a$. The *Perron–Frobenius dimension* $d_{PF}(G)$ of G is defined to be b if $\left\| v\lceil_{\ell^2(\Lambda_n)} \right\| \sim n^{b/2}$. The Perron–Frobenius dimension might depend on the chosen Perron–Frobenius weight v, the last might be not unique if the graph is transient. In our situation, v is the pointwise limit of the finite volume Perron–Frobenius eigenvectors.

scenario is that we can construct locally normal states exhibiting BEC even if the mean density of such states is $+\infty$

$\mathbb{Z}^d \dashv \mathbb{Z}$: the critical density is always finite (we have hidden specrtum), but it is transient if and only if $d \geq 3$. Finally, $d = d_{PF} < d_G = d+1$. The resulting scenario is that we can construct locally normal states exhibiting BEC only if $d \geq 3$. All those states have mean density equal to ρ_c.

$\mathbb{N} \dashv \mathbb{Z}^2$: the critical density is finite (we have hidden specrtum) and it is transient. Finally, $d_PF = d_G = 3$. The resulting scenario is that we can construct locally normal states ω exhibiting BEC at any mean density $\rho(\omega) > \rho_c$.