Finite sums of projections

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OT23 –Timisoara, July 1, 2010

Joint work with

- Herbert Halpern, Ping Wong Ng, Shuang Zhang Finite sums of projections in von Neumann algebras.
- Ping Wong Ng, Shuang Zhang Positive combinations and sums of projections in purely infinite simple C*-algebras and their multiplier algebras.

The main question

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  We need first to answer the following question:

- Which (positive) operators are positive combinations of projections? (finite linear combinations of projections with positive coefficients)

$$a = \sum_{1}^{n} \lambda_j p_j \text{ where } \lambda_j \geq 0, \quad p_j \text{ projections} \in \text{algebra}, \ n \in \mathbb{N}.$$

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We need to follow an alternative approach:

The algebra constant V_o

For some C*-algebras $\mathcal{A}$ there is a constant V_o s.t. for all $a \in \mathcal{A}$ there are $\lambda_j \in \mathbb{C}$ and projections $p_j \in \mathcal{A}$ for which

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(ii) $\sum_1^n |\lambda_j| \leq V_o \|a\|$

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- All W* algebras with no finite type I direct summands with infinite dim center. Implicit in the proofs (see Goldstein & Paskiewicz (1992))
- Infinite simple C*-algebras. AF algebras with finite number of extremal traces. Implicit in the proofs (Fack (1982), Marcoux (2002))

Positive combinations of projections &invertibility

Proposition

If an algebra $\mathcal{A}$

(i) *has a constant V_o as above*

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Notice that the condition that positive combinations of projections are dense in $\mathcal{A}_+$ is satisfied by all real rank zero algebras, and in particular by all W*-algebras.

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Sketch of proof

$$v^*v = e, \quad vv^* = f' \leq f, \qquad q_\pm := \begin{pmatrix} b & \pm\sqrt{b-b^2}\,v^* \\ \pm v\sqrt{b-b^2} & v(e-b)v^* \end{pmatrix}$$

$$a = \frac{1}{2}(q_- + q_+) + \underbrace{c - f' + vbv^*}_{\text{positive, invertible, hence pos comb proj}}$$

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Theorem (Halpern, K, Ng, Zhang)

Let M be a properly infinite W-algebra M and let $a \in M_+$ with range projection $R_a = I$. TFAE*

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Corollary

If M is a finite sum of finite factors or of σ-finite type III factors, then every $a \in M_+$ is a positive combination of projections.

The obstruction in terms of ideals

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A simple example: If $M = \bigoplus_1^\infty B(H_n)$ and $J = \bigoplus_1^\infty K(H_n)$, then central essential norm of $a = \bigoplus_1^\infty a_n \in M_+$ is

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Condition (ii) can be reformulated in terms of the central essential norm relative to an ideal "smaller" than R_a:

(ii) $\exists\ \delta > 0$ such that $\chi_a(0,\delta) \prec \chi_a[\delta,\infty) \iff$

(iii) The central essential norm of a is $\geq \nu I$ for some $\nu > 0$.

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ANSWER

- NO
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Infinite sums of projections in $B(H)$ and W*-factors

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Example (B(H))

$$a := \text{diag}\big(1 - \lambda_1, 1 - \lambda_2, \cdots\big) \oplus \text{diag}\big(1 + \mu_1, 1 + \mu_2 \cdots\big)$$

with $0 < \lambda_j < 1, \mu_j > 0$. Then

$$a_- = \text{diag}\big(\lambda_1, \lambda_2, \cdots\big) \quad \text{and} \quad a_+ = \text{diag}\big(\mu_1, \mu_2 \cdots\big).$$

Nec and (sometimes) suff conditions

Theorem (Ng, K & Zhang (09, JFA))

Let M be a $\sigma-$finite factor and $a \in M_+$. Then a is an infinite sum of projections (strong conv) if and only if

(M type I) $\mathrm{tr}(a_+) \geq \mathrm{tr}(a_-)$ *and* $\mathrm{tr}(a_+) - \mathrm{tr}(a_-) \in \mathbb{N} \cup \{0\} \cup \{\infty\}$;

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Consequence For $h =$diag$\left(1+1, 1+\frac{1}{2}, \cdots, 1+\frac{1}{n}, \cdots\right)$, $h_+ =$diag$\left(1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots\right)$ and $h_- = 0$, hence $\mathrm{tr}(_+) - \mathrm{tr}(h_-) = \infty$ and thus h is an infinite sum of projections.

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Conjecture

We conjecture that the diagonalizability hypothesis in the type II case can be removed. How?

What is known about finite sums of projections in B(H).

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$\|a\|_{ess} > 1 \quad \Rightarrow \quad a$ is a finite sum of projections.

  Notice: $\|a\|_{ess} > 1 \quad \Leftrightarrow \quad a_+ \notin K(H)$.

Key Lemma

Lemma

Assume that M is a properly infinite W-algebra and $e, f \in M$ are projections with $e \perp f$, $e \prec f$, f properly infinite, and $M_f (= fMf)$ has no finite type I summands with infinite dim center.*

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Sketch of proof (to simplify, assume M is a factor

$$a = \beta e + \alpha \sum_{1}^{\infty} e_j \qquad \text{where} \quad e_j \sim e \quad \forall j$$

$$= \underbrace{\beta e + \sum_{1}^{n_1 - 1} \alpha e_j + (\alpha - \gamma_1) e_{n_1}}_{\text{finite sum of projections for appropriate } \gamma_1} + \gamma_1 e_{n_1} + \sum_{n_1+1}^{\infty} \alpha e_j$$

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Non-consecutive blocks are orthogonal.

A sufficient condition for the properly infinite case

Theorem

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The central essential norm condition cannot be eliminated: $a := \bigoplus (1 + \frac{1}{n}) I_n \in \bigoplus B(H_n)$ is NOT the sum of finitely many projections because each summand $(1 + \frac{1}{n}) I_n$ requires at least $n + 1$ projections by Kruglyak, Rabanovich & Samoĭlenko.

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In particular, if M is a σ-finite factor

- M is type I: $\|a\|_{ess} > 1$ (usual essential norm: Choi & Wu result, new proof)

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The central essential norm condition cannot be eliminated: $a := \bigoplus(1+\frac{1}{n})I_n \in \bigoplus B(H_n)$ is NOT the sum of finitely many projections because each summand $(1+\frac{1}{n})I_n$ requires at least $n+1$ projections by Kruglyak, Rabanovich & Samoĭlenko.

In particular, if M is a σ-finite factor

- M is type I: $\|a\|_{ess} > 1$ (usual essential norm: Choi & Wu result, new proof)
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- M is type III: $\|a\| > 1$.

A sufficient condition for the type II_1 case

Recall that we had that if M is a type II factor, $a \in M_+$ is diagonalizable, and $\tau(a_+) \geq \tau(a_-)$, then a is a possibly infinite sum of projections. We can improve this result:

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Theorem

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$B(H)$: a necessary condition

Theorem

Let $a \in B(H)_+$ *be a finite sum of projections and assume that* $\|a\|_{ess} = 1$ *(*$\Leftrightarrow$ $a_+ \in K(H)$*.) Then also* $a_- \in K(H)$ *and*

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The result for W*-algebras is similar.

Tools in the proof

- Frame transform methods permit to construct an isometry w such that

$$\exists \quad \sum_{1}^{n} q_j = I \quad \text{and} \quad q_j waw^* q_j = q_j \quad \forall j$$

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Question

Find a necessary and sufficient condition for $a \in B(H)_+$ to be a finite sum of projections.

Some C*-algebra results - a preview

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If $\|a\|_{ess} > 1$, then a is a finite sum of projections in $\mathcal{M}(\mathcal{B})$.
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Theorem

If $a \in (\mathcal{O}_n)_+$ (the Cuntz algebra) with $2 \leq n < \infty$ and $\|a\| > 1$, then a is a finite sum of projections.

THANK YOU!