Superconformal field theory and operator algebras

Yasu Kawahigashi

University of Tokyo

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Quantum field theory and von Neumann algebras

Outline of the talk:

1. $N = 1$ **Super Virasoro algebras and von Neumann algebras**
   (with Carpi, Longo in 2008)
2. $N = 1$ **Super Virasoro algebras and noncommutative geometry**
   (with Carpi, Hillier, Longo in 2010)
3. $N = 2$ **Super Virasoro algebras, superstring theory and von Neumann algebras**
   (with Carpi, Longo, Xu — in progress)

The **Virasoro algebra** is an infinite dimensional Lie algebra generated by $\{L_n \mid n \in \mathbb{Z}\}$ and a central element c with the following relations.

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}.$$

The Lie group $\mathrm{Diff}(S^1)$ gives a Lie algebra generated by $L_n = -z^{n+1}\frac{\partial}{\partial z}$. The Virasoro algebra is a central extension of the complexification of this.
We can classify **irreducible unitary highest weight** representations, where the central charge c is mapped to a positive scalar.

(Some formal similarity to the **Temperley-Lieb algebra**.)

Super version: $N = 1$ Super Virasoro algebras

The infinite dimensional super Lie algebras generated by central element c, even elements L_n, $n \in \mathbb{Z}$, and odd elements G_r, $r \in \mathbb{Z}$ or $r \in \mathbb{Z} + 1/2$, with the following relations:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \\ [L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r}, \\ [G_r, G_s] &= 2L_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}. \end{aligned}$$

Ramond algebra, if $r \in \mathbb{Z}$

Neveu-Schwarz algebra, if $r \in \mathbb{Z} + 1/2$

Fix a vacuum representation π of the $N = 1$ super Virasoro algebra and simply write L_n for $\pi(L_n)$.

Consider $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, the stress-energy tensor, and $G(Z) = \sum_{r\in\mathbb{Z}+1/2} G_r z^{-r-3/2}$, super stress-energy tensor.

These power series with $z \in \mathbb{C}$, $|z| = 1$ give operator-valued distributions on S^1.

Fix an interval I and take a C^∞-function f with $\operatorname{supp} f \subset I$.

We have (unbounded) operators $\langle L, f\rangle$, $\langle G, f\rangle$.

$A(I)$: the von Neumann algebra generated by these operators with various f.

The $\mathbb{Z}_2$-grading of the super Lie algebra passes to the $\mathbb{Z}_2$-grading of the operator algebras.

Quantum Field Theory: (mathematical setting)

1. Spacetime (e.g., Minkowski space)
2. Symmetry group (e.g., Poincaré group)
3. Quantum fields (operator-valued distributions on the spacetime)

Conformal Field Theory

Two-dimensional Minkowski space $\{(x,t) \mid x,t \in \mathbb{R}\}$
$\to$ One of the light rays $x = \pm t$ compactified to S^1
Orientation preserving diffeomorphism group $\mathrm{Diff}(S^1)$.

We have operator-valued distributions acting on a Hilbert space of states having a vacuum vector.

Operator algebraic axioms: (superconformal field theory)

Motivation: Operator-valued distributions $\{T\}$ on S^1.

Fix an interval $I \subset S^1$, consider $\langle T, f \rangle$ with $\mathrm{supp}\, f \subset I$.

$A(I)$: the von Neumann algebra generated by these (possibly unbounded) operators

1. $I_1 \subset I_2 \Rightarrow A(I_1) \subset A(I_2)$.
2. $I_1 \cap I_2 = \varnothing \Rightarrow [A(I_1), A(I_2)] = 0$. (graded commutator)
3. $\mathrm{Diff}(S^1)$-covariance (conformal covariance)
4. Positive energy/Vacuum vector

Such a family $\{A(I)\}$ is called a superconformal net. The even part gives a local conformal net.

Geometric aspects of local conformal nets

Consider the Laplacian Δ on an n-dimensional compact oriented Riemannian manifold. Weyl formula:

$$\mathrm{Tr}(e^{-t\Delta}) \sim \frac{1}{(4\pi t)^{n/2}}(a_0 + a_1 t + \cdots),$$

where the coefficients have a geometric meaning.

The conformal Hamiltonian L_0 of a local conformal net is the generator of the rotation group of S^1.

For a nice local conformal net, we have an expansion

$$\log \mathrm{Tr}(e^{-tL_0}) \sim \frac{1}{t}(a_0 + a_1 t + \cdots),$$

where a_0, a_1, a_2 are explicitly given. (K-Longo)

This gives an analogy of the Laplacian Δ of a manifold and the conformal Hamiltonian L_0 of a local conformal net.

Noncommutative geometry:

Slogan: Noncommutative operator algebras are regarded as function algebras on noncommutative spaces.
In geometry, we need manifolds rather than compact Hausdorff spaces or measure spaces.

The Connes axiomatization of a noncommutative compact Riemannian spin manifold: spectral triple $(\mathcal{A}, H, D)$.

1. $\mathcal{A}$: $*$-subalgebra of $B(H)$, the smooth algebra $C^\infty(M)$.
2. H: a Hilbert space, the space of L^2-spinors.
3. D: an (unbounded) self-adjoint operator with compact resolvents, the Dirac operator.
4. We require $[D, x] \in B(H)$ for all $x \in \mathcal{A}$.

Our construction in superconformal field theory:

We construct a family $(\mathcal{A}(I), H, D)$ of spectral triples parametrized by intervals $I \subset S^1$ from a representation of the Ramond algebra. (Carpi-Hillier-K-Longo)

One of the Ramond relations gives $G_0^2 = L_0 - c/24$.
So G_0 should play the role of the Dirac operator, which is a "square root" of the Laplacian.
The representation space of the Ramond algebra is our Hilbert space H for the spectral triples (without a vacuum vector). The image of G_0 is now the Dirac operator D, common for all the spectral triples.

Then $\mathcal{A}(I) = \{x \in A(I) \mid [D, x] \in B(H)\}$ gives a net of spectral triples $\{\mathcal{A}(I), H, D\}$ parametrized by I.

$N = 2$ super Virasoro algebra (Ramond/N-S for $a = 0, 1/2$)

Generated by central element c, even elements L_n and J_n, and odd elements $G^{\pm}_{n\pm a}$, $n \in \mathbb{Z}$, with the following.

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \\
[J_m, J_n] &= \frac{c}{3} m \delta_{m+n,0} \\
[L_n, J_m] &= -mJ_{m+n}, \\
[L_n, G^{\pm}_{m\pm a}] &= \left(\frac{n}{2} - (m \pm a)\right) G^{\pm}_{m+n\pm a}, \\
[J_n, G^{\pm}_{m\pm a}] &= \pm G^{\pm}_{m+n\pm a}, \\
[G^{+}_{n+a}, G^{-}_{m-a}] &= 2L_{m+n} + (n-m+2a)J_{n+m} + \\
&\quad \frac{c}{3}\left((n+a)^2 - \frac{1}{4}\right)\delta_{m+n,0}.
\end{aligned}
$$

It is known that an irreducible unitary representation maps c to a scalar in the set

$$\left\{ \frac{3m}{(m+2)} \middle| m = 1, 2, 3, \ldots \right\} \cup [3, \infty).$$

We consider only the case $c = 3m/(m+2)$.

The even part of the superconformal net is identified with the coset net for the inclusion $U(1)_{2m+4} \subset SU(2)_m \otimes U(1)_4$.

The irreducible representations of this local conformal net are labeled with triples (j, k, l) with $0 \le j \le m$, $0 \le k < 2m+4$, $0 \le l < 4$ and $j - k + l \in 2\mathbb{Z}$ with the identification $(j, k, l) = (m - j, k + m + 2, l + 2)$,

The chiral ring is given by $\{(j, j, 0)\}$ and the spectral flow is by $(0, 1, 1)$.

We classify all $N = 2$ superconformal nets with $c < 3$. More quantum fields give more operators, hence a larger von Neumann algebra. So look for possible extensions of $\{A(I)\}$, where $A(I)$ is generated by (super) stress energy tensor.

We have general theory for such a classification based on α-induction and modular invariants.

In similar classifications of local conformal nets and $N = 1$ superconformal nets, our classification lists consist of simple current extensions, the coset constructions, and the mirror extensions in the sense of Xu.

In the $N = 2$ superconformal case, we have a mixture of the coset construction and the mirror extension.

Gepner model: Make a fifth tensor power of the $N = 2$ superconformal net with $c = 9/5$.
$\rightarrow$ an example corresponding to a certain 3-dimensional Calabi-Yau manifold arising from a quintic in $\mathbb{CP}^4$.

This construction gives connection to the mirror symmetry. It appears as an isomorphism of two $N = 2$ super Virasoro algebras sending J_n to $-J_n$ and $G^{\pm}_m$ to $G^{\mp}_m$.

cf. Moonshine net: Make a 48th tensor power of the local conformal net with $c = 1/2$ and then make crossed product extension twice. The automorphism group is the Monster group. (K-Longo) [related to the Leech lattice in dimension 24]