A variational principle for actions of sofic groups

David Kerr

Texas A&M University

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Joint work with Hanfeng Li

Recently Lewis Bowen introduced a notion of entropy for measure-preserving actions $G \curvearrowright (X, \mu)$ of a countable sofic group on a probability space. The basic idea is to model the dynamics of a partition of the probability space by means of partitions of a finite space on which the group acts in a local and approximate way according to the definition of soficity.

Our goal is to define a topological analogue of Bowen's entropy for continuous actions $G \curvearrowright X$ on a compact Hausdorff space and establish a variational principle relating the two. We will do this by taking an operator algebra approach that replaces the combinatorics of partitions with an analysis of linear maps which are approximately multiplicative and approximately equivariant.

Kolmogorov-Sinai entropy

The entropy of a partition $\mathcal{P} = \{P_1, \dots, P_n\}$ of a probability space (X, μ) is defined as

$$H(\mathcal{P}) = -\sum_{i=1}^{n} \mu(P_i) \log \mu(P_i).$$

which can be viewed as the integral of the information function $I(x) = -\log \mu(P_i)$ where $x \in P_i$. For a single measure-preserving transformation $T : X \to X$ we set

$$h_\mu(T, \mathcal{P}) = \lim_{n\to\infty} \frac{1}{n} H(\mathcal{P} \vee T^{-1}\mathcal{P} \vee \cdots \vee T^{-n+1}\mathcal{P})$$
$$h_\mu(T) = \sup_{\mathcal{P}} h_\mu(T, \mathcal{P}).$$

For the shift on $(X_0, \mu_0)^{\mathbb{Z}}$ this is equal to the entropy of the base.

Bowen's measure entropy

The averaging used to define Kolmogorov-Sinai entropy works more generally for actions of amenable groups. Ornstein and Weiss showed that much of the theory extends to this setting, including Ornstein's entropy classification of Bernoulli shifts.

Question

Can the theory be extended beyond the realm of amenability?

Bowen: Yes, if the acting group is sofic.

Basic idea

Replace probability and averaging (information theory) by the counting of discrete models (statistical mechanics).

Bowen's measure entropy

Let $\mathcal{P}$ be a partition of X whose atoms have measures $c_1, \dots, c_n$. In how many ways can we approximately model this ordered distribution of measures by a partition of $\{1, \dots, d\}$ for a given $d \in \mathbb{N}$? By Stirling's formula, the number of models is roughly

$$c_1^{-c_1 d} \cdots c_n^{-c_n d}$$

for large d, so that

$$\frac{1}{d} \log(\#\text{models}) \approx -\sum_{i=1}^{n} c_i \log c_i = H(\mathcal{P}).$$

Bowen's measure entropy

Now let $G \curvearrowright (X, \mu)$ be a measure-preserving action, and let Σ be a sequence of maps $\sigma_i : G \to \mathrm{Sym}(d_i)$ into finite permutation groups which are asymptotically multiplicative and free (the existence of such a sequence defines a *sofic* group).

Let $\mathcal{P}$ be an ordered finite partition of X. For a finite set $F \subseteq G$ and $\varepsilon > 0$ we write $\mathrm{AP}(\mathcal{P}, F, \varepsilon, \sigma_i)$ for the number of ordered partitions $\mathcal{Q}$ of $\{1, \dots, d_i\}$ such that the measures of the atoms of $\bigvee_{s \in F} s^{-1}\mathcal{P}$ and $\bigvee_{s \in F} s^{-1}\mathcal{Q}$ which correspond to each other under the dynamics are summably ε-close. Set

$$h_{\Sigma,\mu}(\mathcal{P}) = \inf_F \inf_{\varepsilon > 0} \limsup_{i \to \infty} \frac{1}{d_i} \log |\mathrm{AP}(\mathcal{P}, F, \varepsilon, \sigma_i)|$$

Theorem (Bowen)

$h_{\Sigma,\mu}(\mathcal{P})$ has a common value for generating finite partitions $\mathcal{P}$.

Linear reformulation

On the set of unital positive maps $L^\infty(X, \mu) \to \mathbb{C}^{d_i}$ we define the pseudometric

$$\rho_{\mathcal{P}}(\varphi, \psi) = \max_{f \in \mathcal{P}} \|\varphi(f) - \psi(f)\|_2.$$

For $\delta > 0$ define $\mathrm{UP}_\mu(\mathcal{P}, F, \delta, \sigma_i)$ to be the set of all unital positive maps $L^\infty(X, \mu) \to \mathbb{C}^{d_i}$ which, to within δ, are approximately multiplicative and F-equivariant and approximately send μ to the uniform probability measure on $\{1, \dots, d_i\}$.

Proposition

$$h_{\Sigma,\mu}(\mathcal{P}) = \sup_{\varepsilon>0} \inf_{F} \inf_{\delta>0} \limsup_{i\to\infty} \frac{1}{d_i} \log N_\varepsilon(\mathrm{UP}_\mu(\mathcal{P}, F, \delta, \sigma_i))$$

where $N_\varepsilon(\cdot)$ denotes the maximal cardinality of an ε-separated set.

Linear reformulation

The previous proposition can be used as a definition of $h_{\Sigma,\mu}(\mathcal{P})$ when $\mathcal{P}$ is any finite partition of unity in $L^\infty(X,\mu)$. One can also more generally define $h_{\Sigma,\mu}(\mathcal{S})$ for any bounded sequence $\mathcal{S}$ in $L^\infty(X,\mu)$. We then have the following.

Theorem

$h_{\Sigma,\mu}(\mathcal{S})$ has a common value over all dynamically generating bounded sequences $\mathcal{S}$ in $L^\infty(X,\mu)$.

Definition

The measure entropy $h_{\Sigma,\mu}(X,G)$ of the action $G \curvearrowright X$ is defined as the common value in the above theorem.

In the case that X is a compact metrizable space, μ is a Borel probability measure on X, and G acts by measure-preserving homeomorphisms, we can formulate an equivalent definition of sofic measure entropy by using approximately equivariant unital homomorphisms $C(X) \to \mathbb{C}^{d_i}$. Such a homomorphism corresponds at the spectral level to a map

$$\{1, \ldots, d_i\} \to X$$

which is approximately equivariant for the given sofic approximation for G. The image of this map can be viewed as a system of interlocking partial orbits.

The entropy thus measures the exponential growth as $i \to \infty$ of the number of approximately equivariant maps $\{1, \ldots, d_i\} \to X$ up to an observational ε-error.

Amenable case

In the case that G is amenable, Ornstein and Weiss's quasitiling theory shows that every sofic approximation $\sigma : G \to \mathrm{Sym}(d)$ approximately decomposes into Følner sets. Thus an approximately equivariant map $\{1, \dots, d\} \to X$ approximately decomposes into partial orbits over Følner sets. Using this fact one can show the following.

Theorem

Suppose that G is amenable. Then $h_{\Sigma,\mu}(X, G) = h_\mu(X, G)$.

Application to Bernoulli actions

In the case that G is amenable the following theorem is a well-known consequence of classical entropy theory. In the case that G contains the free group F_2 it was proved by Lewis Bowen. Note that there are countable sofic groups that lie outside of these two classes, as Ershov showed the existence of a countable nonamenable residually finite torsion group.

Theorem

Let G be a countable sofic group. Let (X, μ) be a standard probability space with $H(\mu) = +\infty$. Then there is no generating countable measurable partition $\mathcal{Q}$ for the Bernoulli action $G \curvearrowright (X, \mu)^G$ such that $H_{\mu^G}(\mathcal{Q}) < +\infty$.

Topological entropy

Recall that the entropy of a homeomorphism $T : X \to X$ of a compact space is defined as

$$h_{\text{top}}(T) = \sup_{\mathcal{U}} \limsup_{n\to\infty} \frac{1}{n} \log N(\mathcal{U} \vee T^{-1}\mathcal{U} \vee \cdots \vee T^{-n+1}\mathcal{U})$$

where $\mathcal{U}$ ranges over the open covers of X and $N(\cdot)$ denotes the minimal cardinality of a subcover.

The entropy can also be expressed by replacing the analysis of set intersections with the counting of partial orbits. More precisely,

$$h_{\text{top}}(T) = \sup_{\varepsilon>0} \limsup_{n\to\infty} \frac{1}{n} \log \operatorname{sep}(n, \varepsilon)$$

where $\operatorname{sep}(n, \varepsilon)$ is the maximal cardinality of an ε-separated set of partial orbits over $\{0, \ldots, n-1\}$.

For a partition of unity $\mathcal{P} \subseteq C(X)$ and a $d \in \mathbb{N}$ we define on the set of unital homomorphism $C(X) \to \mathbb{C}^d$ the pseudometric

$$\rho_{\mathcal{P}}(\varphi, \psi) = \max_{p \in \mathcal{P}} \|\varphi(p) - \psi(p)\|_2.$$

Definition

Let $G \curvearrowright X$ be a continuous action of a countable sofic group on a compact space and $\mathcal{P}$ a partition of unity in $C(X)$. We define

$$h_\Sigma(\mathcal{P}) = \sup_{\varepsilon > 0} \inf_F \inf_{\delta > 0} \limsup_{i \to \infty} \frac{1}{d_i} \log N_\varepsilon(\mathrm{Hom}(\mathcal{P}, F, \delta, \sigma_i))$$

where $\mathrm{Hom}(\cdot)$ is the measureless multiplicative version of $\mathrm{UP}_\mu(\cdot)$.

Topological entropy

As in the measurable setting, one can also define $h_\Sigma(\mathcal{S})$ for any bounded sequence $\mathcal{S}$ in $C(X)$.

Theorem

$h_\Sigma(\mathcal{S})$ has a common value over all dynamically generating bounded sequences $\mathcal{S}$ in $C(X)$.

Definition

The topological entropy $h_\Sigma(X, G)$ is defined as the common value in the above theorem.

Example: The entropy of the shift $G \curvearrowright Y^G$ is $\log |Y|$.

Using the Følner decomposition of sofic approximations for amenable G as in the measurable setting, we can show that the sofic topological entropy agrees with the classical topological entropy in the amenable case:

Theorem

Suppose that G is amenable. Then $h_\Sigma(X, G) = h_{\mathrm{top}}(X, G)$.

The above theorem also follows from its measurable analogue in conjunction with the classical and sofic variational principles, which we turn to next.

Variational principle

The classical variational principle asserts that, for a homeomorphism $T : X \to X$ of a compact Hausdorff space,

$$h_{\mathrm{top}}(T) = \sup_{\mu} h_{\mu}(T)$$

where μ ranges over all invariant Borel probability measures on X. This is moreover true for actions of any countable amenable group.

Theorem (variational principle)

Let $G \curvearrowright X$ be a continuous action of a countable sofic group on a compact metrizable space. Then

$$h_{\Sigma}(X, G) = \sup_{\mu} h_{\Sigma,\mu}(X, G)$$

where μ ranges over all invariant Borel probability measures on X.

Algebraic actions

Let f be an element in the group ring $\mathbb{Z}G$. Then G acts on $\mathbb{Z}G/\mathbb{Z}Gf$ by left translation, and this gives rise to an action α_f of G by automorphisms on the compact Abelian dual group $\widehat{\mathbb{Z}G/\mathbb{Z}Gf}$. This provides a rich class of actions which has been extensively studied in the case $G = \mathbb{Z}^d$ with connections to commutative algebra.

Recall that the Fuglede-Kadison determinant of an invertible element $a \in \mathcal{L}G$ is defined by $\det_{\mathcal{L}G} a = \exp \tau(\log |a|)$.

Theorem (Li)

Suppose that G is amenable and f is invertible in $\mathcal{L}G$. Then

$$h_\mu(\alpha_f) = \log \det_{\mathcal{L}G} f.$$

Algebraic actions

Suppose now that G is residually finite. Let $\{G_i\}_{i=1}$ be a sequence of finite-index normal subgroups of G with $\bigcup_{j=1}^{\infty} \bigcap_{i=j}^{\infty} G_i = \{e\}$, and let $\Sigma = \{\sigma_i : G \to \mathrm{Sym}(G/G_i)\}_{i=1}^{\infty}$ be the associated sofic approximation sequence.

Theorem (Bowen)

Suppose that f is invertible in $\ell^1(G)$. Then

$$h_{\Sigma,\mu}(\alpha_f) = \log \det\nolimits_{\mathcal{L}G} f.$$

The invertibility of f in $\ell^1(G)$ is important in Bowen's argument because it implies the existence of a finite generating partition.

Using the variational principle, one can show the following.

Theorem

Suppose that f is invertible in $C^*(G)$. Then

$$h_\Sigma(\alpha_f) = \log \det_{\mathcal{L}G} f.$$

The invertibility of f in $C^*(G)$ is strictly weaker in general that the invertibility of f in $\ell^1(G)$, for instance if G contains F_2. It is not clear whether there exists a generating finite partition if f is merely assumed to be invertible in $C^*(G)$, and so our definition of sofic measure entropy is essential here.