Structure of C^*-algebras generated by mappings

A. Kuznetsova
jointly with S. Grigorian

Kazan Federal University

3.07.2010

C^*-algebras generated by mappings

Let X be an arbitrary countable set. Mapping $\varphi : X \longrightarrow X$ generates oriented graph (X, φ) with vertices in the elements of X and edges $(x, \varphi(x))$.

C^*-algebras generated by mappings

Let X be an arbitrary countable set. Mapping $\varphi : X \longrightarrow X$ generates oriented graph (X, φ) with vertices in the elements of X and edges $(x, \varphi(x))$.

Consider the Hibert space $l^2(X) = \{f : X \to \mathbb{C} : \sum_{x \in X} |f(x)|^2 < \infty\}$ with natural basis $\{e_x\}_{x \in X}$, $\quad e_x(y) = \delta_{x,y}$.

C^*-algebras generated by mappings

Let X be an arbitrary countable set. Mapping $\varphi : X \longrightarrow X$ generates oriented graph (X, φ) with vertices in the elements of X and edges $(x, \varphi(x))$.

Consider the Hibert space $l^2(X) = \{f : X \to \mathbb{C} : \sum_{x \in X} |f(x)|^2 < \infty\}$ with natural basis $\{e_x\}_{x \in X}$, $\quad e_x(y) = \delta_{x,y}$.

Mapping

$$\varphi : X \longrightarrow X$$

induces the mapping

$$T_\varphi : \{e_x\} \to \{e_x\}; \quad T_\varphi e_x = e_{\varphi(x)}.$$

Theorem

The mapping

$$T_\varphi : \{e_x\} \to \{e_x\}$$

can be extended up to the bounded operator

$$T_\varphi : l^2(X) \longrightarrow l^2(X)$$

if and only if

$$\gamma(\varphi) = \sup_{y \in X} \mathrm{card} \varphi^{-1}(y) = m < \infty.$$

Theorem

The mapping

$$T_\varphi : \{e_x\} \to \{e_x\}$$

can be extended up to the bounded operator

$$T_\varphi : l^2(X) \longrightarrow l^2(X)$$

if and only if

$$\gamma(\varphi) = \sup_{y \in X} \mathrm{card}\varphi^{-1}(y) = m < \infty.$$

We will consider mappings satisfying this condition.
Let's denote via $\mathfrak{A}_\varphi$ the C^*-algebra generated by operator T_φ.

Theorem

The mapping

$$T_\varphi : \{e_x\} \to \{e_x\}$$

can be extended up to the bounded operator

$$T_\varphi : l^2(X) \longrightarrow l^2(X)$$

if and only if

$$\gamma(\varphi) = \sup_{y \in X} \mathrm{card}\varphi^{-1}(y) = m < \infty.$$

We will consider mappings satisfying this condition.
Let's denote via $\mathfrak{A}_\varphi$ the C^*-algebra generated by operator T_φ.

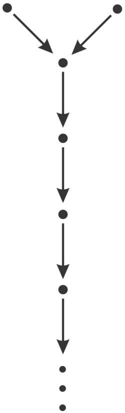
We call $\mathfrak{A}_\varphi$ the C^*-algebra generated by mapping φ.

Some examples of C^*-algebras generated by mappings

pic. 1

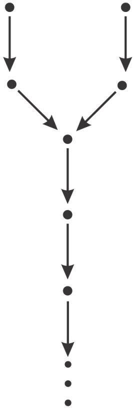
$\mathfrak{T}$

pic. 2



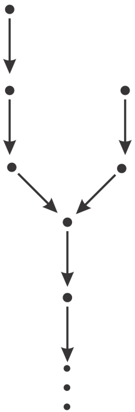
$\mathfrak{T}$

pic. 3



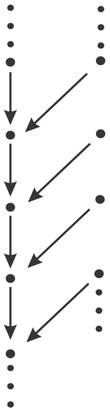
$\mathfrak{T} \bigoplus M_2(\mathbb{C})$

pic. 4



$\mathfrak{T}$

pic. 5



$C(S^1, M_2(\mathbb{C}))$

pic. 6

$\mathfrak{T}$

Structure of operator T_φ

Positive operators $T_\varphi T_\varphi^*$ and $T_\varphi^* T_\varphi$ have the following decomposition:

$$T_\varphi T_\varphi^* = P_1 + 2P_2 + \ldots + mP_m;$$

$$T_\varphi^* T_\varphi = Q_1 + 2Q_2 + \ldots + mQ_m.$$

Structure of operator T_φ

Positive operators $T_\varphi T_\varphi^*$ and $T_\varphi^* T_\varphi$ have the following decomposition:

$$T_\varphi T_\varphi^* = P_1 + 2P_2 + \ldots + mP_m;$$

$$T_\varphi^* T_\varphi = Q_1 + 2Q_2 + \ldots + mQ_m.$$

Operators P_k and Q_k are projections onto the corresponding subspaces:

$$P_k : l^2(X) \longrightarrow l^2(X_k) = \{f \in l^2(X) : T_\varphi T_\varphi^* f = kf\};$$

$$Q_k : l^2(X) \longrightarrow l_k^2 = \{f \in l^2(X) : T_\varphi^* T_\varphi f = kf\}.$$

These operators do not commute.

Lemma

Operator T_φ can be represented as a linear combination of partial isometries,

$$T_\varphi = U_1 + \sqrt{2}U_2 + \ldots + \sqrt{m}U_m.$$

Here U_k are the partial isometries from the space l_k^2 to the space $l^2(X_k)$.

Lemma

Operator T_φ can be represented as a linear combination of partial isometries,

$$T_\varphi = U_1 + \sqrt{2}U_2 + \ldots + \sqrt{m}U_m.$$

Here U_k are the partial isometries from the space l^2_k to the space $l^2(X_k)$.

Theorem

$\mathfrak{A}_\varphi$ is isomorphic to C^*-algebra generated by the finite set of partial isometries satisfying the equalities:

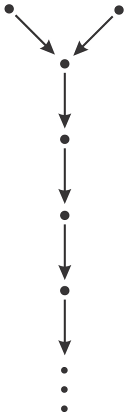
$$U_1^*U_1 + U_2^*U_2 + \ldots + U_m^*U_m = Q;$$

$$U_1U_1^* + U_2U_2^* + \ldots + U_mU_m^* = P;$$

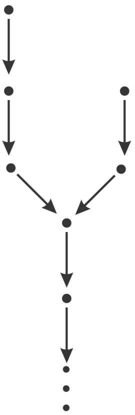
where P and Q are projections.

Examples of Toeplitz algebra generated by a pare of partial isometries

pic. 2



pic. 4



pic. 6

Monomials

We call $x \in X$ the *cyclic* element if there is k such that $\varphi^k(x) = x$.

Monomials

We call $x \in X$ the *cyclic* element if there is k such that $\varphi^k(x) = x$.

Until further notice we assume that X has no cyclic elements for mapping φ.

We call V the *monomial* if it is a product of a finite number of partial isometries,

$$V = U'_{j_1} U'_{j_2} \dots U'_{j_k}, \quad U'_{j_l} \in \{U_{j_l}, U^*_{j_l}\}.$$

Monomials

We call $x \in X$ the *cyclic* element if there is k such that $\varphi^k(x) = x$.

Until further notice we assume that X has no cyclic elements for mapping φ.
We call V the *monomial* if it is a product of a finite number of partial isometries,

$$V = U'_{j_1} U'_{j_2} \dots U'_{j_k}, \quad U'_{j_l} \in \{U_{j_l}, U^*_{j_l}\}.$$

By the term *index* of monomial V ($\mathrm{ind} V$) we mean the difference between the number of partial isometries from sets $\{U^*_k\}_{k=1}^m$ and $\{U_k\}_{k=1}^m$ in its representation.

Monomials

We call $x \in X$ the *cyclic* element if there is k such that $\varphi^k(x) = x$.

Until further notice we assume that X has no cyclic elements for mapping φ.

We call V the *monomial* if it is a product of a finite number of partial isometries,

$$V = U'_{j_1} U'_{j_2} \dots U'_{j_k}, \quad U'_{j_l} \in \{U_{j_l}, U^*_{j_l}\}.$$

By the term *index* of monomial V ($\mathrm{ind} V$) we mean the difference between the number of partial isometries from sets $\{U^*_k\}_{k=1}^m$ and $\{U_k\}_{k=1}^m$ in its representation.

Lemma

The index of monomial V does not depend on its representation.

Graduation of $\mathfrak{A}_\varphi$

Let $\mathfrak{A}_{\varphi,n}$ be closed subspace generated by monomials of index n.

Graduation of $\mathfrak{A}_\varphi$

Let $\mathfrak{A}_{\varphi,n}$ be closed subspace generated by monomials of index n.

Theorem

$\mathfrak{A}_\varphi$ is $\mathbb{Z}$-graduated algebra,

$$\mathfrak{A}_\varphi = \sum_{n=-\infty}^{\infty} \mathfrak{A}_{\varphi,n},$$

and the subalgebra $\mathfrak{A}_{\varphi,0}$ is AF-algebra.

Covariant systems generated by mappings

Algebra $C(S^1, \mathfrak{A}_\varphi)$ is a
C^*-algebra with respect to pointwise multiplication, natural involution and uniform norm $((fg)(e^{i\theta}) = f(e^{i\theta})g(e^{i\theta})$, $(f^*)(e^{i\theta}) = f(e^{i\theta})^*$,
$\|f\|_\infty = \sup\limits_{S^1} \|f\|)$.

Covariant systems generated by mappings

Algebra $C(S^1, \mathfrak{A}_\varphi)$ is a
C^*-algebra with respect to pointwise multiplication, natural involution and uniform norm $((fg)(e^{i\theta}) = f(e^{i\theta})g(e^{i\theta})$, $(f^*)(e^{i\theta}) = f(e^{i\theta})^*$,
$\|f\|_\infty = \sup\limits_{S^1} \|f\|)$.

Let's introduce for every monomial V the *generalized monomial* — $\mathfrak{A}_\varphi$-valued function, defined by $\widetilde{V}(e^{i\theta}) = e^{in\theta}V$, where $n = \mathrm{ind}V$.
It is obvious that $C(S^1, \mathfrak{A}_\varphi) \supset \widetilde{\mathfrak{A}}_\varphi$ — C^*-algebra generated by generalized monomials.

Covariant systems generated by mappings

Algebra $C(S^1, \mathfrak{A}_\varphi)$ is a
C^*-algebra with respect to pointwise multiplication, natural involution and uniform norm $((fg)(e^{i\theta}) = f(e^{i\theta})g(e^{i\theta})$, $(f^*)(e^{i\theta}) = f(e^{i\theta})^*$,
$\|f\|_\infty = \sup\limits_{S^1} \|f\|)$.

Let's introduce for every monomial V the *generalized monomial* — $\mathfrak{A}_\varphi$-valued function, defined by $\widetilde{V}(e^{i\theta}) = e^{in\theta}V$, where $n = \mathrm{ind}\, V$.
It is obvious that $C(S^1, \mathfrak{A}_\varphi) \supset \widetilde{\mathfrak{A}}_\varphi$ — C^*-algebra generated by generalized monomials.

Theorem

$\mathfrak{A}_\varphi$ is isomorphic to $\widetilde{\mathfrak{A}}_\varphi$

Covariant systems generated by mappings

Theorem

There is a covariant system

$$(\mathfrak{A}_\varphi, S^1, \gamma),$$

where γ is embedding of S^1 into $\mathrm{Aut}\mathfrak{A}_\varphi$, and also

$$\mathfrak{A}_{\varphi,n} = \{A \in \mathfrak{A}_\varphi : \gamma(e^{i\theta})(A) = e^{in\theta}A\}.$$

Covariant systems generated by mappings

Theorem

There is a covariant system

$$(\mathfrak{A}_\varphi, S^1, \gamma),$$

where γ *is embedding of* S^1 *into* $\mathrm{Aut}\mathfrak{A}_\varphi$*, and also*

$$\mathfrak{A}_{\varphi,n} = \{A \in \mathfrak{A}_\varphi : \gamma(e^{i\theta})(A) = e^{in\theta}A\}.$$

It is obvious that $\gamma(e^{i\theta})(A) = A \quad \text{if} \quad A \in \mathfrak{A}_{\varphi,0}$.

Nuclear algebras

Let $\mathfrak{B}$ be the arbitrary C^*-algebra.
Mapping φ generates the covariant system $(\mathfrak{A}_\varphi, S^1, \gamma)$ and hence the covariant systems

$$(\mathfrak{A}_\varphi \bigotimes_{\min} \mathfrak{B}, S^1, \gamma \otimes_{\min} I) \quad \text{and} \quad (\mathfrak{A}_\varphi \bigotimes_{\max} \mathfrak{B}, S^1, \gamma \otimes_{\max} I).$$

Let $\mathfrak{A}_\varphi \odot \mathfrak{B}$ be the algebraic tensor product with the identical mapping $I : \sum\limits_{i=1}^{n} A_i \otimes B_i \longrightarrow \sum\limits_{i=1}^{n} A_i \otimes B_i$.

Let's extend I up to the surjective *-homomorphism

$$\Phi : \mathfrak{A}_\varphi \bigotimes_{\max} \mathfrak{B} \longrightarrow \mathfrak{A}_\varphi \bigotimes_{\min} \mathfrak{B},$$

setting

$$\Phi(\sum_{i=1}^{n} A_i \otimes B_i) = \sum_{i=1}^{n} A_i \otimes B_i$$

for every $\sum_{i=1}^{n} A_i \otimes B_i \in \mathfrak{A}_\varphi \odot \mathfrak{B}$.

Let's extend I up to the surjective *-homomorphism

$$\Phi : \mathfrak{A}_\varphi \bigotimes_{\max} \mathfrak{B} \longrightarrow \mathfrak{A}_\varphi \bigotimes_{\min} \mathfrak{B},$$

setting

$$\Phi(\sum_{i=1}^{n} A_i \otimes B_i) = \sum_{i=1}^{n} A_i \otimes B_i$$

for every $\sum_{i=1}^{n} A_i \otimes B_i \in \mathfrak{A}_\varphi \odot \mathfrak{B}$.

Considering the covariant systems mentioned above and using that $\mathfrak{A}_\varphi$ is $\mathbb{Z}$-graduated algebra we obtain that Φ is isomorphism.

Let's extend I up to the surjective *-homomorphism

$$\Phi : \mathfrak{A}_\varphi \bigotimes_{\max} \mathfrak{B} \longrightarrow \mathfrak{A}_\varphi \bigotimes_{\min} \mathfrak{B},$$

setting

$$\Phi(\sum_{i=1}^{n} A_i \otimes B_i) = \sum_{i=1}^{n} A_i \otimes B_i$$

for every $\sum\limits_{i=1}^{n} A_i \otimes B_i \in \mathfrak{A}_\varphi \odot \mathfrak{B}$.

Considering the covariant systems mentioned above and using that $\mathfrak{A}_\varphi$ is $\mathbb{Z}$-graduated algebra we obtain that Φ is isomorphism.

Theorem

$\mathfrak{A}_\varphi$ is a nuclear algebra.

Mappings allowing the cyclic elements

We now turn to the mappings allowing the cyclic elements. In this case there can be different representations of the same monomial V with different indices:

$$V = U'_{j_1} U'_{j_2} \dots U'_{j_k}; \quad V = U''_{i_1} U''_{i_2} \dots U''_{i_l},$$

and

$$\sum_{s=1}^{l} \operatorname{ind} U'_{j_s} \neq \sum_{s=1}^{k} \operatorname{ind} U''_{i_s}.$$

Mappings allowing the cyclic elements

We now turn to the mappings allowing the cyclic elements. In this case there can be different representations of the same monomial V with different indices:

$$V = U'_{j_1} U'_{j_2} \dots U'_{j_k}; \quad V = U''_{i_1} U''_{i_2} \dots U''_{i_l},$$

and

$$\sum_{s=1}^{l} \operatorname{ind} U'_{j_s} \neq \sum_{s=1}^{k} \operatorname{ind} U''_{i_s}.$$

Lemma

If there is such monomial with different indices it must be compact.

Let's consider algebra

$$\mathfrak{B}_\varphi = \mathfrak{A}_\varphi / I_\varphi,$$

where I_φ is the ideal of compact operators.
For this algebra we have similar results.

Let's consider algebra

$$\mathfrak{B}_\varphi = \mathfrak{A}_\varphi / I_\varphi,$$

where I_φ is the ideal of compact operators.
For this algebra we have similar results.

Theorem

$\mathfrak{B}_\varphi$ is $\mathbb{Z}$-graduated algebra,

$$\mathfrak{B}_\varphi = \sum_{n=-\infty}^{\infty} \mathfrak{B}_{\varphi,n},$$

and the subalgebra $\mathfrak{B}_{\varphi,0}$ is AF-algebra.

Let's consider algebra

$$\mathfrak{B}_\varphi = \mathfrak{A}_\varphi / I_\varphi,$$

where I_φ is the ideal of compact operators.
For this algebra we have similar results.

Theorem

$\mathfrak{B}_\varphi$ is $\mathbb{Z}$-graduated algebra,

$$\mathfrak{B}_\varphi = \sum_{n=-\infty}^{\infty} \mathfrak{B}_{\varphi,n},$$

and the subalgebra $\mathfrak{B}_{\varphi,0}$ is AF-algebra.

Note that $\mathfrak{A}_\varphi$ also can be shown to have the *AF*-subalgebra in case of mappings allowing the cyclic elements.

Theorem

There is a covariant system $(\mathfrak{B}_\varphi, S^1, \gamma)$, *and*

$$\mathfrak{B}_{\varphi,n} = \{B \in \mathfrak{B}_\varphi : \gamma(e^{i\theta})(B) = e^{in\theta} B\}.$$

Theorem

There is a covariant system $(\mathfrak{B}_\varphi, S^1, \gamma)$, *and*

$$\mathfrak{B}_{\varphi,n} = \{B \in \mathfrak{B}_\varphi : \gamma(e^{i\theta})(B) = e^{in\theta}B\}.$$

Theorem

$\mathfrak{A}_\varphi$ *is a nuclear algebra.*

Thank you