Square function estimates for analytic operators and applications

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Besançon

Timisoara (OT 23), July 1, 2010

(Joint work with Quanhua Xu)

Ergodic Maximal inequalities

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Assume that T is a contraction (i.e. $\|T\| \leqslant 1$) and T is positive (i.e. $T(x) \geqslant 0$ for any $x \geqslant 0$ in $L^p(\Omega)$). Then

$$\Big\| \sup_{n \geqslant 0} \frac{1}{n+1} \Big| \sum_{k=0}^{n} T^k(x) \Big| \Big\|_p \lesssim \|x\|_p, \qquad x \in L^p(\Omega).$$

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Hopf-Dunford-Schwartz '56 : case when T is an absolute contraction, i.e.

$$\|T\colon L^1 \longrightarrow L^1\| \leqslant 1 \qquad \text{and} \qquad \|T\colon L^\infty \longrightarrow L^\infty\| \leqslant 1.$$

Stronger Maximal inequalities

Stein's Theorem, '61

Assume that T is a positive absolute contraction, and that

$$T\colon L^2 \longrightarrow L^2$$

is selfadjoint and positive in the Hilbertian sense (that is, $\sigma(T) \subset [0,1]$).
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Question :
Which (less restrictive) conditions imply this stronger maximal inequality ?

Semigroups

Let $(T_t)_{t\geqslant 0}$ be a strongly continuous semigroup on $L^p(\Omega)$.

For any $t > 0$, define

$$M_t(x) = \frac{1}{t}\int_0^t T_s(x)\,ds\,, \qquad x \in L^p(\Omega).$$

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Similarly, we have :

- Assume that T_t is a positive contraction for any $t \geqslant 0$. Then we have

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- Assume that T_t is a positive contraction for any $t \geqslant 0$. Then we have

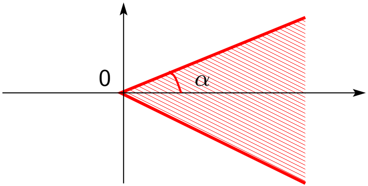
$$\Big\| \sup_{t>0} \big| M_t(x) \big| \Big\|_p \lesssim \|x\|_p, \qquad x \in L^p(\Omega).$$

- Assume that each T_t is a positive absolute contraction and that $(T_t)_{t \geqslant 0}$ is a selfadjoint strongly continuous semigroup on L^2. Then we have

$$\Big\| \sup_{t>0} \big| T_t(x) \big| \Big\|_p \lesssim \|x\|_p, \qquad x \in L^p(\Omega).$$

Analyticity I

For any angle $0 < \alpha < \frac{\pi}{2}$, let : $\Sigma_\alpha = \left\{ z \in \mathbb{C}^* \ : \ |\mathrm{Arg}(z)| < \alpha \right\}$.



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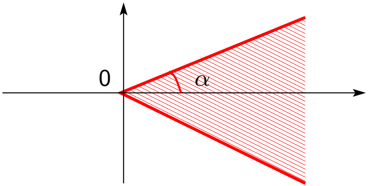

Let $(T_t)_{t \geqslant 0}$ be a strongly continuous semigroup of contractions on X. It is called **analytic** if $(T_t)_{t>0}$ has a bounded holomorphic extension

$$z \in \Sigma_\alpha \mapsto T_z \in B(X),$$

for some $0 < \alpha < \frac{\pi}{2}$.

A necessary and sufficient condition is that

$$\exists\, C > 0 \quad \Big| \quad \forall t > 0,\ x \in X, \qquad \left\| t \frac{d}{dt}\big(T_t(x)\big) \right\| \leqslant C \|x\|.$$

Theorem I

Let $(T_t)_{t\geqslant 0}$ be a strongly continuous semigroup of positive contractions on $L^p(\Omega)$, with $1 < p < \infty$, and assume that $(T_t)_{t\geqslant 0}$ is analytic. Then we have

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Remark.

(1) Let $(T_t)_{t\geqslant 0}$ be a selfadjoint strongly continuous semigroup of contractions on $L^2(\Omega)$. Then $(T_t)_{t\geqslant 0}$ is analytic, by spectral theory.

$$T_t = e^{-tA} \quad \text{with} \quad A = \text{ positive selfadjoint operator.}$$

Extends to $T_z = e^{-zA}$ for $\mathrm{Re}(z) > 0$ with $\|T_z\| \leqslant 1$.

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(2) If further each T_t is an absolute contraction for any $t > 0$, then for any $1 < p < \infty$, the realization of $(T_t)_{t\geqslant 0}$ on $L^p(\Omega)$ is analytic. Here

$$\alpha_p = \frac{\pi}{2} - \pi\Big|\frac{1}{p} - \frac{1}{2}\Big|.$$

Analyticity II

Let $T: X \to X$ be a contraction on Banach space. It is called **analytic** if

$$\exists\, C > 0 \quad \Big| \quad \forall\, n \geqslant 1, \qquad n \big\| T^n - T^{n-1} \big\| \leqslant C.$$

(Coulhon, Saloff-Coste, '85)

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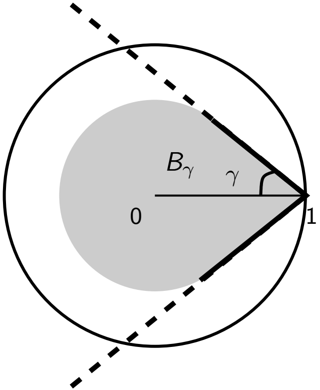
This is equivalent (for a contraction) to the so-called Ritt condition :

$$\sigma(T) \subset \overline{\mathbb{D}} \qquad \text{and} \qquad \|R(\lambda, T)\| \leqslant \frac{K}{|\lambda - 1|}.$$

For any angle $0 < \gamma < \frac{\pi}{2}$, let B_γ be the convex hull of 1 and the disc of center 0 and radius $\sin\gamma$.



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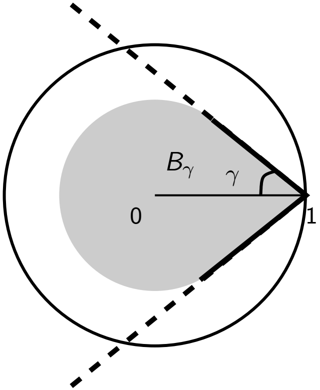

Then analyticity implies that

$$\exists\, \gamma \in \left(0, \frac{\pi}{2}\right) \mid \qquad \sigma(T) \subset B_\gamma .$$

Theorem II

Let T be a positive contraction on $L^p(\Omega)$, with $1 < p < \infty$, and assume that T is analytic. Then we have

$$\Big\| \sup_{n \geqslant 1} \big| T^n(x) \big| \Big\|_p \lesssim \|x\|_p, \qquad x \in L^p(\Omega).$$

Theorem II

Let T be a positive contraction on $L^p(\Omega)$, with $1 < p < \infty$, and assume that T is analytic. Then we have

$$\Big\| \sup_{n \geqslant 1} \big| T^n(x) \big| \Big\|_p \lesssim \|x\|_p, \qquad x \in L^p(\Omega).$$

This extends Stein's Theorem.

Square functions (semigroup case)

Proposition I

Let $(T_t)_{t\geqslant 0}$ be a strongly continuous semigroup of positive contractions on $L^p(\Omega)$, with $1 < p < \infty$, and assume that $(T_t)_{t\geqslant 0}$ is analytic. Then we have an estimate

$$\left\| \left(\int_0^\infty t \left| \frac{d}{dt} \big(T_t(x) \big) \right|^2 dt \right)^{\frac{1}{2}} \right\|_p \lesssim \|x\|_p, \qquad x \in L^p(\Omega).$$

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Such square function estimates appeared in Stein's work for diffusion semigroups and then in H^∞ functional calculus theory (Doust, Cowling, McIntosh, Yagi).

The proof of Proposition I relies on H^∞ calculus and a result of L. Weis.

How to go from Proposition I to Theorem I ?

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Hence

$$\begin{aligned} |T_t(x)| &\leqslant |M_t(x)| \;+\; \frac{1}{t}\Big|\int_0^t sT_s'(x)\, ds\Big| \\ &\leqslant |M_t(x)| \;+\; \frac{1}{t}\Big(\int_0^t s\, ds\Big)^{\frac{1}{2}} \;\Big(\int_0^t s\big|T_s'(x)\big|^2\, ds\Big)^{\frac{1}{2}} \\ &\leqslant |M_t(x)| \;+\; \Big(\int_0^\infty s\big|T_s'(x)\big|^2\, ds\Big)^{\frac{1}{2}}. \end{aligned}$$

Square functions (discrete case)

Proposition II

Let T be a positive contraction on $L^p(\Omega)$, with $1 < p < \infty$, and assume that T is analytic. Then we have an estimate

$$\Big\|\Big(\sum_{n=1}^{\infty} n\,\big|T^n(x) - T^{n-1}(x)\big|^2\Big)^{\frac{1}{2}}\Big\|_p \lesssim \|x\|_p, \qquad x \in L^p(\Omega).$$

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Passing from Proposition II to Theorem II is easy.

Functional calculus for analytic operators

FC Theorem

Let T be a positive contraction on $L^p(\Omega)$, with $1 < p < \infty$, and assume that T is analytic.

(1) Then there exists an angle $\gamma \in \left(0, \frac{\pi}{2}\right)$ and a constant $C \geqslant 1$ such that

$$\|F(T)\| \leqslant C \sup\left\{ \left| F(z) \right| \, : \, z \in B_\gamma \right\}$$

for any polynomial F.

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(2) More generally, for any sequence $(F_n)_{n\geqslant 1}$ of polynomials and any $x \in L^p(\Omega)$, we have

$$\left\|\left(\sum_{n=1}^{\infty}\left|F_n(T)x\right|^2\right)^{\frac{1}{2}}\right\|_p \leqslant C\,\|x\|_p \sup\left\{\left(\sum_{n=1}^{\infty}\left|F_n(z)\right|^2\right)^{\frac{1}{2}} \,:\, z \in B_\gamma\right\}.$$

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For any $z \in \mathbb{D}$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \left|F_n(z)\right|^2 &= \sum_{n=1}^{\infty} n|z|^{2(n-1)}|z-1|^2 \\
&= |1-z|^2 \frac{1}{\left(1-|z|^2\right)^2} \\
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This upper bound is bounded on B_γ.

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- For any integer $m \geqslant 2$, there is an estimate

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- For any integer $m \geqslant 1$, we have a maximal inequality for the m-th derivative,

$$\Big\| \sup_{n \geqslant 0} (n+1)^m \big| T^n (T-I)^m (x) \big| \Big\|_p \lesssim \|x\|_p, \qquad x \in L^p(\Omega).$$

Noncommutative L^p-spaces

Let M be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace τ. For any $1 \leqslant p < \infty$, define

$$\|x\|_p = \left(\tau(|x|^p)\right)^{\frac{1}{p}}$$

on a suitable subspace $\mathcal{S} \subset M$.

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This includes :

- Commutative L^p-spaces $L^p(\Omega, \mu)$, associated to $M = L^\infty(\Omega, \mu)$.
- Schatten spaces $S^p(H)$, associated to $B(H)$.

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Given $1 \leqslant p < \infty$, $L^p(M;\ell^\infty)$ is defined as the space of all sequences $(x_n)_{n\geqslant 0}$ in $L^p(M)$ for which there exist $a, b \in L^{2p}(M)$ and a bounded sequence $(z_n)_{n\geqslant 0}$ in M such that

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For such a sequence, set

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Then $L^p(M;\ell^\infty)$ is a Banach space (Pisier, Junge).

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Assume that T is a positive absolute contraction, that is, $\|T\colon L^1(M) \to L^1(M)\| \leqslant 1$ and $\|T\colon M \longrightarrow M\| \leqslant 1$.

(1) For any $1 < p < \infty$, there is an estimate

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(2) If further $T\colon L^2(M) \longrightarrow L^2(M)$ is selfadjoint and positive in the Hilbertian sense, then there is a (better ! !) estimate

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Analyticity of noncommutative absolute contractions

Let $T \colon M \to M$ be an absolute contraction. For any $1 < p < \infty$, provisionaly denote its L^p-realization by

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–> Notion of analytic absolute contraction.

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Thm A

For any integer $m \geqslant 1$, we have an estimate

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However using interpolation, the above L^2-estimates suffice to lead to :

Thm B

For any $1 < p < \infty$, we have an estimate

$$\Big\|\big(T^n x\big)_{n\geqslant 0}\Big\|_{L^p(M;\ell^\infty)} \lesssim \|x\|_p, \qquad x \in L^p(M).$$

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$$v_q \subset \ell^\infty$$

The shift operator on $\mathbb{Z}$

For any $1 < p < \infty$, consider

$$s_p : \ell^p_{\mathbb{Z}} \longrightarrow \ell^p_{\mathbb{Z}}, \qquad s_p\big((c_j)_j\big) = (c_{j-1})_j,$$

the shift operator on $\ell^p_{\mathbb{Z}}$.

Theorem (Bourgain, Jones-Kaufman-Rosenblatt-Wierdl, '98)

For any $2 < q < \infty$, there is an estimate

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For $c = (c_j)_j$, then $M_n(s_p)c$ is given by

$$\big(M_n(s_p)c\big)_j = \frac{1}{n+1} \sum_{k=0}^{n} c_{j+k} \,.$$

Final Results

Questions.

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Let $T\colon L^p(\Omega) \to L^p(\Omega)$ be a positive contraction and let $2 < q < \infty$.

Theorem

(1) We have an estimate

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(1) is 'direct', (2) follows from (1) and square function estimates.