# Weakly wandering vectors

Vladimir Müller

Timisoara, 2010

joint work with Yu. Tomilov, Torun

**Definition**

Let $T \in B(H)$. A vector $x \in H$ is called wandering for $T$ if $T^n x \perp T^m x$ for all $n \neq m$.

**Definition**

Let $T \in B(H)$. A vector $x \in H$ is called wandering for $T$ if $T^n x \perp T^m x$ for all $n \neq m$.

**Definition**

Let $T \in B(H)$. A vector $x \in H$ is weakly wandering for $T$ if the orbit $\{T^n x : n = 0, 1, \dots\}$ contains infinitely many mutually orthogonal vectors.

## Theorem

*(Krengel 1972) Let* $U \in B(H)$ *be a unitary operator. The following statements are equivalent:*

## Theorem

*(Krengel 1972) Let $U \in B(H)$ be a unitary operator. The following statements are equivalent:*

*(i) there exists a dense subset of weakly wandering vectors;*

## Theorem

*(Krengel 1972) Let $U \in B(H)$ be a unitary operator. The following statements are equivalent:*

*(i) there exists a dense subset of weakly wandering vectors;*

*(ii) the spectral measure of $U$ is continuous,*

**Theorem**

*(Krengel 1972) Let $U \in B(H)$ be a unitary operator. The following statements are equivalent:*

*(i) there exists a dense subset of weakly wandering vectors;*

*(ii) the spectral measure of $U$ is continuous,*

*i.e., $\sigma_p(U) = \emptyset$.*

### Example

Let $T = \operatorname{diag}\left\{\frac{n}{n+1} : n = 1, 2, \dots\right\}$.

**Example**

Let $T = \operatorname{diag}\left\{\frac{n}{n+1} : n = 1, 2, \dots\right\}$.

Then no orbit of $T$ for $x \neq 0$ contains two orthogonal vectors.

**Example**

Let $k \in \mathbb{N}$, $\mu = e^{2\pi i/k}$,

## Example

Let $k \in \mathbb{N}$, $\mu = e^{2\pi i/k}$, $S = \bigoplus_{j=1}^{k} \mu^j T$.

**Example**

Let $k \in \mathbb{N}$, $\mu = e^{2\pi i/k}$, $S = \bigoplus_{j=1}^{k} \mu^j T$.

Then $\operatorname{card} \sigma(T) \cap \mathbb{T} = k$ and no orbit of $T$ for $x \neq 0$ contains $k+1$ mutually orthogonal vectors.

Conjecture: if $\operatorname{card}\sigma(T)\cap\mathbb{T}=k$ then there are orbits containing $k$ mutually orthogonal vectors.

Conjecture: if $\operatorname{card}\sigma(T)\cap\mathbb{T}=k$ then there are orbits containing $k$ mutually orthogonal vectors.

NO

Conjecture: if $\operatorname{card} \sigma(T) \cap \mathbb{T} = k$ then there are orbits containing $k$ mutually orthogonal vectors.

NO

**Example**

Let $\mu = e^{2\pi i/7}$,

Conjecture: if $\operatorname{card}\sigma(T)\cap\mathbb{T}=k$ then there are orbits containing $k$ mutually orthogonal vectors.

NO

**Example**

Let $\mu=e^{2\pi i/7}$, $V=T\oplus\mu T\oplus\mu^3 T$.

Conjecture: if $\operatorname{card} \sigma(T) \cap \mathbb{T} = k$ then there are orbits containing $k$ mutually orthogonal vectors.

NO

### Example

Let $\mu = e^{2\pi i/7}$, $V = T \oplus \mu T \oplus \mu^3 T$.

Then $\operatorname{card} \sigma(T) \cap \mathbb{T} = 3$ but no orbit of $T$ for a nonzero vector $x$ contains two orthogonal vectors.

### Theorem

*Let* $T \in B(H)$ *be a power bounded operator,* $\operatorname{card} \sigma(T) \cap \mathbb{T}$ *infinite and* $\sigma_p(T) \cap \mathbb{T} = \emptyset$.

**Theorem**

*Let $T \in B(H)$ be a power bounded operator, $\operatorname{card} \sigma(T) \cap \mathbb{T}$ infinite and $\sigma_p(T) \cap \mathbb{T} = \emptyset$. Then there exists a dense subset consisting of weakly wandering vectors.*

### Theorem

*(Jacobs, de Leeuw, Glicksberg) Let $T \in B(H)$ be power bounded, $\sigma_p(T) \cap \mathbb{T} = \emptyset$. Then*

$$D - \lim \langle T^n x, y \rangle = 0$$

*for all $x, y \in H$.*

**Theorem**

*(Jacobs, de Leeuw, Glicksberg) Let* $T \in B(H)$ *be power bounded,* $\sigma_p(T) \cap \mathbb{T} = \emptyset$*. Then*

$$D - \lim \langle T^n x, y \rangle = 0$$

*for all* $x, y \in H$*.*

The density of a set $A \subset \mathbb{N}$ is

$$\mathrm{Dens}\,(A) = \lim_{n \to \infty} n^{-1} \mathrm{card}\,(A \cap \{1, \ldots, n\})$$

## Theorem

*(Jacobs, de Leeuw, Glicksberg) Let $T \in B(H)$ be power bounded, $\sigma_p(T) \cap \mathbb{T} = \emptyset$. Then*

$$D - \lim \langle T^n x, y \rangle = 0$$

*for all $x, y \in H$.*

The density of a set $A \subset \mathbb{N}$ is

$$\mathrm{Dens}\,(A) = \lim_{n\to\infty} n^{-1}\mathrm{card}\,(A \cap \{1, \dots, n\})$$

$D - lim\ a_n = a \iff$ there exists $A \subset \mathbb{N}$ of density 0 such that $\lim_{n\to\infty, n\notin A} \ a_n = a$.

## Lemma

*Let* $T \in B(H)$ *be power bounded,* $\sigma_p(T) \cap \mathbb{T} = \emptyset$, $\lambda \in \sigma(T) \cap \mathbb{T}$,

## Lemma

*Let* $T \in B(H)$ *be power bounded,* $\sigma_p(T) \cap \mathbb{T} = \emptyset$*,* $\lambda \in \sigma(T) \cap \mathbb{T}$*,* $\varepsilon > 0$*,* $M \subset H$*,* $\operatorname{codim} M < \infty$*,* $n \in \mathbb{N}$*.*

## Lemma

*Let* $T \in B(H)$ *be power bounded,* $\sigma_p(T) \cap \mathbb{T} = \emptyset$*,* $\lambda \in \sigma(T) \cap \mathbb{T}$*,* $\varepsilon > 0$*,* $M \subset H$*,* $\operatorname{codim} M < \infty$*,* $n \in \mathbb{N}$*. Then there exists* $x \in M$*,* $\|x\| = 1$ *such that*

$$\|T^j x - \lambda^j x\| < \varepsilon \qquad (j = 1, \dots, n).$$

## Lemma

*Let $A \subset \mathbb{T}$ be an infinite subset, $\alpha_1, \dots, \alpha_k \in \mathbb{T}$, $\varepsilon > 0$.*

**Lemma**

*Let $A \subset \mathbb{T}$ be an infinite subset, $\alpha_1, \dots, \alpha_k \in \mathbb{T}$, $\varepsilon > 0$. Then there exists $n \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_k \in A$ such that*

$$|\lambda_j^n - \alpha_j| < \varepsilon \qquad (j = 1, \dots, k).$$

## Lemma

*Let $A \subset \mathbb{T}$ be an infinite subset, $\alpha_1, \dots, \alpha_k \in \mathbb{T}$, $\varepsilon > 0$. Then there exists $n \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_k \in A$ such that*

$$|\lambda_j^n - \alpha_j| < \varepsilon \qquad (j = 1, \dots, k).$$

*Moreover, the set of such $n$ is of positive density.*