Finite rank operators in Lie ideals of nest algebras

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- I. Notation
- II. Rank 1 operators
- III. Finite rank operators
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I. Notation

• $\mathcal{H}$ is a complex Hilbert space; $B(\mathcal{H})$ is the set of bounded linear operators on $\mathcal{H}$

• projection P in $B(H)$

$$P^2 = P \quad \text{and} \quad P^* = P$$

• P, Q projections

$$P \leq Q \qquad \text{if} \qquad PQ = P (= QP)$$

• The set of projections together with the partial order relation "$\leq$" is a complete lattice.

I. Notation

- **Nest $\mathcal{N}$**
a totally ordered family of projections $\mathcal{N} \subseteq B(\mathcal{H})$ containing 0 and the identity I

- **Complete nest $\mathcal{N}$**
if $\mathcal{N}$ is a complete sublattice of the lattice of projections in $B(\mathcal{H})$

- $P \in \mathcal{N}$

$$P_- = \bigvee \{Q \in \mathcal{N} : Q < P\}$$

- **Continuous nest $\mathcal{N}$**

$$P_- = P \qquad \text{for all} \quad P \in \mathcal{N}$$

I. Notation

- **Nest algebra $\mathcal{T}(\mathcal{N})$**

  all operators $T \in B(\mathcal{H})$ such that, for all $P \in \mathcal{N}$,

  $$T(P(\mathcal{H})) \subseteq P(\mathcal{H})$$

  equivalently

  $$P^{\perp}TP = 0$$

  where

  $$P^{\perp} = I - P$$

- **Continuous nest algebra $\mathcal{T}(\mathcal{N})$** – nest $\mathcal{N}$ is continuous

  (From now on all nests considered will be continuous nests)

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I. Notation

Nest algebra $\mathcal{T}(\mathcal{N})$ with product

$$[T, S] = TS - ST$$

is Lie algebra

- **Lie ideal $\mathcal{L}$**

complex subspace $\mathcal{L}$ of the nest algebra $\mathcal{T}(\mathcal{N})$ s. t.

$$[\mathcal{L}, \mathcal{T}(\mathcal{N})] \subseteq \mathcal{L}$$

II. Rank 1 operators

- **rank 1 operator** $\quad x \otimes y : \mathcal{H} \to \mathcal{H}$

$$z \mapsto \langle z, x \rangle y \qquad x, y, z \in \mathcal{H}$$

- $x \otimes y \in \mathcal{T}(\mathcal{N})$ iff $P\_x = 0$ and $Py = y$ $(P \in \mathcal{N})$

  where

$$P = \bigwedge \{ Q \in \mathcal{N} : Qy = y \}$$

  (cf. [3])

- Consequence: $x \perp y$ (since the nest $\mathcal{N}$ is continuous)

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Projections associated to $x \otimes y$

- $$\hat{P}_x = \bigvee \{Q \in \mathcal{N} : Qx = 0\}$$
- $$P_y = \bigwedge \{Q \in \mathcal{N} : Qy = y\}$$

Consequences:

1. $P_y y = y$ and $\hat{P}_x x = 0$
2. $x \otimes y \in \mathcal{T}(\mathcal{N})$ iff $P_y \leq \hat{P}_x$
3. $x \otimes y \in \mathcal{T}(\mathcal{N}) \Rightarrow P_y x = 0$

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II. Operators of rank 1

Theorem

$\mathcal{T}(\mathcal{N})$ continuous nest algebra, $\mathcal{L}$ norm closed Lie ideal,
$x \otimes y \in \mathcal{L}$ and $w \otimes z \in \mathcal{T}(\mathcal{N})$ satisfying

$$\hat{P}_x \le \hat{P}_w \qquad \text{and} \qquad P_z \le P_y.$$

Then, $w \otimes z \in \mathcal{L}$.

The "corner" of $x \otimes y$

$$\begin{bmatrix} 0 & P_y \mathcal{T}(\mathcal{N}) \hat{P}_x^{\perp} \\ 0 & 0 \end{bmatrix}$$

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$\mathcal{T}(\mathcal{N})$ continuous nest algebra, $\mathcal{L}$ norm closed Lie ideal,
$x \otimes y \in \mathcal{L}$ *and* $w \otimes z \in \mathcal{T}(\mathcal{N})$ *satisfying*

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II. Operators of rank 1

Sketch of Proof.

- Proving that $x \otimes z \in \mathcal{L}$ when $P_z < P_y$

Define $y' = P_z^{\perp} y$ $(\Rightarrow y' \neq 0)$

$$P_z y' = 0 \quad \Rightarrow \quad P_z \leq \hat{P}_{y'} \quad \Rightarrow y' \otimes z \in \mathcal{T}(\mathcal{N})$$

Therefore

$$\mathcal{L} \ni [x \otimes y, y' \otimes z] = < z, x > (y' \otimes y) - < y, y' > (x \otimes z)$$
$$= - < y, y' > (x \otimes z) = -\|y'\|^2 (x \otimes z)$$

Hence, $x \otimes z \in \mathcal{L}$.

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Hence, $x \otimes z \in \mathcal{L}$.

Sketch of Proof (continuation).

- $x \otimes z \in \mathcal{L}$ when $P_z < P_y$ (proved)
- Proving that $x \otimes z \in \mathcal{L}$ when $P_z = P_y$

$$P_y(\mathcal{H}) = P_z(\mathcal{H}) = \overline{\bigcup_{P \in \mathcal{N}, P < P_z} P(\mathcal{H})}$$

There exists a sequence (z_n)

$$(z_n) \quad \text{lies in} \quad \bigcup_{P \in \mathcal{N}, P < P_z} P(\mathcal{H}) \quad \text{with} \quad z_n \longrightarrow z$$

Therefore

$$x \otimes z_n \longrightarrow x \otimes z \quad \text{and} \quad x \otimes z \in \mathcal{L} \quad (\text{note: } x \otimes z_n \in \mathcal{L})$$

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1. $x \otimes z \in \mathcal{L}$ (proved)
2. $w \otimes y \in \mathcal{L}$ (similar)
3. $w \otimes z$

By 1. above,

$$x \otimes z \in \mathcal{L}$$

Applying 2. to $x \otimes z$

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II. Operators of rank 1

Recall that a mapping $\varphi : \mathcal{N} \to \mathcal{N}$, defined on a nest $\mathcal{N}$, is called a *homomorphism* if, for all projections P and Q in $\mathcal{N}$,

$$P \leq Q \quad \Longrightarrow \quad \varphi(P) \leq \varphi(Q).$$

A homomorphism φ is said to be *left order continuous* if, for all subsets $\mathcal{M}$ of the nest $\mathcal{N}$, the projection $\varphi(\bigvee \mathcal{M})$ is equal to the supremum $\bigvee \varphi(\mathcal{M})$.

II. Operators of rank 1

Proposition

$\mathcal{T}(\mathcal{N})$ continuous nest algebra; $\mathcal{L}$ norm closed Lie ideal
Let, for all $P \in \mathcal{N}$,

$$P' = \bigvee \left\{ P_y \in \mathcal{N} : x \otimes y \in \mathcal{L} \ \wedge \ \hat{P}_x < P \right\} \tag{1}$$

Then

- *the mapping $P' \mapsto P$ is a left order continuous homomorphism*
- $$P \leq P' \qquad\qquad \text{for all} \quad P \in \mathcal{N}$$

II. Operators of rank 1

Characterisation of the rank 1 operators in $\mathcal{L}$

Lemma

$\mathcal{T}(\mathcal{N})$ continuous nest algebra, $\mathcal{L}$ norm closed Lie ideal

Then

$x \otimes y \in \mathcal{L}$ *if and only if, for all projections* $P \in \mathcal{N}$,

$$P'^{\perp}(x \otimes y)P = 0$$

Here $P \mapsto P'$ is the left order continuous homomorphism defined above.

Decomposability of the finite rank operators in $\mathcal{L}$

Theorem

$\mathcal{T}(\mathcal{N})$ continuous nest algebra, $\mathcal{L}$ norm closed Lie ideal,
$T \in \mathcal{L}$ finite rank operator

Then

T can be written as a finite sum of rank one operators lying in $\mathcal{L}$.

Sketch of Proof.

- Assertion holds if $T = 0$ or if T is a rank one operator.
- $T \in \mathcal{L}$ operator of rank $n \geq 2$

  It is possible to write

$$T = \sum_{i=1}^{n} x_i \otimes y_i$$

  where, for all $i = 1, \ldots, n$,

$$x_i \otimes y_i \in \mathcal{T}(\mathcal{N}) \qquad \text{(cf. [1, 3])}$$

  Suppose that (without loss of generality)

$$\hat{P}_{x_1} \leq \hat{P}_{x_2} \leq \cdots \leq \hat{P}_{x_n}$$

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Sketch of Proof (continuation).

1. $\hat{P}_{x_1} < \cdots < \hat{P}_{x_n}$
2. $\hat{P}_{x_1} = \hat{P}_{x_2} = \cdots = \hat{P}_{x_n}$
3. $\hat{P}_{x_1} \leq \hat{P}_{x_2} \leq \cdots \leq \hat{P}_{x_n}$

Proof of case

1. $\hat{P}_{x_1} < \cdots < \hat{P}_{x_n}$

$$\begin{aligned}\mathcal{L} \ni \left[\hat{P}_{x_n}, T\right] &= \hat{P}_{x_n}\Big(\sum_{i=1}^{n} x_i \otimes y_i\Big) - \Big(\sum_{i=1}^{n} x_i \otimes y_i\Big)\hat{P}_{x_n} \\ &= \sum_{i=1}^{n} x_i \otimes (\hat{P}_{x_n} y_i) - \sum_{i=1}^{n} (\hat{P}_{x_n} x_i) \otimes y_i \\ &= T - \sum_{i=1}^{n-1} (\hat{P}_{x_n} x_i) \otimes y_i\end{aligned}$$

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$$= \sum_{i=1}^{n} x_i \otimes (\hat{P}_{x_n} y_i) - \sum_{i=1}^{n} (\hat{P}_{x_n} x_i) \otimes y_i$$

$$= T - \sum_{i=1}^{n-1} (\hat{P}_{x_n} x_i) \otimes y_i$$

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$$\mathcal{L} \ni \left[\hat{P}_{x_n}, T\right] = \hat{P}_{x_n}\left(\sum_{i=1}^{n} x_i \otimes y_i\right) - \left(\sum_{i=1}^{n} x_i \otimes y_i\right)\hat{P}_{x_n}$$

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Sketch of Proof (continuation).

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2. $\hat{P}_{x_1} = \hat{P}_{x_2} = \cdots = \hat{P}_{x_n}$
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Hence

$$T_1 = \sum_{i=1}^{n-1} (\hat{P}_{x_n} x_i) \otimes y_i$$

has rank equal to $n-1$ (not difficult to see) and lies in $\mathcal{L}$. If $n = 2$, the proof ends.

If $n > 2$, analogously,

$$\mathcal{L} \ni \left[\hat{P}_{x_{n-1}}, T_1\right] = T_1 - \sum_{i=1}^{n-2} (\hat{P}_{x_{n-1}} x_i) \otimes y_i$$

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III. Finite rank operators

$\hat{P}_x = \bigvee\{Q \in \mathcal{N} : Qx = 0\}$

Repeating the process

$$\mathcal{L} \ni T_{n-1} = (\hat{P}_{x_2} x_1) \otimes y_1$$

Since

$$\hat{P}_{\hat{P}_{x_i} x_1} = \hat{P}_{x_1} \qquad \text{for all} \quad i \in \{2, \dots, n\},$$

it follows that $\quad x_1 \otimes y_1 \in \mathcal{L} \quad$ and $\quad (\hat{P}_{x_i}) x_1 \otimes y_1 \in \mathcal{L}$

recall the "corner" theorem

$$\left[\begin{array}{c|c} 0 & x_1 \otimes y_1 \\ \hline 0 & 0 \end{array}\right]$$

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$$T_{n-2} = (\hat{P}_{x_3}x_1) \otimes y_1 + (\hat{P}_{x_3}x_2) \otimes y_2,$$

similarly yields that,

$$x_2 \otimes y_2 \in \mathcal{L} \qquad\qquad (\hat{P}_{x_i}x_2) \otimes y_2 \in \mathcal{L}$$

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Go back again...

Proof complete after repeating this reasoning sufficiently many times.

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Characterisation of the finite rank operators in $\mathcal{L}$

Theorem

$\mathcal{T}(\mathcal{N})$ continuous nest algebra, $\mathcal{L}$ norm closed Lie ideal, T finite rank operator

Then

$T \in \mathcal{L}$ *if and only if, for all projections* $P \in \mathcal{N}$,

$$P'^{\perp} T P = 0$$

III. Finite rank operators

$P \mapsto P' = \bigvee \left\{ P_{\mathbf{y}} \in \mathcal{N} : \mathbf{x} \otimes \mathbf{y} \in \mathcal{L} \ \wedge \ \hat{P}_{\mathbf{x}} < P \right\}$

Proof. Consequence of the decomposability of the finite rank operators and the characterisation of rank 1 operators in $\mathcal{L}$.

recall the lemma

$$x \otimes y \in \mathcal{L} \quad \text{iff} \quad P'^{\perp}(x \otimes y)P = 0$$

III. Finite rank operators

Let

- $\mathcal{F}_{\mathcal{L}}$ - set of finite rank operators in $\mathcal{L}$
- $\mathcal{B} = \{S \in B(\mathcal{H}) : P'^{\perp} SP = 0\}$ associative ideal of $\mathcal{T}(\mathcal{N})$
- $\mathcal{F}_{\mathcal{L}}$ associative ideal (since $\mathcal{F}_{\mathcal{L}} \subseteq \mathcal{B}$)

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IV. Example

Continuity of the nest is important

Let

- $\mathcal{N}$ - nest such that

$$\exists_{P \in \mathcal{N}} \qquad \dim(P - P_-)(\mathcal{H}) \geq 2.$$

- $\mathcal{L}$ - norm closed subspace generated by the projection $P - P_-$ and

$$\left\{ S \in \mathcal{T}(\mathcal{N}) : S = P_- S P_-^\perp + (P - P_-) S P^\perp \right\} \quad \left(\text{associative ideal}\right)$$

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V. Compact operators

Define, for all projections P in the nest $\mathcal{N}$,

$$\mathcal{Z}_P = P(\mathcal{H}).$$

Recall for a $\mathcal{T}(\mathcal{N})$-bimodule $\mathcal{J}$

- **support function** $\Phi_{\mathcal{J}}$

$$\mathcal{Z}_P \mapsto \Phi_{\mathcal{J}}(\mathcal{Z}_P) \qquad \text{with} \qquad \Phi_{\mathcal{J}}(\mathcal{Z}_P) = \overline{\mathcal{J}(\mathcal{Z}_P)}$$

- $\mathcal{T}(\mathcal{N})$-bimodule

$$\mathrm{Bim}(\Phi_{\mathcal{J}}) = \{T \in B(\mathcal{H}) : T\mathcal{Z}_P \subseteq \Phi_{\mathcal{J}}(\mathcal{Z}_P)\}$$

(cf. [2]).

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V. Compact operators

Denote by $\mathcal{K}(\mathcal{H})$ the associative ideal of compact operators in $B(\mathcal{H})$.

Corollary

$\mathcal{T}(\mathcal{N})$ continuous nest algebra, $\mathcal{L}$ norm closed Lie ideal, $\mathcal{F}_{\mathcal{L}}$ set of finite rank operators in $\mathcal{L}$

Then

- $$\overline{\mathcal{F}_{\mathcal{L}}} = Bim(\Phi_{\mathcal{F}_{\mathcal{L}}}) \cap \mathcal{K}(\mathcal{H})$$
- *$Bim(\Phi_{\mathcal{F}_{\mathcal{L}}}) \cap \mathcal{K}(\mathcal{H})$ is an associative ideal of $\mathcal{T}(\mathcal{N})$*

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