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KMS states on groupoid C*-algebras

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KMS states

Motivated by examples of Gibbs states in statistical mechanics and quantum field theory, Kubo, Martin and Schwinger have introduced the following definition.

Definition

Let A be a C*-algebra, σ_t a strongly continuous one-parameter group of automorphisms of A and $\beta \in \mathbb{R}$. A state of A φ is called KMS$_\beta$ for σ if for all $a, b \in A$, there is a function F bounded continuous on the strip $0 \leq Imz \leq \beta$ and analytic on $0 < Imz < \beta$ such that:

- $F(t) = \varphi(a\sigma_t(b))$ for all $t \in \mathbb{R}$;
- $F(t + i\beta) = \varphi(\sigma_t(b)a)$ for all $t \in \mathbb{R}$.

A state φ is called tracial if $\varphi(ab) = \varphi(ba)$ for all $a, b \in A$. Thus KMS states generalize tracial states (obtained when $\beta = 0$ or when σ is trivial).

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Gibbs states

In the basic framework of statistical quantum mechanics, the KMS states are exactly the Gibbs states.

The relevant C*-algebra is the algebra $A = \mathcal{K}(\mathcal{H})$ of compact operators on the Hilbert space $\mathcal{H}$. Time evolution is given by a self-adjoint operator H called the hamiltonian.

Definition

The state $\varphi = \dfrac{Tr(\ .\ e^{-\beta H})}{Tr(e^{-\beta H})}$ is called Gibbs state (for the hamiltonian$H$ and at inverse temperature β).

Note that this definition requires $e^{-\beta H}$ to be trace-class. Gibbs states maximize the free energy: $F(\varphi) = S(\varphi) - \beta\varphi(H)$, where the entropy $S(\varphi) = -Tr(\Phi \log \Phi)$, with $\varphi = Tr(\ .\ \Phi)$.

Properties of KMS states

Let A be a C*-algebra and let (σ_t) be a strongly continuous one-parameter group of automorphisms of A.

- KMS states are invariant under σ_t for all t.
- Given $\beta \in \mathbb{R}$, the set Σ_β of KMS$_\beta$ states is a Choquet simplex in A^*, i.e. a $*$-weakly closed convex subset of A^* and each of its elements is the barycenter of a unique probability measure supported on the extremal elements.
- The extremal KMS$_\beta$ states are factorial.

Problem. Given a C*-dynamical system $(A, (\sigma_t))$ as above, determine its KMS$_\beta$. The discontinuities of the map $\beta \mapsto \Sigma_\beta$ are interpreted as phase transitions.

Groupoids

Definition

A groupoid is a small category $(G, G^{(0)})$ whose arrows are invertible.

The elements of $G^{(0)}$ are the units and denoted by $x, y, \ldots$. The elements of G are the arrows and denoted by $\gamma, \gamma', \ldots$. The range and source maps are denoted by $r, s : G \to G^{(0)}$. The inverse map is denoted by $\gamma \mapsto \gamma^{-1}$. The product $(\gamma, \gamma') \mapsto \gamma\gamma'$ is defined on the set $G^{(2)}$ of composable arrows.
A topological groupoid is a groupoid endowed with a topology compatible with the groupoid structure. We shall be chiefly concerned with second countable locally compact Hausdorff groupoids.

Examples

Group actions. Let the group Γ act on the space X through the action map $(x,t)\in X\times\Gamma\mapsto xt\in X$. Then define $G^{(0)}=X$ and

$$G=X\rtimes\Gamma=\{(x,t,y)\in X\times\Gamma\times X : xt=y\}.$$

The range and source maps are $r(x,t,y)=x$ and $s(x,t,y)=y$. The product map is $(x,t,y)(y,t',z)=(x,tt',z)$ and the inverse is $(x,t,y)^{-1}=(y,t^{-1},x)$.

Endomorphisms. Suppose that we have a map T of X into itself, not necessarily invertible. We define $G^{(0)}=X$ and the groupoid of the endomorphism

$$G=G(X,T)=\{(x,m-n,y)\in X\times\mathbf{Z}\times X : m,n\in\mathbf{N}, T^mx=T^ny\}.$$

The maps and operations are the same as above.

Haar systems

Definition

A Haar system for the locally compact groupoid G is a continuous and invariant r-system $(\lambda^x), x \in G^{(0)}$ for the left G-space G. We implicitly assume that all λ^x are non-zero.

If the range map $r : G \rightarrow G^{(0)}$ has countable fibers, the counting measures on the fibers form a Haar system if and only if r is a local homeomorphism. One then says that G is an *étale* groupoid. This is a large and interesting class of groupoids including groupoids of discrete group actions, groupoids of endomorphisms provided they are themselves local homeomorphisms and transverse holonomy groupoids.

Quasi-invariant measures

Let (G, λ) be a locally compact groupoid with Haar system. Given a measure μ on $G^{(0)}$, one defines the measure $\mu \circ \lambda$ (and similarly the measure $\mu \circ \lambda^{-1}$) on G by

$$\int fd(\mu \circ \lambda) = \int \int f(\gamma) d\lambda^x d\mu(x), \quad f \in C_c(G)$$

Proposition

Let μ be a measure on $G^{(0)}$ such that $\mu \circ \lambda$ and $\mu \circ \lambda^{-1}$ are equivalent. Then $D_\mu = d(\mu \circ \lambda)/d(\mu \circ \lambda^{-1})$ is a cocycle with values in $\mathbb{R}_+^*$.

Definition

- One says that a measure μ sur $G^{(0)}$ is quasi-invariant if $\mu \circ \lambda$ and $\mu \circ \lambda^{-1}$ are equivalent. The cocycle D_μ is called its Radon-Nikodym derivative.
- Given a cocycle D with values in $\mathbb{R}_+^*$, we say that μ is a D-measure or is D-invariant if it is quasi-invariant with $D_\mu = D$.

Quasi-invariant measures and KMS states have similar properties. For example, for $G^{(0)}$ compact, the set M_D of D-probability measures is a Choquet simplex in the dual of $C(G^{(0)})$. Its extremal elements are ergodic measures.

Groupoid C*-algebras

Let (G, λ) be a locally compact groupoid with Haar system. The following operations turn $C_c(G)$ into a $*$-algebra:

$$f * g(\gamma) = \int f(\gamma\gamma')g(\gamma'^{-1})d\lambda^{s(\gamma)}(\gamma');$$

$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

The full norm is $\|f\| = \sup \|L(f)\|$ where L runs over all representations in Hilbert spaces. Its completion is the full C*-algebra $C^*(G)$.

The reduced norm is $\|f\|_r = \sup \|\pi_x(f)\|$ where $\pi_x(f)\xi = f * \xi$ for $\xi \in L^2(G_x, \lambda_x)$. Its completion is the reduced C*-algebra $C^*_r(G)$.

Diagonal automorphism groups

Let (G,λ) be a locally compact groupoid with Haar system and let A be one of its C*-algebras, reduced or full.

Proposition

Let c be a continuous cocycle on G with values in $\mathbb{R}$. Then the formula

$$\sigma_t(f)(\gamma) = e^{itc(\gamma)} f(\gamma)$$

defines a one-parameter automorphism group σ of A

KMS weights and quasi-invariant measures

A measure μ on $G^{(0)}$ defines a weight φ_μ on A according to $\varphi_\mu(f) = \int f_{|G^{(0)}} d\mu$ for $f \in C_c(G)$. If G is étale and if μ is a probability measure, φ_μ is a state.

Theorem (R80)

Let G be a groupoid and let c be a cocycle as above. For a measure μ on $G^{(0)}$, the following conditions are equivalent:

1. *The weight φ_μ is KMS_β for the diagonal automorphism group σ defined by the cocycle c.*
2. *The measure μ is quasi-invariant with Radon-Nikodym derivative $D_\mu = e^{-\beta c}$.*

Are all KMS weights of a diagonal automorphism group given by a quasi-invariant measure?

Theorem (Kumjian-R06)

Let G be a groupoid and let c be a cocycle as above.

1. *Let φ be a KMS_β-weight for σ. Then its restriction to the subalgebra $C_c(G^{(0)})$ is a quasi-invariant measure with $D_\mu = e^{-\beta c}$.*
2. *If $c^{-1}(0)$ is principal, every KMS-state for σ is of the form φ_μ.*

Thus we are led to the problem of finding all D-measures, where $D = e^{-\beta c}$ is given.

Gauge group of the Cuntz algebra

Recall the definition of the Cuntz algebra and its gauge automorphism group:

Definition

Let n be an integer ≥ 2. The Cuntz algebra O_n is defined by n generators $S_1, \ldots, S_n$ satisfying the Cuntz relations $S_i^* S_j = \delta_{i,j} \mathrm{I}$ and $\sum_{i=1}^n S_i S_i^* = \mathrm{I}$.

Let e^{it} be a complex number of module 1. Note that $e^{it} S_1, \ldots, e^{it} S_n$ satisfy the Cuntz relations and generate O_n. There exists a unique automorphism σ_t of O_n such that $\sigma_t(S_j) = e^{it} S_j$ for all $j = 1, \ldots, n$. This defines a strongly continuous one-parameter group of automorphisms of O_n called the gauge group of the Cuntz algebra O_n.

its KMS state

This example fits our groupoid framework: let $X = \{1, \ldots, n\}^{\mathbf{N}}$ and let T be the one-sided shift: $T(x_0x_1x_2\ldots) = x_1x_2\ldots$ Introduce $G = G(X, T) = \{(x, m-n, y) : T^m x = T^n y\}$ as before and the cocycle $c : G \to \mathbf{Z}$ given by $c(x, m-n, y) = m - n$.

It is not difficult to see that $C^*(G) = O_n$ and that the gauge group σ is the diagonal automorphism group defined by the cocycle c.

A probability measure on X which is $e^{-\beta c}$-invariant is necessarily invariant under $c^{-1}(0)$. But there is one and only probability measure on X invariant under the tail equivalence relation $c^{-1}(0)$, the product measure $\{1/n, \ldots, 1/n\}^{\mathbf{N}}$. This gives:

Theorem (Olesen-Pedersen, Elliott, Evans)

The gauge group of the Cuntz algebra O_n has a unique KMS state. It occurs at the inverse temperature $\beta = \log n$.

Existence of quasi-invariant measures

As we have seen, the problem of finding KMS states reduces in some cases to finding quasi-invariant measures with a prescribed Radon-Nikodym derivative.

Finding invariant measures for a given dynamical system is a classical problem. In fact, there are two problems, according to whether the measure is finite or not. Along the same lines, the existence of a quasi-invariant measure with a prescribed Radon-Nikodym derivative has recently been studied in

B. Miller, The existence of measures of a given cocycle, I: atomless, ergodic σ-finite measures, II: probability measures, Ergod. Th. & Dynam. Sys. (2008).

references

The existence of atomless, ergodic σ-finite invariant measures is related to the famous Mackey-Glimm dichotomy; see for example

A. Ramsay, The Mackey-Glimm dichotomy for foliations and other Polish groupoids, JFA (1990).

The existence of finite invariant measures has been studied by E. Hopf and more recently by

M. Nadkarni, On the existence of a finite invariant measure, Proc. Indian Acad. Sci. Math. (1990).

The existence of quasi-invariant measures with a given cocycle had been considered earlier by

K. Schmidt, Lectures on cocycles of ergodic transformation groups, Macmillan Lecture Notes in Mathematics, Delhi (1977).

AP equivalence relations

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of compact spaces and for each $n \in \mathbb{N}$, let $\pi_{n+1,n} : X_n \to X_{n+1}$ be a surjective local homeomorphism. Define $\pi_n = \pi_{n,n-1} \circ \cdots \circ \pi_{2,1} \circ \pi_{1,0}$ from $X = X_0$ onto X_n. Consider the equivalence relation $R_n = \{(x, x') \in X \times X \mid \pi_n(x) = \pi_n(x')\}$ endowed with the product topology and the equivalence relation $R = \cup R_n$ endowed with the inductive limit topology. We say that R is an approximately proper equivalence relation.

The Dobrushin-Lanford-Ruelle scheme

Theorem (R05)

Let R be an AP equivalence relation on a compact space X and let $D : R \to \mathbf{R}_+^*$ be a continuous cocycle. Then, there exists at least one D-probability measure.

The idea of the proof is straightforward. Consider first the case of a proper equivalence relation R with quotient map $\pi : X \to \Omega$. The cocycle D can be uniquely written $D(x,y) = \rho(x)/\rho(y)$ where the potential ρ is normalized by $\sum_{\pi(x)=\omega} \rho(x) = 1$ for all $\omega \in \Omega$. Introduce the conditional expectation $E : C(X) \to C(\Omega)$ such that

$$E(f)(\omega) = \sum_{\pi_(x)=\omega} \rho(x) f(x).$$

We make the following observation:

Proposition

A probability measure μ on X is a D-measure if and only there exists a probability measure Λ on Ω such that $\mu = \Lambda \circ E$.

Consider now the case of an AP equivalence relation $R = \cup R_n$. Construct the sequence of compatible expectations $E_n : C(X) \to C(X_n)$. Then, a probability measure μ on X is a D-measure if and only it factors through each E_n. This realizes the set of D-probability measures as a decreasing intersection of non-empty compact convex sets.

The Mackey-Glimm dichotomy

The existence of σ-finite invariant, or more generally D-invariant, ergodic measures is trivial: just consider measures supported on a single orbit. The Mackey-Glimm dichotomy is concerned with the existence of non-trivial ergodic measures. It is usually stated for a Borel countable equivalence relation R on a standard Borel space X.

One says that R is smooth if there is a countable Borel cover (B_i) of X such that $\cup_i R_{|B_i}$ is reduced to the diagonal Δ.

Theorem (Mackey-Glimm dichotomy)

Let R be as above. Then the following conditions are equivalent

1. *R is not smooth.*
2. *There exists an atomless invariant ergodic σ-finite measure.*

The main step for $1 \Rightarrow 2$ is the construction of a Borel subset $Y \subset X$ such that $R_{|Y}$ is a non-proper AP equivalence relation. Then pick an atomless ergodic probability measure μ on Y invariant under $R_{|Y}$ and propagate it to a measure on X invariant under R.

A measure μ on Y which is D_Y-invariant, where D_Y is the restriction of a cocycle D to $R_{|Y}$, can be propagated to σ-finite D-measure in a similar fashion. The previous result on AP equivalence relations does not ensures the existence of μ since D_Y is not necessarily continuous (even if D is continuous).

B. Miller gives the following D-version of the Mackey-Glimm dichotomy. Let $D : R \rightarrow \mathbf{R}_+^*$ be a Borel cocycle on R. Let us say that D is σ-discrete if there is a countable Borel cover (B_i) of X and open neighborhoods U_i of 1 in $\mathbf{R}_+^*$ such that $\cup_i R_{|B_i} \cap D^{-1}(U_i)$ is reduced to the diagonal Δ.

Theorem (Miller 08)

Let R be as above. Then the following conditions are equivalent:

1. *D is not σ-discrete.*
2. *There exists an atomless D-invariant ergodic σ-finite measure.*

Compressibility

Let us turn now to the existence of D-invariant probability measures. The classical obstruction to the existence of an invariant probability measure is compressibility.

Definition

A Borel groupoid G on $X = G^{(0)}$ is compressible if there is a Borel bisection S such that $s(S) = X$ and $[X \setminus r(S)] = X$.

Theorem (Nadkarni 90)

Let G be as above. Then the following conditions are equivalent

1. *G is not compressible.*
2. *There exists an invariant probability measure.*

B. Miller gives a D-version of the theorem of Nadkarni. He introduces several equivalent definitions of D-compressibility which reduce to usual compressibility when $D = 1$. These definitions are rather technical and are not reproduced here. They use the conditional expectations E_S associated to a proper sub-equivalence relation $S \subset R$.

Theorem (Miller 08)

Let R be as above. Then the following conditions are equivalent

1. *D is not compressible.*
2. *There exists a D-invariant probability measure.*

Endomorphisms

Here is an explicit construction of a quasi-invariant measure with a prescribed cocycle. This nice example is due to

M. Ionescu and A. Kumjian, Hausdorff measures and KMS states, arXiv:1002.0790v1.

As previously, we consider the Deaconu groupoid

$$G(X,T)=\{(x,m-n,y): x,y\in X;\ m,n\in\mathbb{N} \text{ et } T^m x = T^n y\}$$

where X is a locally compact Hausdorff space and $T: X\to X$ is a local homeomorphism.

A continuous cocycle $D: G\to \mathbf{R}_+^*$ is given by a continuous function $\psi: X\to\mathbf{R}_+^*$ according to the formula

$$D(x,m-n,y)=\frac{\psi(x)\psi(Tx)\dots\psi(T^{m-1}x)}{\psi(y)\psi(Ty)\dots\psi(T^{n-1}y)}$$

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Conformal maps

Proposition

Assume that there exists a conformal metric for T, i.e. a metric d defining the topology of X such that for all $x \in X$,

$$(*) \qquad \lim_{y \to x} \frac{d(Tx, Ty)}{d(x, y)} = \psi(x).$$

Then, the Hausdorff measure μ of d is D^{-s}-invariant, where s is the Hausdorff dimension of (X, d).

Proof. As a consequence of its definition and of $(*)$, the s-Hausdorff measure μ satisfies $dT^*\mu/d\mu = \psi^s$.

An example

Given $0 < r_1, \ldots, r_n < 1$, there is a unique metric d on $X = \{1, \ldots, n\}^{\mathbf{N}}$ such that $\mathrm{diam}(Z(x_0 x_1 \ldots x_N)) = r_{x_0} r_{x_1} \ldots r_{x_N}$ and $\mathrm{diam}(X) = 1$. It easy to check that its Hausdorff dimension is the unique solution of the equation $(**) \quad \sum_{i=1}^{n} r_i^s = 1$ and that the Hausdorff measure satisfies $\mu(Z(x_0 x_1 \ldots x_N)) = r_{x_0}^s r_{x_1}^s \ldots r_{x_N}^s$.

The one-sided shift T is conformal with respect to d: $\lim_{y \to x} \frac{d(Tx,Ty)}{d(x,y)} = \psi(x) = 1/r_{x_0}$. Therefore the pair (μ, s) satisfies $dT^*\mu/d\mu = \psi^s$. Moreover, since T is expansive (i.e. there exists $\epsilon > 0$ such that for all $x \neq y$, there exists n such that $d(T^n x, T^n y) \geq \epsilon$) and exact (for all non-empty open set $U \subset X$, there is n such that $T^n(U) = X$), it is the only solution.

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Given $0 < r_1, \dots, r_n < 1$, there is a unique metric d on $X = \{1, \dots, n\}^{\mathbf{N}}$ such that $\mathrm{diam}(Z(x_0 x_1 \dots x_N)) = r_{x_0} r_{x_1} \dots r_{x_N}$ and $\mathrm{diam}(X) = 1$. It easy to check that its Hausdorff dimension is the unique solution of the equation $(**)$ $\quad \sum_{i=1}^n r_i^s = 1$ and that the Hausdorff measure satisfies $\mu(Z(x_0 x_1 \dots x_N)) = r_{x_0}^s r_{x_1}^s \dots r_{x_N}^s$.

The one-sided shift T is conformal with respect to d: $\lim_{y \to x} \frac{d(Tx,Ty)}{d(x,y)} = \psi(x) = 1/r_{x_0}$. Therefore the pair (μ, s) satisfies $dT^*\mu/d\mu = \psi^s$. Moreover, since T is expansive (i.e. there exists $\epsilon > 0$ such that for all $x \neq y$, there exists n such that $d(T^n x, T^n y) \geq \epsilon$) and exact (for all non-empty open set $U \subset X$, there is n such that $T^n(U) = X$), it is the only solution.

Generalized gauge group of the Cuntz algebra

Let us give a C*-algebraic translation of the above result: define the generalized gauge group σ of the Cuntz algebra by $\sigma_t(S_j) = r_j^{-it} S_j$ for all $j = 1, \ldots, n$.

Theorem (Evans)

The above generalized gauge group of the Cuntz algebra O_n has a unique KMS state. It occurs at the inverse temperature s determined by $(**)$ and it is given by the above measure μ.