Composition operators in the Dirichlet space

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$$\mathcal{D} := \{f \in \mathrm{Hol}(\mathbb{D}) : \mathcal{D}(f) := \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) \, < \, \infty\},$$

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- Hence $\mathcal{D} \subsetneq H^2(\mathbb{D})$, where

$$H^2(\mathbb{D}) := \left\{ f(z) = \sum_{n=0}^{\infty} a_n \, z^n : \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}.$$

Theorem (Fatou)

If $f \in H^2$, then

$$f^*(e^{i\theta}) := \lim_{r\to 1} f(re^{i\theta}), \qquad (\textit{exists a.e. on } \mathbb{T}).$$

Moreover,

$$\|f^*\|_{L^2} = \|f\|_{H^2}.$$

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Theorem (Beurling)

If $f \in \mathcal{D}$, then f^* exists n.e. on $\mathbb{T}$.

- Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic. The *composition operator* C_φ is defined by

$$C_\varphi(f) = f \circ \varphi, \qquad (f \in \mathrm{Hol}(\mathbb{D})).$$

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Problem

When is C_φ bounded, compact or *Hilbert-Schmidt* on $\mathcal{D}$?

Boundedness

Theorem (Littlewood's Subordination Principle)

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic. Then C_φ is bounded *on H^2 and*

$$\left\| C_\varphi(f) \right\|_{H^2} \leq \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}\; \|f\|_{H^2}. \tag{1}$$

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- $$C_\varphi \text{ bounded on } \mathcal{D} \Longrightarrow \varphi \in \mathcal{D}, \; (\text{ Take } f(z) = z).$$

- $$\varphi \in \mathcal{D} \not\Longrightarrow C_\varphi \text{ bounded on } \mathcal{D}.$$

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Let $\varphi \in \mathcal{D}$ *and let* $d\mu(z) = |\varphi'(z)|^2 \, dA(z)$.

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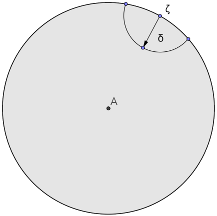
$$C_\varphi \text{ bounded on } \mathcal{D} \iff \mu \circ \varphi^{-1}(S(\zeta,\delta)) = O(\delta^2),$$

where $S(\zeta,\delta) = \{z \in \mathbb{D} : |z-\zeta| < \delta\}$, *for* $\zeta \in \mathbb{T}$ *and* $0 < \delta \leq 2$.



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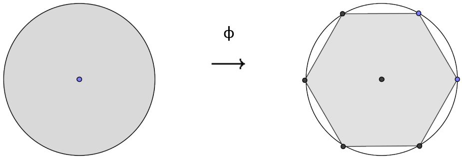
- the image of $\mathbb{D}$ under φ is contained in a polygon inscribed in $\mathbb{D}$:



Proposition

$$C_\varphi \text{ compact on } H^2 \iff f_n \longrightarrow 0 \text{ weakly in } H^2 \Longrightarrow \|C_\varphi(f_n)\|_2 \longrightarrow 0.$$

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- Since $z^n \longrightarrow 0$ weakly in H^2, we have:

$$\begin{aligned}\|C_\varphi(z^n)\|_2^2 &= \|\varphi^n\|_2^2 \\ &= \frac{1}{2\pi}\int_0^{2\pi} |\varphi(e^{i\theta})|^{2n}\, d\theta \longrightarrow 0.\end{aligned}$$

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- In particular,

$$C_\varphi \text{ compact on } H^2 \Longrightarrow \left|\{e^{i\theta} : |\varphi(e^{i\theta})| = 1\}\right| = 0.$$

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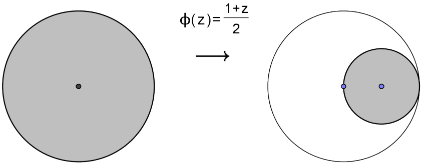
Answer: No. For example : $\varphi(z) = (1+z)/2$ (Schwartz, '69).



- Consider the normalized reproducing kernels for H^2 :

$$k_z(\omega) := \frac{K_z(\omega)}{\|K_z\|} = \frac{\sqrt{1-|z|^2}}{1-\bar{z}\omega}, \qquad (z, \omega \in \mathbb{D}).$$

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$$k_z(\omega) := \frac{K_z(\omega)}{\|K_z\|} = \frac{\sqrt{1-|z|^2}}{1-\bar{z}\omega}, \qquad (z,\omega \in \mathbb{D}).$$

- We easily see that $(k_z)_{z\in\mathbb{D}} \longrightarrow 0$ weakly in H^2, as $|z| \longrightarrow 1$.

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$$\lim_{|z|\to 1} \frac{1-|\varphi(z)|}{1-|z|} = \infty.$$

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Theorem (Julia-Carathéodory)

Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be holomorphic, and let $\zeta \in \mathbb{T}$. The following are equivalent.

(i) $\liminf_{z\to\zeta} \dfrac{1-|\varphi(z)|}{1-|z|} = \delta < \infty$;

(ii) $\angle \lim_{z\to\zeta} \dfrac{\varphi(z)-\lambda}{z-\zeta}$ *exists for some* $\lambda \in \mathbb{T}$;

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Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be holomorphic.

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C_φ *compact on* $H^2 \Longrightarrow \varphi$ *has an angular derivative at no point of* $\mathbb{T}$.

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$$\left\{\begin{array}{l} \varphi \text{ has an angular derivative at no point of } \mathbb{T} \\ \\ \text{and} \\ \\ \left|\{e^{i\theta} : |\varphi(e^{i\theta})| = 1\}\right| = 0 \end{array}\right. \Longrightarrow C_\varphi \text{ compact on } H^2?$$

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Answer: No (Shapiro-MacCluer, '86).

For $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$, *Nevanlinna's counting function* N_φ is defined by

$$N_\varphi(\omega) = \begin{cases} \displaystyle\sum_{z \in \varphi^{-1}(\omega)} \log \frac{1}{|z|}, & \omega \in \varphi(\mathbb{D}); \\ 0, & \omega \notin \varphi(\mathbb{D}). \end{cases}$$

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Theorem (Shapiro, '87)

C_φ is compact on H^2 if and only if

$$\lim_{|\omega| \to 1} \frac{N_\varphi(\omega)}{1 - |\omega|} = 0.$$

General criteria

- For $p \geq 1$ and $\alpha > -1$, the *weighted Bergman space* $\mathcal{A}^p_\alpha$ is defined by

$$\mathcal{A}^p_\alpha := \{f \in \mathrm{Hol}(\mathbb{D}) : \|f\|^p_{p,\alpha} := \int_{\mathbb{D}} |f(z)|^p \, dA_\alpha(z) < \infty\},$$

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- We set

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- The classical Besov space $\mathcal{B}_p = \mathcal{D}^p_{p-2}$.

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- The classical Besov space $\mathcal{B}_p = \mathcal{D}^p_{p-2}$.
- The classical Dirichlet space $\mathcal{D} = \mathcal{D}^2_0$.

- Given $p > 1$ and $\alpha > -1$, take $\beta \geq 0$ such that

$$\delta := \delta(p, \alpha, \beta) = 2 + \beta - (2 + \alpha)/p > 0.$$

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- For $\lambda \in \mathbb{D}$, we set

$$F_\lambda(z) = \frac{(1 - |\lambda|^2)^\delta}{(1 - \overline{\lambda} z)^{1+\beta}}, \qquad (z \in \mathcal{D}).$$

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Theorem (EKSY)

Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$. Then

(a) *C_φ bounded on $\mathcal{D}^p_\alpha \iff \|C_\varphi(F_\lambda)\|_{\mathcal{D}^p_\alpha} = O(1)$;*

(b) *C_φ compact on $\mathcal{D}^p_\alpha \iff \|C_\varphi(F_\lambda)\|_{\mathcal{D}^p_\alpha} = o(1)$, as $|\lambda| \to 1$.*

Corollary

Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$. *Then*

(a) C_φ *bounded on* $H^2 \iff \|C_\varphi(k_\lambda)\|_2 = O(1)$;

(b) C_φ *compact on* $H^2 \iff \|C_\varphi(k_\lambda)\|_2 = o(1)$, *as* $|\lambda| \to 1$.

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Corollary (Tjani, '03)

Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ *and let* $p > 1$. *Then*

(a) C_φ *bounded on* $\mathcal{B}_p \iff \|C_\varphi(b_\lambda)\|_{\mathcal{B}_p} = O(1)$;

(b) C_φ *compact on* $\mathcal{B}_p \iff \|C_\varphi(b_\lambda)\|_{\mathcal{B}_p} = o(1)$, *as* $|\lambda| \to 1$,

where, for $\lambda \in \mathbb{D}$,

$$b_\lambda(z) = \frac{\lambda - z}{1 - \overline{\lambda} z}, \qquad (z \in \mathbb{D}).$$

Corollary (Zhu, '01)

Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$. *Then*

(a) C_φ *bounded on* $\mathcal{D} \iff \|C_\varphi(b_\lambda)\|_{\mathcal{D}} = O(1)$;

(b) C_φ *compact on* $\mathcal{D} \iff \|C_\varphi(b_\lambda)\|_{\mathcal{D}} = 1 + o(1)$, *as* $|\lambda| \to 1$.

Hilbert-Schmidt membership

- Let H be a separable Hilbert space. An operator $T : H \longrightarrow H$ is called *Hilbert-Schmidt* if
$$\sum_{n=0}^{\infty} \|Te_n\|^2 < \infty,$$
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- C_φ is Hilbert-Schmidt on H^2 if and only if

$$\begin{aligned}\sum_{n=0}^{\infty} \|C_\varphi(z^n)\|^2 &= \sum_{n=0}^{\infty} \|\varphi^n\|^2 \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{1 - |\varphi(e^{i\theta})|^2} < \infty.\end{aligned}$$

Hence

$$C_\varphi \text{ Hilbert-Schmidt on } H^2 \Longrightarrow \left|\{e^{i\theta} : |\varphi(e^{i\theta})| = 1\}\right| = 0.$$

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Theorem (EKSY)

Let $E \subset \mathbb{T}$ be closed, and $|E| = 0$. There exists $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$, with $\varphi \in A(\mathbb{D})$ such that C_φ is a Hilbert-Schmidt operator on H^2 and

$$E = \{e^{i\theta} : |\varphi(e^{i\theta})| = 1\}.$$

C_φ Hilbert-Schmidt on H^2, if and only if

$$\int_0^{2\pi} \frac{d\theta}{1 - |\varphi(e^{i\theta})|^2} < \infty \iff \int_0^1 \frac{|E_\varphi(s)|}{(1-s)^2}\, ds < \infty,$$

where

$$E_\varphi(s) := \{e^{i\theta} : |\varphi(e^{i\theta})| > s\}, \qquad (0 < s < 1).$$

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Theorem (EKSY)

If C_φ Hilbert-Schmidt on $\mathcal{D}$, then

$$\int_0^1 \frac{cap\,(E_\varphi(s))}{1-s} \log\frac{1}{1-s}\,ds < \infty. \tag{2}$$

Corollary (Gallardo-Gonzáles, '03)

$$C_\varphi \textit{ Hilbert-Schmidt on } \mathcal{D} \Longrightarrow cap\left(\{e^{i\theta} : |\varphi(e^{i\theta})| = 1\}\right) = 0.$$