Almost Invariant Half-Spaces

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This is joint work with

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Motivation

Invariant subspace problem

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- Lomonosov - for operators commuting with a compact operator
- Enflo - first example of a bounded operator without invariant subspaces
- Read - bounded operator on ℓ_1 without invariant subspaces
- Argyros and Haydon - example of a Banach space such that every bounded operator is a compact perturbation of a multiple of identity

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Does every bounded linear operator on a Banach space have almost invariant half-spaces?

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Proposition

Let $T \in \mathcal{L}(X)$ and $H \subseteq X$ be a half-space. Then H is almost invariant under T if and only if H is invariant under $T + K$ for some finite rank operator K.

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Donoghue operators

A Donoghue operator is a weighted shift $D : l_2 \to l_2$, $De_1 = 0$, $De_i = w_i e_{i-1}$ for $i > 1$ where $(w_i)_i$ is a sequence of non-zero complex numbers such that $(|w_i|)_i$ is monotone decreasing and in l_2.

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D has only invariant subspaces of finite dimension and D^* has only invariant subspaces of finite codimension.

The Method (sketch)

For a nonzero vector $e \in X$ and for $\lambda \in \rho(T)^{-1}$ define a vector $h(\lambda, e)$ in X by

$$h(\lambda, e) := \left(\lambda^{-1} I - T\right)^{-1}(e).$$

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$$T h(\lambda, e) = \lambda^{-1} h(\lambda, e) - e$$

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Lemma

Let X be a Banach space, $T \in \mathcal{L}(X)$ and $e \in X$ be an arbitrary non-zero vector. Let $A \subseteq \rho(T)^{-1}$. Then the closed subspace Y of X defined by

$$Y = \overline{\operatorname{span}}\{h(\lambda, e) \colon \lambda \in A\}$$

is a T-almost invariant subspace (which is not not necessarily a half-space).

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Let $T \in \mathcal{L}(X)$ is such that T has no eigenvalues. Then, for any nonzero vector $e \in X$ the set $\{h(\lambda, e) : \lambda \in \rho(T)^{-1}\}$ is linearly independent.

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Thus, for any $A \subseteq \rho(T)^{-1}$ with infinite cardinality we have that

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is infinite dimensional and T-almost invariant with 1-dimensional "error".

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How can we choose $A \subseteq \rho(T)^{-1}$ in such a way that Y is also infinite codimensional?

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Condition (2) is satisfied by many important classes of operators. For example:

- if 0 is an isolated point of $\sigma(T)$ (in particular, if T is quasinilpotent)
- if 0 belongs in the unbounded component of $\rho(T)$

Main Result

Theorem

Let X be a Banach space and $T \in \mathcal{L}(X)$ satisfy the following:

1. *T has no eigenvalues.*
2. *$\rho(T)^{-1}$ has a connected component $\mathcal{C}$ such that $0 \in \overline{\mathcal{C}}$ and $\mathcal{C}$ contains a neighbourhood of ∞.*
3. *There is a vector whose orbit is a minimal sequence.*

Then T has an almost invariant half-space.

Corollary

If $X = \ell_p$ $(1 \leq p < \infty)$ or c_0 and $T \in \mathcal{L}(X)$, is a weighted right shift operator with weights converging to zero then both T and T^* have almost invariant half-spaces.

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$D \in \mathcal{L}(\ell_2)$ is defined by:

$$De_1 = 0, \quad De_i = w_i e_{i-1}, \quad i > 1,$$

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If D is a Donoghue operator then both D and D^* have almost invariant half-spaces with one dimensional "error".

Donoghue operators do not have invariant half-spaces, yet they have almost-invariant half-spaces with one dimensional "error".

Proof (sketch) - construction of the subspace

Let $e \in X$ be such that $(T^i e)_{i=0}^{\infty}$ is minimal.

Let (c_i) be a sequence of positive real numbers so that c_i converges to 0 "fast" and in particular $\sqrt[i]{c_i} \to 0$.

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For every $k = 0, 1, \ldots,$ put $F_k(z) = z^k F(z)$. Then

$$F_k(z) = \sum_{i=0}^{\infty} c_i^{(k)} z^i = \sum_{i=k}^{\infty} c_{i-k} z^i$$

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As before, f_k is bounded, so it can be extended to $\overline{\text{span}}\{T^i e \colon i \in \mathbb{N}\}$, and then by Hahn-Banach to a bounded functional on all of X.

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$$\lambda_n F_k(\lambda_n) = \lambda_n^{k+1} F(\lambda_n) = 0.$$

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However $f_N(T^M e) = 0$ by definition of f_N while $\sum_{k=M}^{N-1} a_k f_k(T^M e) = a_M c_0 \neq 0$, contradiction.

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