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THE GRAVITATIONAL FIELD OF AN ELECTRICALLY CHARGED MASS POINT AND THE CAUSALITY PRINCIPLE IN THE RTG

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We find the effective Riemannian space–time corresponding to the gravitational field generated by a charged mass point in the framework of the relativistic theory of gravity. The causality principle plays an important role in solving this problem. The analytic form and the domain of definition, i.e., the gravitational radius, of the obtained solution differ from the corresponding results in Einstein's general relativity theory.

Keywords: relativistic theory of gravity, gravitational fields, causality principle

I dedicate this paper to the memory of my teacher, Professor E. Soós

1. Introduction

The problem in this paper is to find the effective Riemannian space–time corresponding to the gravitational field generated by a charged mass point in the framework of the relativistic theory of gravity (RTG).

In Sec. 2, we present the basics of this new RTG elaborated by Logunov and his coworkers (see [1]–[3]). In the RTG, a gravitational field is determined unambiguously by solving the complete system of equations of the RTG. The causality principle (CP) in the RTG permits selecting those RTG solutions that can have a physical meaning.

In Sec. 3, we present a formulation of the basic laws of electromagnetism in the vacuum in the presence of the gravitational field. The presentation is based on the behavior of an electromagnetic field being formulatable in a four-dimensional manifold without a supplementary mathematical structure (see Secs. 266–275 in [4] and Secs. 1–3 in [5]).

The problem of finding the field of an electrically charged mass point was solved in the framework of Einstein's general relativity theory (GRT) by Nordström and Jeffrey (see Sec. 56 in [6]). At the beginning of Sec. 4, we present the solution they obtained. But as was shown by Logunov and his coworkers, a unique solution of the problem cannot be obtained in the GRT without introducing prior assumptions (see Sec. 12 in [1]). Moreover, the gravitational radius of the source point as a function depending on q^2 and m^2, where q and m are the respective electric charge and mass of the point source, has a discontinuity at $q^2 = m^2$. In the RTG, this problem was first analyzed by Karabut and Chugreev [7], but only for $m^2 \geq q^2$. In Sec. 4.1, we present their solution and verify that it satisfies the CP in the RTG and is therefore a physically meaningful solution.

Soós and we [8] reanalyzed the problem in the RTG, also considering the case $q^2 > m^2$. It is important to analyze this case because just this variant holds for the electron. The analytic form of our solution as well as its domain of definition, i.e., the gravitational radius r_{g}, depend essentially on the relation between

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q^2 and m^2. But in Sec. 4.2, we show that this solution does not satisfy the CP in the RTG. Therefore, this solution cannot be an acceptable solution in this theory. In Sec. 4.2, we find a unique solution for the gravitational field generated by a charged mass point in accordance with the RTG. The obtained solution has the same analytic form for all possible relations between q^2 and m^2. The gravitational radius depends on this relation, but it is a continuous function depending on q^2 and m^2.

2. Equations of the RTG and the causality principle

The RTG was constructed by Logunov and his coworkers (see [1]–[3]) as a field theory of the gravitational field in the framework of special relativity. The Minkowskian space–time is a fundamental space that incorporates all physical fields including gravitation. The length element of this space is

$$d\sigma^2 = \gamma_{mn}(x)\, dx^m\, dx^n, \tag{2.1}$$

where x^m, $m = 1, 2, 3, 4$, is an admissible coordinate system in the underlying Minkowski space–time and $\gamma_{mn}(x)$ are the components of the Minkowskian metric in the assumed coordinate system.

The gravitational field is described by a second-order symmetric tensor $\phi^{mn}(x)$ whose action generates an effective Riemannian space–time. A basic assumption of the RTG is that the behavior of matter in the Minkowskian space–time with the metric $\gamma_{mn}(x)$ under the influence of the gravitational field $\phi^{mn}(x)$ is identical to its behavior in the effective Riemannian space–time with the metric $g_{mn}(x)$ determined according to the rules

$$\tilde{g}^{mn} = \sqrt{-g}g^{mn} = \sqrt{-\gamma}\gamma^{mn} + \sqrt{-\gamma}\phi^{mn}, \qquad g = \det g_{mn}, \qquad \gamma = \det \gamma_{mn}. \tag{2.2}$$

Such an interaction of the gravitational field with matter was called the geometrization principle of the RTG.

The behavior of the gravitational field is governed by the differential equations of the RTG (relativistic units are used in all equations)

$$R^m_n - \frac{1}{2}\delta^m_n R + \frac{m_{\mathrm{g}}^2}{2}\left(\delta^m_n + g^{mk}\gamma_{kn} - \frac{1}{2}\delta^m_n g^{kl}\gamma_{kl}\right) = 8\pi T^m_n, \tag{2.3}$$

$$D_m\tilde{g}^{mn} = \tilde{g}^{mn}_{,m} + \gamma^n_{mp}\tilde{g}^{mp} = 0, \quad m, n, k, l = 1, 2, 3, 4. \tag{2.4}$$

Here, R^m_n is the Ricci tensor corresponding to g_{mn}, $R = R^m_m$ is the scalar curvature, δ^m_n is the Kronecker symbol, m_{g} is the graviton mass, T^m_n denotes the energy–momentum tensor of the sources of the gravitational field, D_m is the operator of covariant differentiation with respect to the metric γ_{mn}, γ^n_{mp} are the components of the metric connection generated by γ_{mn}, and the comma marks the derivative with respect to the corresponding coordinate. Equations (2.3) and (2.4) are covariant under arbitrary coordinate transformations with a nonzero Jacobian and are form invariant only with respect to coordinate transformations that leave the Minkowskian metric (2.1) form invariant. These equations imply that the gravitational field admits only states with spin 0 and spin 2. This assumption is one of the main postulates of the RTG in [1]. In [2] and [3], Eqs. (2.4), which determine the polarization states of the field, followed because the source of the gravitational field is the universal conserved density of the energy–momentum tensor of the entire matter including the gravitational field.

Because the graviton mass is extremely small ($m_{\mathrm{g}} \simeq 10^{-66}$ grams), we analyze the problem of finding the effective Riemannian space–time corresponding to the gravitational field generated by a charged mass point in the framework of the RTG by considering Eqs. (2.3) without mass terms.

The CP in the RTG (see Sec. 6 in [2]) affirms that any motion of a pointlike test body must occur within the causality light cone of the Minkowski space–time. According to Logunov's analysis, the CP is satisfied if and only if for any isotropic Minkowskian vector u^m, i.e., for any vector u^m satisfying the condition

$$\gamma_{mn} u^m u^n = 0, \tag{2.5}$$

the metric of the effective Riemannian space–time satisfies the restriction

$$g_{mn} u^m u^n \leq 0. \tag{2.6}$$

According to the CP of the RTG, only those solutions of system (2.3), (2.4) that satisfy this restriction can have a physical meaning.

We stress that the CP in the above form can be formulated only in the RTG because only then is the space–time Minkowskian and the gravitational field is described by a second-order symmetric tensor field $\phi_{mn}(x)$, where x^m are the admissible coordinates in the underlying Minkowski space–time, x^1, x^2, and x^3 are the spacelike variables, and x^4 is the timelike variable.

3. Electromagnetic field equations in the RTG

The theory of electromagnetism is a very elegant and formally simple part of physics. The two principles of conservation set up as the basis for this theory are the conservation of charge and the conservation of magnetic flux (see Secs. 266–270 in [4] and also Secs. 1–3 in [5]). These two conservation laws can be formulated in a four-dimensional manifold independent of any space–time geometry. The differential form of the so-called Maxwell–Bateman laws is (see Secs. 1–3 in [5])

$$\begin{aligned} &\Psi^{mn}_{,m} = K^n, \\ &F^{mn}_{,m} = 0, \quad m,n,p = 1,2,3,4, \end{aligned} \tag{3.1}$$

where F^{mn} is a contravariant axial two-vector density representing the electromagnetic field, Ψ^{mn} is a contravariant two-vector density representing the electromagnetic induction, and K^n is a contravariant one-vector density representing the electric current and the charge density. All the fields involved in Eqs. (3.1) depend on an admissible coordinate system (x^n) in the four-dimensional manifold.

The concept of conservation thus formulated has a topological meaning, being independent of the mathematics of length, time, and angles. To interpret Eqs. (3.1) in the familiar terms of electromagnetic theory, a metric structure must be considered on the four-dimensional manifold.

We now assume that our manifold is the Minkowski space–time of the RTG. In this case, we assume that

$$\begin{aligned} &F^{\alpha\beta} = \varepsilon^{\alpha\beta\delta} E_\delta, \qquad F^{\alpha 4} = B^\alpha, \\ &\Psi^{\alpha\beta} = \varepsilon^{\alpha\beta\delta} H_\delta, \qquad \Psi^{\alpha 4} = -D^\alpha, \\ &K^\alpha = j^\alpha, \qquad\qquad K^4 = \rho, \end{aligned} \tag{3.2}$$

where $\alpha, \beta, \delta = 1, 2, 3$, $\varepsilon^{\alpha\beta\delta}$ are the contravariant Ricci antisymmetric symbols, E_δ are the covariant components of the electric field, B^α are the contravariant components of the magnetic field corresponding to the selected Minkowskian coordinate system, which may be inertial or noninertial, H_δ are the covariant components of the magnetic induction, D^α are the contravariant components of the electric induction, and j^α and ρ represent the respective contravariant components of the electric current and the electric charge density.

As a fundamental hypothesis of the relativistic electromagnetism in the vacuum (without gravitation), we assume that F^{mn} and Ψ^{mn} are connected by the Maxwell–Lorentz ether relation (see Sec. 280 in [4] and Sec. 24 in [5]):

$$\Psi^{mn} = \sqrt{-\gamma}\gamma^{mp}\gamma^{nq}\widehat{F}_{pq}, \qquad \widehat{F}_{pq} = \frac{1}{2}\varepsilon_{mnpq}F^{mn}, \quad m,n,p,q = 1,2,3,4, \tag{3.3}$$

where ε_{mnpq} are the Ricci covariant permutation symbols and $\widehat{F}_{pq}$ is the dual of F^{mn}, being a skew-symmetric absolute tensor. From (3.3) in a Minkowskian inertial frame, we obtain the familiar relations characterizing the vacuum,

$$D^\alpha = E_\alpha, \qquad H_\alpha = B^\alpha.$$

We now assume the presence of the gravitational field. Equations (3.1) then remain valid. Taking the geometrization principle of the RTG into account, we assume that the Maxwell–Lorentz ether relation takes the form

$$\Psi^{mn} = \sqrt{-g}g^{mp}g^{nq}\widehat{F}_{pq}, \qquad \widehat{F}_{pq} = \frac{1}{2}\varepsilon_{mnpq}F^{mn}, \quad m,n,p,q = 1,2,3,4, \tag{3.4}$$

in the presence of the gravitational field.

Introducing (3.4) in (3.1) and taking into account that for any skew-symmetric tensor X^{mn} we have the formula

$$\nabla_m X^{mn} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^m}\left(\sqrt{-g}X^{mn}\right),$$

where ∇_m is the operator of covariant differentiation consistent with the metric g_{mn}, we can write the laws describing the behavior of an electromagnetic field in the RTG in the form

$$\begin{aligned} &\nabla_m \widehat{F}^{mn} = J^n, \\ &\nabla_i \widehat{F}^{mn} + \nabla_m \widehat{F}^{ni} + \nabla_n \widehat{F}^{im} = 0, \end{aligned} \tag{3.5}$$

where $J^n \equiv K^n/\sqrt{-g}$.

To obtain the energy–momentum tensor T^m_n of the electromagnetic field (in the vacuum) in the RTG, we start with its expression in the relativistic electromagnetism excluding gravitation (see Sec. 7 in [9]):

$$\sqrt{-\gamma}T^m_n = -\frac{1}{4\pi}\widehat{F}_{np}\Psi^{mp} + \frac{1}{16\pi}\widehat{F}_{pq}\Psi^{pq}\delta^m_n. \tag{3.6}$$

Again using the geometrization principle of the RTG and the assumed Maxwell–Lorentz ether relations (3.4), we can conclude that T^m_n in the RTG has the form

$$T^m_n = -\frac{1}{4\pi}\widehat{F}_{np}\widehat{F}^{mp} + \frac{1}{16\pi}\widehat{F}_{pq}\widehat{F}^{pq}\delta^m_n, \qquad \widehat{F}^{mp} = g^{mi}g^{pj}\widehat{F}_{ij}. \tag{3.7}$$

In GRT, electromagnetic field equations (3.5) and the electromagnetic field energy–momentum tensor T^m_n given by (3.7) have the same form; the important difference is that in the RTG all fields depend on the coordinates in the underlying Minkowski space–time.

4. The gravitational field generated by a charged mass point in the RTG

In the framework of GRT, the problem of finding the field produced by a charged mass point with mass m and electric charge q was solved by Nordström and Jeffrey (see Sec. 56 in [6]). The problem is static and spherically symmetric. The nonzero components of the Riemannian metric, which represent the electrogravitational field, are

$$
\begin{aligned}
g_{11} &= -\frac{1}{1-2m/r+q^2/r^2}, \qquad & g_{22} &= -r^2, \\
g_{33} &= -r^2\sin^2\theta, & g_{44} &= 1-\frac{2m}{r}+\frac{q^2}{r^2},
\end{aligned}
\tag{4.1}
$$

where r and θ are two of the spherical coordinates $\{r,\theta,\varphi,t\}$ centered at the charged mass point and we write the components of the metric in this system. The domains of definition for these coordinates are $0 \le r_{\rm g} < r < \infty$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$, and $-\infty < t < \infty$, where $r_{\rm g}$ is the gravitational radius of the point source. According to GRT, the value of this gravitational radius depends on the relation between q^2 and m^2 as

$$
r_{\rm g} = \begin{cases} m+\sqrt{m^2-q^2}, & q^2 \le m^2, \\ 0, & q^2 > m^2. \end{cases}
\tag{4.2}
$$

We observe that the function $r_{\rm g}$ depending on q^2 and m^2 has a discontinuity at $q^2 = m^2$.

We now consider the problem of finding the effective Riemannian space–time corresponding to the gravitational field generated by a charged mass point in accordance with the RTG. We must solve system of equations (2.3), (2.4), (3.5) for the coordinates of the underlying Minkowski space–time. Only those solutions that satisfy the CP can represent physically acceptable fields.

We assume that the spherical coordinates $\{r,\theta,\varphi,t\}$ centered at the charged mass point are the coordinates in the underlying Minkowski space–time. Solution (4.1) satisfies Eqs. (2.3) without mass terms and (3.5). We now check whether this solution satisfies (2.4).

The metric of the Minkowskian space–time, which corresponds to switching off the gravitational field, is

$$
d\sigma^2 = dt^2 - dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\varphi^2.
\tag{4.3}
$$

From (4.1), we find that the nonzero components of the tensor g^{mn} in this case are

$$
\begin{aligned}
g^{11} &= -\left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right), \qquad & g^{22} &= -\frac{1}{r^2}, \\
g^{33} &= -\frac{1}{r^2\sin^2\theta}, & g^{44} &= \frac{1}{1-2m/r+q^2/r^2}
\end{aligned}
\tag{4.4}
$$

and the determinant of the metric tensor g_{mn} is

$$
g = -r^4\sin^2\theta.
\tag{4.5}
$$

From (4.4), (4.5), and (2.2), we obtain the nonzero components of $\tilde{g}^{mn}$:

$$
\begin{aligned}
\tilde{g}^{11} &= -\sin\theta(r^2-2mr+q^2), \qquad & \tilde{g}^{22} &= -\sin\theta, \\
\tilde{g}^{33} &= -\frac{1}{\sin\theta}, & \tilde{g}^{44} &= \frac{r^2\sin\theta}{1-2m/r+q^2/r^2}.
\end{aligned}
\tag{4.6}
$$

Taking (4.3) into account, we obtain the coefficients of the metric connection generated by γ_{mn}:

$$\gamma_{22}^1 = -r, \qquad \gamma_{33}^1 = -r\sin^2\theta, \qquad \gamma_{12}^2 = \gamma_{13}^3 = \frac{1}{r},$$
$$\gamma_{33}^2 = -\sin\theta\cos\theta, \qquad \gamma_{23}^3 = \cot\theta. \tag{4.7}$$

Taking (4.6) and (4.7) into account and setting $n = 1$ in system (2.4), we obtain an equation that is not satisfied.

Therefore, solution (4.1) written in the spherical coordinates $\{r, \theta, \varphi, t\}$ with the metric tensor of the Minkowski space–time having form (4.3) does not satisfy Eq. (2.4); hence, it is not an admissible solution in the RTG.

To find an admissible solution in RTG, we use the same procedure as in Chap. 13 in [1]. We therefore seek a system of coordinates $\{\eta^i\} = \{R, \Theta, \Phi, T\}$, in which Eqs. (2.3) and (3.5) are satisfied and Eqs. (2.4) establish a one-to-one correspondence with a nonzero Jacobian between the sets of coordinates $\{\eta^i\}$ and $\{\xi^i\} = \{r, \theta, \varphi, t\}$ in the Minkowski space–time. We pass from the variables $\{\xi^i\}$ to the variables $\{\eta^i\}$ assuming that

$$R = R(r), \qquad \Theta = \theta, \qquad \Phi = \varphi, \qquad T = t. \tag{4.8}$$

The function $R(r)$ must satisfy the restrictions

$$R(r) > 0 \quad \text{for } r > r_{\text{g}}, \qquad \frac{dR}{dr} > 0 \quad \text{for } r > r_{\text{g}}. \tag{4.9}$$

The transformation is performed such that when the gravitational field disappears, the metric of the underlying Minkowski space–time becomes

$$d\sigma^2 = dt^2 - dR^2 - R^2\,d\theta^2 - R^2\sin^2\theta\,d\varphi^2. \tag{4.10}$$

System of equations (2.4) allows finding the explicit form of the function $R(r)$. Equations (2.4) can be written in the form (see formulas (13.17) and (13.22) in [1])

$$\frac{1}{\sqrt{-g(\xi)}}\frac{\partial}{\partial\xi^m}\left(\sqrt{-g(\xi)}g^{mn}(\xi)\frac{\partial\eta^p}{\partial\xi^n}\right) = -\gamma_{mn}^p(\eta)\frac{\partial\eta^m}{\partial\xi^i}\frac{\partial\eta^n}{\partial\xi^j}g^{ij}(\xi). \tag{4.11}$$

From (4.10), we obtain the nonzero components $\gamma_{mn}^p(\eta)$:

$$\gamma_{22}^1 = -R, \qquad \gamma_{33}^1 = -R\sin^2\theta, \qquad \gamma_{12}^2 = \gamma_{13}^3 = \frac{1}{R},$$
$$\gamma_{33}^2 = -\sin\theta\cos\theta, \qquad \gamma_{23}^3 = \cot\theta. \tag{4.12}$$

For $p = 2, 3, 4$ with (4.4), (4.5), and (4.12) taken into account, Eqs. (4.11) become identities, while for $p = 1$, the equation becomes

$$\frac{d^2R(r)}{dr^2}(r^2 - 2mr + q^2) + 2(r - m)\frac{dR(r)}{dr} - 2R(r) = 0, \quad r > r_{\text{g}}. \tag{4.13}$$

The gravitational radius r_{g} here is given by expressions (4.2). The solution of this equation depends essentially on the relation between q^2 and m^2 (see Sec. 4 in [8]).

4.1. Case $q^2 \leq m^2$. The solution of Eq. (4.13) obtained in the case $q^2 \leq m^2$ is (see Sec. 4 in [8])

$$R(r) = r - m, \quad r > r_{\rm g} = m + \sqrt{m^2 - q^2}, \tag{4.14}$$

with the admissible minimum value for $R(r)$ being

$$R_{\rm g} = \sqrt{m^2 - q^2}. \tag{4.15}$$

The quantity $R_{\rm g}$ is the gravitational radius of our point source in the system of coordinates $\{\eta^i\}$.

From the tensor transformation law, the nonzero components of the effective Riemannian metric g_{mn} in the new coordinate system $\{\eta^i\} = \{R, \theta, \varphi, t\}$ are

$$\begin{aligned} g_{11} &= -\frac{(R+m)^2}{R^2 - m^2 + q^2}, \qquad & g_{22} &= -(R+m)^2, \\ g_{33} &= -(R+m)^2 \sin^2\theta, \qquad & g_{44} &= \frac{R^2 - m^2 + q^2}{(R+m)^2}. \end{aligned} \tag{4.16}$$

This solution was first obtained in [7]; Soós and we [8] obtained the same result for this case. The analysis of the problem in the framework of the RTG does not stop here. A necessary condition for solution (4.16) to have a physical sense is satisfaction of the CP. Assuming expression (4.10) for the underlying Minkowski space–time, we set the Minkowskian isotropic vector to be $u = (0, 1, 0, R(r))$. Condition (2.6) then becomes

$$(R^2 - m^2 + q^2)R^2 \leq (R+m)^4, \quad R > R_{\rm g}, \tag{4.17}$$

where $R_{\rm g}$ is given by (4.15). It is easy to verify that this restriction is satisfied.

Also, for the Minkowskian isotropic vector $u = (1, 0, 0, 1)$, the causality condition

$$(R^2 - m^2 + q^2)^2 \leq (R+m)^4, \quad R > R_{\rm g}, \tag{4.18}$$

must be satisfied. Because (4.17) holds, inequality (4.18) is obviously satisfied. Therefore, solution (4.16) is a physically acceptable solution in the RTG.

4.2. Case $q^2 > m^2$. According to the empirical values of q and m, exactly the case $q^2 > m^2$ holds for a single electron. Therefore, it is important to see what the solution is in this case. This case was first analyzed in RTG the in [8].

Integrating Eq. (4.13) and choosing one of the two arbitrary integration constants such that $R(r)/r \to 1$ as $r \to \infty$ is ensured, we obtain

$$R(r) = r - m + C\left(1 - \frac{r-m}{p} \arctan \frac{p}{r-m}\right), \quad r > r_{\rm g}, \tag{4.19}$$

where $p = \sqrt{q^2 - m^2}$ and C is a real constant.

The components of the effective Riemannian space–time in the system of coordinates $\{\eta^i\}$ are

$$\begin{aligned} g_{11} &= \left(\frac{dr(R)}{dR}\right)^2 \left(-\frac{1}{1 - 2m/r(R) + q^2/r^2(R)}\right), \qquad & g_{22} &= -r^2(R), \\ g_{33} &= -r^2(R) \sin^2\theta, & g_{44} &= 1 - \frac{2m}{r(R)} + \frac{q^2}{r^2(R)}, \end{aligned} \tag{4.20}$$

where the function $r(R)$ is given implicitly by (4.19).

Because Eqs. (2.4) are general covariant, the system of coordinates $\{\eta^i\}$ is not privileged. Therefore, solution (4.1) is also a solution of this system of equations, but the system of Minkowskian coordinates in which this solution is written is the system in which the Minkowskian metric has the form

$$d\sigma^2 = dt^2 - \left(\frac{dR(r)}{dr}\right)^2 dr^2 - R^2(r)\, d\theta^2 - R^2(r) \sin^2\theta\, d\varphi^2, \tag{4.21}$$

not form (4.3), and $R(r)$ is explicitly given by (4.19).

Therefore, the solution thus obtained can be written either in form (4.20) in the system of coordinates for which the underlying Minkowskian metric has form (4.10) or in form (4.1) in the system of coordinates for which the underlying Minkowskian metric is (4.21). Soós and we [8] considered form (4.20), (4.10).

Apparently, our problem has a family of solutions depending on the parameter C. But the obtained solutions must satisfy the CP in the RTG. Because we do not have the explicit form of the inverse function $r(R)$, we check whether this principle is satisfied using form (4.1), (4.21) of the solutions.

For the Minkowskian isotropic vector $u = (0, 1, 0, R(r))$, causality condition (2.6) becomes

$$(r^2 - 2mr + q^2)R^2(r) \le r^4, \quad r > r_{\mathrm{g}}, \quad R > R_{\mathrm{g}}, \tag{4.22}$$

where $R_{\mathrm{g}} \ge 0$ is the lower admissible bound for the function $R(r)$, whose value must also be determined. Also, for the Minkowskian isotropic vector $u = \bigl(1, 0, 0, dR(r)/dr\bigr)$, the inequality

$$(r^2 - 2mr + q^2)^2 \left(\frac{dR(r)}{dr}\right)^2 \le r^4, \quad r > r_{\mathrm{g}}, \quad R > R_{\mathrm{g}}, \tag{4.23}$$

must hold.

For the considered case $q^2 > m^2$, $r_{\mathrm{g}} = 0$ according to (4.2). We now show that if we impose causality conditions (4.22) and (4.23), then r_{g} cannot be zero. Therefore, the solution obtained in [8] does not satisfy the CP in the RTG. This is the main reason for reanalyzing this problem.

Indeed, if $r_{\mathrm{g}} = 0$ and we let r tend to zero in (4.22), we obtain $R(0) = 0$. This yields

$$C = \frac{m}{1 - (m/p) \arctan(p/m)} \equiv C_1. \tag{4.24}$$

Letting r tend to zero in (4.23), we find $dR(0)/dr = 0$, which implies

$$C = \frac{1}{-(1/p) \arctan(p/m) + m/(m^2 + p^2)} \equiv C_2. \tag{4.25}$$

It is easy to see that $C_1 < C_2$. Hence, we obtain a contradiction because we must have $C_1 = C_2$ from (4.24) and (4.25). Restrictions (4.22) and (4.23) can therefore be satisfied only if $r_{\mathrm{g}} > 0$.

We now find the value of this r_{g}. We return to the conditions that must be satisfied by the function $R(r)$. The function $R(r)$ has analytic expression (4.19) and must satisfy conditions (4.9), (4.22), and (4.23). The real constant C must be such that the positive, increasing function $R(r)$ takes its minimum possible value, denoted by R_{g}, at the minimum possible value $r = r_{\mathrm{g}}$.

From analytic expression (4.19) for $R(r)$, we note that this function has the straight line $R(r) = r - m$ as its asymptote at ∞. Because the function

$$f(r) = 1 - \frac{r - m}{p} \arctan \frac{p}{r - m} \ge 0, \quad r > 0, \tag{4.26}$$

is positive, we find from (4.19) that

$$R(r) \leq r - m \qquad \text{if and only if} \qquad C \leq 0 \tag{4.27}$$

and

$$R(r) > r - m \qquad \text{if and only if} \qquad C > 0. \tag{4.28}$$

We now consider these two possibilities separately.

1. In the case $C \leq 0$, taking expression (4.19) for the function $R(r)$, the first of conditions (4.9) to be satisfied by this function, and relation (4.26) into account, we conclude that

$$r > m. \tag{4.29}$$

Differentiating function (4.19), we obtain

$$\frac{dR(r)}{dr} = 1 + C\biggl(-\frac{1}{p}\arctan\frac{p}{r-m} + \frac{r-m}{(r-m)^2+p^2}\biggr). \tag{4.30}$$

Because the function

$$h(r) = -\frac{1}{p}\arctan\frac{p}{r-m} + \frac{r-m}{(r-m)^2+p^2} \leq 0, \quad r > 0, \tag{4.31}$$

is negative, we obtain

$$\frac{dR(r)}{dr} \geq 1, \quad r > r_{\mathrm{g}}, \tag{4.32}$$

in the considered case. Therefore, the second condition in (4.9) is obviously satisfied.

Causality condition (4.23) must also hold for $r > r_{\mathrm{g}}$, $R > R_{\mathrm{g}}$, and assuming (4.32), we find

$$r \geq \frac{q^2}{2m}. \tag{4.33}$$

In view of (4.27), (4.33), and $r - m < r$, causality condition (4.22) is satisfied.

Conditions (4.29) and (4.33) are necessary conditions. To ensure satisfaction of the second condition in (4.9) and (4.23), the real constant C must respectively satisfy

$$C > \frac{m-r}{1-((r-m)/p)\arctan(p/(r-m))}, \quad r > r_{\mathrm{g}}, \tag{4.34}$$

and

$$C \geq \frac{2mr-q^2}{(r^2-2mr+q^2)\bigl(-(1/p)\arctan(p/(r-m)) + (r-m)/((r-m)^2+p^2)\bigr)}, \quad r > r_{\mathrm{g}}. \tag{4.35}$$

According to (4.29) and (4.33), we find

$$r_{\mathrm{g}} = \begin{cases} m, & m^2 < q^2 < 2m^2, \\ \dfrac{q^2}{2m}, & 2m^2 \leq q^2. \end{cases} \tag{4.36}$$

Taking the criterion for choosing the real constant C and inequalities (4.34) and (4.35), where r_{g} has value (4.36), into account, we finally obtain $C = 0$. Substituting this in (4.19), we obtain

$$R(r) = r - m, \quad r > r_{\mathrm{g}}. \tag{4.37}$$

2. In the case $C > 0$, r also cannot take values smaller than m. Indeed, if r took values smaller than m, then, as follows from (4.31), the point $r = m$ would be a return point for $R(r)$, and $R(r)$ would therefore not be a strictly increasing function.

Moreover, in this case, the second condition in (4.9) is not satisfied. Indeed, because the function $h(r)$ in (4.31) is a monotonically increasing function tending to zero as r tends to ∞, we find that there exists $r > r_{\mathrm{g}} \geq m$ such that $dR(r)/dr = 0$. Therefore, for $C > 0$, the function $R = R(r)$ cannot be strictly increasing function for $r > r_{\mathrm{g}}$.

Summarizing, if the relation between q^2 and m^2 is $q^2 > m^2$, the only function of form (4.19) that satisfies restrictions (4.9), (4.22), and (4.23) is function (4.37), where r_{g} is given by (4.36). Then, the analytic expression for $R = R(r)$ is the same as in the case $q^2 \leq m^2$, but its domain of definition is different.

5. Conclusion

We can conclude that the solution of our problem in accordance with the RTG is unique for all possible relations between m^2 and q^2. The analytic expression of the solution is independent of the relation between m^2 and q^2, but its domain of definition, i.e., r_{g}, depends on this relation. On the basis of (4.19), (4.36), and (4.37), we can write

$$R(r) = r - m, \quad r > r_{\mathrm{g}}, \tag{5.1}$$

where

$$r_{\mathrm{g}} = \begin{cases} m + \sqrt{m^2 - q^2}, & q^2 \leq m^2, \\ m, & m^2 < q^2 < 2m^2, \\ \dfrac{q^2}{2m}, & 2m^2 \leq q^2. \end{cases} \tag{5.2}$$

We note that the function r_{g} depending on q^2 and m^2 is a continuous function.

Both here and in GRT, the expression of the effective Riemannian metric has the same form (4.1) in the system of coordinates $\{\xi^i\} = \{r, \theta, \varphi, t\}$. But in the RTG, these coordinates are the space–time coordinates in the Minkowski universe. The length element of this universe has form (4.21), not form (4.3). Substituting (5.1) in (4.21), we find this form:

$$d\sigma^2 = dt^2 - dr^2 - (r - m)^2\, d\theta^2 - (r - m)^2 \sin^2\theta\, d\varphi^2. \tag{5.3}$$

Comparing (4.2) and (5.2), we emphasize the difference between the gravitational radius in these two theories.

We can also write the components of the effective Riemannian metric in the system of coordinates $\{\eta^i\} = \{R, \theta, \varphi, t\}$, in which the Minkowskian length element has form (4.10):

$$\begin{aligned} g_{11} &= -\frac{(R+m)^2}{R^2 - m^2 + q^2}, \qquad & g_{22} &= -(R+m)^2, \\ g_{33} &= -(R+m)^2 \sin^2\theta, \qquad & g_{44} &= \frac{R^2 - m^2 + q^2}{(R+m)^2}. \end{aligned} \tag{5.4}$$

It follows from (5.1) and (5.2) that the gravitational radius in the system of coordinates $\{\eta^i\}$ becomes

$$R_{\rm g} = \begin{cases} \sqrt{m^2 - q^2}, & q^2 \le m^2, \\ 0, & m^2 < q^2 < 2m^2, \\ \dfrac{q^2 - 2m^2}{2m}, & 2m^2 \le q^2. \end{cases} \tag{5.5}$$

The function $R_{\rm g}$ is also a continuous function of q^2 and m^2.

The result obtained again shows the important role played by the CP in the RTG. Indeed, the analytic expression of the function $R = R(r)$ is much simpler than that obtained in [8].

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