Universal occurrence of localization in continuum random Schrödinger Hamiltonians

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The continuous Anderson Hamiltonian

The Anderson Hamiltonian is the random Schrödinger operator

$$H_\omega := -\Delta + V_\omega \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^d), \tag{1}$$

with

$$V_\omega(x) := \sum_{\zeta \in \mathbb{Z}^d} \omega_\zeta \, u(x-\zeta), \tag{2}$$

where

- The single-site potential $u \geq 0$ is a bounded measurable function on $\mathbb{R}^d$ with compact support, $u \geq c\chi_{\Lambda_\delta}$, $c, \delta > 0$, i.e. u uniformly bounded away from zero in a neighborhood of the origin.
- $\omega = \{\omega_\zeta\}_{\zeta \in \mathbb{Z}^d}$ is a family of independent, identically distributed random variables with common probability distribution μ, such that
  - μ is non-degenerate with compact support $\subset [0,\infty[$.
  - $0 \in \operatorname{supp} \mu$.

  Without loss of generality we may just assume

$$\{0,1\} \in \operatorname{supp} \mu \subset [0,1].$$

Basic properties

- H_ω is a random nonnegative self-adjoint operator.
- H_ω is $\mathbb{Z}^d$-ergodic: there exists an ergodic family $\{\tau_y;\ y \in \mathbb{Z}^d\}$ of measure preserving transformations on the underlying probability space $(\Omega, \mathbb{P})$ such that

$$U(y) H_\omega U(y)^* = H_{\tau_y(\omega)} \quad \text{for all} \quad y \in \mathbb{Z}^d$$

where $(U(y)f)(x) = f(x-y)$. It follows that

  - The spectrum is nonrandom:

$$\sigma(H_\omega) = [0, \infty[\quad \text{with probability one.}$$

  - The pure point, absolutely continuous, and singular continuous components of $\sigma(H_\omega)$ are also nonrandom (i.e., equal to fixed sets) with probability one.

The continuous Poisson Hamiltonian

The Poisson Hamiltonian is the random Schrödinger operator

$$H_{\omega)} := -\Delta + V_\omega \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^d), \tag{3}$$

with

$$V_\omega(x) := \sum_{\zeta \in X(\omega)} u(x-\zeta), \tag{4}$$

where

- The single-site potential $u \geq 0$ is a bounded measurable function on $\mathbb{R}^d$ with compact support, $u \geq c\chi_{\Lambda_\delta}$, $c, \delta > 0$.
- $\omega \to X(\omega) \subset \mathbb{R}^d$ is a Poisson process with density $\rho > 0$.

The family is $\mathbb{R}^d$-ergodic and $\sigma(H_\omega) = [0, \infty[$ a.s.

Related models and generalizations

One may replace $-\Delta$ by

- $-\Delta + V_{\rm per}$ [Kirsch Stolz Stolmann '98] or $-\Delta + V_{\rm bg}$
- $-\nabla \frac{1}{\rho_\omega} \nabla$ [Figotin, Klein '96]
- $(-i\nabla + A)^2$, $d = 2$, constant magnetic field, QHE [Combes Hislop'95, Wang '97, G. Klein'03] [G. Klein Schenker'07]
- $(-i\nabla + A_\omega)^2$ [Ghribi, Hislop, Klopp '07]

Other possibilities

- Replace $u \leq 0$ or u non sign definite [Klopp'95]
- Replace iid random variables ω_i by independant rv.
- Locate the impurities on a Delone set.
- Study the random displacement model [Klopp '93]
- Consider several interacting particles in a random potential [Chulaevski-Suhov '07, Kirsch '07]

Anderson Localization

Definition: The Anderson Hamiltonian H_ω exhibits Anderson localization at the bottom of the spectrum if there exist $E_0 > 0$ and $m > 0$, such that the following holds with probability one:

- H_ω has pure point spectrum in $[0, E_0]$.
- If ϕ is an eigenfunction of H_ω with eigenvalue $E \in [0, E_0]$,

  $$\|\chi_x \phi\| \le C_{\omega,\phi}\, e^{-m|x|} \qquad \text{for all} \quad \mathbf{x} \in \mathbb{R}^d$$

  (χ_x is the characteristic function of a cube of side one centered at x.)
- There exist $\tau > 1$ and $s \in]0,1[$ such that for all eigenfunctions ψ, ϕ (possibly equal) with the same eigenvalue $E \in [0, E_0]$,

  $$\|\chi_x \psi\|\,\|\chi_y \phi\| \le C_{\mathbf{x}}\,\|\psi\|_-\,\|\phi\|_-\, e^{|y|^\tau} e^{-|x-y|^s} \;\; \text{for } x, y \in \mathbb{Z}^d \quad (5)$$

- The eigenvalues of H_ω in $[0, E_0]$ have finite multiplicity.

Dynamical Localization

Definition: The Anderson Hamiltonian H_ω exhibits strong dynamical localization at the bottom of the spectrum if there exist $E_0 > 0$ and $s > 0$ such that

$$\mathbb{E}\left\{\sup_{t\in\mathbb{R}}\left\||x|^{\frac{p}{2}}e^{-itH_\omega}\chi_{[0,E_0]}(H_\omega)\chi_0\right\|_2^{\frac{2s}{p}}\right\} < \infty \quad \text{for all } p \geq 1$$

Existence of localization at the bottom of the spectrum: Poisson

The following theorem is a joint work:

- Germinet, Hislop and Klein [J. Europ. Math. Soc. '07]

Theorem

Let H_ω be the Poisson Hamiltonian on $\mathrm{L}^2(\mathbb{R}^d)$ with Poisson density $\rho > 0$. Then there exists $E(\rho) > 0$ such that H_ω exhibits Anderson localization as well as strong dynamical localization in $[0, E(\rho)]$.

Related results:

- Lifshitz tails [Donsker-Varadhan '75]
- Localization in dimension $d = 1$ [Stolz '95]
- Any d, $u \leq 0$ (then $\sigma(H_\omega) = \mathbb{R}$) [G. Hislop Klein, CRM '07]

Existence of localization at the bottom of the spectrum: Anderson

The following theorem is based on joint work in progress:

- Germinet and Klein
- Aizenman, G., Klein and Warzel [Preprint, available on Arxiv]

Theorem

Let H_ω be the Anderson Hamiltonian on $\mathrm{L}^2(\mathbb{R}^d)$ with single-site probability distribution μ, where

$$\{0,1\} \in \operatorname{supp}\mu \subset [0,1],$$

but μ is otherwise arbitrary. Then H_ω exhibits Anderson localization as well as strong dynamical localization at the bottom of the spectrum.

μ continuous with some regularity

Localization at the bottom of the spectrum in the multi-dimensional case was known in the following cases:

- μ absolutely continuous with a bounded density.
  - Anderson localization: [Combes, Hislop 1994; Klopp 1995; Kirsch, Stollmann, Stolz 1998; Germinet, Klein 2001; Klopp 2002; Germinet, Klein 2003; Aizenman, Elgart, Naboko, Schenker, Stolz 2006, ...]
  - Dynamical localization: [Germinet, De Bièvre 1998; Damanik, Stollmann 2001; Germinet, Klein 2001; Aizenman, Elgart, Naboko, Schenker, Stolz 2006]
- μ Hölder continuous and some log-Hölder continuous: Improvements on the Wegner estimate [Stollmann 2000; Combes, Hislop, Klopp 2007] allow the extension of the proof of localization by a multiscale analysis as in [Germinet, Klein 2001].

The Bernoulli-Anderson Hamiltonian

The ω_ζ's are Bernoulli random variables:

$$\mu(\{0\}) = \mu(\{1\}) = \frac{1}{2}$$

- Anderson localization: [Bourgain, Kenig 2005]
- Dynamical localization: [Germinet, Klein]

How to prove localization

Localization can be proved by a multiscale analysis if we have

- A priori finite volume estimates: we can see the signature of localization at large enough scales.
- A Wegner estimate: control of the size of the finite volume resolvents with sufficient probability.
  - μ regular: Wegner estimate known at all scales.
  - μ Bernoulli and general case: Wegner estimate proved in each scale in the multiscale analysis.

The Theorem is proved by a multiscale analysis as in [Bourgain-Kenig 2005], using free sites, a quantitative unique continuation principle, classes of equivalence of configurations, and a new concentration bound for functions of i.i.d.r.v.'s.

Finite volume operators and free sites

Given a box $\Lambda = \Lambda_L(x)$ in $\mathbb{R}^d$, $Y \subset \Lambda \cap \mathbb{Z}^d$, and $t_Y = \{t_\zeta\}_{\zeta \in Y} \in [0,1]^Y$, let

$$H_{\omega,(Y,t_Y),\Lambda} := -\Delta_\Lambda + V_{\omega,(Y,t_Y),\Lambda} \quad \text{on} \quad \mathrm{L}^2(\Lambda) \tag{6}$$

$$V_{\omega,(Y,t_Y),\Lambda} := \sum_{\zeta \in \Lambda \cap (\mathbb{Z}^d \setminus Y)} \omega_\zeta \, u(x-\zeta) + \sum_{\zeta \in Y} t_\zeta \, u(x-\zeta) \tag{7}$$

$$R_{\omega,(Y,t_Y),\Lambda}(z) := (H_{\omega,(Y,t_Y),\Lambda} - z)^{-1} \tag{8}$$

where $\Delta_\Lambda :=$ Laplacian on Λ with Dirichlet boundary condition.
Definition: A box Λ_L is said to be (ω, Y, E, m)-good if for all $t_Y \in [0,1]^Y$ we have

$$\|R_{\omega,(Y,t_Y),\Lambda}(E)\| \leq \mathrm{e}^{L^{1-}} \tag{9}$$

$$\|\chi_x R_{\omega,(Y,t_Y),\Lambda}(E) \chi_y\| \leq \mathrm{e}^{-m|x-y|} \ \text{if} \ |x-y| \geq \tfrac{L}{10} \tag{10}$$

In this case Y consists of (ω, E)-free sites for the box Λ_L.

"A priori" finite volume estimates

Proposition: Let H_ω be the μ-Anderson Hamiltonian on $\mathrm{L}^2(\mathbb{R}^d)$, fix $p > 0$. Take $q \in \mathbb{N}$ and let $S_\Lambda = \Lambda \cap q\mathbb{Z}^d$ for a box Λ. Then there exists a finite scale $\tilde{L}_{u,\mu,d,p,q}$ and a constant $C_{u,\mu,d,p,q} > 0$, such that for all scales $L \geq \tilde{L}_{u,\mu,d,p,q}$, setting

$$E_L = C_{u,\mu,d,p,q} \left(\log L\right)^{-2} \quad \text{and} \quad m_L = \tfrac{1}{2}\sqrt{E_L}, \tag{11}$$

we have

$$\mathbb{P}\{\Lambda_L \text{ is } (\omega, S_{\Lambda_L}, E, m_L)\text{-good}\} \geq 1 - L^{-pd} \tag{12}$$

for all energies $E \in [0, E_L]$. In fact, for all energies $E \in [0, E_0]$, scales $L \geq \tilde{L}_{u,\mu,d,p,q}$, and boxes Λ_L, we have

$$\|R_{\omega,t_{S_{\Lambda_L}},\Lambda_L}(E)\| \leq E_L^{-1} \tag{13}$$

and

$$\|\chi_y R_{\omega,t_{S_{\Lambda_L}},\Lambda_L}(E)\chi_{y'}\| \leq 2E_L^{-1}\mathrm{e}^{-\sqrt{E_L}|y-y'|} \text{ for } y, y' \in \Lambda_L,\ \left|y - y'\right| \geq 4\sqrt{d} \tag{14}$$

for all $t_{S_{\Lambda_L}} \in [0,1]^{S_{\Lambda_L}}$ with probability $\geq 1 - L^{-pd}$.

The multiscale analysis

Proposition

Fix an energy $E_0 > 0$. *Pick*

$$p = \frac{3}{8}d- \ , \quad \rho_1 = \frac{3}{4}- \ , \quad \text{and} \quad \rho_2 = 0+ ,$$

more precisely, pick $p, \rho_1, \rho_2 = \rho_1^{n_1}$ *with* $n_1 \in \mathbb{N}$ *such that*

$$\tfrac{8}{11} < \tfrac{d}{d+p} < \rho_1 < \tfrac{3}{4} \qquad \text{and} \qquad p < d\left(\tfrac{\rho_1}{2} - \rho_2\right) \tag{15}$$

Let $E \in [0, E_0]$, *and suppose* L *is* (E, m_0)*-localizing for all* $L \in [L_0^{\rho_1 \rho_2}, L_0^{\rho_1}]$, *where*

$$m_0 \geq L_0^{-\tau_0} \quad \text{with} \quad \tau_0 = 0+ < \rho_2 \tag{16}$$

and L_0 *is some sufficiently large scale.*

Then L *is* $(E, \frac{m_0}{2})$*-localizing for all* $L \geq L_0$.

Quantitative Unique Continuation Principle

Lemma (Bourgain-Kenig)

Assume $\Delta\varphi = V\varphi$ *on* $B(0,L) \subset \mathbb{R}^d$ *with* $L \gg 1$, *such that* $\|\chi_0 \varphi\| = 1$, $\|\chi_x \varphi\| \leq C$, $\|V\|_\infty \leq C$. *Let* $|x_0| = R > 1$. *Then*

$$\|\chi_{x_0} \varphi\| \geq c\, \mathrm{e}^{-c\, R^{\frac{4}{3}} (\log R)} \tag{17}$$

What is needed for the Wegner estimate

To obtain the Wegner estimate from the Bourgain-Kenig's quantitative UCP one needs to prove the following:

Consider a box $\Lambda = \Lambda_L$, *let* $\ell = L^\rho$ *with* $\rho = \frac{3}{4}-$ *(so* $L^{\frac{4}{3}\rho} = L^{1-}$*).*
Let $S \subset \Lambda \cap \mathbb{Z}^d$ *with* $|S| = \ell^{d-}$, *fix* $\omega \in [0,1]^{(\Lambda \cap \mathbb{Z}^d) \setminus S}$, *and set*

$$H(t_S) := H_{\omega, t_S, \Lambda} \quad \text{for all} \quad t_S \in [0,1]^S. \tag{18}$$

Consider an energy E_0, *set* $I = (E_0 - \mathrm{e}^{-c_1 \ell}, E_0 + \mathrm{e}^{-c_1 \ell})$. *Let* $E_\tau(t_S)$ *be a continuous eigenvalue parametrization of* $\sigma(H(t_S))$ *such that* $E_\tau(0) \in I$ *(a finite family). Let* $E(t_S) = E_{\tau_0}(t_S)$ *for some* τ_0.
Suppose

$$\mathrm{e}^{-c_3 \ell^{\frac{4}{3}} \log \ell} \leq \frac{\partial}{\partial t_j} E(t_S) \leq \mathrm{e}^{-c_2 \ell} \quad \text{for all} \quad j \in S \quad \text{if} \quad E(t_S) \in I. \tag{19}$$

Let $\omega_S = \{\omega_j\}_{j \in S}$ *be iid random variables with common probability distribution* μ. *Then for all large* L

$$\mathbb{P}\left\{ E(\omega_S) \in (E_0 - \mathrm{e}^{-2c_3 \ell^{\frac{4}{3}} \log \ell}, E_0 + \mathrm{e}^{-2c_3 \ell^{\frac{4}{3}} \log \ell}) \right\} \leq \frac{C}{\ell^{\frac{d}{2}-}} \tag{20}$$

The concentration bound

Theorem (AGKW)

Let F be a real-valued Borel function on $\mathbb{R}^n$ such that for some $\alpha > 0$ we have

$$\alpha t \leq F(\mathbf{t} + t\mathrm{e}_j) - F(\mathbf{t}) \tag{21}$$

for all $t \geq 0$, $\mathbf{t} \in \mathbb{R}^n$, $j = 1, 2, \ldots, n$.
Given random a variable X with non-degenerate probability distribution μ, consider the random variable $Z = F(X_1, X_2, \ldots, X_n)$, where $\{X_i\}_{i=1,\ldots,n}$ are independent copies of X.
Then there exist constants Θ_μ and $s_\mu > 0$ such that

$$\sup_{r \in \mathbb{R}} \mathbb{P}\{Z \in [r, r+s]\} \leq \frac{\Theta_\mu}{\sqrt{n}} \quad \textit{for all } s < s_\mu. \tag{22}$$

Bernoulli decompositions

Let X be a real random variable with distribution μ.

Definition

A representation $X \overset{\mathscr{D}}{=} Y(t) + \delta(t)\,\eta$, where

- η is a $\{0,1\}$-Bernoulli rv with $p := \mathbb{P}(\eta = 1) \in (0,1)$,
- t an independent rv with the uniform distribution on $(0,1)$,
- $Y : (0,1) \to \mathbb{R}$,
- $\delta : (0,1) \to [0,\infty)$,

is called a Bernoulli decomposition of X.

Theorem (AGKW)

Any non-degenerate rv has a Bernoulli decomposition. One may even choose $\inf \delta > 0$.

Application: I. Concentration inequalities

Let $\{\eta_j\}$ be independent copies of a $\{0,1\}$-Bernoulli rv.

Classical Littlewood-Offord inequality:

Let $p = \frac{1}{2}$ and $a_1, \ldots, a_N \in \mathbb{R}$ with $|a_j| > 1$. Then for any interval I of length at most one

$$\mathbb{P}\Big(\sum_{j=1}^{N} a_j \eta_j \in I\Big) \le \frac{\mathrm{const}}{\sqrt{N}}. \qquad \text{Erdös '49}$$

Consequence of

Probabilistic Sperner/LYM inequalities:

Let $\mathscr{A} \subset \{0,1\}^N$ be an **antichain**, i.e., any two $\eta, \eta' \in \mathscr{A}$ are not comparable in the sense of partial order on $\{0,1\}^N$.

Then $\boxed{\mathbb{P}(\eta \in \mathscr{A}) \le \frac{\mathrm{const}}{\sigma_p \sqrt{N}},}$ where $\sigma_p := \sqrt{p(1-p)}$.

New concentration inequality

Theorem

Let $X_1, \ldots X_N$ *independent rv's & pick* $x_- < x_+$ *and* $p_\pm > 0$ *s.t.*

$$\mathbb{P}\{\, X_j \le x_- \,\} \ge p_- \quad \textit{and} \quad \mathbb{P}\{\, X_j \ge x_+ \,\} \ge p_+ \,.$$

Let $\Phi : \mathbb{R}^N \to \mathbb{R}$ *be monotone s.t. for some* $\varepsilon > 0$

$$\Phi(\boldsymbol{t} + v\boldsymbol{e}_j) - \Phi(\boldsymbol{t}) \ge \varepsilon$$

for all $v > x_+ - x_-$, $\boldsymbol{t} \in \mathbb{R}^N$ *and* $j \in \{1, \ldots, N\}$. *Then*

$$\sup_{u \in \mathbb{R}} \mathbb{P}\{\Phi(X_1, \ldots, X_N) \in [u, u + \alpha]\} \le \frac{4}{\sqrt{N}} \sqrt{\frac{1}{p_+} + \frac{1}{p_-}} \,.$$

Remark: For $\Phi(X_1, \ldots, X_N) = \sum_{j=1}^N X_j$ such inequalities go back to Doeblin/Lévy '36, Erdös '49, Kolmogorov '58, Rogozin '61, Esseen '68, Kesten '69.

Application: II. Singularity of random matrices

Theorem (Bruneau, G. '07)

Let $M_N = (m_{ij})$ be a matrix, whose entries are independent rv's. Suppose there is $p \in (0, \frac{1}{2})$ s.t.

$$\mathbb{P}(m_{ij} < x_{ij}^-) > p \quad \text{and} \quad \mathbb{P}(m_{ij} > x_{ij}^+) > p$$

for all $i, j \in \{1, \ldots N\}$ and some $x_{ij}^- < x_{ij}^+$. Then

$$\boxed{\mathbb{P}(M_N \text{ is singular}) \leq \frac{\text{const}}{\sqrt{N}}.}$$

Based on results by Komlós '68.
Improves on a remark of Tao and Vu '06.

The continuous Delone Hamiltonian

Definition

Let $0 < r < R$ be given. A countable subset Q of $\mathbb{R}^d$ is a (r, R)-Delone set iff

- $\mathrm{Card}(Q \cap \Lambda_r) \leq 1$, for any Λ_r;
- $\mathrm{Card}(Q \cap \Lambda_R) \geq 1$, for any Λ_R.

We set $\mathscr{D}_{r,R}$ to be the set of all (r, R)-Delone sets.

The Delone Hamiltonian is the random Schrödinger operator

$$H_Q := -\Delta + V_Q \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^d), \tag{23}$$

with

$$V_Q(x) := \sum_{\zeta \in Q} u(x - \zeta), \tag{24}$$

where

- The single-site potential $\mathrm{L}^\infty(\mathbb{R}^d) \ni u \geq c\chi_{\Lambda_\delta}$, $c, \delta > 0$, is a measurable function on $\mathbb{R}^d$ with compact support.
- Q is a (r, R)-Delone set.

Note that $\inf\{\sigma(H_Q), Q \in \mathscr{D}_{r,R}\} > 0$.

Existence of loc. at the bottom of the spectrum: Delone

The topology in $\mathscr{D}_{r,R}$ is generated by the set of neighborhoods:

$$N(Q,\varepsilon,L) = \{Q', \forall q \in Q \cap \Lambda_L, \mathrm{dist}(q, Q' \cap \Lambda_L) \leq \varepsilon, \text{ and } \longleftrightarrow\}.$$

Theorem (G., Müller - in progress)

- *Let Q be in $\mathscr{D}_{r,R}$. There exists (r,R)-Delone sets Q' arbitrarily close to Q such that $\inf\sigma(H_Q) = \inf\sigma(H_{Q'})$, and $H_{Q'}$ exhibits Anderson localization as well as strong dynamical localization at the bottom of its spectrum.*
- *There exists in $\mathscr{D}_{r,R}$ a dense union of G_δ, such that associated Delone Hamiltonians exhibits Anderson localization as well as strong dynamical localization at the bottom of their spectrum.*

Related result:

- There is a dense G_δ of (r,R)-Delone sets in $\mathscr{D}_{r,R}$ such that the associated Delone Hamiltonian has a singular continuous component in its spectrum [Lenz-Stollmann '06]