Repeated interaction quantum systems: deterministic and random *

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* Joint work with

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The Deterministic Model

Quantum system $\mathcal{S}$:

- Finite dimensional system, driven by Hamiltonian $H_{\mathcal{S}}$ on $\mathfrak{H}_{\mathcal{S}}$, s.t. $\sigma(H_{\mathcal{S}}) = \{e_1, \cdots , e_d\}$.

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Interaction

- W_k operator on $\mathfrak{H}_{\mathcal{S}} \otimes \mathfrak{H}_{\mathcal{E}_k}$, $k \geq 1$.

Repeated Interactions Quantum Systems

Evolution Let $\{\tau_k\}_{k\in\mathbb{N}^*}$, with $\inf_k \tau_k > 0$ be a set of durations.

For $t = \tau_1 + \tau_2 + \cdots + \tau_{m-1} + s,\ \ 0 \le s < \tau_m$,

- $\mathcal{S}$ and $\mathcal{E}_m$ are driven by $H_{\mathcal{S}} + H_{\mathcal{E}_m} + W_m$
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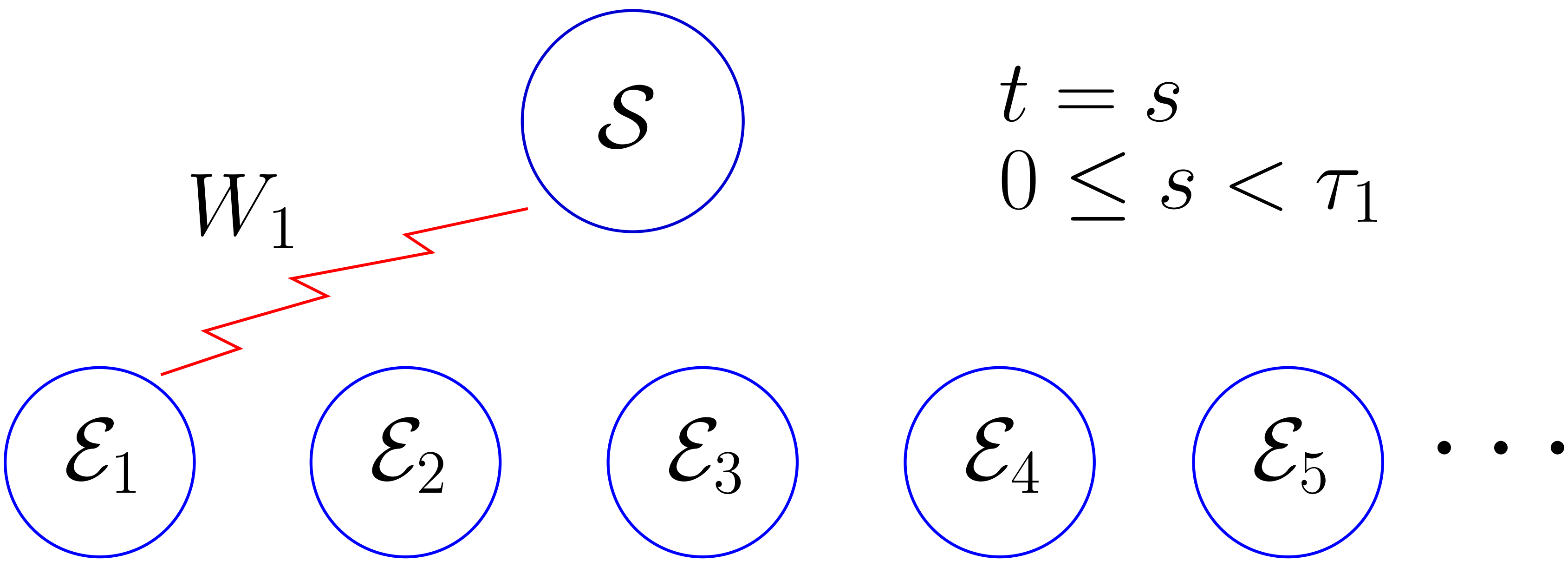

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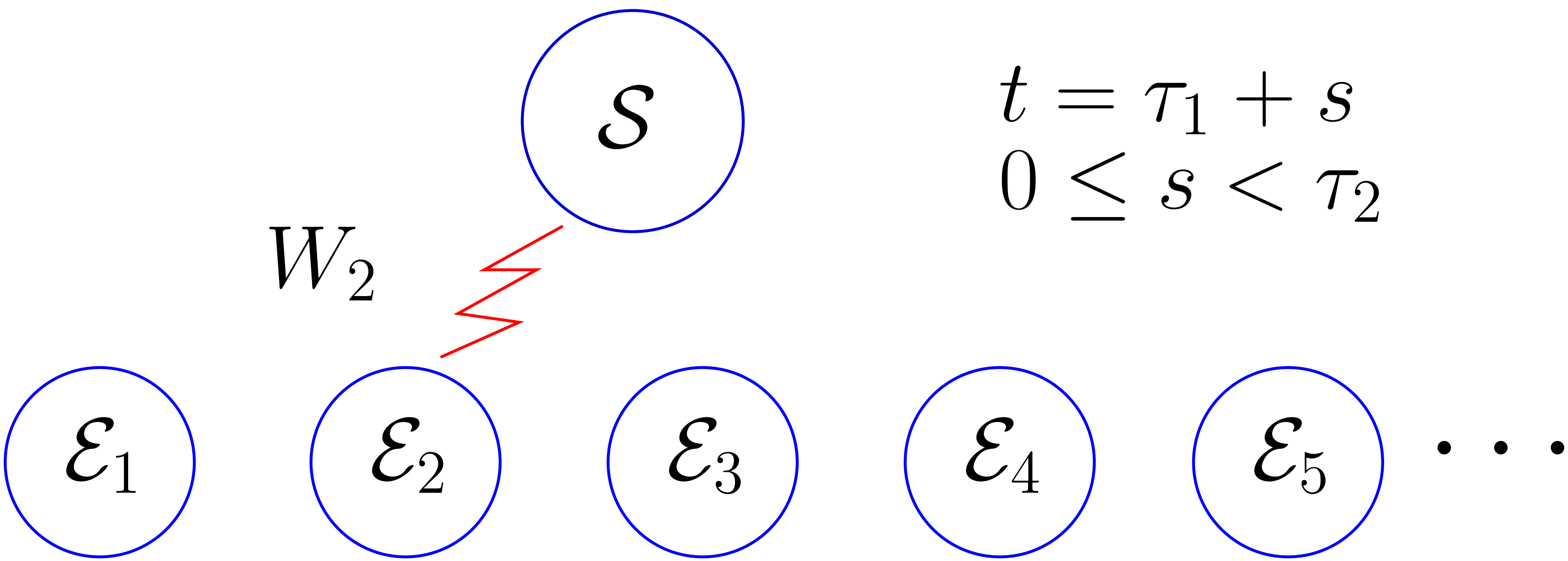

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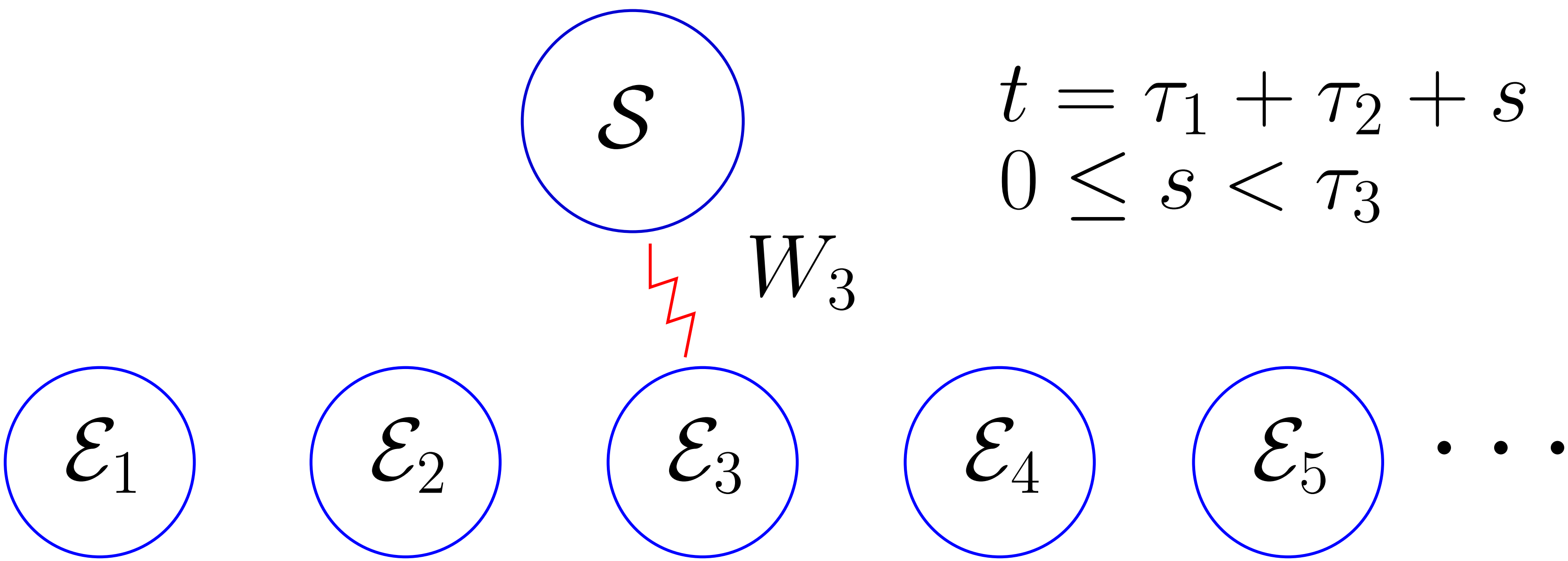

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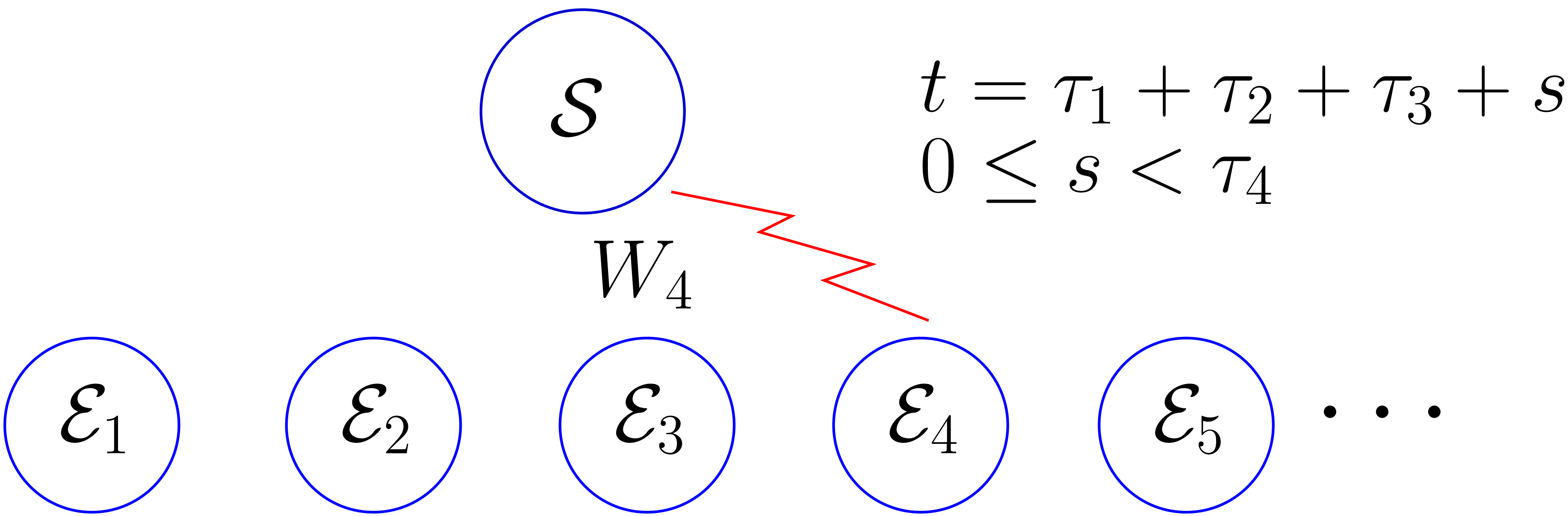

Questions about Open Quantum Systems

Large times asymptotics

Let $A = A_{\mathcal{S}} \otimes \mathbb{I}_{\mathcal{C}} \in \mathcal{B}(\mathfrak{H}_{\mathcal{S}} \otimes \mathfrak{H}_{\mathcal{C}})$ be an observable on acting on $\mathcal{S}$ only

Let $\tau^t(A)$ be its Heisenberg evolution, at time $t = \tau_1 + \tau_2 + \cdots + \tau_m$

Let $\rho : \mathcal{B}(\mathfrak{H}_{\mathcal{S}} \otimes \mathfrak{H}_{\mathcal{C}}) \to \mathbb{C}$ be a state (“density matrix”) on observables of $\mathcal{S} + \mathcal{C}$

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  Non trivial entropy production ?
  Asymptotic 2 nd law of thermodynamics ?

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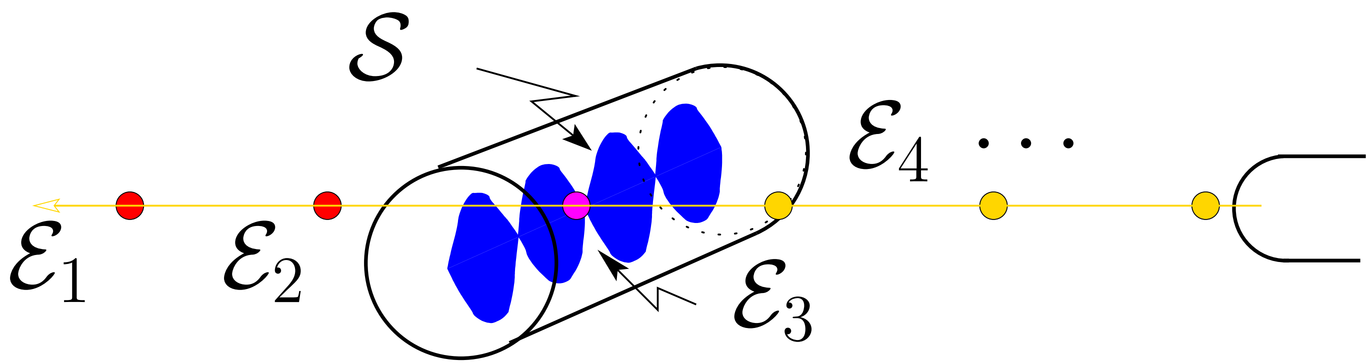
- Non-trivial examples ?

Motivation

Experimental reality

One-atom maser — Walther et al ’85, Haroche et al ’92



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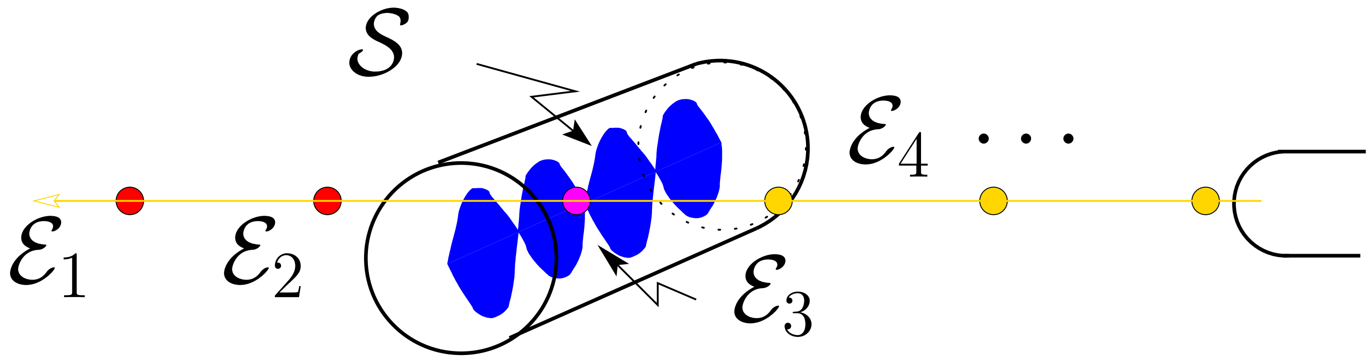

- $\mathcal{S}$: one mode of E-M field in a cavity
- $\mathcal{E}_k$: atom $\#k$ interacting with the mode
- $\mathcal{C}$: sequence of atoms passing through the cavity

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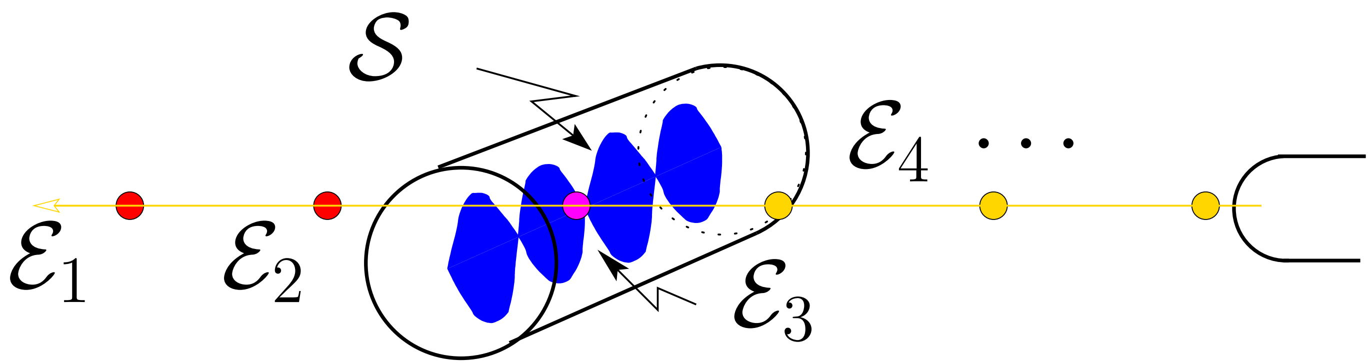

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Ideal RIQS used as models — Vogel et al ’93, Wellens et al ’00

$\Rightarrow$ Random RIQS model fluctuations w.r.t. ideal situation

Mathematical Framework

GNS representation

Let $\rho \in \mathcal{B}_1(\mathfrak{H})$ be a density matrix on $\mathfrak{H}$

$$0 < \rho = \sum \lambda_j |\varphi_j\rangle\langle\varphi_j| \text{ and } \mathrm{Tr}\rho = 1$$

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- $\mathfrak{H} \to \mathcal{H} = \mathfrak{H} \otimes \mathfrak{H}$
- $\rho \in \mathcal{B}_1(\mathfrak{H}) \to \Psi_\rho = \sum_j \sqrt{\lambda_j} \varphi_j \otimes \varphi_j \in \mathcal{H}$
- $A \in \mathcal{B}(\mathfrak{H}) \to \Pi(A) = A \otimes \mathbb{I}_\mathfrak{H} \in \mathcal{B}(\mathcal{H})$

$$\Rightarrow \mathrm{Tr}_\mathfrak{H}(\rho A) = \langle\Psi_\rho|A \otimes \mathbb{I}_\mathfrak{H} \Psi_\rho\rangle_\mathcal{H} = \mathrm{Tr}_\mathcal{H}(|\Psi_\rho\rangle\langle\Psi_\rho|\Pi(A))$$

- $\mathbb{I}_\mathfrak{H} \otimes B \in \mathcal{B}(\mathcal{H})$ don't play any role
- For gases with ∞-ly many particles, GNS is required and non-trivial

Liouvillean

Evolution of observables

$$\tau^t(A) = e^{itH} A e^{-itH} \in \mathcal{B}(\mathfrak{H})$$

Evolution of states

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Given ρ invariant, $\exists$ a unique self-adjoint L on $\mathcal{H} = \mathfrak{H} \otimes \mathfrak{H}$ s.t.

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Simple setup

$$L = H \otimes \mathbb{I}_{\mathfrak{H}} - \mathbb{I}_{\mathfrak{H}} \otimes H$$

Formalization

After GNS (writing A for $\Pi(A)$)

- Hilbert spaces $\mathcal{H}_\mathcal{S}$, $\mathcal{H}_{\mathcal{E}_k}$, and $\mathcal{H}_\mathcal{C} = \mathcal{H}_{\mathcal{E}_1} \otimes \mathcal{H}_{\mathcal{E}_2} \otimes \mathcal{H}_{\mathcal{E}_3} \otimes \cdots$

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$\Psi_{\mathcal{S}} \in \mathcal{H}_{\mathcal{S}}$ and $\Psi_{\mathcal{E}_k} \in \mathcal{H}_{\mathcal{E}_k}$ s.t.

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Entire system

$$\mathcal{S} + \mathcal{C} \text{ on } \mathcal{H} = \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{C}}, \text{ driven by } L_{\text{free}} = L_{\mathcal{S}} + \sum_k L_{\mathcal{E}_k}$$

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Interaction

$$V_k \in \mathfrak{M}_\mathcal{S} \otimes \mathfrak{M}_{\mathcal{E}_k}, \text{ the GNS repres. of } W_k \qquad + \text{ tech. hyp.}$$

Dynamics

Repeated interaction Schrödinger dynamics

For any $m \in \mathbb{N}$, if $t = \tau_1 + \tau_2 + \cdots + \tau_m$ and $\psi \in \mathcal{H}$,

$$U(m)\psi := e^{-i\tilde{L}_m} e^{-i\tilde{L}_{m-1}} \cdots e^{-i\tilde{L}_1} \psi$$

where the generator for the duration τ_m is

$$\tilde{L}_m = \tau_m L_m + \tau_m \sum_{k \neq m} L_{\mathcal{E},k}$$

with

$$\begin{cases} L_m = L_{\mathcal{S}} + L_{\mathcal{E}_m} + V_m & \text{on } \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}_m} \quad \text{coupled} \\ L_{\mathcal{E},k} & \text{on } \mathcal{H}_{\mathcal{E}_k} \quad \text{free} \end{cases}$$

Dynamics

Repeated interaction Schrödinger dynamics

For any $m \in \mathbb{N}$, if $t = \tau_1 + \tau_2 + \cdots + \tau_m$ and $\psi \in \mathcal{H}$,

$$U(m)\psi := e^{-i\tilde{L}_m} e^{-i\tilde{L}_{m-1}} \cdots e^{-i\tilde{L}_1} \psi$$

where the generator for the duration τ_m is

$$\tilde{L}_m = \tau_m L_m + \tau_m \sum_{k \neq m} L_{\mathcal{E},k}$$

with $\begin{cases} L_m = L_{\mathcal{S}} + L_{\mathcal{E}_m} + V_m & \text{on } \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}_m} \quad \text{coupled} \\ L_{\mathcal{E},k} & \text{on } \mathcal{H}_{\mathcal{E}_k} \quad \text{free} \end{cases}$

To be studied

Let $\varrho \in \mathcal{B}_1(\mathcal{H})$ be a state on $\mathcal{H}$ and $A_{\mathcal{S}} \in \mathfrak{M}$ an observable on $\mathcal{S}$

$$m \mapsto \varrho(U^*(m) A_{\mathcal{S}} U(m)) \equiv \varrho(\alpha^m(A_{\mathcal{S}})), \quad \text{as} \quad m \to \infty$$

Reduction to a Product of Matrices

Special state

$\varrho_0 = \langle \Psi_0 | \cdot \Psi_0 \rangle$ where

$\Psi_0 = \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{C}}$ and $\Psi_{\mathcal{C}} = \Psi_{\mathcal{E}_1} \otimes \Psi_{\mathcal{E}_2} \otimes \cdots \in \mathcal{H}_{\mathcal{C}}$

$P = \mathbb{I}_{\mathcal{H}_{\mathcal{S}}} \otimes |\Psi_{\mathcal{C}}\rangle\langle\Psi_{\mathcal{C}}|$ is the projector on $\mathcal{H}_{\mathcal{S}} \otimes \mathbb{C}\Psi_{\mathcal{C}} \simeq \mathcal{H}_{\mathcal{S}}$

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$C-$ Liouvillean Given $L_{\mathcal{S}}$, $L_{\mathcal{E}_m}$ and $V_m \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{E}_m}$,

$$\exists \quad K_m \text{ s.t. } \begin{cases} e^{i\tilde{L}_m} A e^{-i\tilde{L}_m} = e^{iK_m} A e^{-iK_m} \quad \forall A \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{C}} \\ K_m \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{C}} = 0. \end{cases}$$

K_m is not self-adjoint, not even normal ! Jaksic, Pillet '02

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$$K_m = \tau_m(L_{\text{free}} + V_m - V_m'), \quad V_m' = J_m \Delta_m^{\frac{1}{2}} V_m \Delta_m^{-\frac{1}{2}} J_m$$ Tomita-Takesaki '57

Reduction to a Product of Matrices

Evolution of ϱ_0

$$
\begin{aligned}
\varrho_0(\alpha^m(A_S)) &= \langle \Psi_0 | e^{i\tilde{L}_1} \cdots e^{i\tilde{L}_m} A_S e^{-i\tilde{L}_m} \cdots e^{-i\tilde{L}_1} \Psi_0 \rangle \\
&= \langle \Psi_0 | e^{iK_1} \cdots e^{iK_m} A_S e^{-iK_m} \cdots e^{-iK_1} \Psi_0 \rangle \\
&= \langle \Psi_0 | P e^{iK_1} \cdots e^{iK_m} A_S P \Psi_0 \rangle \\
&= \langle \Psi_0 | (P e^{iK_1} P)(P e^{iK_2} P) \cdots (P e^{iK_m} P) A_S \Psi_0 \rangle \\
&\equiv \langle \Psi_S | M_1 M_2 \cdots M_m A_S \Psi_S \rangle
\end{aligned}
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where $M_j \simeq P e^{iK_j} P$ on $\mathcal{H}_S$.

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Reduced Dynamical Operators

$\{M_j \in L(\mathcal{H}_{\mathcal{S}})\}_{j\in\mathbb{N}}$ s.t. $\exists C < \infty$ and $\Psi_{\mathcal{S}} \in \mathcal{H}_{\mathcal{S}}$ with

$$\begin{cases} M_j\Psi_{\mathcal{S}} = \Psi_{\mathcal{S}} \\ \|M_{j_1}M_{j_2}\cdots M_{j_k}\| \le C, \quad \forall\{j_i\}_{i\in\mathbb{N}} \end{cases}$$

Note: Unitarity of evolution and $\dim\mathcal{H}_{\mathcal{S}} < \infty$.

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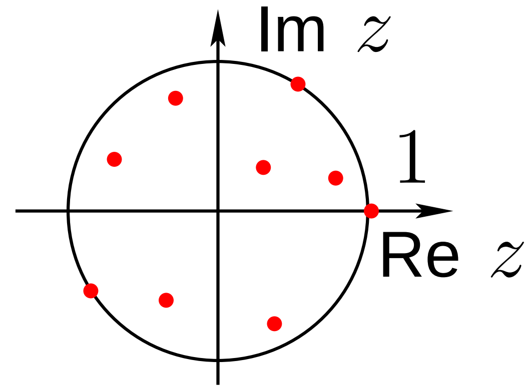
Consequence

$$\forall j \in \mathbb{N} \quad \begin{cases} \text{spec } M_j \subset \{|z| \leq 1\} \\ 1 \in \text{spec } M_j \end{cases}$$



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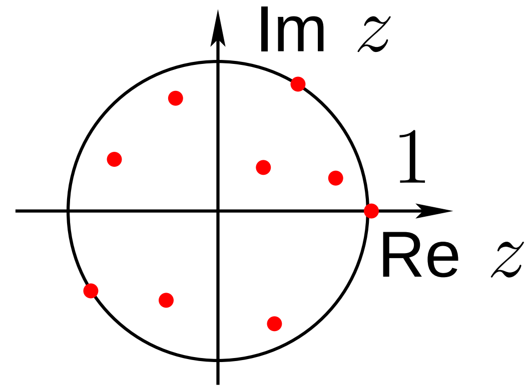

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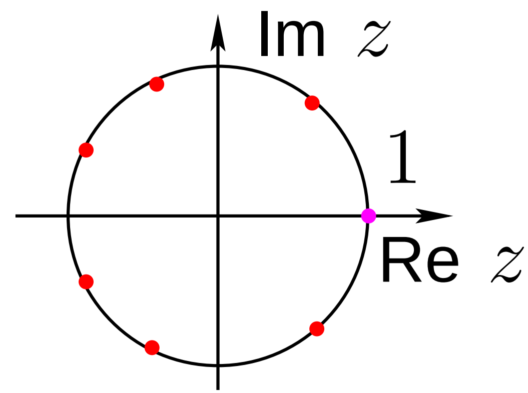
Uncoupled case

$V_j = 0 \;\Rightarrow\; M_j = e^{-i\tau_j L_\mathcal{S}}$ unitary,

$$\begin{cases} \text{spec } M_j = \{e^{-i\tau_j(e_k - e_l)}\}_{k,l} \\ 1 \text{ is } \dim \mathfrak{h}_\mathcal{S}\text{-fold degenerate} \end{cases}$$



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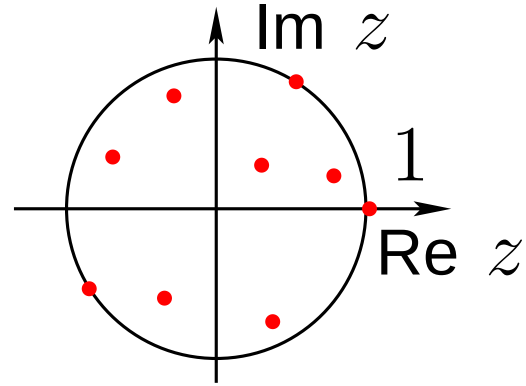

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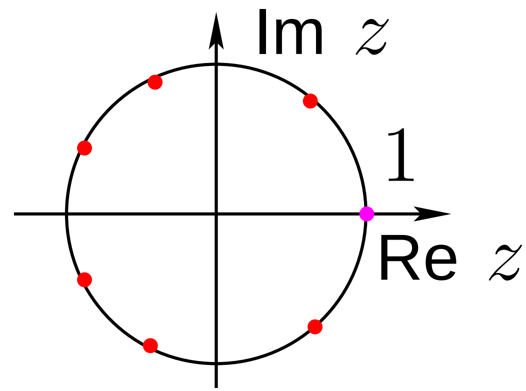


Fact: In general, $M_1 M_2 \cdots M_n$ does not converge as $n \to \infty$!

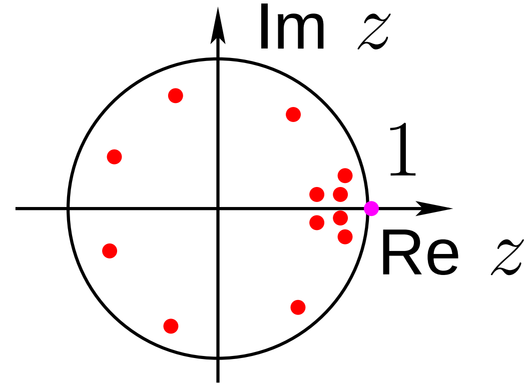
Ideal RIQS

Identical RDO's

$$M_j = M, \quad \forall j \in \mathbb{N}$$

Assumption (E)

$$\begin{cases} \text{spec } M \cap \mathbb{S}^1 = \{1\} \\ 1 \text{ is simple} \end{cases}$$



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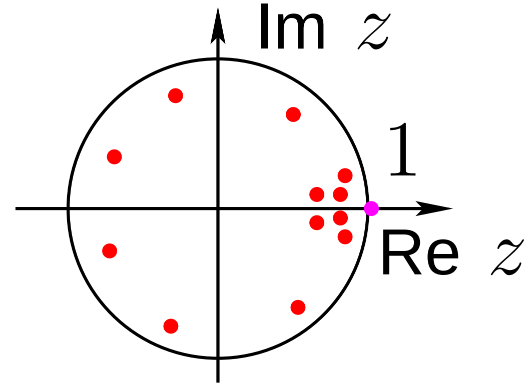

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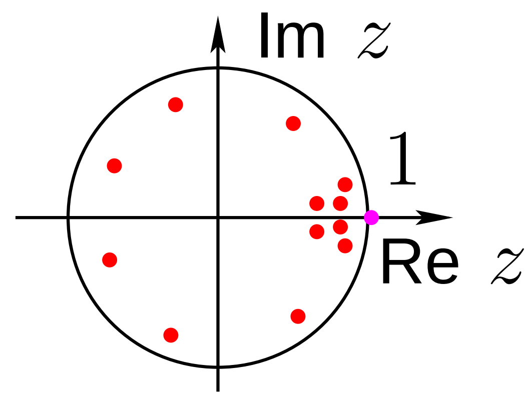

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Repeated Interaction Asymptotic State — Bruneau, J., Merkli '06

Under (E), for any state ϱ on $\mathfrak{M}$ and any observable $A_{\mathcal{S}}$ on $\mathcal{S}$

$$\lim_{m\to\infty} \varrho(\alpha^m(A_{\mathcal{S}})) = \varrho_+(A_{\mathcal{S}}), \qquad \varrho_+(\cdot) = \langle\psi| \cdot \Psi_{\mathcal{S}}\rangle, \text{ a state on } \mathcal{H}_{\mathcal{S}}$$

Actually, if $t = m(t)\tau + s(t)$, $0 \le s(t) < \tau$,

$$\lim_{t\to\infty} \left| \varrho(\alpha^t(A_{\mathcal{S}})) - \varrho_+(P\alpha^{s(t)}(A_{\mathcal{S}})P) \right| = 0, \qquad \text{i.e. “RIAS”}$$

Random Repeated Interaction Quantum Systems

Setup Let $(\Omega_0, \mathcal{F}, dp)$ be a probability space

Let $\Omega_0 \ni \omega_0 \mapsto M(\omega_0) \in \mathcal{H}$ be an RDO valued random variable:

$$\begin{cases} M(\omega_0)\Psi_{\mathcal{S}} = \Psi_{\mathcal{S}}, & \forall\, \omega_0 \in \Omega_0 \\ \|M(\omega_1)M(\omega_2)\cdots M(\omega_m)\| \leq C & \forall(\omega_1, \omega_2, \cdots, \omega_m) \in \Omega_0^m, \quad \forall m \in \mathbb{N}. \end{cases}$$

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Product of i.i.d. random matrices

Let $\Omega = \Omega_0^{\mathbb{N}}$, $d\mathbb{P} = \Pi_{j\geq 1} dp$ and $\omega = (\omega_1, \omega_2, \omega_3, \dots) \in \Omega$.

$\Rightarrow$

$$\chi(m, \omega) = M(\omega_1)M(\omega_2)\cdots M(\omega_m), \text{ as } m \to \infty.$$

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Simplification: The RRDO $M(\omega_0)$ satisfies (E) on Ω_0.

Thus

$$M(\omega_j) = M_j = |\Psi_S\rangle\langle\psi_j| + M_{Q_j}$$

where

$P_j = |\Psi_{\mathcal{S}}\rangle\langle\psi_j|$ spectral projector on 1, $Q_j = (\mathbb{I} - P_j)$
$M_{Q_j} = Q_j M_j Q_j$ s.t. spec$M_{Q_j} \subset \{|z| < 1\}$.

Product of Random matrices

Properties

$$\begin{cases} P_j P_k = P_k = |\Psi_S\rangle\langle\psi_k|, & \|\psi_j\| \leq C \\ Q_j P_k = 0, & \|M_{Q_1} \cdots M_{Q_m}\| \leq C \end{cases}$$

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Fluctuating + decaying parts

$$\begin{cases} \chi(n,\omega) & = & |\Psi_{\mathcal{S}}\rangle\langle\theta_n| + M_{Q_1}\cdots M_{Q_n} \\ \theta_n & = & \psi_n + M_{Q_n}^*\psi_{n-1} + M_{Q_n}^* M_{Q_{n-1}}^*\psi_{n-2} + \cdots + M_{Q_n}^*\cdots M_{Q_2}^*\psi_1 \end{cases}$$

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$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi(n,\omega) \quad \text{ergodic limit}$$

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General hypothesis

$$p(M(\omega_0) \text{ satisfies } (E)) > 0.$$

Main Results

Theorem (Decay) Bruneau, J., Merkli '07

$\exists \alpha > 0, \quad C > 0$ s.t.

i) $\|M_{Q_1} \cdots M_{Q_n}\| \leq Ce^{-\alpha n}, \quad \forall n \geq 1,$ almost surely

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Theorem (Fluctuation)

Recall $\chi(n, \omega) = M(\omega_1) M(\omega_2) \cdots M(\omega_n)$

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Theorem (Ergodic limit of RRIQS)

For any state ϱ on $\mathfrak{M}$ and any observable $A_{\mathcal{S}}$ on $\mathcal{S}$,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \varrho(\alpha_\omega^n(A_{\mathcal{S}})) = \langle\bar{\theta}|A_{\mathcal{S}}\Psi_{\mathcal{S}}\rangle,$$ almost surely

Comments

- Beck-Schwarz '57 Existence of erg. lim. of $\chi(n,\omega)$, but no value
- Order matters

$$M(\omega_n)M(\omega_{n-1})\cdots M(\omega_1) = |\Psi_{\mathcal{S}}\rangle\langle\eta_\infty(\omega)| + O(e^{-\alpha n}) \quad \text{a.s}$$

$$\eta_\infty(\omega) = \psi(\omega_1) + M_Q^*(\omega_1)\psi(\omega_2) + M_Q^*(\omega_1)M_Q^*(\omega_2)\psi(\omega_3) + \cdots$$

- Numerous results on pdts of random matrices (e.g. Guivarc'h, Kifer-Liu,), focus on
  1. Lyapunov exponents
  2. Limit in law, or pdt. in the "wrong" order
  3. Stochastic or invertible matrices

- $\varrho_0 \mapsto \varrho$ via cyclicity and separability of Ψ_0.
- More general observables can be considered.

  $\Rightarrow$ for RRIQS, positive entropy production in the RIAS and (averaged) 2^{nd} law of thermodynamics if $\mathcal{C}$ at thermal equilibrium.

Applications

Spin-spin

- $\mathcal{S}$ and $\mathcal{E}_j$ spins with e.v. $\{0, E_S\}$, resp. $\{0, E_j\}$
- $W_j = \lambda(a_S \otimes a_j^* + a_S^* \otimes a_j)$
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$$M_j \text{ s.t. (E) true} \Leftrightarrow \tau_j \neq T_j\mathbb{Z},$$

where $T_j = 2\pi/\sqrt{(E_S - E_j)^2 + 4\lambda^2}$

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Random energies and durations

Let $\omega_0 \mapsto (E(\omega_0), \tau(\omega_0))$, s.t. $p(\tau(\omega_0) \neq T(\omega_0)\mathbb{Z}) > 0$

Then, for any ϱ and a.s.,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \varrho(\alpha_\omega^m(A_\mathcal{S})) \simeq \varrho_{\tilde{\beta},\mathcal{S}}(A_\mathcal{S}), \qquad \tilde{\beta} \text{ explicit and complicated}$$

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Spin-spin

- $\mathcal{S}$ and $\mathcal{E}_j$ spins with e.v. $\{0, E_S\}$, resp. $\{0, E_j\}$
- $W_j = \lambda(a_S \otimes a_j^* + a_S^* \otimes a_j)$
- Ψ_S tracial, $\Psi_{\mathcal{E}_j} \simeq \varrho_{\beta,\mathcal{E}_j} = e^{-\beta H_{\mathcal{E}_j}} / Z_{\beta,j}$

Explicit computations

$$M_j \text{ s.t. (E) true} \Leftrightarrow \tau_j \neq T_j \mathbb{Z},$$

where $T_j = 2\pi / \sqrt{(E_S - E_j)^2 + 4\lambda^2}$

Random energies and durations

Let $\omega_0 \mapsto (E(\omega_0), \tau(\omega_0))$, s.t. $p(\tau(\omega_0) \neq T(\omega_0)\mathbb{Z}) > 0$

Then, for any ϱ and a.s.,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \varrho(\alpha_\omega^m(A_\mathcal{S})) \simeq \varrho_{\tilde{\beta},\mathcal{S}}(A_\mathcal{S}), \qquad \tilde{\beta} \text{ explicit and complicated}$$

Ideal RIQS $\quad \tilde{\beta} = \beta E_\mathcal{E} / E_\mathcal{S}$ i.e. no thermalization