# Spectral analysis of isolated Bloch families in regular magnetic fields

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# Intro-1

Let us begin by presenting a very brief view of the 'scenary' in which our work is situated.

- To start with, let us say that we are interested in mathematical models for describing different phenomena in solids and mainly concerning their interaction with the electromagnetic fields.
- In fact, we focus on phenomena at the atomic scale, as governed by quantum physics and this is why the name of Felix Bloch appears.

## The Solid

For the class of phenomena we are interested in, the solid is described as a crystalline structure given by a regular lattice of atoms, or rather positive ions having a certain effective charge 'visible' at the scale we consider and a gas of electrons moving around.

# Intro-2

As physicists always do, let us start with the 0-th approximation order and consider the atoms as charged point particles fixed on the vertices of the regular lattice and neglect the interaction between the different electrons in the 'electron gas' moving around.
Then our physical system is a family of independent electrons, also considered as point particles, moving in the periodic field of the atoms on the lattice and a possible 'external field'. Thus it is enough, for the beginning to consider one such electron moving in the given fields.

# Intro-3

Each electron has its own:

well-defined physical characteristics: and

- mass: $m$,
- negative electric charge: $e$,
- spin: $1/2$

a family of physical observables through which we describe its movement:

- position: q,
- momentum: p,
- spin projections: $\sigma_j$,
- energy: $E = (1/2m)|p|^2 + V(q)$,
- angular momentum: l,
- etc ...

For the moment we shall neglect the spin variations.

# Intro-4

**The quantum principles imply that**

There is a complex Hilbert space $\mathscr{H}$ such that:

- The physical states are described by the rank 1 orthogonal projections in $\mathscr{H}$: $P \in \mathbb{P}(\mathscr{H})$,
- The physical observables are described by (possibley unbounded) self-adjoint operators on $\mathscr{H}$: $T = T^* : \mathscr{D}(T) \to \mathscr{H}$.
- $\sigma(T) = \{$the set of values of the observable $T\}$.
- $\mathtt{Tr}(PT)$ measures the mean value of the observable $T$ in the state $P$.

By "quantization", one replaces the abelian algebra of classical observables (defined as functions on the phase space) with a non-abelian algebra of operators in a complex Hilbert space.

# Intro-5

A pointwise particle of mass $m$ in the $d$-dimensional Euclidean space $\mathcal{X}$:

- One has to fix an origin and a frame, so that $\mathcal{X} \cong \mathbb{R}^d$.
- $\mathcal{H} = L^2(\mathbb{R}^d)$,
- $P \in \mathbb{P}(\mathcal{H}) \rightsquigarrow \psi \in L^2(\mathbb{R}^d),\ \|\psi\|^2 = \int_{\mathbb{R}^d} dx\, |\psi(x)|^2 = 1.$
- $q \equiv \{q_j,\ 1 \leq j \leq d\} \rightsquigarrow \{Q_j,\ 1 \leq j \leq d\}$, $\big(Q_j f\big)(x) := x_j f(x),\ \forall x \in \mathbb{R}^d, \forall f \in L^2(\mathbb{R}^d)$.
- $p \equiv \{p_j,\ 1 \leq j \leq d\} \rightsquigarrow \{D_j,\ 1 \leq j \leq d\}$, $D_j : \{f \in \mathcal{H},\ -i\partial_{x_j} f \in L^2(\mathbb{R}^d)\}$.
- $E \rightsquigarrow H, \quad Hf = -(1/2m)\Delta f + V(Q)f$
  where:
  - $V : \mathbb{R}^d \to \mathbb{R}$ is the potential energy,
  - $\big(V(Q)f\big)(x) := V(x)f(x),\ \forall x \in \mathbb{R}^d, \forall f \in L^2(\mathbb{R}^d)$.
- The time evolution: "the state $P_0 \in \mathbb{P}(\mathcal{H})$ at $t = 0$ becomes $P_t := e^{-itH} P_0 e^{itH} \in \mathbb{P}(\mathcal{H})$ at time $t \in \mathbb{R}$.

For $d \geq 2$ we consider the $d$-dimensional real Euclidean space $\mathcal{X} \cong \mathbb{R}^d$ and:

- the dual $\mathcal{X}^*$ with the canonical bilinear duality map: $< \cdot, \cdot >: \mathcal{X}^* \times \mathcal{X} \to \mathbb{R}$;
- the phase space defined by its cotangent bundle $\Xi \cong \mathcal{X} \times \mathcal{X}^*$;
- Let $\mathfrak{E} := \{\mathfrak{e}_j\}_{j \in \underline{d}}$ be the canonical linear basis of $\mathbb{R}^d$.
- $\Gamma := \Big\{ \sum_{1 \leq j \leq d} \gamma_j \mathfrak{e}_j \in \mathcal{X}, \ \gamma_j \in \mathbb{Z}, \forall j \Big\} \cong \mathbb{Z}^d$ - *the regular lattice*.
- $V : \mathcal{X} \to \mathbb{R}$ will be a $\Gamma$-periodic function (the periodic potential).
- $\Gamma_* := \Big\{ \sum_{1 \leq j \leq d} \gamma_j^* \mathfrak{e}_j^* \in \mathcal{X}^*, < \mathfrak{e}_j^*, \mathfrak{e}_k >= 2\pi \delta_{j,k}, \ \gamma_j^* \in \mathbb{Z}, \forall j \Big\} \cong \mathbb{Z}^d$ *the dual lattice*.

$\tau_v$ denotes translation by $v \in \mathcal{V}$ on any function space on a linear space $\mathcal{V}$.

Any $\Gamma$-periodic function may be considered as a function defined on $\mathcal{X}/\Gamma$ that has a non-trivial topological structure.

Let us recall the quotient group: $\mathbb{R}/\mathbb{Z} =: \mathbb{S} \cong \{\mathfrak{z} \in \mathbb{C} \ |\mathfrak{z}| = 1\}$ with the canonical projection $\mathfrak{p}^1 : \mathbb{R} \rightarrowtail \mathbb{S}$ given by $\mathfrak{p}^1(x) := e^{-2\pi i x}$.

Then:

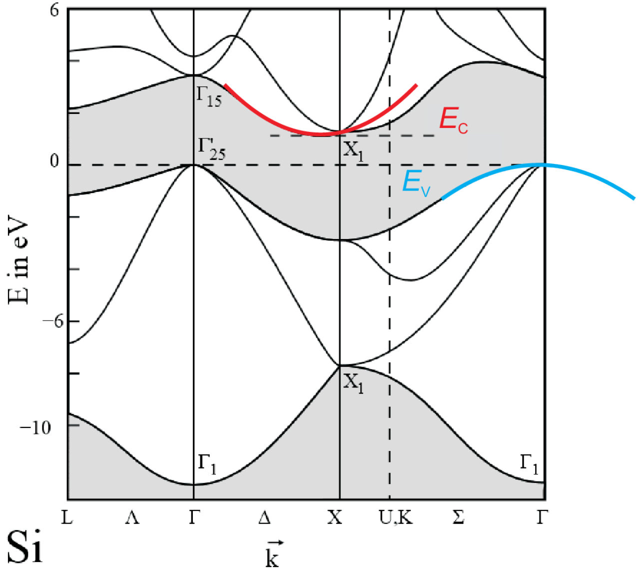
- $\mathcal{X}/\Gamma =: \mathbb{T} \cong [\mathbb{S}]^d$, $\mathfrak{p} := [\mathfrak{p}^1]^d : \mathcal{X} \rightarrowtail \mathbb{T}$;
- $\mathcal{X}^*/\Gamma_* =: \mathbb{T}_* \cong [\mathbb{S}]^d$, $\mathfrak{p}_* := [\mathfrak{p}^1]^d : \mathcal{X}^* \rightarrowtail \mathbb{T}_*$.

We shall frequently use the sections:

- $\mathfrak{s} : \mathbb{T} \to \mathcal{E} := \{x \in \mathcal{X},\ -1/2 \leq x_j < 1/2,\ 1 \leq j \leq d\} \subset \mathcal{X}$, $\mathfrak{p} \circ \mathfrak{s} = \mathbf{I}\mathrm{d}_{\mathbb{T}},\ \mathfrak{s} \circ \mathfrak{p} = \mathbf{I}\mathrm{d}_{\mathcal{E}}$;
- $\mathfrak{s}_* : \mathbb{T}_* \to \mathcal{B} := \{\xi \in \mathcal{X}^*,\ -1/2 \leq \xi_j < 1/2,\ 1 \leq j \leq d\} \subset \mathcal{X}^*$, $\mathfrak{p}_* \circ \mathfrak{s}_* = \mathbf{I}\mathrm{d}_{\mathbb{T}_*},\ \mathfrak{s}_* \circ \mathfrak{p}_* = \mathbf{I}\mathrm{d}_{\mathcal{B}}$.

# Intro-6

Considering the physical system we are interested in, we notice that we have to work with "Γ-periodic partial differential operators" or even more, with "Γ-periodic pseudo-differential operators.
As a consequence of the Schur lemma, a Γ-periodic operator $H$ is decomposable in the Fourier transformed representation as an operator-valued function $\mathbb{T}_* \ni \theta \mapsto \widehat{H}(\theta)$.
Moreover, for the Hamiltonian operators we are dealing with, the 'fibre operators' $\widehat{H}(\theta)$ have compact resolvent and thus a discrete spectrum, defining a family of real functions on $\mathbb{T}_*$ the so-called Bloch levels, like in the following picture.

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# Intro-7

An essential feature concerning the above structure, comes from the spin of the electrons: being half-integer, they are fermions and one of the quantum principles does not allow for two fermions of the same type to be in the same state $P \in \mathbb{P}(\mathcal{H})$. Thus, depending on its density, the particles in the electron gas in the solid must occupy a number of Bloch levels up to some energy, called the Fermi energy and its position with respect to the different Bloch levels and the possible spectral gaps has very significant consequences on its time evolution and interaction with other external fields.

In this work, we shall be interested in the interaction of the electrons of the solid, with an external magnetic field that is smooth and bounded together with all its derivatives, but is not supposed to vanish at infinity.

*Let us now be more precise!*

# Plan of the talk

1. The framework
   - The unperturbed Hamiltonian
   - The Bloch-Floquet theory
2. The Problem
   - The perturbed magnetic Hamiltonian.
   - The main results
   - Main lines of the proof
3. References

The Hörmander classes of symbols.

For $p \in \mathbb{R}$ and $\rho \in [0,1]$ we define:

$$S^p_\rho(\mathcal{X}^*, \mathcal{X}) := \left\{F \in C^\infty(\Xi)\,,\, \nu^{p,\rho}_{n,m}(F) < \infty\, \forall (n,m) \in \mathbb{N} \times \mathbb{N}\right\},$$

$$\nu^{p,\rho}_{n,m}(F) := \max_{|\alpha|\leq n} \max_{|\beta|\leq m} \sup_{(x,\xi)\in\Xi} <\xi>^{-p+\rho m} \left|\left(\partial^\alpha_x \partial^\beta_\xi F\right)(x,\xi)\right|.$$

Definition

A symbol $F \in S^p_\rho(\mathcal{X}^*, \mathcal{X})$ is called *elliptic*, when:

$$\exists (C,R) \in \mathbb{R}_+ \times \mathbb{R}_+,\ |\xi| \geq R \Rightarrow \left|F(x,\xi)\right| \geq C <\xi>^p .$$

Notation: ($\Gamma$-periodic symbols)

$$S^p_\rho(\mathcal{X}^*, \mathcal{X})_\Gamma := \left\{F \in S^p_\rho(\mathcal{X}^*, \mathcal{X}),\ F(x+\gamma) = F(x),\, \forall (x,\gamma) \in \mathcal{X} \times \Gamma\right\}$$

# Comments-1

- As you know very well, we have the Weyl-Hörmander prescription (or 'quantization procedure') to associate to these symbols some operators $\mathscr{S}(\mathcal{X}) \rightarrow \mathscr{S}'(\mathcal{X})$, in such a way that polynomials of order $p$ in the dual variable $\xi \in \mathcal{X}^*$ are taken into differential operators of order $p$.
- In collaboration with Viorel Iftimie and late Marius Măntoiu and later on with Horia Cornean and Bernard Helffer, we have constructed a "twisted" Weyl-Hörmander calculus in which we embodied the "magnetic field" by replacing the usual differential operators $D_j$" $= -i\partial_{x_j}$ by some covariant derivations $D_j^A := -i\partial_{x_j} - A_j(x)$ associated to the 1-form $A$ defining the magnetic field $B = dA$.
- The main technical problem arises from the fact that without imposing a vanishing condition on the magnetic field $B$, one has to do with vector potentials $A$ that grow (linearly) at infinity!

# The magnetic field

Let us denote by:

- $\mathbf{\Lambda}^p_{\mathtt{bd}}(\mathcal{X})$ the real space of smooth $p$-forms on $\mathcal{X}$ having components of class $BC^\infty(\mathcal{X})$
- $\mathbf{\Lambda}^p_{\mathtt{pol}}(\mathcal{X})$ the real space of smooth $p$-forms on $\mathcal{X}$ having components with polynomial growth together with all their derivatives
- $d : \mathbf{\Lambda}^p_{\mathtt{pol}}(\mathcal{X}) \to \mathbf{\Lambda}^{(p+1)}_{\mathtt{pol}}(\mathcal{X})$ the exterior differential that restricts to a map $\mathbf{\Lambda}^p_{\mathtt{bd}}(\mathcal{X}) \to \mathbf{\Lambda}^{(p+1)}_{\mathtt{bd}}(\mathcal{X})$

**The magnetic field:** $B \in \mathbf{\Lambda}^2_{\mathtt{bd}}(\mathcal{X}), \quad dB = 0,$

**The vector potential:**

There exists $A \in \mathbf{\Lambda}^1_{\mathtt{pol}}(\mathcal{X})$ such that $B = dA$.

# The magnetic phase.

The magnetic phase.

$$\begin{aligned}\Lambda^A(x,y) \; &:= \; \exp\Big(-i\int_{[x,y]} A\Big) \\ &= \; \exp\Big(-i\int_0^1 ds\,\big\langle A_j\big(x+s(y-x)\big)\,,\,(y-x)\big\rangle\Big)\end{aligned} \tag{1}$$

defining the magnetic translations $[U^A(z)f](x) := \Lambda^A(x,x+z)f(x+z)$ having as self-adjoint generators the covariant derivations $D_j^A = -i\partial_{x_j} - A_j(x)$.

This is *the singular 'magnetic' phase* put into evidence by H. Cornean and G. Nenciu.

# Magnetic quantization.

We denote by $\mathcal{L}(\mathcal{V}_1;\mathcal{V}_2)$ the space of linear continuous operators between the topological vector spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ with its strong topology (of uniform convergence on bounded sets).

**Magnetic Weyl quantization:**

$$\mathfrak{Op}^A:\ \mathscr{S}(\Xi)\overset{\sim}{\longrightarrow}\mathcal{L}\big(\mathscr{S}'(\mathcal{X});\mathscr{S}(\mathcal{X})\big):$$

$$\big(\mathfrak{Op}^A(\Phi)\psi\big)(x):=(2\pi)^{-d/2}\int_{\mathcal{X}}dy\int_{\mathcal{X}^*}d\eta\, e^{i<\eta,x-y>}\,\Lambda^A(x,y)\,\Phi\big((x+y)/2,\eta\big)\,\psi(y)$$

$$\forall\psi\in\mathscr{S}(\mathcal{X})$$

- By duality we get a map $\mathfrak{Op}^A:\ \mathscr{S}'(\Xi)\overset{\sim}{\longrightarrow}\mathcal{L}\big(\mathscr{S}(\mathcal{X});\mathscr{S}'(\mathcal{X})\big)$.

**It is gauge covariant:** if $dA=dA'=B$, then $\exists U\in\mathbb{U}\big(L^2(\mathcal{X})\big)$ such that $\mathfrak{Op}^{A'}(\Phi)=U\mathfrak{Op}^A(\Phi)U^{-1}$.

# The 'unperturbed' Hamiltonian:

## The Hamiltonian function:

$$h \in S^p_1(\mathcal{X}^*, \mathcal{X})_\Gamma,\ p > 0 \qquad h > 0,\ \text{elliptic.}$$

## The 'residual' periodic magnetic field:

$$B_\circ \in \mathbf{\Lambda}^2_{\mathrm{bd}}(\mathcal{X}), \quad \text{such that:} \quad B_\circ(x+\gamma) = B_\circ(x)\ \forall(x,\gamma) \in \mathcal{X} \times \Gamma,$$

$$\int_{\mathfrak{e}_j \wedge \mathfrak{e}_k} B_\circ = 0.$$

$$\exists A_\circ \in \mathbf{\Lambda}^1_{\mathrm{bd}}(\mathcal{X}), \quad A_\circ(x+\gamma) = A_\circ(x)\ \forall(x,\gamma) \in \mathcal{X} \times \Gamma.$$

## The free Hamiltonian:

$$H^\Gamma := \overline{\mathfrak{Op}^{A_\circ}(h)}.$$

# The Bloch-Floquet representation.

The space $L^2(\mathcal{X})$ with the unitary action of $\mathbb{R}^d \cong \mathcal{X}$ by translations is unitary equivalent with the Hilbert direct integral:

$$\mathscr{F} := \int_{\mathbb{T}_*}^{\oplus} d\theta\, \mathscr{F}_\theta$$

where: $\mathscr{F}_\theta := \big\{ f \in L^2_{\mathtt{loc}}(\mathcal{X}),\ \tau_\gamma f = e^{i<\theta,\gamma>} f, \forall \gamma \in \Gamma \big\} \cong e^{i<\theta,\cdot>} L^2(\mathbb{T})$
with the quadratic norm: $\|f\|_\theta^2 := \int_{\mathcal{E}} d\hat{x}\, |f(\hat{x})|^2$.

$$\mathscr{U}_\Gamma : L^2(\mathcal{X}) \xrightarrow{\sim} \mathscr{F}, \quad \big(\mathscr{U}_\Gamma u\big)_\theta(x) := \sum_{\gamma \in \Gamma} e^{-i<\theta,\gamma>} u(\gamma + x).$$

Let $\mathfrak{f} \in C^\infty(\mathcal{X}^* \times \mathcal{X}))$ be defined by $\mathfrak{f}(\xi, x) := e^{i<\xi,x>}$ and $\mathscr{G} := \mathfrak{f}^{-1}\mathscr{F}$.
In fact $\mathscr{G}$ with the induced action of $\mathcal{X}^* \cong \mathbb{R}^d$ is isomorphic to the space of sections of the vector bundle $\mathfrak{F} \rightarrowtail \mathbb{T}_*$ associated to the principal bundle $\mathcal{X}^* \rightarrowtail \mathbb{T}_*$ by the representation $\hat{\mathcal{U}}^\dagger : \Gamma_* \to \mathbb{U}\big(L^2(\mathbb{T})\big)$:
$\big(\hat{\mathcal{U}}^\dagger(\gamma^*)\phi\big)(\omega) := e^{-i<\gamma^*,\omega>}\phi(\omega)$.

# The Bloch-Floquet Theorem.

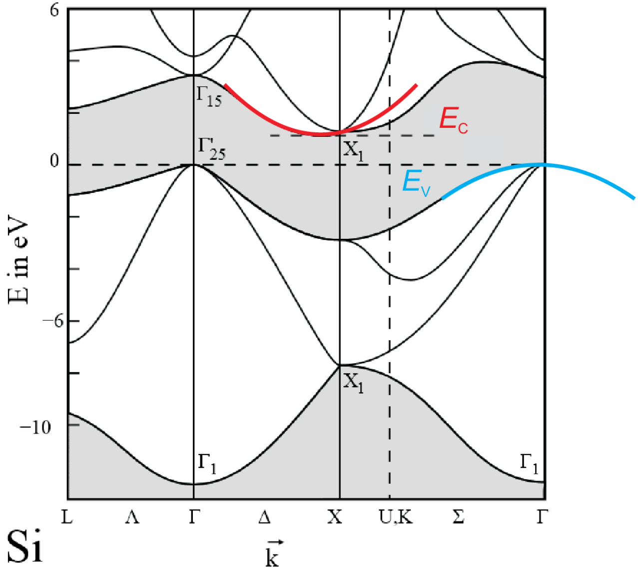
The $\Gamma$-periodicity of $h \in S^p_1(\mathcal{X}^*, \mathcal{X})_\Gamma$ and $A^\circ \in \mathbf{\Lambda}^1_{\text{bd}}(\mathcal{X})$ implies the Floquet decomposability of $H^\Gamma$ and we may define:

$$\widetilde{H}^\Gamma(\xi) = \mathfrak{f}(\xi, \cdot)^{-1} \widehat{H}^\Gamma_{\mathfrak{p}_*(\xi)} \mathfrak{f}(\xi, \cdot) \in \mathbb{B}\big(L^2(\mathbb{T})\big), \ \forall \xi \in \mathcal{X}^*.$$

Theorem A

- For each $\xi \in \mathcal{X}^*$ the operator $\widetilde{H}^\Gamma(\xi)$ is self-adjoint on the Sobolev space $\mathscr{H}^p(\mathbb{T})$, has compact resolvent and defines an *analytic family of type A, in the sense of Kato* with respect to the variable $\xi \in \mathcal{X}^*$.
- There exists a family of continuous functions $\mathbb{T}_* \ni \theta \mapsto \lambda_j(\theta) \in \mathbb{R}$ indexed by $j \in \mathbb{N}_\bullet$, called *the Bloch eigenvalues*, such that

$$\lambda_j(\theta) \leq \lambda_{j+1}(\theta), \ \forall (j, \theta) \in \mathbb{N} \times \mathbb{T}_*, \quad \sigma\big(\widetilde{H}_\Gamma(\xi)\big) = \bigcup_{j \in \mathbb{N}_\bullet} \lambda_j\big(\mathfrak{p}_*(\xi)\big).$$

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# The Bloch Projections.

**For each fixed $\boldsymbol{\theta} \in \mathbb{T}_*$** we can define:

- $\nu(k,\theta) = \inf\{j \in \mathbb{N}\,,\ \lambda_j(\theta) = \lambda_k(\theta)\}$,
- a closed contour $\mathscr{C}_n$ isolating $\lambda_n(\theta)$ from the rest of the spectrum of $\widehat{H}^\Gamma(\theta)$,
- *the Riesz spectral projections* :

$$\widehat{\pi}_k(\theta) := \begin{cases} \frac{1}{2\pi i}\oint_{\mathscr{C}_k(\theta)} d\mathfrak{z}\,\big(\widehat{H}(\theta) - \mathfrak{z}\mathbf{1}\big)^{-1} \in \mathbb{B}\big(\mathscr{F}_\theta\big) & \text{if } \nu(k,\theta) = k\,, \\ 0 & \text{if } \nu(k,\theta) < k\,, \end{cases}$$

### Theorem B

For any $\theta \in \mathbb{T}_*$ and any $k \in \mathbb{N}_\bullet$:

$$\widehat{\pi}_k(\theta)\mathscr{F}_\theta \subset C^\infty(\mathcal{X}) \bigcap \mathscr{F}_\theta =: \mathscr{F}^\infty_\theta.$$

# The isolated Bloch family hypothesis.

We suppose fixed the Hamiltonian $H^\Gamma$ as above.

## Hypothesis

Given the B-F decomposition of $H^\Gamma$ with Bloch eigenvalues $\{\lambda_k(\theta)\}_{k\in\mathbb{N}_\bullet}$, there exist $k_0 \in \mathbb{N}_\bullet$ and $N \in \mathbb{N}$ such that:

1. $\lambda_{k_0-1}(\theta) < \lambda_{k_0}(\theta), \quad \lambda_{k_0+N}(\theta) < \lambda_{k_0+N+1}(\theta), \quad \forall\theta \in \mathbb{T}^d_*$, (where by convention $\lambda_0 := -\infty$).
2. $\mathrm{d}_0 := \inf_{\theta\in\mathbb{T}_*} \lambda_{k_0+N+1}(\theta) - \sup_{\theta\in\mathbb{T}_*} \lambda_{k_0-1}(\theta) > 0$

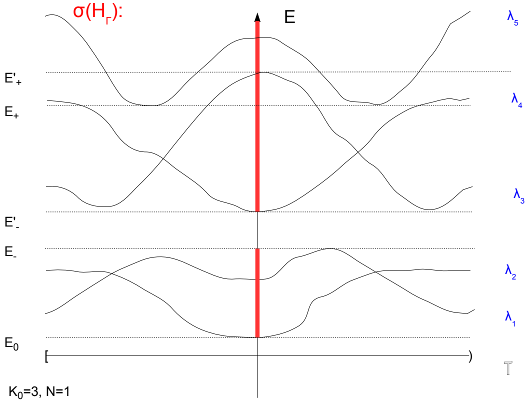
   so that we have the following non-void interval:

$$J_{\mathfrak{B}} := \Big[\sup_{\theta\in\mathbb{T}_*} \lambda_{k_0-1}(\theta)\,,\, \inf_{\theta\in\mathbb{T}_*} \lambda_{k_0+N+1}(\theta)\Big] \equiv [E_-, E_+] \subset \mathbb{R},$$

We shall denote by: $E'_- := \inf_{\theta\in\mathbb{T}_*} \lambda_{k_0}(\theta)$ and $E'_+ := \sup_{\theta\in\mathbb{T}_*} \lambda_{k_0+N}(\theta)$.

# An isolated Bloch family.

- spectral islands with absolutely continuous spectrum.
- possible spectral gaps



### Definition

We call *isolated Bloch family* for the Hamiltonian $H^\Gamma$, a family of Bloch levels $\mathfrak{B} := \{\lambda_k : \mathbb{T}^d_* \to \mathbb{R}\,,\ k_0 \leq k \leq k_0 + N,\}$ that satisfy the above Hypothesis.

Let us emphasize that we do not assume the existence of any spectral gap below or above the image of the isolated family $\mathfrak{B}$ in the spectrum of $H^\Gamma$,

i.e. we do not suppose satisfied the following

### Spectral gap condition:

$$\sup_{\theta\in\mathbb{T}_*}\lambda_{k_0-1}(\theta) < \inf_{\theta\in\mathbb{T}_*}\lambda_{k_0}(\theta), \text{ and } \sup_{\theta\in\mathbb{T}_*}\lambda_{k_0+N}(\theta) < \inf_{\theta\in\mathbb{T}_*}\lambda_{k_0+N+1}(\theta).$$

For some technical reasons that will become clear further, we shall choose our energy scale in order to have:

$$\inf_{\theta\in\mathbb{T}^d_*}\lambda_1(\theta) =: E_0 > 0\,.$$

# The $J_{\mathfrak{B}}$ dynamics

Let us consider an interval $J \subset \mathring{J}_{\mathfrak{B}}$ and $u \in E_{H^\Gamma}(J_{\mathfrak{B}})L^2(\mathcal{X})$ with $\|u\| = 1$, where

$E_{H^\Gamma}(J)$ is the spectral projection of $H^\Gamma$ on the Borel subset $J \subset \mathbb{R}$.

Then: $$H^\Gamma\, u \,=\, H^\Gamma\, E_{H^\Gamma}(J)u \,=\, H^\Gamma_{\mathfrak{B}}\, u,$$

where:

$$H^\Gamma_{\mathfrak{B}} = \mathscr{U}_\Gamma^{-1}\Big(\int_{\mathbb{T}_*}^{\oplus} d\theta\, \widehat{H}^\Gamma_{\mathfrak{B}}(\theta)\Big)\mathscr{U}_\Gamma \quad \widehat{H}^\Gamma_{\mathfrak{B}}(\theta) := \sum_{k_0\leq k\leq k_0+N} \lambda_k(\theta)\, \widehat{\pi}_k(\theta) \in \mathbb{B}\big(L^2(\mathcal{E})\big).$$

# *The Problem*

Our objective in this work is to analyze the influence of a perturbing magnetic field on *the dynamics associated to the isolated Bloch family* $\mathfrak{B}$ of the unperturbed (magnetic, periodic) Hamiltonian $H^{\Gamma}$.

We shall not impose to our perturbing magnetic field to vanish at infinity and thus the perturbation it produces is not relatively bounded with respect to the unperturbed Hamiltonian $H^{\Gamma}$.

# The perturbing magnetic field

We perturb $H^\Gamma$ by adding a magnetic field $B^{\epsilon,c} \in \mathbf{\Lambda}^2_{\mathtt{bd}}(\mathcal{X})$ with $dB^{\epsilon,c} = 0$, controlled by two parameters $(\epsilon, c) \in [0, \epsilon_0] \times [0, 1]$
(for some $\epsilon_0 > 0$ usually small enough)

and having the form: $B^{\epsilon,c} := \epsilon B^\bullet + c\epsilon B^\epsilon \in \mathbf{\Lambda}^2_{\mathtt{bd}}(\mathcal{X})$,

where:

- $B^\bullet$ is a constant magnetic field,
- $B^\epsilon \in \mathbf{\Lambda}^2_{\mathtt{bd}}(\mathcal{X})$ with uniform estimates for $\epsilon \in [0, \epsilon_0]$.

Moreover we choose

- $A^\bullet_k(x) := (1/2) \sum_{1 \leq j \leq d} B^\bullet_{j,k} x_j$,
- $A^\epsilon \in \mathbf{\Lambda}^1_{\mathtt{pol}}(\mathcal{X})$ such that $B^\epsilon = dA^\epsilon$

and $B^{\epsilon,c} = dA^{\epsilon,c}$ for $A^{\epsilon,c}(x) := \epsilon A^\bullet(x) + c\epsilon A^\epsilon(x)$.

# The perturbed Hamiltonian

Let us emphasize that **the total magnetic field** of our problem is:

$$B := B^\circ + B^{\epsilon,c} = B^\circ + \epsilon(B^\bullet + c\,B^\epsilon) = d(A^\circ + A^{\epsilon,c}) \equiv dA.$$

**The perturbed magnetic Hamiltonian** is then defined by

$$H^{\epsilon,c} := \overline{\mathfrak{Op}^A(h)} \text{ on } L^2(\mathcal{X})$$

and has **the domain** $\mathscr{H}^p_A(\mathcal{X}) = \mathscr{H}^p_{A^{\epsilon,c}} \subset L^2(\mathcal{X})$.

An important role will be played by the following two resolvents:

- $R^\circ_\mathfrak{z} := \big(H^\Gamma - \mathfrak{z}\mathbf{1}\big)^{-1} =: \mathfrak{Op}^{A^\circ}(\mathfrak{r}^\circ_\mathfrak{z})$ for $\mathfrak{z} \in \mathbb{C} \setminus \sigma(H^\Gamma)$,
- $R^{\epsilon,c}_\mathfrak{z} := \big(H^{\epsilon,c} - \mathfrak{z}\mathbf{1}\big)^{-1} =: \mathfrak{Op}^{A}(\mathfrak{r}^{\epsilon,c}_\mathfrak{z})$ for $\mathfrak{z} \in \mathbb{C} \setminus \sigma(H^{\epsilon,c})$.

The magnetic pseudo-differential calculus implies that both $\mathfrak{r}^\circ_\mathfrak{z}$ and $\mathfrak{r}^{\epsilon,c}_\mathfrak{z}$ belong to $S^{-p}_1(\Xi)$ with $\mathfrak{r}^\circ_\mathfrak{z}$ $\Gamma$-periodic.

# The main results

## The Main Theorem

There exist $\epsilon_0 > 0$ and $C > 0$ such that for any pair $(\epsilon, c) \in [0, \epsilon_0] \times [0, 1]$,
if we denote by $\vartheta(\epsilon) := \epsilon \ln(\epsilon^{-1})$
and by $J^{\epsilon}_{\mathfrak{B}}$ the interval $\big(E_- + C\vartheta(\epsilon)\,,\, E_+ - C\vartheta(\epsilon)\big)$,

there exists an orthonormal projection $P^{\epsilon,c}_{\mathfrak{B}}$ and
an effective magnetic Hamiltonian $\mathfrak{H}^{\epsilon,c}_{\mathfrak{B}} \in \mathbb{B}\big(L^2(\mathcal{X})\big)$ commuting with $P^{\epsilon,c}_{\mathfrak{B}}$
and satisfying the following properties:

1. For any compact interval $J \subset J^{\epsilon}_{\mathfrak{B}}$:
$$\mathrm{d_H}\Big(J \cap \sigma(H^{\epsilon,c})\,,\, J \cap \sigma\big(\mathfrak{H}^{\epsilon,c}_{\mathfrak{B}}\big|_{P^{\epsilon,c}_{\mathfrak{B}} L^2(\mathcal{X})}\big)\Big) \leq C\epsilon^2.$$
2. $E^{\epsilon,c}_h(J) \leq P^{\epsilon,c}_{\mathfrak{B}}$ and for any $t \in \mathbb{R}_+$ there exists $C > 0$ such that:
$$\forall\, v \in E_J(H^{\epsilon,c})L^2(\mathcal{X}) : \ \big\|e^{-itH^{\epsilon,c}}v - e^{-it\mathfrak{H}^{\epsilon,c}_{\mathfrak{B}}}v\big\|_{L^2(\mathcal{X})} \leq Ct^3\,\epsilon^2\,\|v\|_{L^2(\mathcal{X})}.$$

# The main results

Let us consider

- the unitary equivalence $L^2(\mathcal{X}) \cong \ell^2\big(\Gamma; L^2(\mathcal{E})\big)$;
- the orthonormal basis $\{\mathfrak{e}_\gamma,\ \gamma \in \Gamma\}$ of $\ell^2(\Gamma)$ defined by $\mathfrak{e}_\gamma(\alpha) := \delta_{\alpha,\gamma}$.

## A Generalised Peierls-Onsager formula

There exist $\epsilon_0 > 0$ and $C > 0$ such that for any $\epsilon \in [0, \epsilon_0]$, there exists a smooth map $\widehat{\mathfrak{H}}^\epsilon : \mathbb{T}_* \to \mathbb{B}\big(L^2(\mathcal{E})\big)$, independent of $B^\epsilon$ verifying the estimations:

- $\forall a \in \mathbb{N}^d,\ \exists C_a > 0,\ \sup_{\theta\in\mathbb{T}_*}\left\|\left[\partial_\theta^a\big(\widehat{\mathfrak{H}}^\epsilon_{\mathfrak{B}} - \widehat{H}^\Gamma_{\mathfrak{B}}\big)\right](\theta)\right\|_{\mathbb{B}(L^2(\mathcal{E}))} \leq C_a\epsilon,$
- $\forall c \in [0,1]$ there exist an unitary operator $\mathfrak{U}^{\epsilon,c} \in \mathbb{U}\big(L^2(\mathcal{X})\big)$ and some $C > 0$ such that:
$$\left\|\big(\mathfrak{e}_\alpha\,,\,\mathfrak{U}^{\epsilon,c}\mathfrak{H}^{\epsilon,c}_{\mathfrak{B}}[\mathfrak{U}^{\epsilon,c}]^{-1}\mathfrak{e}_{\alpha-\gamma}\big) - \Lambda^{\epsilon,c}(\alpha,\gamma)\big(\mathcal{F}_{\mathbb{T}_*}\widehat{\mathfrak{H}}^\epsilon_{\mathfrak{B}}\big)(\gamma)\right\|_{\mathbb{B}(L^2(\mathcal{E}))} \leq C\,c\epsilon.$$

# *Main lines of the proof.*

# Orthogonal decomposition induced by the isolated Bloch family

Given the isolated Bloch family $\mathfrak{B} := \{\lambda_k : \mathbb{T}_* \to \mathbb{R},\ k_0 \leq k \leq k_0 + N\}$,

let: $\mathfrak{n}_0 := \underline{[k_0]}$, $\mathfrak{n}_{\mathfrak{B}} := \underline{[k_0 + N]} \setminus \underline{[k_0 - 1]}$ and $\mathfrak{n}_\infty := \mathbb{N}_\bullet \setminus \underline{[k_0 + N]}$

and define the operators (with $\mathfrak{a} \in \{0, \mathfrak{B}, \infty\}$):

- $P_{\mathfrak{a}} := \mathscr{U}_\Gamma^{-1}\Big(\int_{\mathbb{T}_*}^{\oplus} d\theta \sum_{k \in \mathfrak{n}_{\mathfrak{a}}} \widehat{\pi}_k(\theta)\Big)\mathscr{U}_\Gamma, \quad P_\perp := P_0 + P_\infty,$
- $H^\Gamma_{\mathfrak{a}} := \mathscr{U}_\Gamma^{-1}\Big(\int_{\mathbb{T}_*}^{\oplus} d\theta \sum_{k \in \mathfrak{n}_{\mathfrak{a}}} \lambda_k(\theta)\,\widehat{\pi}_k(\theta)\Big)\mathscr{U}_\Gamma, \quad H^\Gamma_\perp := H^\Gamma_0 + H^\Gamma_\infty.$

verifying the identities:

- $P_0 \oplus P_{\mathfrak{B}} \oplus P_\infty = \mathbb{1}_{L^2(\mathcal{X})}$,
- $H^\Gamma_0 \oplus H^\Gamma_{\mathfrak{B}} \oplus H^\Gamma_\infty = H^\Gamma$,
- $\sigma(H^\Gamma_0) \subset [E_0, E_-]$, $\sigma(H^\Gamma_{\mathfrak{B}}) \subset [E'_-, E'_+]$, $\sigma(H^\Gamma_\infty) \subset [E_+, +\infty)$

# Riesz formulae for the orthogonal decomposition

- We can find a smooth curve $\mathscr{C}_{\mathfrak{B}}(\theta) \subset \mathbb{C}$ diffeomorphic to a circle, containing $\sigma_{\mathfrak{B}}(\theta)$ in its inner domain and remaining at a distance greater than some $\mathrm{d} \in \big(0, \mathrm{d}_{\mathfrak{B}}/2\big)$ from the spectrum of $\widehat{H}^{\Gamma}(\theta)$.
- Then we have the formula:

$$\widehat{P}_{\mathfrak{B}}(\theta) = \sum_{k_0 \leq k \leq k_0+N} \widehat{\pi}_k(\theta) = (2\pi i)^{-1} \int_{\mathscr{C}_{\mathfrak{B}}(\theta)} \big(\widehat{H}^{\Gamma}(\theta) - \mathfrak{z}\mathbf{1}\big)^{-1} d\mathfrak{z}.$$

- For any $\tilde{\theta} \in \mathbb{T}^d_*$, we can find a small open neighbourhood $\tilde{O} \subset \mathbb{T}^d_*$ such that we may take $\mathscr{C}_{\mathfrak{B}}(\theta) = \tilde{\mathscr{C}_{\mathfrak{B}}}$ constant for all $\theta \in \tilde{O}$.
- It follows that the following application is smooth:

$$\tilde{O} \ni \theta \mapsto (2\pi i)^{-1} \int_{\tilde{\mathscr{C}_{\mathfrak{B}}}} \big(\widehat{H}^{\Gamma}(\theta) - \mathfrak{z}\mathbf{1}\big)^{-1} d\mathfrak{z} \in \mathbb{B}(\mathscr{F}_{\theta}).$$

# Regularity of the orthogonal decomnposition

- Due to the above analysis and Theorems A & B:

  $\exists p_{\mathfrak{B}} \in S^{-\infty}(\mathcal{X}^*, \mathcal{X})_\Gamma$, such that $P_{\mathfrak{B}} := \mathscr{U}_\Gamma^{-1} \widehat{P}_{\mathfrak{B}} \mathscr{U}_\Gamma = \mathfrak{Op}^{A^\circ}(p_{\mathfrak{B}})$.

- Identical arguments imply that:

  $\exists p_0 \in S^{-\infty}(\mathcal{X}^*, \mathcal{X})_\Gamma$, such that $P_0 := \mathscr{U}_\Gamma^{-1} \widehat{P}_0 \mathscr{U}_\Gamma = \mathfrak{Op}^{A^\circ}(p_0)$;

  $\exists h_0 \in S^{-\infty}(\mathcal{X}^*, \mathcal{X})_\Gamma$, such that $H_0^\Gamma := \mathscr{U}_\Gamma^{-1} \widehat{H}_0^\Gamma \mathscr{U}_\Gamma = \mathfrak{Op}^{A^\circ}(h_0)$;

  $\exists h_{\mathfrak{B}} \in S^{-\infty}(\mathcal{X}^*, \mathcal{X})_\Gamma$, such that $H_{\mathfrak{B}} := \mathscr{U}_\Gamma^{-1} \widehat{H}_{\mathfrak{B}}^\Gamma \mathscr{U}_\Gamma = \mathfrak{Op}^{A^\circ}(h_{\mathfrak{B}})$.

- It follows that:

  $p_\infty := 1 - (p_0 + p_{\mathfrak{B}}) \in S^0_1(\mathcal{X}^*, \mathcal{X})_\Gamma, \ p_\perp := 1 - p_{\mathfrak{B}} \in S^0_1(\mathcal{X}^*, \mathcal{X})_\Gamma,$

  $h_\infty := h - (h_0 + h_{\mathfrak{B}}) \in S^p_1(\mathcal{X}^*, \mathcal{X})_\Gamma, \ h_\perp := h - h_{\mathfrak{B}} \in S^0_1(\mathcal{X}^*, \mathcal{X})_\Gamma$

  $h_\infty$ and $h_\perp$ are elliptic symbols.

  $H^\Gamma_\infty = \mathfrak{Op}^{A^\circ}(h_\infty), \ H^\Gamma_\perp = \mathfrak{Op}^{A^\circ}(h_\perp)$.

# The perturbed isolated Bloch family

**We notice that:**

$$P_{\mathfrak{B}} = P_{\ker H_\perp^\Gamma} = (2\pi i)^{-1} \int_{\mathscr{C}_{\mathfrak{B}}} d\mathfrak{z} \left(H_\perp^\Gamma - \mathfrak{z}\mathbf{1}\right)^{-1}.$$

where $\mathscr{C}_{\mathfrak{B}} \subset \mathbb{C}$ is a circle enclosing $0 \in \mathbb{C}$ in its interior domain and having a radius $r_0 := E_0/2$

Definition

- $H_{\mathfrak{a}}^{\epsilon,c} := \mathfrak{Op}^A(h_{\mathfrak{a}})$ for $\mathfrak{a} \in \{0, \mathfrak{B}, \infty\}$.
- $P_{\mathfrak{B}}^{\epsilon,c} := (2\pi i)^{-1} \int_{\mathscr{C}_{\mathfrak{B}}} d\mathfrak{z} \left(H_\perp^{\epsilon,c} - \mathfrak{z}\mathbf{1}\right)^{-1}$ with $\mathscr{C}_{\mathfrak{B}}$ as above.
- $\mathfrak{H}_{\mathfrak{B}}^{\epsilon,c} := P_{\mathfrak{B}}^{\epsilon,c} H^{\epsilon,c} P_{\mathfrak{B}}^{\epsilon,c}$ - *the effective Hamiltonian of the isolated Bloch family in magnetic field*.

# Perturbing the orthogonal decomposition

**Our results on spectral regularity imply that:**

If we denote by $\vartheta(\epsilon) := \epsilon \ln(\epsilon^{-1})$, there exists $\epsilon_0 > 0$ and $c > 0$ such that:

$$\forall(\epsilon, c) \in [0, \epsilon_0] \times [0, 1] \quad \begin{aligned} &\sigma(H_0^{\epsilon,c}) \subset [E_0 - c\vartheta(\epsilon)\,,\, E_- + c\vartheta(\epsilon)], \\ &\sigma(H_{\mathfrak{B}}^{\epsilon,c}) \subset [E'_- - c\vartheta(\epsilon)\,,\, E'_+ + c\vartheta(\epsilon)], \\ &\sigma(H_\infty^{\epsilon,c}) \subset [E_+ - c\vartheta(\epsilon)\,,\, +\infty). \end{aligned}$$

Proposition P

With the above notations and hypothesis, there exists $\epsilon_0 > 0$ and $C > 0$ such that the interval $J_{\mathfrak{B}}^\delta := \big(E_- + C\vartheta(\epsilon)\,,\, E_+ - C\vartheta(\epsilon)\big)$ is not void and for any pair $(\epsilon, c) \in [0, \epsilon_0] \times [0, 1]$ and any closed interval $J \subset J_{\mathfrak{B}}^\delta$, the spectral projection $E_h^{\epsilon,c}(J)$ of $H^{\epsilon,c}$ associated with $J$ satisfies

$$E_h^{\epsilon,c}(J) \subseteq P_{\mathfrak{B}}^{\epsilon,c} L^2(\mathcal{X}).$$

# A Feshbach-Schur quotient argument.

## Hypothesis

In a separable complex Hilbert space $\mathcal{H}$
we consider a family of pairs $(H_\kappa, P_\kappa)$ indexed by $\kappa \in [0, \kappa_0]$ for some $\kappa_0 > 0$, where for any $\kappa \in [0, \kappa_0]$:

- $H_\kappa : \mathcal{D}(H_\kappa) \to \mathcal{H}$ is a lower-semibounded self-adjoint operator,
- $\Pi_\kappa = \Pi_\kappa^* = \Pi_\kappa^2$ is an orthogonal projection, such that, with $\Pi_\kappa^\perp := \mathbb{1} - \Pi_\kappa$, we have the properties:
  1. $\exists C > 0$ such that for any $\kappa \in [0, \kappa_0]$ and any $u \in \mathcal{H}$ we have that $\Pi_\kappa \mathcal{H} \subset \mathcal{D}(H_\kappa)$ and $\|\Pi_\kappa^\perp H_\kappa \Pi_\kappa\|_{\mathbb{B}(\mathcal{H})} \leq C\kappa$;
  2. $\exists J \subset \mathbb{R}$ with non-void interior, such that $\forall (\kappa, t) \in [0, \kappa_0] \times J$, the operator $\Pi_\kappa^\perp H_\kappa \Pi_\kappa^\perp - t\Pi_\kappa^\perp$ is invertible as operator in $\Pi_\kappa^\perp \mathcal{H}$ with the inverse $R_\kappa^\perp(t)$ being uniformly bounded on $J$.

Under the above Hypothesis we have the estimate (for some $C > 0$):

$$\left\| \Pi_\kappa H_\kappa \Pi_\kappa^\perp R_\kappa^\perp(t) \Pi_\kappa^\perp H_\kappa \Pi_\kappa \right\|_{\mathbb{B}(\mathcal{H})} \leq C\kappa^2 \left\| R_\kappa(t) \right\|_{\mathbb{B}(\mathcal{H})}.$$

# Theorem C

Under the above Hypothesis we have that:

- $t \in J \cap \sigma(H)$ if and only if $t \in J \cap \sigma\big(\Pi_\kappa H_\kappa \Pi_\kappa - \Pi_\kappa H R_\kappa^\perp(t) H_\kappa \Pi_\kappa\big)$
- if we denote by $\mathring{R}_\kappa(t) := \big(\Pi_\kappa(H_\kappa - t)\Pi_\kappa - \Pi_\kappa H_\kappa R_\kappa^\perp(t) H_\kappa \Pi_\kappa\big)^{-1}$ as operator in $\Pi_\kappa \mathcal{H}$, for any $\kappa \in [0, \kappa_0]$ we have the following block structure with respect to the decomposition $\mathbb{1} = \Pi_\kappa \oplus \Pi_\kappa^\perp$:

$$\big(H_\kappa - t\mathbb{1}_{\mathcal{H}}\big)^{-1} = \begin{pmatrix} \Pi_\kappa(H_\kappa - t)\Pi_\kappa & \Pi_\kappa H_\kappa \Pi_\kappa^\perp \\ \Pi_\kappa^\perp H_\kappa \Pi_\kappa & \Pi_\kappa^\perp (H_\kappa - t)\Pi_\kappa^\perp \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} \mathring{R}_\kappa(t) & -\mathring{R}_\kappa(t) H R_\kappa^\perp(t) \\ -R_\kappa^\perp(t) H \mathring{R}_\kappa(t) & R_\kappa^\perp(t) + R_\kappa^\perp(t) H \mathring{R}_\kappa(t) H R_\kappa^\perp(t) \end{pmatrix}.$$

- The operator $[H_\kappa] := \Pi_\kappa H_\kappa \Pi_\kappa$ is a bounded self-adjoint operator and there exists $C > 0$ such that:

$$\max\Big\{ \sup_{\lambda \in J \cap \sigma(H_\kappa)} \operatorname{dist}\Big(\lambda, \sigma\big([H_\kappa]\big)\Big), \sup_{\lambda \in J \cap \sigma([H_\kappa])} \operatorname{dist}\Big(\lambda, \sigma\big(H_\kappa\big)\Big) \Big\} \leq C\,\kappa^2 \left\| R_\kappa^\perp(t) \right\|_{\mathbb{B}(\mathcal{H})}.$$

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*Thank you for your attention.*