# The Algebra of Observables in a Magnetic Field


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## 1 Introduction

We shall introduce a $C^*$-algebra containig the functional calculus of a large class of Schrödinger operators with variable magnetic fields. This is motivated by some recent operator algebraic methods for analyzing the essential spectrum and the regions of non-propagation ([AMP], [GI1], [GI2], [M1], [M2]) and in the same time by the interest of elaborating a gauge-invariant pseudodifferential calculus in the presence of a variable magnetic field ([N], [CN], [P]).

Our results (Theorems 3.9 and 3.10) are the main technical ingredients for the type of developments mentioned above and we shall elaborate them in some future works. For reasons of space we shall not aim at optimal results, the arguments will be rather condensed and we send to references for some justifications and for further details. The concept and use of observables affiliated to $C^*$-algebras are borrowed from [ABG], [GI1], [GI2]. In this setting magnetic fields, especially constant ones, are considered in [GI2]. In this paper, for the variable magnetic fields the authors propose the use of group extensions, but we feel that twisted crossed products are better suited both from the technical and the conceptual points of view. The general theory of twisted crossed products has been developped mainly in ([PR1], [PR2]). It is well known that $C^*$-algebraic methods lead to very interesting results in the case of periodic Hamiltonians with a constant magnetic field; for this and many related topics we refer to [B], [BES] (and the refferences therein).

In $X = \mathbb{R}^n$ we consider a non-relativistic particle subject to a potential $V$ and a magnetic field $B$, both assumed to be bounded and uniformly continuous. We intend to study the Schrödinger Hamiltonian $H := (\Pi^A)^2 + V$ in $\mathcal{H} := L^2(X)$; here $A$ is a vector potential for $B$ and $\Pi^A := -i\nabla - A$ is the magnetic momentum. It is well known that $H$ can be defined as a self-adjoint operator and that $C_0^\infty(X)$ is a core for it ([AHS], [LS]). We shall define a $C^*$-algebra of bounded operators in $\mathcal{H}$ containing the resolvent family $\{(H-z)^{-1} \mid z \in \mathbb{C}\setminus\mathbb{R}\}$ and thus the entire $C_\infty$-class functional calculus of $H$ (we denote by $C_\infty(\mathbb{R})$ the set of continuous functions on $\mathbb{R}$ vanishing at infinity). This $C^*$-algebra has a remarkable structure. We consider the natural action by translations $\theta$ of $X$ on $BC_u(X)$ (the $C^*$-algebra of bounded, uniformly continuous functions on $X$) and the imaginary exponential of the flux of the magnetic field, that defines a cocycle $\omega_B$ on $X \times X$ with values in the group of unitary elements in $BC_u(X)$. From these one can define the $C^*$-algebra $\mathfrak{C} := BC_u(X) \rtimes_\theta^{\omega_B} X$ (the twisted crossed product) that mixes in a subtle non-commutative way $BC_u(X)$ and $L^1(X)$. Then any representation of $\mathfrak{C}$ contains at the same time multiplication operators (by functions of class $BC_u(X)$) and $C_\infty$-functions of the magnetic momenta, the commutation formulae: $i[Q_j, Q_k] = 0$, $i[\Pi_j^A, Q_k] = \delta_{jk}$, $i[\Pi_j^A, \Pi_k^A] = B_{kj}$ being taken into account. We prove in this paper that any such representation also contains the resolvent family of the magnetic Schrödinger Hamiltonian $H$.

## 2 The Twisted Crossed Product Algebra Associated to a Magnetic Field

We consider a magnetic field as being defined by a bounded, uniformly continuous, matrix-valued function $B : X \to \mathbb{M}_{n,n}(\mathbb{R})$ satisfying the conditions: $B_{jk} = -B_{kj}$, $\partial_j B_{kl} + \partial_k B_{lj} + \partial_l B_{jk} = 0$.

The physical description of a particle moving in a magnetic field $B$ is obtained by replacing the usual momentum $p$ of the particle by the expression $\pi := p - A(x)$ where $A$ is a vector potential for our magnetic field, i.e. a vector function $A : X \to X$ satisfying: $B_{jk} = \partial_j A_k - \partial_k A_j$. Under our hypothesis on the function $B$ such a vector potential always exists but is not unique (for example any gradient of a regular scalar function can be added to $A$).

The unitary groups associated to the self-adjoint operators $\Pi_j^A$ are the magnetic translations [S], [Z]. They do not commute and their composition puts into evidence the following imaginary exponential of the flux of the magnetic field:

$$\omega_B(x,y;q) := \exp\left\{-i\int_{<q,q+x,q+x+y>} B(\xi)d\sigma(\xi)\right\},$$

where $< q, q+x, q+x+y >$ is the triangle defined by the points: $q, q+x, q+x+y$. We consider this function as a mapping $X \times X \ni (x,y) \mapsto \omega_B(x,y;.) \in C_u(X;\mathbb{T}^1)$ (where $\mathbb{T}^1$ is the multiplicative group of complex numbers of modulus one) and observe that:

$$\omega_B(x,y)\omega_B(x+y,z) = \theta(x)\{\omega_B(y,z)\}\omega_B(x,y+z), \tag{2.1}$$

$$\omega_B(x,0) = \omega_B(0,x) = 1, \tag{2.2}$$

$$\omega(x,-x) = \omega(-x,x) = 1, \tag{2.3}$$

where $(\theta(x)f)(y) := f(y+x)$ denotes the action of $X$ by translations on $BC_u(X)$.

**Definition 2.1.** We shall call a **Twisted Quantum Dynamical System**, shortened TQDS, a quadruplet $\{X, \mathcal{A}, \theta, \omega\}$ where: $X$ is a second-countable locally compact abelian group, $\mathcal{A}$ is a separable abelian unital $C^*$-algebra, $\theta : X \to Aut(\mathcal{A})$ is a continuous group homomorphism (taking the topology of simple convergence on $Aut(\mathcal{A})$) and $\omega : X \times X \to \mathcal{U}(\mathcal{A})$ is a continuous mapping into the group of unitary elements of $\mathcal{A}$ satisfying conditions (2.1, 2.2). We say that $\omega$ is a $\boldsymbol{\theta}$**-2-cocycle**. If the $\theta$-2-cocycle $\omega$ also satisfies (2.3) we say that we have a **Magnetic Quantum Dynamical System**, shortened a MQDS.

**Definition 2.2.** A **covariant representation** of a TQDS $\{X, \mathcal{A}, \theta, \omega\}$ is a triple $\{\mathcal{H}, U, \rho\}$ where $\mathcal{H}$ is a Hilbert space, $U : X \to \mathcal{U}(\mathcal{H})$ is a strongly continuous mapping into the group of unitary operators on $\mathcal{H}$ and $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ a non-degenerate representation of $\mathcal{A}$ on $\mathcal{H}$ such that: $U(x)U(y) = \rho[\omega(x,y)]U(x+y)$, $U(x)\rho(A)U(x)^* = \rho[\theta(x)A]$ (by *non-degenerate* we mean that the linear space generated by the family $\{\rho(A)u \mid \forall A \in \mathcal{A}, \forall u \in \mathcal{H}\}$ is dense in $\mathcal{H}$).

**The Twisted Convolution Algebra.** Let us fix a Haar measure $dx$ on $X$ and consider the complex linear space $L^1(X;\mathcal{A})$ of Bochner integrable vector functions on $X$ with values in $\mathcal{A}$, with the $L^1$-norm $\|f\|_{1,\mathcal{A}} := \int_X dx \|f(x)\|_{\mathcal{A}}$. We define the composition given by the following 'twisted convolution':

$$(f \star_\theta^\omega g)(x) := \int_X dy \left\{\theta\left(\frac{y-x}{2}\right)[f(y)]\right\}\left\{\theta\left(\frac{y}{2}\right)[g(x-y)]\right\}\left\{\theta\left(-\frac{x}{2}\right)\omega(y,x-y)\right\}$$

and an involution defined by $f^\star(x) := f(-x)^*$. We shall denote the structure thus defined by $L^1_\theta(X;\mathcal{A})^\omega$ and call it *the twisted convolution algebra* associated to the TQDS; it is not difficult to verify that it forms a Banach $^*$-algebra. Let us observe that we use an isomorphic form of the usual twisted crossed product, that in the absence of the magnetic field leads to the Weyl form of the symbolic calculus.

Given a Banach $^*$-algebra $\mathcal{B}$, any $C^*$-seminorm on it is bounded by the given norm, so that the supremum of these $C^*$-seminorms exists and satisfies the same bound. We call the $C^*$-algebra obtained by separation and completion its *enveloping $C^*$-algebra* [D], denoted by $\boldsymbol{C}^*[\mathcal{B}]$; let $\mathfrak{j} : \mathcal{B} \to \boldsymbol{C}^*[\mathcal{B}]$ be the natural morphism thus obtained. Then $\boldsymbol{C}^*\left[L^1_\theta(X;\mathcal{A})^\omega\right] \equiv \mathcal{A} \rtimes_\theta^\omega X$ is called **the twisted crossed product of $\mathcal{A}$ by $X$.** In this case the application $\mathfrak{j}$ is injective so that $L^1_\theta(X;\mathcal{A})^\omega$ is isomorphic to a dense $^*$-subalgebra of $\mathcal{A} \rtimes_\theta^\omega X$. Let us observe that in the literature there are equivalent definitions of this structure but we do not want to insist upon this point. It is known that the non-degenerate representations of $\mathcal{A} \rtimes_\theta^\omega X$ are in a one-to-one correspondence with the covariant representations of the TQDS $\{X, \mathcal{A}, \theta, \omega\}$. For a covariant representation $\{\mathcal{H}, U, \rho\}$ we denote by $\rho \rtimes U$ the associated representation of the twisted crossed product and we have:

$$(\rho \rtimes U)(f) := \int_X dx\, \rho\left(\theta(\frac{x}{2})f(x)\right) U(x), \qquad \forall f \in L^1(X;\mathcal{A}).$$

**Proposition 2.3.** *For any TQDS:*

1. *The representation $U : X \to \mathcal{U}(\mathcal{H})$ induces a linear contraction $\tilde{U} : L^1(X) \to \mathcal{B}(\mathcal{H})$, with $L^1(X)$ considered as a complex Banach space $\tilde{U}(\phi) := \int_X dx\, \phi(x)U(x)$;*

2. *The image* $(\rho \rtimes U)\{\mathcal{A} \rtimes_\theta^\omega X\}$ *is equal to the norm closure of the linear space generated by the set* $\{\rho(a)\tilde{U}(\phi) \mid \forall a \in \mathcal{A},\ \forall \phi \in L^1(X)\}$. *The statement remains true for the linear space generated by the set of products taken in the reversed order.*

The first point is obvious; the second one follows by rephrasing the proof in [GI1] for the untwisted case.

**The Schrödinger Representation.** The arguments below may be generalized to any TQDS due to the triviality of the $\theta$-2-cohomology group discussed in [GI2], but for space reasons we shall concentrate on the physical case of interest.

To *a quantum particle in a magnetic field* one can associate in a natural way a MQDS and, once a vector potential $A$ is chosen, a covariant representation of it. In fact one takes: $X := \mathbb{R}^n$ the group of translations; $\mathcal{A} := BC_u(\mathbb{R}^n)$ the algebra of observables associated to the position operator; $(\theta(x)a)(y) := a(y+x)$ the standard representation of translations on this algebra; $\omega := \omega_B$ the $\theta$-2-cocycle defined by the magnetic field; $\mathcal{H} := L^2(\mathbb{R}^n)$, $(\rho(a)u)(x) \equiv (a(Q)u)(x) := a(x)u(x)$, $U_A(x) = \Lambda_A(x)T(x)$ (the magnetic translation), where: $(T(x)u)(y) := u(y+x)$, $\Lambda_A(x) := \exp\{-i\Gamma_A[Q, Q+x]\} \in \mathcal{B}(\mathcal{H})$ with $\Gamma_A[x,y] := \int_{[x,y]} A(\xi)\cdot d\xi$ (the circulation of $A$ along the line segment $[x,y]$). We call this covariant representation **the Schrödinger representation with vector potential** $A$. We denote $\mathfrak{R}_A \equiv \rho \rtimes U_A$ and we have

$$\mathfrak{R}_A(f)u := \int_X dx f\left(x, Q + \frac{x}{2}\right)\Lambda_A(x)T(x)u. \qquad (2.4)$$

It is easy to verify that this representation is injective and the following 'gauge covariance' relation holds:

$$\rho(e^{i\lambda})\mathfrak{R}_A(f)\rho(e^{-i\lambda}) = \mathfrak{R}_{A+\nabla\lambda}(f), \qquad \forall \lambda \in C^1(X;\mathbb{R}).$$

# 3 The Resolvent of the Magnetic Schrödinger Hamiltonian

Let us denote by $h: X \to \mathbb{R}$ the analytic function $h(p) := \sum_{j=1}^n p_j^2$ and let $\mathfrak{h} := \mathcal{F}h$ be its Fourier transform that defines a compactly supported distribution of second order. Given a magnetic field $B$ and an associated vector potential $A$, the operator $h(\Pi^A)$ gives the corresponding Schrödinger Hamiltonian on $L^2(X)$. Our aim is to show that its resolvent belongs to the image through the Schrödinger representation of $\mathfrak{C} = \mathcal{A} \rtimes_\theta^{\omega_B} X$ with $\mathcal{A} = BC_u(X)$. This is an instance of the following abstract setting.

Let $\mathfrak{C}$ be a $C^*$-algebra; we call **an observable affiliated to** $\mathfrak{C}$ [ABG] a morphism $\Phi : C_\infty(\mathbb{R}) \to \mathfrak{C}$. If $\mathfrak{C}$ is a $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$, a self-adjoint operator $H$ on $\mathcal{H}$ defines an observable affiliated to $\mathfrak{C}$ iff $\eta(H)$, defined by functional calculus, belongs to $\mathfrak{C}$ for any $\eta \in C_\infty(X)$. By an usual density argument this follows if one has $(H-z)^{-1} \in \mathfrak{C}$ for any $z$ with $\Im z \neq 0$.

Let $z \in \mathbb{C}\setminus\mathbb{R}$ be fixed and let us define: $h_z := h - z1$, $\mathfrak{h}_z := \mathcal{F}h_z$ and $G_z := \mathcal{F}h_z^{-1}$. Then $G_z$ is a function that solves the equation $(-\Delta - z)G_z = \delta$ or equivalently $\mathfrak{h}_z * G_z = \delta$ (with $*$ the usual convolution of distributions). Our aim is to 'deform' this equation in order to obtain an inverse of $\mathfrak{h}_z$ for the $\star_\theta^{\omega_B}$-operation and prove that it is in fact an element of the algebra $BC_u(X) \rtimes_\theta^{\omega_B} X$ and thus the affiliation of $\mathfrak{h}$ to this algebra. In order to do this we have to extend $\star_\theta^{\omega_B}$ to a slightly larger class of distributions.

First of all let us observe that any function $f \in L^1_\theta(X,\mathcal{A})^{\omega_B}$ defines a linear continuous map $\mathcal{S}(X) \to \mathcal{A}$ by the formula $< f, \varphi > := \int_X dx f(x)\varphi(x)$, $\forall \varphi \in \mathcal{S}(X)$. In a similar way, any $h \in L^1(X)$ defines a map $\mathcal{S}(X;\mathcal{A}) \to \mathcal{A}$ that we shall denote by $<< .,. >>$ the associated pairing. Then, taking $g$ and $h$ elements of $L^1(X)$, one has the relations: $< h \star_\theta^{\omega_B} g, \varphi > = << h, \Psi_g^B(\varphi) >>$ and $< g \star_\theta^{\omega_B} h, \varphi > = << h, \tilde{\Psi}_g^B(\varphi) >>$ with

$$\Psi_g^B(\varphi)(y) := \int_X dx\ g(x)\varphi(x+y)\left\{\theta\left(-\frac{x+y}{2}\right)\omega_B(y,x)\right\} \qquad (3.5)$$

$$\tilde{\Psi}_g^B(\varphi)(y) := \int_X dx\ g(x)\varphi(x+y)\left\{\theta\left(-\frac{x+y}{2}\right)\omega_B(x,y)\right\}. \qquad (3.6)$$

These formulae allows us to extend the $\star_\theta^{\omega_B}$-operation to the Dirac measure $\delta$ in $0 \in X$ that becomes a unit for the algebra $L^1_\theta(X;\mathcal{A})^{\omega_B}$. Let $L^1_\theta(X;\mathcal{A})^{\omega_B} \oplus \mathbb{C}\delta$ be the minimal unital extension of $L^1_\theta(X;\mathcal{A})^{\omega_B}$ and we shall continue to denote by $\star_\theta^{\omega_B}$ the associative operation on this larger algebra. By the same formula (3.5) we may extend the $\star_\theta^{\omega_B}$-operation to the pair $(\mathfrak{h}_z, f)$ with $f \in L^1_\theta(X;\mathcal{A})^{\omega_B}$, under some regularity assumptions on the magnetic field $B$.

**Hypothesis 3.4.** *The magnetic field is given by a bounded matrix-valued $C^2$-function with bounded derivatives up to second order.*

It is evident that for $\Im z \neq 0$ the function $h_z^{-1}$ is a symbol of class $S^{-2}$ (in the sense defined in \$1.1 of [ABG]) and that for $z = \pm i\sigma$, with $\sigma > 0$, we have $h_{\pm i\sigma}(p)^{-1} = \sigma^{-1} h_{\pm i}(p/\sigma^{1/2})^{-1}$.

**Lemma 3.5.** *For $m > 0$ suppose given a familly of functions $\{f_\sigma\}_{\sigma>0} \subset S^{-m}(\mathbb{R}^n)$, satisfying the homogeneity condition $f_\sigma(p) = \sigma^a f_1(p/\sigma^b)$. Then $\mathcal{F}f_\sigma \in L^1(\mathbb{R}^n)$ for any $\sigma$ and we have the estimate $\|\mathcal{F}f_\sigma\|_{L^1} = \sigma^a \|\mathcal{F}f_1\|_{L^1}$.*

This lemma follows easily from Proposition 1.3.6 of [ABG] and the fact that for $n < 2m$ any symbol of class $S^{-m}$ is in $L^2$, so that its Fourier transform is also in $L^2 \subset L^1_{\mathrm{loc}}$. A straightforward computation leads from the homogeneity of $f_\sigma$ to that of the $L^1$-norm of its Fourier transform.

Using the Lemma above for $\{x^\alpha \partial^\beta G_{i\sigma}\}_\sigma$ (with $|\alpha| \leq 1,\ |\beta| \leq 1$) we get:

$$\|x^\alpha \partial^\beta G_{i\sigma}\|_{L^1} \leq \sigma^{-1-(|\alpha-|\beta|)/2} \|x^\alpha \partial^\beta G_i\|_{L^1}. \tag{3.7}$$

For space reasons we shall continue the arguments in a minimal version, suitable for our explicit form of $\mathfrak{h}$ and leave the general problem of extending the $\star_\theta^{\omega_B}$-operation for a forthcoming paper. We shall consider a mollifier family $\{\mathfrak{h}_z^\epsilon\}_{\epsilon>0}$ for $\mathfrak{h} \in \mathcal{E}_2'(X)$ (the space of second order compactly supported distributions on $X$). Then for any $g \in L^1(X)$ we define for $\varphi \in \mathcal{S}(X)$:

$$\begin{aligned} <\mathfrak{h}_z \star_\theta^{\omega_B} g, \varphi> &:= \lim_{\epsilon\to 0} << \mathfrak{h}_z^\epsilon, \Psi_g^B(\varphi) >>, \\ < g \star_\theta^{\omega_B} \mathfrak{h}_z, \varphi> &:= \lim_{\epsilon\to 0} << \mathfrak{h}_z^\epsilon, \tilde{\Psi}_g^B(\varphi) >> . \end{aligned}$$

For $g = G_z$ we have

$$\begin{aligned} &\lim_{\epsilon\to 0} << \mathfrak{h}_z^\epsilon, \Psi_g^B(\varphi) >>= \varphi(0) + <\mu_z, \varphi>, \\ &\lim_{\epsilon\to 0} << \mathfrak{h}_z^\epsilon, \tilde{\Psi}_g^B(\varphi) >>= \varphi(0) + <\nu_z, \varphi>, \end{aligned}$$

where

$$\mu_z(x) := -\left(\nabla G_z\right)(x) \cdot \left[\left(\underset{1}{\nabla} - \underset{2}{\nabla}\right)\omega_B\right](0,x) + G_z(x)\left[\left(\underset{1}{\nabla} - \underset{2}{\nabla}\right)^2 \omega_B\right](0,x),$$

$$\nu_z(x) := -\left(\nabla G_z\right)(x) \cdot \left[\left(\underset{2}{\nabla} - \underset{1}{\nabla}\right)\omega_B\right](x,0) + G_z(x)\left[\left(\underset{2}{\nabla} - \underset{1}{\nabla}\right)^2 \omega_B\right](x,0)$$

as maps in $\mathcal{B}(\mathcal{S}(X);\mathcal{A})$ (here $\underset{j}{\nabla}$ signifies the gradient with respect to the j-th argument). For a magnetic field $B$ satisfying the Hypothesis 3.4, all the factors containing derivatives of the $\theta$-2-cocycle $\omega_B$ in the above formulae are continuous functions from $X \times X$ into $BC_u(X)$, bounded by quadratic polynomials in the variable $x$. Using these facts and the estimate (3.7) we obtain:

**Lemma 3.6.** *We have $\mathfrak{h}_z \star_\theta^{\omega_B} G_z = \delta + \mu_z$, $G_z \star_\theta^{\omega_B} \mathfrak{h}_z = \delta + \nu_z$. For any $z \in \mathbb{C}\setminus\mathbb{R}$ the two maps $\mu_z$ and $\nu_z$ belong to $L^1_\theta(X;\mathcal{A})^{\omega_B}$ and we have the estimates:*

$$\|\mu_{\pm i\sigma}\|_{L^1} \leq \sigma^{-1} C, \quad \|\nu_{\pm i\sigma}\|_{L^1} \leq \sigma^{-1} C.$$

Let us define:

$$\mathfrak{r}_{\pm i\sigma} := G_{\pm i\sigma} \star_\theta^{\omega_B} \left\{\sum_{j=0}^{\infty} (\mu_{\pm i\sigma})^{\star j}\right\}; \qquad \tilde{\mathfrak{r}}_{\pm i\sigma} := \left\{\sum_{j=0}^{\infty} (\nu_{\pm i\sigma})^{\star j}\right\} \star_\theta^{\omega_B} G_{\pm i\sigma}. \tag{3.8}$$

Here $f^{\star j}$ means the $\star_\theta^{\omega_B}$-product of $j$ factors $f$. We define $\mathfrak{h}_{\pm i\sigma} \star_\theta^{\omega_B} \mathfrak{r}_{\pm i\sigma}$ and $\tilde{\mathfrak{r}}_{\pm i\sigma} \star_\theta^{\omega_B} \mathfrak{h}_{\pm i\sigma}$ by the same regularization method as above. Using the associativity property of the $\star_\theta^{\omega_B}$-product and some obvious algebraic manipulations, we get: $\mathfrak{h}_{\pm i\sigma} \star_\theta^{\omega_B} \mathfrak{r}_{\pm i\sigma} = \delta$, $\tilde{\mathfrak{r}}_{\pm i\sigma} \star_\theta^{\omega_B} \mathfrak{h}_{\pm i\sigma} = \delta$, $\tilde{\mathfrak{r}}_{\pm i\sigma} = \tilde{\mathfrak{r}}_{\pm i\sigma} \star_\theta^{\omega_B} \mathfrak{h}_{\pm i\sigma} \star_\theta^{\omega_B} \mathfrak{r}_{\pm i\sigma} = \mathfrak{r}_{\pm i\sigma}$, $\mathfrak{r}_{i\sigma} = \mathfrak{r}^\star_{-i\sigma}$

and $\mathfrak{r}_{i\sigma} - \mathfrak{r}_{-i\sigma} = 2i\sigma\mathfrak{r}_{i\sigma}\mathfrak{r}_{-i\sigma}$. By the Neumann series, Lemma 3.6 and the last two formulae above, we can define by analytic extension a resolvent function $\mathbb{C} \setminus \mathbb{R} \ni z \to \mathfrak{r}_z \in \mathfrak{C}$ (extending $\mathfrak{r}_{\pm i\sigma}$). This gives the following:

**Theorem 3.7.** *Let $B$ be a magnetic field satisfying our Hypothesis 3.4. There exists an observable $\Phi_\mathfrak{h}$ affiliated to $BC_u(X) \rtimes_\theta^{\omega_B} X$, such that for the extended operation defined above we have $\mathfrak{h}_z \star_\theta^{\omega_B} \Phi_\mathfrak{h}(r_z) = \Phi_\mathfrak{h}(r_z) \star_\theta^{\omega_B} \mathfrak{h}_z = \delta$ with $r_z(t) := (t-z)^{-1}$.*

It is easy to see that for a magnetic field satisfying Hypothesis 3.4 one can always choose a vector potential satisfying

**Hypothesis 3.8.** The vector potential $A$ is in $L^4_{\text{loc}}(X)$ and $divA$ is in $L^2_{\text{loc}}(X)$.

**Theorem 3.9.** *For any magnetic field $B$ satisfying Hypothesis 3.4 and for any associated vector potential satisfying Hypothesis 3.8, the formula (2.4) defining the Schrödinger representation may be extended to the distribution $\mathfrak{h}$, giving an essentially self-adjoint operator $\mathfrak{R}_A(\mathfrak{h})$ on $C_0^\infty(X) \subset L^2(X)$. Its closure $H^A$ satisfies:*

$$H^A = \sum_{j=1}^n (\Pi_j^A)^2, \qquad \mathfrak{R}_A(\mathfrak{r}_z) = (H^A - z)^{-1}.$$

**Conclusion.** The magnetic Schrödinger Hamiltonian defines an observable affiliated to the $C^*$-algebra $\mathfrak{R}_A \left\{BC_u(X) \rtimes_\theta^{\omega_B} X\right\}$.

Proof of Proposition 3.9. Let $u \in C_0^\infty(X)$ and $v \in L^2(X)$; we have $u \in \mathcal{D}(\Pi_j^A)$, $u \in \cap_{j=1}^n \mathcal{D}((\Pi_j^A)^2)$ [LS] and $-\Delta_x < v, U^A(x)u >= \sum_{j=1}^n < v, (\Pi_j^A)^2 u >$. Using the mollifier family for $\mathfrak{h}$ and the Dominated Convergence Theorem we extend formula (2.4) to obtain $\mathfrak{R}_A(\mathfrak{h}) = \sum_{j=1}^n (\pi_j^A)^2$ on the domain $C_0^\infty(X)$. Using once again the results in [LS] we obtain the essential self-adjointness of $\mathfrak{R}_A(\mathfrak{h})$ and thus the first stated equality. Observing that $\mathfrak{R}_A$ extends naturally to the minimal unital extension of $L^1_\theta(X;\mathcal{A})^{\omega_B}$ and that $\mathfrak{R}_A(\delta) = 1$ (the identity operator on $L^2(X)$), once again by a mollifier family for $\mathfrak{h}$ and the remarks above we obtain the second stated equality.

■

**Theorem 3.10.** *Let $V \in BC_u(X)$ and $V(Q)$ the operator of multiplication by $V$ in $L^2(X)$. Then $H := H^A + V(Q)$ defines a self-adjoint operator on $\mathcal{D}(H^A)$ and an observable affiliated to $\mathfrak{R}_A \left\{BC_u(X) \rtimes_\theta^{\omega_B} X\right\}$.*

Proof. The sum $H := H^A + V(Q)$ is obviously self-adjoint on $\mathcal{D}(H^A)$ and for $\Im z$ large enough the Neumann series for its resolvent $(H-z)^{-1} = (H_A - z)^{-1} \sum_{j\geq 0} (-1)^j \left[V(Q)(H_A - z)^{-1}\right]^j$ is convergent in norm. Thus the conclusion of the Theorem follows once we know that the product $V(Q)(H_A - z)^{-1} = \rho(V)\mathfrak{R}_A(\mathfrak{r}_z)$ belongs to the $C^*$-algebra $\mathfrak{C}$ and this follows from the second point of Proposition 2.3. ■

A comparison of our formulae (3.8) and relation (5.30) from [N] shows that our method leads to a non-perturbative, representation independent definition of the resolvent of a large class of magnetic Schrödinger Hamiltonians.

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