The Mathematical Formalism of a Particle in a Magnetic Field

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1 Introduction

In this review article we develop a basic part of the mathematical theory involved in the description of a particle (classical and quantal) placed in the Euclidean space $\mathbb{R}^N$ under the influence of a magnetic field B, emphasising the structure of the family of observables.

The classical picture is known, see for example [21]; we present it here for the convenience of the reader, in a form well-fitted for the passage to the quantum counterpart. In doing this we shall emphasize a manifestly gauge invariant Hamiltonian description [27] that is less used, although it presents many technical advantages and a good starting point for quantization.

The main contribution concernes the quantum picture. Up to our knowledge, until recently the single right attitude towards defining quantum observables when a nonconstant magnetic field is present can be found in a remarkable old paper of Luttinger [14] (we thank Gh. Nenciu for pointing it out to us); still this was undeveloped and with a limited degree of generality.

In recent years, the solution to this problem appeared in two related forms: (1) a gauge covariant pseudodifferential calculus in [8, 9, 17, 19] and (2) a C^*-algebraic formalism in [16] and [19]. We cite here also the results in [22], where a gauge independent perturbation theory is elaborated for the resolvent of a magnetic Schrödinger Hamiltonian, starting from an observation in [3].

For the classical picture, we define a perturbed symplectic form on phase space [27] and study the motions defined by classical Hamiltonians with respect to the associated perturbed Poisson algebra. The usual magnetic momenta appear then as momentum map for the associated 'symplectic translations'.

The quantum picture is treated in detail; two points of view are addopted: The first is to preserve (essentially) the same set of functions as observables, but with a different algebraic structure. The main input is a new, $(B,\hbar)$-dependent multiplication law associated to the perturbed symplectic form defined for the classical theory. This new product converges in a suitable sense to poinwise (classical) multiplication when $\hbar \to 0$. And it collapses for $B = 0$ to the symbol multiplication of Weyl and Moyal, familiar from pseudodifferential theory. It depends on no choice of a vector potential, so it

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is explicitely gauge invariant. Asside the pseudodifferential form, we present also a form comming from the theory of twisted crossed product C^*-algebras and justified by interpreting our physical system as a dynamical system given by an action twisted by the magnetic field.

The second point of view, more conventional, is in terms of self-adjoint operators in some Hilbert space. One achieves this by representing the previously mentioned intrinsic structures, and this is done by choosing vector potentials A generating the magnetic field B. For different but equivalent choices one gets unitarily equivalent representations, a form of what is communly called "gauge covariance". The represented form is best-suited to the interpretation in terms of magnetic canonical commutation relations. A functional calculus is associated to this highly non-commutative family of operators. Actually, the twisted dynamical system mentioned above (a sort of twisted imprimitivity system) is equivalent to these commutation rules.

The limit $\hbar \to 0$ of the quantum system was studied in [18], in the framework of Rieffel's strict deformation quantization.

To show that the formalism is useful in applications, we dedicate a section to spectral theory for anisotropic magnetic operators, following [20]. This relies heavily on an affiliation result, saying that the resolvent family of a magnetic Schrödinger Hamiltonian belongs to a suitable C^*-algebra of magnetic pseudodifferential operators.

Recently, the usual pseudodifferential theory has been generalized to a groupoid setting, cf. [13, 23] and references therein; this is in agreement with modern trends in deformation quantization, cf. [10] for example. The right concept to include magnetic fields should be that of twisted groupoid, as appearing in [30], accompanied by the afferent C^*-algebras. Let us also mention here the possibility to use our general framework in dealing with nonabelian gauge theories.

2 The Classical Particle in a Magnetic Field

In this section we shall give a classical bakground for our quantum formalism. We use the setting and ideas in [21] but develop the gauge invariant Poisson algebra feature. We begin by very briefly recalling the usual Hamiltonian formalism for classical motion in a magnetic field and then change the point of view by perturbing the canonical symplectic structure.

2.1 Two Hamiltonian Formalisms

The basic fact provided by physical measurements is that the magnetic field in $\mathbb{R}^3$ may be described by a function $B : \mathbb{R}^3 \to \mathbb{R}^3$ with $divB = 0$, such that the motion $\mathbb{R} \ni t \mapsto q(t) \in \mathbb{R}^3$ of a classical particle (mass m and electric charge e) is given by the equation of motion defined by the *Lorentz* force:

$$m\ddot{q}(t) = e\dot{q}(t) \times B(q(t)) \tag{1}$$

where $\times$ is the antisymmetric vector product in $\mathbb{R}^3$ and the point denotes derivation with respect to time. An important fact about this equation of motion is that it can be derived from a Hamilton function, the price to pay being the necessity of a vector potential, i.e. a vector field $A : \mathbb{R}^3 \to \mathbb{R}^3$ such that $B = rotA$, that is unfortunately not uniquely determined.

Let us very briefly recall the essential facts concerning the Hamiltonian formalism. Given a smooth manifold X we associate to it its "phase space" defined as the cotangent bundle $\mathbb{T}^*X$ on which we have a canonical symplectic form, that we shall denote by σ. If we set $\Pi : \mathbb{T}[\mathbb{T}^*X] \to \mathbb{T}^*X$ and $\tilde{\pi} : \mathbb{T}^*X \to X$ the canonical projections and $\tilde{\pi}_* : \mathbb{T}[\mathbb{T}^*X] \to \mathbb{T}X$ the tangent map of $\tilde{\pi}$, then $\sigma := d\beta$ where $\beta(\boldsymbol{\xi}) := [\Pi(\boldsymbol{\xi})](\tilde{\pi}_*(\boldsymbol{\xi}))$, for $\boldsymbol{\xi}$ a smooth section in $\mathbb{T}[\mathbb{T}^*X]$. A Hamiltonian system is determined by a Hamilton function $\mathrm{h} : \mathbb{T}^*X \to \mathbb{R}$ (supposed to be smooth) such that the vector field associated to the law of motion of the system ($\mathbb{R} \ni t \mapsto \mathfrak{x}(t) \in \mathbb{T}^*X$) is given by the following first order differential equation $\boldsymbol{\xi} \lrcorner \sigma - d\mathrm{h} = 0$, where $\boldsymbol{\xi} \lrcorner \sigma$ is the one-form defined by $(\boldsymbol{\xi} \lrcorner \sigma)(\boldsymbol{\eta}) := \sigma(\boldsymbol{\xi}, \boldsymbol{\eta})$, for any $\boldsymbol{\eta}$ smooth section in $\mathbb{T}[\mathbb{T}^*X]$.

Let us take $X = \mathbb{R}^3$ such that all the above bundles are trivial and we have canonical isomorphisms $\mathbb{T}^*X \cong X \times X^*$ (that we shall also denote by Ξ) and $\mathbb{T}[\mathbb{T}^*X] \cong (X \times X^*) \times (X \times X^*)$, defined by the usual transitive action of translations on X; we can view any two sections $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ as functions $\boldsymbol{\xi}(q,p) = (x(q,p), k(q,p))$, $\boldsymbol{\eta}(q,p) = (y(q,p), l(q,p))$ and we can easily verify that $\sigma(\boldsymbol{\xi}, \boldsymbol{\eta}) = k \cdot y - l \cdot x$, with $\xi \cdot y$ the canonical pairing $X^* \times X \to \mathbb{R}$. Moreover, the equations of motion defined by a Hamilton function h become:

$$\begin{cases} \dot{q}_j = \ \ \partial \mathrm{h}/\partial p_j \ , \\ \dot{p}_j = -\partial \mathrm{h}/\partial q_j . \end{cases} \tag{2}$$

Then (1) may be written in the above form if one chooses a vector potential A such that $B = rotA$ and defines the Hamilton function

$$\mathrm{h}_A(q,p) := (2m)^{-1} \sum_{j=1}^{3} (p_j - eA_j(q))^2 \ .$$

Although very useful, this Hamiltonian description has the drawback of involving the choice of a vector potential. Two different choices A and A' have to satisfy $rot(A - A') = 0$. Since $\mathbb{R}^3$ is simply connected, there exists a function $\varphi : \mathbb{R}^3 \to \mathbb{R}$ with $A' = A + \nabla\varphi$ and any such choice is admissible. We call these changes of descriptions "gauge transformations"; the "gauge group" is evidently $C^\infty(X)$ and the action of the gauge group is given by $\mathrm{h}_A \to \mathrm{h}_{A'}$.

An interesting fact is that we can actually obtain an explicitely gauge invariant description by using a perturbed symplectic form on $\mathbb{T}^*X$ [27]. For that it is important to notice that the magnetic field may in fact be described as a 2-form (a field of antisymmetric bilinear functions on $\mathbb{R}^3$), due to the obvious isomorphism between $\mathbb{R}^3$ and the space of antisymmetric matrices on $\mathbb{R}^3$ (just take $B_{jk} := \epsilon_{jkl} B_l$ with ϵ_{jkl} the completely antisymetric tensor of rank 3 on $\mathbb{R}^3$). Thus from now on we shall consider the magnetic field B given by a smooth section of the vector bundle $\Lambda^2 X \to X$ (the fibre at x being $\mathbb{T}^*_x X \wedge \mathbb{T}^*_x X \cong [\mathbb{T}_x X \wedge \mathbb{T}_x X]^*$). Due to the canonical global trivialisation discussed above (defined by translations) we can view B as a smooth map $B : X \to X^* \wedge X^* \cong (X \wedge X)^*$. Then a vector potential is described by a 1-form $A : X \to X^*$ such that $B = dA$ where d is the exterior differential. This also allow us to consider the case $X = \mathbb{R}^N$ for any natural number N.

Any k-form on X may be considered as a k-form on $\mathbb{T}^* X$. Explicitely, using the projection $\tilde{\pi} : \mathbb{T}^* X \to X$, we may canonically define the pull-back $\tilde{\pi}^* B$ of B and the "perturbed symplectic form" on $\mathbb{T}^* X$ defined by the magnetic field B as $\sigma_B := \sigma + \tilde{\pi}^* B$.

Now let us briefly recall the construction of the Poisson algebra associated to a symplectic form. We start from the trivial fact that any nondegenerate bilinear form Σ on the vector space Ξ defines a canonical isomorphism $\mathfrak{i}_\Sigma : \Xi \to \Xi^*$ by the equality $[\mathfrak{i}_\Sigma(x)](y) := \Sigma(x, y)$. Then we define the following composition law on $C^\infty(X)$: $\{f, g\}_B := \sigma_B(\mathfrak{i}^{-1}_{\sigma_B}(df), \mathfrak{i}^{-1}_{\sigma_B}(dg))$, called the Poisson braket. The case $B = 0$ gives evidently the canonical Poisson braket $\{.,.\}$ on the cotangent bundle. A computation gives immediately

$$\{f, g\}_B = \sum_{j=1}^{N} \left(\partial_{p_j} f \ \partial_{q_j} g - \partial_{q_j} f \ \partial_{p_j} g \right) + e \sum_{j,k=1}^{N} B_{jk}(\cdot) \ \partial_{p_j} f \ \partial_{p_k} g \ . \tag{3}$$

For the usual Hamilton function of the free classical particle $\mathrm{h}(p) := (2m)^{-1} \sum\limits_{j=1}^{N} p_j^2$, we can write down the Poisson form of the equation of motion:

$$\begin{cases} \dot{q}_j = \ \ \{\mathrm{h}, q_j\}_B = \frac{1}{m} p_j \ , \\ \dot{p}_j = -\{\mathrm{h}, p_j\}_B = \frac{e}{m} \sum\limits_{k=1}^{N} B_{kj}(q) p_k , \end{cases} \tag{4}$$

that combine to the equation of motion (1) defined by the Lorentz force.

We remark finally that in the present formulation the Hamilton function of the free particle $\mathrm{h}(q, p) = (2m)^{-1} \sum p_j^2$ is no longer privileged; any Hamilton function is now a candidate for a Hamiltonian system in a magnetic field just by considering it on the phase space endowed with the magnetic symplectic form. The relativistic kinetic energy $\mathrm{h}(p) := (p^2 + m^2)^{1/2}$ is a physically interesting example.

Remark. The real linear space $C^\infty(\Xi; \mathbb{R})$ endowed with the usual product of functions and the magnetic Poisson braket $\{.,.\}_B$ form a *Poisson algebra* (see [10, 18]), i.e. $(C^\infty(\Xi; \mathbb{R}), \cdot)$ is a real abelian algebra and $\{.,.\}_B : C^\infty(\Xi; \mathbb{R}) \times C^\infty(\Xi; \mathbb{R}) \to C^\infty(\Xi; \mathbb{R})$ is an antisymetric bilinear composition law that satisfies the Jacobi identity and is a derivation with respect to the usual product.

2.2 Magnetic Translations

For the perturbed symplectic form on $\mathbb{T}^*X$, the usual translations are no longer symplectic. We intend to define "magnetic symplectic translations" and compute the associated momentum map. Using the canonical global trivialisation, we are thus looking for an action $X \ni x \mapsto \alpha_x \in \mathit{Diff}(X \times X^*)$ having the form $\alpha_x(q,p) = (q + x, p + \tau_x(q,p))$. A *group action* clearly imposes the 1-cocycle condition: $\tau_{x+y}(q,p) = \tau_x(q,p) + \tau_y(q + x, p + \tau_x(q,p))$. The symplectic condition reads: $(\alpha_{-x})^*\sigma_B = \sigma_B$. A simple computation gives us for any $(q,p) \in \Xi$:

$$[\alpha_{-x}]^* = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ [\tau_{-x}]^*_X & \mathbf{1} + [\tau_{-x}]^*_{X^*} \end{pmatrix}^{\wedge 2} : \Lambda^2_{(q,p)}(\Xi) \to \Lambda^2_{(q+x,p+\tau_x(q,p))}(\Xi) \,, \quad (5)$$

where we identified all the cotangent fibres

$$\mathbb{T}^*_{(q,p)}\Xi \cong \mathbb{T}^*_q X \oplus \mathbb{T}^*_p(\mathbb{T}^*_q X) \cong \mathbb{T}^*_q X \oplus \mathbb{T}^*_p X^* \quad (6)$$

$$[\tau_{-x}]^*_X : \mathbb{T}^*X^* \to \mathbb{T}^*X, \qquad [\tau_{-x}]^*_{X^*} : \mathbb{T}^*X^* \to \mathbb{T}^*X^* \,. \quad (7)$$

Finally we obtain:

$$\left\{[\alpha_{-x}]^* \,\sigma_B - \sigma_B\right\}\Big|_{(q+x,p+\tau_x(q,p))} = \quad (8)$$

$$\sum_{j,k=1}^{N} \left\{[\boldsymbol{T}_{-x}(q,p)]_{jk} dq_j \wedge dq_k + [\boldsymbol{S}_{-x}(q,p)]_{jk} dq_j \wedge dp_k\right\},$$

with $\left(\boldsymbol{T}_x(q,p)\right)_{jk} =$

$$= (\partial/\partial q_j)(\tau_x(q,p))_k - (\partial/\partial q_k)(\tau_x(q,p))_j + eB(q)_{jk} - eB(q+x)_{jk} \,, \quad (9)$$

$$\left(\boldsymbol{S}_x(q,p)\right)_{jk} = (\partial/\partial p_j)(\tau_x(q,p))_k \,. \quad (10)$$

Asking for α_x to be symplectic implies that $\boldsymbol{S} = 0$, hence τ_x does not depend on p. If we fix a point $q_0 \in X$ we can define the function $a(x) := \tau_x(q_0) \in X^*$ and the condition imposed on $\tau_x(q)$ for having a group action leads to $\tau_x(q + q_0) = a(x + q) - a(q)$. Chosing $q_0 = 0$ and a vector potential A for B, the first equation in (9) implies $(\tau_x(q)) := eA(q + x) - eA(q)$.

Let us compute the associate differential action. We set $[(DA(q)) \cdot x]_j := \sum_k [\partial_k A_j(q)] x_k$ and for $x \in X$ we define the vector field in $\mathbb{T}(X \times X^*)$:

$$\mathfrak{t}_x(q,p) := (\partial/\partial t)\mid_{t=0} \alpha_{-tx}(q,p) = \quad (11)$$

$$= (-x, (\partial/\partial t)\mid_{t=0} \tau_{-tx}(q)) = (-x, e(DA(q)) \cdot x) \,.$$

Let us find the associated momentum map. A computation using the definition above (see also [18]) gives: $[\mathfrak{i}_{\sigma_B}](x,l) = (l + e x \lrcorner B, -x)$, where $(x \lrcorner B)(y) := B(x,y)$. Then we obtain

$$[\mathsf{i}_{\sigma_B}](\mathsf{t}_x^B)_{(q,p)} = (e(DA(q)) \cdot x - ex \lrcorner B, x)_{(q,p)} = (-d(eA(q) \cdot x), x)_{(q,p)},$$

with $A(q) \cdot x = \sum_{j=1}^{N} A_j(q) x_j$. It follows then that $[\mathsf{i}_{\sigma_B}](\mathsf{t}_x^B) = d\gamma_x^A$, where $\gamma_x^A(q,p) := x \cdot p - eA(q) \cdot x$ and thus for any direction $\nu \in X$ ($|\nu| = 1$) we have defined the infinitesimal observable magnetic momentum along ν to be $\gamma_\nu^A(q,p) := \nu \cdot (p - eA(q))$. The momentum map ([21]) is thus given by

$$\boldsymbol{\mu}^A : \mathbb{T}^* X \to X^*, \qquad [\boldsymbol{\mu}^A(q,p)](x) := \gamma_x^A(q,p)\,, \tag{12}$$

i.e. $\boldsymbol{\mu}^A(q,p) = p - eA(q)$.

3 The Quantum Picture

A guide in guessing a quantum multiplication for observables is the Weyl-Moyal product of symbols, valid for $B = 0$ and underlying the Weyl form of pseudodifferential theory. A replacement of σ by σ_B, as suggested by Sect. 2.1, triggers a formalism which will be exposed in the following sections. Here we examine a way to extend the multiplication, put it into a form suited for dynamical systems and C^*-norms and study how unbounded observables may be expressed by means of bounded ones.

3.1 The Magnetic Moyal Product

The well-known formula of symbol composition in the usual Weyl quantization can be expressed in terms of the canonical symplectic form. Assume for simplicity that $f, g \in \mathcal{S}(\Xi)$; then Weyl and Moyal proposed the multiplication

$$(f \circ^\hbar g)(\xi) = (2/\hbar)^{2N} \int_\Xi d\eta \int_\Xi d\zeta \; \exp\{-(2i/\hbar)\sigma(\eta,\zeta)\}\, f(\xi - \eta) g(\xi - \zeta),$$

where $\xi = (q,p),\ \eta = (y,k),\ \zeta = (z,l)$. By a simple calculation, one gets

$$(f \circ^\hbar g)(\xi) = (2/\hbar)^{2N} \int_\Xi d\eta \int_\Xi d\zeta \; \exp\left\{ -(i/\hbar) \int_{\mathcal{T}(\xi,\eta,\zeta)} \sigma \right\} f(\xi - \eta) g(\xi - \zeta),$$

in terms of the flux of σ through the triangle in phase space

$$\mathcal{T}(\xi,\eta,\zeta) := < (q-y-z, p-k-l), (q+y-z, p+k-l), (q+z-y, p+l-k) > \;.$$

A magnetic field B is turned on, with components supposed of class $C^\infty_{\text{pol}}(X)$, *i.e.* indefinitely derivable and each derivative polynomially bounded. Taking into account the formalism of Sect. 2.1, it is natural to replace σ by σ_B:

$$(f\circ^\hbar_B g)(\xi)=(2/\hbar)^{2N}\int_\Xi d\eta\int_\Xi d\zeta\ \exp\left\{-(i/\hbar)\int\limits_{\mathcal{T}(\xi,\eta,\zeta)}\sigma_B\right\}f(\xi-\eta)g(\xi-\zeta)\ . \tag{13}$$

This leads readily to the formula.

$$(f\circ^\hbar_B g)(\xi)= \tag{14}$$

$$=(2/\hbar)^{2N}\int_\Xi d\eta\int_\Xi d\zeta\ e^{-(2i/\hbar)\sigma(\eta,\zeta)}\exp\left\{-(i/\hbar)\int\limits_{T(q,y,z)}B\right\}f(\xi-\eta)g(\xi-\zeta)\ ,$$

where the triangle $T(q,y,z):=<q-y-z,q+y-z,q+z-y>$ is the projection of $\mathcal{T}(\xi,\eta,\zeta)$ on the configuration space. We call the composition law $\circ^\hbar_B:\mathcal{S}(\Xi)\times\mathcal{S}(\Xi)\to\mathcal{S}(\Xi)$ *the magnetic Moyal product.* It is well-defined, associative, non-commutative and satisfies $\overline{f\circ^\hbar_B g}=\overline{g}\circ^\hbar_B\overline{f}$. It offers a way to compose observables in a quantum theory of a particle placed in the magnetic field. It is expressed only in terms of B; no vector potential is needed.

3.2 The Magnetic Moyal Algebra

The $*$-algebra $\mathcal{S}(\Xi)$ is much too small for most of the applications. Extensions by absolutely convergent integrals still give rather poor results. One method to get much larger algebras (classes of Hörmander symbols) is by oscillatory integrals. This requires somewhat restricted conditions on the magnetic field, but leads to a powerful filtred symbolic calculus that we intend to develop in a forthcoming paper. Here we indicate an approach by duality.

So let us keep the mild assumption that the components of the magnetic field are $C^\infty_{\rm pol}(X)$-functions. The duality approach is based on the observation [17, Lem. 14] : For any f,g in the Schwartz space $\mathcal{S}(\Xi)$, we have

$$\int_\Xi d\xi\,(f\circ^\hbar_B g)(\xi)=\int_\Xi d\xi\,(g\circ^\hbar_B f)(\xi)=\int_\Xi d\xi\,f(\xi)\,g(\xi)=\langle\overline{f},g\rangle\equiv(f,g)\ .$$

As a consequence, if f,g and h belong to $\mathcal{S}(\Xi)$, the equalities $(f\circ^\hbar_B g,h)=(f,g\circ^\hbar_B h)=(g,h\circ^\hbar_B f)$ hold. This suggests

Definition 1. *For any distribution $F\in\mathcal{S}'(\Xi)$ and any function $f\in\mathcal{S}(\Xi)$ we define*

$$(F\circ^\hbar_B f,h):=(F,f\circ^\hbar_B h),\quad (f\circ^\hbar_B F,h):=(F,h\circ^\hbar_B f)\quad \textit{for all } h\in\mathcal{S}(\Xi)\ .$$

The expressions $F\circ^\hbar_B f$ and $f\circ^\hbar_B F$ are *a priori* tempered distributions. The Moyal algebra is precisely the set of elements of $\mathcal{S}'(\Xi)$ that preserves regularity by composition.

Definition 2. *The Moyal algebra* $\mathcal{M}(\Xi) \equiv \mathcal{M}_B^\hbar(\Xi)$ *is defined by*

$$\mathcal{M}(\Xi) := \left\{F \in \mathcal{S}'(\Xi) \mid F \circ_B^\hbar f \in \mathcal{S}(\Xi) \text{ and } f \circ_B^\hbar F \in \mathcal{S}(\Xi) \text{ for all } f \in \mathcal{S}(\Xi)\right\}.$$

For two distributions F and G in $\mathcal{M}(\Xi)$, the Moyal product is extended by $(F \circ_B^\hbar G, h) := (F, G \circ_B^\hbar h)$ for all $h \in \mathcal{S}(\Xi)$.

The set $\mathcal{M}(\Xi)$ with this composition law and the complex conjugation $F \mapsto \overline{F}$ is a unital *-algebra. Actually, this extension by duality also gives compositions $\mathcal{M}(\Xi) \circ_B^\hbar \mathcal{S}'(\Xi) \subset \mathcal{S}'(\Xi)$ and $\mathcal{S}'(\Xi) \circ_B^\hbar \mathcal{M}(\Xi) \subset \mathcal{S}'(\Xi)$. An important result [17, Prop. 23] concerning the Moyal algebra is that it contains $C^\infty_{\mathrm{pol,u}}(\Xi)$, the space of infinitely derivable complex functions on Ξ having polynomial growth at infinity uniformly for all the derivatives.

This duality strategy is often substantiated in calculations by regularization techniques. Further properties of $\circ_B^\hbar$ and $\mathcal{M}(\Xi)$ can be found in [17].

3.3 The Twisted Crossed Product

One thing missing in the pseudodifferential setting is a "good norm" on suitable subclasses of $\mathcal{M}(\Xi)$. We can introduce some useful norms after a partial Fourier transformation $1 \otimes \mathcal{F} : \mathcal{S}(\Xi) \equiv \mathcal{S}(X \times X^\star) \to \mathcal{S}(X \times X)$. Setting $(1 \otimes \mathcal{F})(f \circ_B^\hbar g) =: [(1 \otimes \mathcal{F})f] \diamond_B^\hbar [(1 \otimes \mathcal{F})f]$, one gets for $\varphi = (1 \otimes \mathcal{F})f$, $\psi = (1 \otimes \mathcal{F})g$ in $\mathcal{S}(X \times X)$ the multiplication law

$$\left(\varphi \diamond_B^\hbar \psi\right)(q;x) := \int_X dy\, \varphi\left(q - \frac{\hbar}{2}(x-y); y\right) \psi\left(q + \frac{\hbar}{2}y; x-y\right) e^{-(i/\hbar)\Phi_B^\hbar(q,x,y)} \tag{15}$$

where $\Phi_B^\hbar(q,x,y)$ is the flux of B through the triangle defined by the points $q - \frac{\hbar}{2}x$, $q - \frac{\hbar}{2}x + \hbar y$ and $q + \frac{\hbar}{2}x$. The partial Fourier transformation also converts the complex conjugation $f \mapsto \overline{f}$ into the involution $\varphi \mapsto \varphi^\diamond$, with $\varphi^\diamond(q;x) := \overline{\varphi(q;-x)}$. Thus one gets a new $*$-algebra $\left(\mathcal{S}(X \times X), \diamond_B^\hbar, {}^\diamond\right)$, isomorphic with the previous one. This also can be extended in various ways; in particular, there are Moyal type algebras $\mathcal{M}(X \times X) \equiv \mathcal{M}_B^\hbar(X \times X)$ in this setting too. But it is important to note that (15) is just a particular instance of a general mathematical object, *the twisted crossed product.* We give here the main ideas and refer to [25] and [26] for the full theory and to [16] and especially [19] for a comprehensive treatment of its relevance to quantum magnetic fields.

Let $\mathcal{A}$ be a unital C^*-algebra composed of bounded, uniformly continuous functions on X; this algebra is supposed to contain the "admissible" potentials. The idea behind this algebra is that for many problems it is more adequate to consider the whole algebra generated by a potential function and its translations. We shall always assume that $\mathcal{A}$ contains the constant functions as well as the ideal $C_0(X) := \{a : X \to \mathbb{C} \mid f \text{ is continuous and } a(x) \to 0 \text{ for } x \to \infty\}$ (in fact this hypothesis is not necessary everywhere) and is stable by translations, *i.e.* $\theta_x^\hbar(a) := a(\cdot + \hbar x) \in \mathcal{A}$ for all $a \in \mathcal{A}$ and $x \in X$.

Such a C^*-algebra will be called *admissible*. Thus, for any $\hbar \neq 0$, one can define the continuous action of X by automorphisms of $\mathcal{A}$:

$$\theta^\hbar : X \to \mathrm{Aut}(\mathcal{A}), \quad \left[\theta^\hbar_x(a)\right](y) := a(y + \hbar x) \ .$$

$\theta^\hbar$ is a group morphism and the maps $X \ni x \mapsto \theta^\hbar_x(a) \in \mathcal{A}$ are all continuous.

We suppose B to have components B_{jk} in $\mathcal{A}$ and we define the map:

$$(q, x, y) \mapsto \omega^\hbar_B(q; x, y) := e^{-(i/\hbar)\Gamma_B(<q, q+\hbar x, q+\hbar x+\hbar y>)} \ ,$$

where $\Gamma_B(< q, q+\hbar x, q+\hbar x+\hbar y >)$ denotes the flux of the magnetic field B through the triangle defined by the vertices $q, q+\hbar x, q+\hbar x+\hbar y$ in X. It can be interpreted as a map $\omega^\hbar_B : X \times X \to C(X; \mathbb{T})$, $\left[\omega^\hbar_B(x, y)\right](q) := \omega^\hbar_B(q; x, y)$ with values in the set of continuous functions on X taking values in the 1-torus $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$. It is easy to see by Stokes Theorem and the equation $dB = 0$ that $\omega^\hbar_B$ satisfies *the 2-cocycle condition*

$$\omega^\hbar_B(x, y)\omega^\hbar_B(x + y, z) = \theta^\hbar_x\left[\omega^\hbar_B(y, z)\right]\omega^\hbar_B(x, y + z), \quad \forall x, y, z \in X \ .$$

It is also *normalized*, i.e. $\omega^\hbar_B(x, 0) = 1 = \omega^\hbar_B(0, x)$, $\forall x \in X$.

The quadruplet $(\mathcal{A}, \theta^\hbar, \omega^\hbar_B, X)$ is a *magnetic* example of *an abelian twisted C^*-dynamical system* $(\mathcal{A}, \theta, \omega, X)$. In the general case X is an abelian second countable locally compact group, $\mathcal{A}$ is an abelian C^*-algebra, θ is a continuous morphism from X to the group of automorphisms of $\mathcal{A}$ and ω is a continuous 2-cocycle with values in the group of all unitary elements of $\mathcal{A}$.

Given any abelian twisted C^*-dynamical system, a natural C^*-algebra can be defined. We recall its construction. Let $L^1(X; \mathcal{A})$ be the set of Bochner integrable functions on X with values in $\mathcal{A}$, with the L^1-norm $\|\varphi\|_1 := \int_X dx \ \|\varphi(x)\|_{\mathcal{A}}$. For any $\varphi, \psi \in L^1(X; \mathcal{A})$ and $x \in X$, we define the product

$$(\varphi \diamond \psi)(x) := \int_X dy \ \theta_{\frac{y-x}{2}}[\varphi(y)] \ \theta_{\frac{y}{2}}[\psi(x - y)] \ \theta_{-\frac{x}{2}}[\omega(y, x - y)]$$

and the involution $\phi^\diamond(x) := \theta_{-\frac{x}{2}}[\omega(x, -x)^{-1}]\phi(-x)^*$. In this way, one gets a Banach $*$-algebra.

Definition 3. *The enveloping C^*-algebra of $L^1(X, \mathcal{A})$ is called* the twisted crossed product *and is denoted by $\mathcal{A} \rtimes^\omega_\theta X$. It is the completion of $L^1(X; \mathcal{A})$ under the C^*-norm*

$$\| \varphi \| := \sup\{\| \pi(\varphi) \|_{B(\hbar)} \mid \pi : L^1(X; \mathcal{A}) \to B(\mathcal{H}) \text{ representation}\} \ .$$

It is easy to see that, with $\theta = \theta^\hbar$, $\omega = \omega^\hbar_B$, one gets exactly the structure exposed above restricted to $\mathcal{S}(X \times X) \subset L^1(X; \mathcal{A})$. The C^*-algebra $\mathcal{A} \rtimes^{\omega^\hbar_B}_{\theta^\hbar} X$ will be denoted simply by $\mathfrak{C}^\hbar_B(\mathcal{A})$. In the magnetic case $\omega^h_B(x, -x) = 1$.

After a partial Fourier transformation we get the C^*-algebra $\mathfrak{B}^\hbar_B(\mathcal{A}) := (1 \otimes \mathcal{F}^{-1})\mathfrak{C}^\hbar_B(\mathcal{A})$, which is another extension of the $*$-subalgebra $\mathcal{S}(\Xi)$ endowed with complex conjugation and the multiplication (14).

3.4 Abstract Affiliation

When working with a self-adjoint operator H in a Hilbert space $\mathcal{H}$, it might be useful to know that the functional calculus of H (its resolvent for example) belongs to some special C^*-algebra of $B(\mathcal{H})$. Our representation-free approach forces us to use an abstract version, borrowed from [1].

Definition 4. An observable affiliated to a C^*-algebra $\mathfrak{C}$ *is a morphism* $\Phi : C_0(\mathbb{R}) \to \mathfrak{C}$.

Recall that a function $\mathrm{h} \in C^\infty(X^*)$ is called *an elliptic symbol of type* $s \in \mathbb{R}$ if (with $\langle p \rangle := \sqrt{1+p^2}$) $|(\partial^\alpha \mathrm{h})(p)| \le c_\alpha \langle p \rangle^{s-|\alpha|}$ for all $p \in X^*, \alpha \in \mathbb{N}^N$ and there exist $R > 0$ and $c > 0$ such that $c\,\langle p \rangle^s \le \mathrm{h}(p)$ for all $p \in X^*$ and $|p| \ge R$. Such a function is naturally contained in $C^\infty_{\mathrm{pol,u}}(\Xi)$, thus in $\mathcal{M}(\Xi)$. For any $z \notin \mathbb{R}$, we also set $r_z : \mathbb{R} \to \mathbb{C}$ by $r_z(t) := (t-z)^{-1}$. $BC^\infty(X)$ is the space of all functions in $C^\infty(X)$ with bounded derivatives of any order.

Theorem 1. *Assume that B is a magnetic field whose components belong to $\mathcal{A} \cap BC^\infty(X)$. Then each real elliptic symbol* h *of type $s > 0$ defines an observable $\Phi^\hbar_{B,\mathrm{h}}$ affiliated to $\mathfrak{B}^\hbar_B(\mathcal{A})$, such that for any $z \notin \mathbb{R}$ one has*

$$(\mathrm{h} - z) \circ^\hbar_B \Phi^\hbar_{B,\mathrm{h}}(r_z) = 1 = \Phi^\hbar_{B,\mathrm{h}}(r_z) \circ^\hbar_B (\mathrm{h} - z)\,. \tag{16}$$

In fact one has $\Phi^\hbar_{B,\mathrm{h}}(r_z) \in (1 \otimes \mathcal{F})\big(L^1(X;\mathcal{A})\big) \subset \mathcal{S}'(\Xi)$, so the compositions can be interpreted as $\mathcal{M}(\Xi) \times \mathcal{S}'(\Xi) \to \mathcal{S}'(\Xi)$ and $\mathcal{S}'(\Xi) \times \mathcal{M}(\Xi) \to \mathcal{S}'(\Xi)$.

The proof can be found in [20] and consists in starting with the usual inverse for function multiplication and control the corrections using the L^1-norm in the algebra $\mathfrak{C}^\hbar_B(\mathcal{A})$. This result is basic for our approach to spectral analysis for Hamiltonians with magnetic fields in Sections 6.1 and 6.2. A represented version will be found in Sect. 5.3.

4 The Limit $\hbar \to 0$

The quantum and classical descriptions we have given for a particle in a magnetic field, can be gathered into a common "continuous" structure indexed by the Plank' constant $\hbar \in [0, \hbar_0]$, by the procedure of strict deformation quantization. Our strategy follows [10] and the details may be found in our paper [18]. The main idea is to define for each value of $\hbar \in [0, \hbar_0]$ an algebra of bounded observables and using a common dense subalgebra, to prove that the family is in fact a continuous field of C^*-algebras (see [28, 29]).

So far we have defined for $\hbar > 0$ a C^*-algebra $\mathfrak{B}^\hbar_B(\mathcal{A})$ describing the observables of the quantum particle in a magnetic field B. Let us define now for $\hbar = 0$ the C^*-algebra $\mathfrak{B}^0_B(\mathcal{A}) := C(X^*; \mathcal{A})$ with the usual commutative product of functions ($f \circ^0_B g := fg$) and the involution defined by complex conjugation. Setting $\mathcal{A}^\infty := \{a \in \mathcal{A} \cap C^\infty(X) \mid \partial^\alpha a \in \mathcal{A}, \forall \alpha \in \mathbb{N}^N\}$ one verifies that the linear space $\mathfrak{A} := \mathcal{S}(X^*; \mathcal{A})$ is closed for any Moyal product $\circ_B^\hbar$, also for $\hbar = 0$. For $\hbar \in [0, \hbar_0]$ we denote by $\|.\|_\hbar$ the C^*-norm in $\mathfrak{B}_B^\hbar(\mathcal{A})$.

Moreover let us remark that the real algebra $\mathfrak{A}_0 := \{f \in \mathfrak{A} \mid \overline{f} = f\}$ is a Poisson sub-algebra of $C^\infty(\Xi; \mathbb{R})$ endowed with the magnetic Poisson braket associated to the magnetic field B. It is easy to verify that one has the

- Completness condition: $\mathfrak{A} = \mathbb{C} \otimes \mathfrak{A}_0$ is dense in each C^*-algebra $\mathfrak{B}_B^\hbar(\mathcal{A})$.

The following convergences are proved by direct computation [18]:

- von Neumann condition: For f and g in $\mathfrak{A}_0$ one has

$$\lim_{\hbar \to 0} \| \frac{1}{2} \left(f \circ_B^\hbar g + g \circ_B^\hbar f \right) - fg \|_\hbar = 0 \, .$$

- Dirac condition: For f and g in $\mathfrak{A}_0$ one has

$$\lim_{\hbar \to 0} \| \frac{1}{i\hbar} \left(f \circ_B^\hbar g - g \circ_B^\hbar f \right) - \{f, g\}_B \|_\hbar = 0 \, .$$

An argument using a theorem in [24], concerning continuous fields of twisted crossed-products, allows to prove the following continuity result [18]:

- Rieffel condition: For $f \in \mathfrak{A}_0$ the map $[0, \hbar_0] \ni \hbar \mapsto \|f\|_\hbar \in \mathbb{R}$ is continuous.

Following [10,28,29] we say that we have a *strict deformation quantization* of the Poisson algebra $\mathfrak{A}_0$.

5 The Schrödinger Representation

A complete overview of the formalism is achieved only after representations in Hilbert spaces are also outlined. This will put forward magnetic potentials, but in a gauge covariant way. We obtain integrated forms of covariant representations as well as the magnetic version of pseudodifferential operators. Unbounded pseudodifferential operators have their resolvents in well-controlled C^*-algebras composed of bounded ones, as a consequence of Sect. 3.4; this is basic to the spectral results of Sect. 6.

5.1 Representations of the Twisted Crossed Product

Fortunately, non-degenerate representations of twisted crossed product C^*-algebras admit a complete classification. We recall that the representation $\rho : \mathfrak{C} \to B(\mathcal{H})$ of the C^*-algebra $\mathfrak{C}$ in the Hilbert space $\mathcal{H}$ is called *non-degenerate* if $\rho(\mathfrak{C})\mathcal{H}$ generates $\mathcal{H}$. Since $\mathcal{A} \rtimes_\theta^\omega X$ was obtained from the twisted C^*-dynamical system $(\mathcal{A}, \theta, \omega, X)$, one may expect that the representations of $\mathcal{A} \rtimes_\theta^\omega X$ can be deduced from a certain kind of Hilbert representations of the system $(\mathcal{A}, \theta, \omega, X)$.

Definition 5. *Given a twisted dynamical system* $(\mathcal{A}, \theta, \omega, X)$*, we call* covariant representation *a Hilbert space* $\mathcal{H}$ *together with two maps* $r : \mathcal{A} \to B(\mathcal{H})$ *and* $U : X \to \mathcal{U}(\mathcal{H})$ *satisfying:*
- r *is a non-degenerate representation,*
- U *is strongly continuous and* $U(x)U(y) = r[\omega(x, y)]U(x+y) \ \ \forall x, y \in X,$
- $U(x)r(a)U(x)^* = r[\theta_x(a)], \quad \forall x \in X, \ a \in \mathcal{A}.$

It can be shown that there is a one-to-one correspondence between covariant representations of $(\mathcal{A}, \theta, \omega, X)$ and non-degenerate representations of $\mathcal{A} \rtimes_\theta^\omega X$. The following evident statement will be needed.

Lemma 1. *For* $(\mathcal{H}, r, U)$ *covariant representation of* $(\mathcal{A}, \theta, \omega, X)$*, the map* $r \rtimes U$ *defined on* $L^1(X; \mathcal{A})$ *by the formula*

$$(r \rtimes U)\varphi := \int_X dx \, r\left[\theta_{x/2}\big(\varphi(x)\big)\right] U(x)$$

extends to a representation of $\mathcal{A} \rtimes_\theta^\omega X$*, called* the integrated form of (r, U).

For our magnetic C^*-dynamical systems one constructs covariant representations by choosing vector potentials. We shall call them and their integrated forms *Schrödinger representations*, inspired by the case $B = 0$. For A such that $B = dA$ and for points $x, y \in X$, we define $\Gamma_A([x, y]) := \int_{[x,y]} A$ the circulation of A through the segment $[x, y] := \{sx + (1 - s)y \mid s \in [0, 1]\}$. By Stokes Theorem we have

$$\Gamma_B(< q, q + \hbar x, q + \hbar x + \hbar y >) =$$

$$= \Gamma_A([q, q + \hbar x])\Gamma_A([q + \hbar x, q + \hbar x + \hbar y])\Gamma_A([q + \hbar x + \hbar y, q]) \ ,$$

leading to

$$\omega_B^\hbar(q; x, y) = \lambda_A^\hbar(q; x)\lambda_A^\hbar(q + \hbar x; y) \left[\lambda_A^\hbar(q; x + y)\right]^{-1} \ , \qquad (17)$$

where we set $\lambda_A^\hbar(q; x) := \exp\left\{-(i/\hbar)\Gamma_A([q, q + \hbar x])\right\}$. We define $\mathcal{H} := L^2(X)$, $r : \mathcal{A} \to B[L^2(X)]$, $r(a)$:=the operator of multiplication by $a \in \mathcal{A}$ and

$$\left[U_A^\hbar(x)u\right](q) := \lambda_A^\hbar(q; x)u(q + \hbar x), \quad \forall q, x \in X, \quad \forall u \in L^2(X) \ .$$

It follows easily that $(\mathcal{H}, r, U_A^\hbar)$ is a covariant representation of $(\mathcal{A}, \theta^\hbar, \omega_B^\hbar, X)$. The integrated form associated to $(\mathcal{H}, r, U_A^\hbar)$ is $\mathfrak{Rep}_A^\hbar \equiv r \rtimes U_A^\hbar : \mathfrak{C}_B^\hbar(\mathcal{A}) \to B\left[L^2(X)\right]$, given explicitely on $L^1(X; \mathcal{A})$ by

$$\left[\mathfrak{Rep}_A^\hbar(\varphi)u\right](x) = \hbar^{-N} \int_X dy \ e^{(i/\hbar)\Gamma_A([x,y])} \varphi\left(\frac{x + y}{2}, \frac{y - x}{\hbar}\right) u(y) \ . \qquad (18)$$

5.2 Pseudodifferential Operators

Let us compose $\mathfrak{Rep}^\hbar_A$ with the partial Fourier transformation in order to get a representation $\mathfrak{Op}^\hbar_A := \mathfrak{Rep}^\hbar_A \circ (1 \otimes \mathcal{F}) : \mathfrak{B}^\hbar_B(\mathcal{A}) \to B(\mathcal{H})$. A calculation on suitable subsets of $\mathfrak{B}^\hbar_B(\mathcal{A})$ (on $\mathcal{S}(\Xi)$ for example) gives the explicit action

$$\left[\mathfrak{Op}^\hbar_A(f)u\right](x) = \tag{19}$$

$$= \hbar^{-N} \int_X \int_{X^\star} dy\ dk\ e^{(i/\hbar)(x-y)\cdot k} e^{-(i/\hbar)\Gamma_A([x,y])} f\left(\frac{x+y}{2}, k\right) u(y)\ .$$

We call $\mathfrak{Op}^\hbar_A(f)$ *the magnetic pseudodifferential operator associated to the symbol f*. A posteriori, one may say that *la raison d'être* of the composition (13) is to ensure the equality: $\mathfrak{Op}^\hbar_A(f)\mathfrak{Op}^\hbar_A(g) = \mathfrak{Op}^\hbar_A(f \circ^\hbar_B g)$. One also has $\mathfrak{Op}^\hbar_A(f)^* = \mathfrak{Op}^\hbar_A(\overline{f})$. Some properties of $\mathfrak{Op}^\hbar_A$ can be found in [17] and [19].

Now it is easy to see what gauge covariance is at the level of the two representations $\mathfrak{Rep}^\hbar_A$ and $\mathfrak{Op}^\hbar_A$. If two 1-forms A and A' are equivalent ($A' = A + d\rho$) then one will get unitarily equivalent representations:

$$\mathfrak{Op}^\hbar_{A'}(f) = e^{(i/\hbar)\rho}\mathfrak{Op}^\hbar_A(f)e^{(-i/\hbar)\rho} \quad \text{and} \quad \mathfrak{Rep}^\hbar_{A'}(\varphi) = e^{(i/\hbar)\rho}\mathfrak{Rep}^\hbar_A(\varphi)e^{(-i/\hbar)\rho}\ .$$

We refer to [17] for a comparaison with a quantization procedure $f \mapsto \mathfrak{Op}^{\hbar,A}(f)$, combining (in an inappropriate order) the usual, non-magnetic calculus with the minimal coupling rule $(x,p) \mapsto (x, p - A(x))$. It is *not* gauge-covariant, so that it is not suitable as a real quantization procedure.

Finally let us quote a result linking $\mathcal{M}(\Xi)$ with $\mathfrak{Op}^\hbar_A$ [17, Prop. 21] : For any vector potential A in $C^\infty_{\rm pol}(X)$, $\mathfrak{Op}^A$ is an isomorphism of *-algebras between $\mathcal{M}(\Xi)$ and $\mathcal{L}[\mathcal{S}(X)] \cap \mathcal{L}[\mathcal{S}'(X)]$, where $\mathcal{L}[\mathcal{S}(X)]$ and $\mathcal{L}[\mathcal{S}'(X)]$ are, respectively, the spaces of linear continuous operators on $\mathcal{S}(X)$ and $\mathcal{S}'(X)$.

5.3 A New Justification: Functional Calculus

We give here a new justification of our formalism. It is obvious that if one gives some convincing reason for working with (19), then the remaining part can be deduced as a necessary consequence, by reversing the arguments.

Let us accept that our quantum particle placed in a magnetic field is described by the family of elementary operators $Q_1, \dots, Q_N; (\Pi^\hbar_A)_1, \dots, (\Pi^\hbar_A)_N$, where Q_j is the operator of multiplication by x_j and $(\Pi^\hbar_A)_j := P^\hbar_j - A_j = -i\hbar\partial_j - A_j$ is the j'th component of the magnetic momentum defined by a vector potential A with $dA = B$ (these may be considered as quantum observables associated to the position and the momentum map for the translation group). Then $\mathfrak{Op}^\hbar_A$ should be a functional calculus $f \mapsto \mathfrak{Op}^\hbar_A(f) \equiv f(Q, \Pi^\hbar_A)$ for this family of non-commuting self-adjoint operators. The scheme is: (i) consider the commutation relations satisfied by $Q, \Pi^\hbar_A$, (ii) condense them in a global, exponential form, (iii) define $\mathfrak{Op}^\hbar_A(f)$ by decomposing f as a continuous linear combination of exponentials. We mention that exactly this argument leads to the usual Weyl calculus ($B = 0$).

So let us take into account the following commutation relations, easy to check: $i[Q_j, Q_k] = 0$, $i[\Pi^\hbar_{A,j}, Q_k] = \hbar\delta_{j,k}$, $i[\Pi^\hbar_{A,j},\, \Pi^\hbar_{A,k}] = \hbar B_{kj}(Q)$, $\forall j, k = 1, \dots, N$. A convenient global form may be given in terms of *the magnetic Weyl system*. Recall the unitary group $\left(e^{iQ\cdot p}\right)_{p\in X^\star}$ of the position as well as *the magnetic translations* $\left(U^\hbar_A(q) := e^{iq\cdot\Pi^\hbar_A}\right)_{q\in X}$, given explicitely in the Hilbert space $\mathcal{H} := L^2(X)$ by

$$U^\hbar_A(x) = e^{-(i/\hbar)\Gamma_A([Q,Q+\hbar x])} e^{ix\cdot P^\hbar} \;, \tag{20}$$

which is just another way to write (17). The family $\left(U^\hbar_A(x)\right)_{x\in X}$ satisfies

$$U^\hbar_A(x) U^\hbar_A(x') = \omega^\hbar_B(Q; x, x') U^\hbar_A(x + x'), \quad x, x' \in X \;,$$

where we set $\omega^\hbar_B(q; x, x') := e^{-(i/\hbar)\Gamma_B(<q,q+\hbar x,q+\hbar x+\hbar x'>)}$.

Now the magnetic Weyl system is the family $\left(W^\hbar_A(q, p)\right)_{(q,p)\in\Xi}$ of unitary operators in $\mathcal{H}$ given by

$$W^\hbar_A(q, p) := e^{-i\sigma\left((q,p),(Q,\Pi^\hbar_A)\right)} = e^{-i(\hbar/2)q\cdot p} e^{-iQ\cdot p} U^\hbar_A(x)$$

and it satisfies for all $(q, p), (q', p') \in \Xi$

$$W^\hbar_A(q, p) W^\hbar_A(q', p') = e^{(i/2)\sigma\left((q,p),(q',p')\right)} \omega^\hbar_B(Q; q, q') W^\hbar_A(q + q', p + p') \;.$$

To construct $\mathfrak{Op}^\hbar_A(f) \equiv f(Q, \Pi^\hbar_A))$ one does not dispose of a spectral theorem. Having the functional calculus with a C_0-group in mind, one proposes

$$\mathfrak{Op}^\hbar_A(f) := \int_\Xi d\xi \left(\mathfrak{F}_\Xi f\right)(\xi) W^\hbar_A(\xi) \;,$$

where $\left(\mathfrak{F}_\Xi f\right)(\xi) := \int_\Xi d\eta\; e^{-i\sigma(\xi,\eta)} f(\eta)$ is the symplectic Fourier transform (with a suitable Haar measure). Some simple replacements lead to (19). Details concerning this construction may be found in [19] together with an analysis of the role of the algebra $\mathcal{A}$.

5.4 Concrete Affiliation

If $\mathcal{H}$ is a Hilbert space and $\mathfrak{C}$ is a C^*-subalgebra of $B(\mathcal{H})$, then a self-adjoint operator H in $\mathcal{H}$ defines an observable Φ_H affiliated to $\mathfrak{C}$ if and only if $\Phi_H(\eta) := \eta(H)$ belongs to $\mathfrak{C}$ for all $\eta \in C_0(\mathbb{R})$. A sufficient condition is that $(H - z)^{-1} \in \mathfrak{C}$ for some $z \in \mathbb{C}$ with $\mathfrak{Im} z \neq 0$. Thus an observable affiliated to a C^*-algebra is the abstract version of the functional calculus of a self-adjoint operator. By combining Theorem 1 with the representations introduced above one gets

Corollary 1. *We are in the framework of Theorem 1. Let A be a continuous vector potential that generates B. Then* $\mathfrak{Op}^\hbar_A(\mathrm{h})$ *defines a self-adjoint operator* $\mathrm{h}(\Pi^\hbar_A)$ *in* $\mathcal{H}$ *with domain given by the image of the operator* $\mathfrak{Op}^\hbar_A\left[(\mathrm{h}-z)^{-1}\right]$ *(which do not depend on $z \notin \mathbb{R}$). This operator is affiliated to* $\mathfrak{Op}^\hbar_A\left[\mathfrak{B}^\hbar_B(\mathcal{A})\right] = \mathfrak{Rep}^\hbar_A\left[\mathfrak{B}^\hbar_B(\mathcal{A})\right]$.

6 Applications to Spectral Analysis

It seems to be common knowledge the fact that "the essential spectrum of partial differential operators depend only on the behaviour at infinity of the coefficients". But precise and general results emerged quite recently; some references are [1, 4–7, 11, 12], [15]. We review here a Theorem of [20] under simplifying assumptions (a scalar potential V can be easily added). Compared with the nice results of [7], it is much better if B (and V) is bounded, but we cannot say anything when B is unbounded towards infinity, case generously treated in [7]. The theory is in terms of C^*-algebras, quasi-orbits of some dynamical systems and asymptotic Hamiltonians associated to these quasi-orbits. The same asymptotic Hamiltonians play a role in localisation results (leading to non-propagation properties for the evolution group), extracted in an abridged form from [20] and [2].

6.1 The Essential Spectrum

We give a description of the essential spectrum of observables affiliated to the C^*-algebra $\mathfrak{B}^\hbar_B(\mathcal{A})$. For the generalised magnetic Schrödinger operators of Theorem 1, this is expressed in terms of the spectra of so-called *asymptotic operators*. The affiliation criterion and the algebraic formalism introduced above play an essential role in the proof of this result. We start by recalling some definitions in relation with topological dynamical systems.

By Gelfand theory, the abelian C^*-algebra $\mathcal{A}$ is isomorphic to the C^*-algebra $C(S_\mathcal{A})$, where $S_\mathcal{A}$ is the spectrum of $\mathcal{A}$. Since $\mathcal{A}$ was assumed unital and contains $C_0(X)$, $S_\mathcal{A}$ is a compactification of X. We shall therefore identify X with a dense open subset of $S_\mathcal{A}$. By stability under translations, the group law $\theta : X \times X \to X$ extends then to a continuous map $\tilde{\theta} : X \times S_\mathcal{A} \to S_\mathcal{A}$. Thus the complement $F_\mathcal{A}$ of X in $S_\mathcal{A}$ is closed and invariant; it is the space of a compact topological dynamical system. For any $\mathfrak{x} \in F_\mathcal{A}$, let us call the set $\{\tilde{\theta}(x, \mathfrak{x}) \mid x \in X\}$ *the orbit generated by* $\mathfrak{x}$, and its closure a *quasi-orbit*. Usually there exist many elements of $F_\mathcal{A}$ that generate the same quasi-orbit. In the sequel, we shall often encounter the restriction $a|_F$ of an element $a \in \mathcal{A} \equiv C(S_\mathcal{A})$ to a quasi-orbit F. Naturally $a|_F$ is an element of $C(F)$, but this algebra can be realized as a subalgebra of $BC_u(X)$. By a slight abuse of notation, we shall identify $a|_F$ with a function defined on X, thus inducing a multiplication operator in $\mathcal{H}$.

The calculation of the essential spectrum may be performed at an abstract level, *i.e.* without using any representation, (see [20] where a potential V is also included). We present, for convenience, a represented version.

Theorem 2. *Let B be a magnetic field whose components belong to $\mathcal{A} \cap BC^\infty(X)$. Assume that $\{F_\nu\}_\nu$ is a covering of $F_\mathcal{A}$ by quasi-orbits. Then for each real elliptic symbol* h *of type $s > 0$, if A, A_ν are continuous vector potentials respectively for B, $B_\nu \equiv B_{F_\nu}$, one has*

$$\sigma_{\text{ess}}\left[\mathrm{h}(\Pi^\hbar_A)\right] = \overline{\bigcup_\nu \sigma[\mathrm{h}(\Pi^\hbar_{A_\nu})]}\,. \tag{21}$$

The operators $\mathrm{h}(\Pi^\hbar_{A_\nu})$ are the asymptotic operators mentioned earlier. All the spectra in (21) are only depending on the respective magnetic fields. Examples may be found in [20], see also [15]. Some related results may be found in the recent paper [11].

6.2 A Non-Propagation Result

We finally describe, following [20], how the localization results proved in [2] in the case of Schrödinger operators without magnetic field can be extended to the situation where a magnetic field is present. Once again, the algebraic formalism and the affiliation criterion introduced above play an essential role in the proofs. For any quasi-orbit F, let $\mathfrak{N}_F$ be the family of sets of the form $W = \mathcal{W} \cap X$, where $\mathcal{W}$ is any element of a base of neighbourhoods of F in $S_\mathcal{A}$. We write χ_W for the characteristic function of W.

Theorem 3. *Let B be a magnetic field whose components belong to $\mathcal{A} \cap BC^\infty(X)$ and let* h *be a real elliptic symbol of type $s > 0$. Assume that $F \subset F_\mathcal{A}$ is a quasi-orbit. Let A, A_F be continuous vector potentials for B and B_F, respectively. If $\eta \in C_0(\mathbb{R})$ with $\mathrm{supp}\,(\eta) \cap \sigma\left[\mathrm{h}(\Pi^\hbar_{A_F})\right] = \emptyset$, then for any $\epsilon > 0$ there exists $W \in \mathfrak{N}_F$ such that $\left\|\chi_W(Q)\,\eta\left[\mathrm{h}(\Pi^\hbar_A)\right]\right\| \le \epsilon$. In particular, the following inequality holds uniformly in $t \in \mathbb{R}$ and $u \in \mathcal{H}$:*

$$\left\|\chi_W(Q)\, e^{-it\mathrm{h}(\Pi^\hbar_A)}\,\eta\left[\mathrm{h}(\Pi^\hbar_A)\right]u\right\| \le \epsilon\|u\|\,.$$

The last statement of this theorem gives a precise meaning to the notion of non-propagation. We refer to [2] for physical explanations and interpretations of this result as well as for some examples.

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References

1. W.O. Amrein, A. Boutet de Monvel, V. Georgescu: *C_0-Groups, Commutator Methods and Spectral Theory of N-Body Hamiltonians*, (Birkhäuser Verlag, 1996).
2. W.O. Amrein, M. Măntoiu, R. Purice: Ann. Henri Poincaré **3**, 1215 (2002).
3. H. D. Cornean, G. Nenciu: Ann. Henri Poincaré **1**, 203 (2000).
4. V. Georgescu, A. Iftimovici: preprint mp-arc01-99.
5. V. Georgescu, A. Iftimovici: Commun. Math. Phys. **228**, 519 (2002).
6. V. Georgescu, A. Iftimovici: C^*-Algebras of Quantum Hamiltonians. In *Operator Algebras and Mathematical Physics, Constanţa, 2001*, ed by J.-M. Combes, J. Cuntz, G. A. Elliott et al, (Theta Foundation, 2003).
7. B. Helffer, A. Mohamed: Ann. Inst. Fourier **38**, 95 (1988).
8. M. V. Karasev, T. A. Osborn: J. Math. Phys. **43**, 756 (2002).
9. M. V. Karasev, T. A. Osborn: J. Phys. A **37**, 2345 (2004).
10. N. P. Landsman: *Mathematical Topics Between Classical and Quantum Mechanics*, (Springer-Verlag, New-York, 1998), 529 pp.
11. Y. Last, B. Simon: preprint mp-arc 05-112.
12. R. Lauter, V. Nistor: Analysis of Geometric Operators on Open Manifolds: a Groupoid Approach. In *Quantization of Singular Symplectic Quotients*, ed by N. P. Landsman, M. Pflaum and M. Schlichenmaier, (Birkhäuser, Basel, 2001).
13. R. Lauter, B. Monthubert and V. Nistor: Documenta Math. **5**, 625 (2000).
14. J. M. Luttinger: Phys. Rev. **84**, 814 (1951).
15. M. Măntoiu: J. reine angew. Math. **550**, 211 (2002).
16. M. Măntoiu, R. Purice: The Algebra of Observables in a Magnetic Field. In: *Mathematical Results in Quantum Mechanics (Taxco, 2001)*, ed by R. Weder, P. Exner and B. Grébert (Amer. Math. Soc., Providence, RI, 2002) pp 239–245.
17. M. Măntoiu, R. Purice: J. Math.Phys. **45**, 1394 (2004).
18. M. Măntoiu, R. Purice: *Strict deformation quantization for a particle in a magnetic field*, to appear in J. Math. Phys. 2005.
19. M. Măntoiu, R. Purice, S. Richard: preprint mp-arc 04-76.
20. M. Măntoiu, R. Purice, S. Richard: preprint mp-arc 05-84.
21. J. E. Marsden, T. S. Raţiu: *Introduction to Mechanics and Symmetry*, 2nd edn, (Springer-Verlag, Berlin, New York, 1999), 582 pp.
22. G. Nenciu: preprint mp-arc 00-96.
23. V. Nistor, A. Weinstein, P. Xu: Pacific J. Math., **189**, 117 (1999).
24. M. Nilsen: Indiana Univ. Math. J. **45**, 436 (1996).
25. J.A. Packer, I. Raeburn: Math. Proc. Camb. Phil. Soc. **106**, 293 (1989).
26. J.A. Packer, I. Raeburn: Math. Ann. **287**, 595 (1990).
27. G. D. Raikov, M. Dimassi: Cubo Mat. Educ. **3**, 317 (2001).

28. M. Rieffel: Math. Ann. **283**, 631 (1989).
29. M. Rieffel: Memoirs of the AMS, **106**, 93 pp (1993).
30. J. Renault: *A Groupoid Approach to C^*-Algebras*, (Springer, Berlin, 1980) 160 pp.