The algebra of quantum observables in a magnetic field.
Spectral continuity with respect to the magnetic field.

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IMAR

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Introduction

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An interesting fact that we pointed out is that the algebra of observables is defined only in terms of the magnetic field without the need of a vector potential.

Moreover, we used some algebraic techniques in order to prove a number of spectral results and I shall present here one of them.

References:

- M. Măntoiu and R. Purice, *The Magnetic Weyl Calculus*, J. Math. Phys. **45**, No 4 (2004), 1394–1417.

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- N. Athmouni, M. Măntoiu and R. Purice: *On the continuity of spectra for families of magnetic pseudodifferential operators*, preprint arXiv

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Magnetic fields and gauge transformations

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These equations can be considered either in $\mathcal{D}'$ or on smaller spaces like C^∞ or C^∞_{pol}.

The magnetic canonical variables

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  defined once we have chosen a vector potential A for B.
- Appearently this prescription is highly non-unique due to the gauge ambiguity.
- In fact, one can easily see that the Hamilton equations of motion only depend on the magnetic field B.

The gauge invariant formalism

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We associate to σ^B a new Poisson bracket:

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where $\mathfrak{j}_B$ is the canonical isomorphism

$$\mathfrak{j}_B : \Xi \rightarrow \Xi^*, \quad < \mathfrak{j}_B(\mathfrak{X}), \mathfrak{Y} > := \sigma^B(\mathfrak{X}, \mathfrak{Y}).$$

The gauge invariant formalism

Using the canonical global coordinates we have:

$$\{f,g\}^B(x,\xi) :=$$

$$= \sum_{j=1}^{n} \left[\big(\partial_{\xi_j} f\big)(x,\xi)\big(\partial_{x_j} g\big)(x,\xi) - \big(\partial_{x_j} f\big)(x,\xi)\big(\partial_{\xi_j} g\big)(x,\xi) \right] +$$

$$\sum_{j,k=1}^{n} B_{jk}(x)\big(\partial_{\xi_j} f\big)(x,\xi)\big(\partial_{\xi_k} g\big)(x,\xi)$$

The Schrödinger representation

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representing the canonical momenta in the magnetic field.

- We can then define the Schrödinger operator:

$$H^A := \sum_{1 \leq j \leq n} \left(\Pi_j^A\right)^2 + V(Q)$$

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One can easily verify that changing the gauge

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$$= H^A - 2\sum_{1\leq j\leq n} (\partial_j\Phi)D_j + i\Delta\Phi + |\nabla\Phi|^2 =$$

$$= H^A + e^{i\Phi}\sum_{1\leq j\leq n} \left[D_j^2, e^{-i\Phi}\right] = e^{i\Phi}H^A e^{-i\Phi}.$$

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- What about effective Hamiltonians and functional calculus?

A gauge covariant functional calculus

The covariant functional calculus

We consider the unitary groups associated to the above $2n$ self-adjoint operators $Q_1, \ldots, Q_n,, \Pi_1^A, \ldots, \Pi_n^A$

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$$W^A((x,\xi)) := e^{-i<\xi,x/2>} V^A(\xi) U^A(x) =$$
$$= e^{-i<\xi,(Q+x/2)>} e^{-i \int_{[Q,Q+x]} A} e^{i<x,D>}$$

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The covariant functional calculus

- For any test function $f : \Xi \to \mathbb{C}$ we define the associated magnetic Weyl operator:

$$\mathfrak{Op}^A(f) := \int_\Xi dX \hat{f}(X) W^A(X) \in \mathbb{B}[\mathcal{H}]$$

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- In fact for any tempered distribution $F \in \mathcal{S}'(\Xi)$ we can define the linear operator:

$$\mathfrak{Op}^A(F) := \int_\Xi dX \hat{F}(X) W^A(X) \in \mathbb{B}[\mathcal{S}(\mathcal{X}); \mathcal{S}'(\mathcal{X})]$$

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- It defines a linear bijection [M.P., *J. Math. Phys. 04*].
- The covariant calculus associated to any two gauge-equivalent vector potentials are unitarily equivalent:

$$A' = A + d\varphi \quad \Rightarrow \quad \mathfrak{Op}^{A'}(f) = e^{i\varphi(Q)} \mathfrak{Op}^A(f) e^{-i\varphi(Q)}.$$

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then

$$\mathfrak{Op}^A(h) = H$$

with H defined previously.

The *magnetic* algebra of quantum observables

The magnetic Moyal product

The above functional calculus induces a *magnetic composition* on the complex linear space of test functions $\mathcal{S}(\Xi)$:

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Explicitely we have:

$$(f\sharp^B g)(X) := 4^n\int_\Xi dY\int_\Xi dZ\, e^{-i\int_{\mathcal{T}_X(Y,Z)}\sigma^B}\, f(X-Y)\, g(X-Z)$$

where $\mathcal{T}_X(Y,Z)$ is the triangle in Ξ having vertices:

$$X-Y-Z,\quad X+Y-Z,\quad X-Y+Z.$$

The *magnetic* algebra of quantum observables

We ca extend the product $\sharp^B$ by duality to bilinear maps:

$$\mathcal{S}'(\Xi)\sharp^B\mathcal{S}(\Xi) \rightarrow \mathcal{S}'(\Xi); \quad \mathcal{S}(\Xi)\sharp^B\mathcal{S}'(\Xi) \rightarrow \mathcal{S}'(\Xi).$$

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We set:

$$\mathfrak{M}^B(\Xi) := \left\{F \in \mathcal{S}'(\Xi) \mid F\sharp^B\phi \in \mathcal{S}(\Xi), \phi\sharp^B F \in \mathcal{S}(\Xi), \forall\phi \in \mathcal{S}(\Xi)\right\}$$

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This defines a *-algebra for the *composition* $\sharp^B$
and the usual complex conjugation as *-*conjugation*.

The norm

- The familly:

$$\mathfrak{C}^B(\Xi) := \left\{ F \in \mathcal{S}'(\Xi) \mid \mathfrak{Op}^A(F) \in \mathbb{B}[L^2(\mathcal{X})] \right\}$$

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- On $\mathfrak{C}^B(\Xi)$ we can define the map:

$$\|F\|_B := \|\mathfrak{Op}^A(F)\|_{\mathbb{B}[L^2(\mathcal{X})]}$$

that does not depend on the choice of A
and is in fact a C^*-norm on $\mathfrak{C}^B(\Xi)$.

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- $\mathfrak{C}^B(\Xi)$ is a C^*-algebra isomorphic to $\mathbb{B}[L^2(\mathcal{X})]$.

Continuity of the spectra

Symbols

We shall use the following Hörmander type symbols:

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$$|F|^{(m)}_{(a,\alpha)} := \sup_{(x,\xi)\in\Xi} <\xi>^{-m+|\alpha|} \left|(\partial^a_x \partial^\alpha_\xi F)(x,\xi)\right|, \quad \forall F \in C^\infty(\Xi)$$

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Hypothesis

The magnetic field B has components of class $BC^\infty(\mathcal{X})$.

Calderon-Vaillancourt type Theorem

By usual oscilatory integrals techniques we prove that:

Proposition [I.M.P., *Proc. RIMS 07*]

For any $m \in \mathbb{R}$ we have $S^m(\Xi) \subset \mathfrak{M}^B(\Xi)$.

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Theorem [I.M.P., *Proc. RIMS 07*]

In any Schrödinger representation of the form $\mathfrak{Op}^A$,
the operator corresponding to an observable F of class $S^0(\Xi)$, defines a bounded operator
and there exist two constants $c(n) \in \mathbb{R}_+$ and $p(n) \in \mathbb{N}$, depending only on the dimension n of the space $\mathcal{X}$, such that we have the estimation:

$$\|\mathfrak{Op}^A(F)\|_{\mathbb{B}(\mathcal{H})} \leq c(n)|F|^{(0)}_{(p(n),p(n))}.$$

Self-adjointness

Definition

For $m > 0$ a symbol $F \in S^m(\Xi)$ is said to be elliptic
if there exist two positive constants R and C such that
for any $(x, \xi) \in \Xi$ with $|\xi| \geq R$ one has that

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Proposition

- if $F \in S^0(\Xi)$ is a real function, $\mathfrak{Op}_\hbar^A(F)$ is a bounded self-adjoint operator on $L^2(\mathcal{X})$ for any vector potential A of B;
- if $F \in S^m(\mathcal{X})$ is a real elliptic symbol with $m > 0$, then $\mathfrak{Op}_\hbar^A(F)$ has a self-adjoint extension in $L^2(\mathcal{X})$ for any vector potential A of B.

The spectral result

The spectral result

Hypothesis 1

Consider a family of Hamiltonians $\{h^\epsilon\}_{\epsilon\in I}$ with $I \subset \mathbb{R}$ a compact interval, such that

- $h^\epsilon \in S^m(\Xi)$ elliptic with $m > 0$, for each $\epsilon \in I$,
- the map $I \ni \epsilon \mapsto h^\epsilon \in S^m(\Xi)$ is continuous for the Fréchet topology on $S^m(\Xi)$.
- there exist $C \in \mathbb{R}$ such that $h^\epsilon \geq -C$, $\forall\ \epsilon \in I$.

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Hypothesis 2

We are given a family of magnetic fields $\{B^\epsilon\}_{\epsilon\in I}$ with the components $B^\epsilon_{jk} \in BC^\infty(\mathcal{X})$ such that the map $I \ni \epsilon \mapsto B^\epsilon_{jk} \in BC^\infty(\mathcal{X})$ is continuous for the Fréchet topology on $BC^\infty(\mathcal{X})$.

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1. The family $\{\sigma^\epsilon\}_{\epsilon\in I}$ is called *outer continuous* at $\epsilon_0 \in I$ if for any compact $K \subset \mathbb{R}$ such that $K \cap \sigma^{\epsilon_0} = \emptyset$, there exists a neighborhood $V_K^{\epsilon_0}$ of ϵ_0 with $K \cap \sigma^\epsilon = \emptyset$, $\forall \epsilon \in V_K^{\epsilon_0}$.

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Definition

Let I be a compact interval and suppose given a family $\{\sigma^\epsilon\}_{\epsilon\in I}$ of closed subsets of $\mathbb{R}$.

1. The family $\{\sigma^\epsilon\}_{\epsilon\in I}$ is called *outer continuous* at $\epsilon_0 \in I$ if for any compact $K \subset \mathbb{R}$ such that $K \cap \sigma^{\epsilon_0} = \emptyset$, there exists a neighborhood $V_K^{\epsilon_0}$ of ϵ_0 with $K \cap \sigma^\epsilon = \emptyset$, $\forall \epsilon \in V_K^{\epsilon_0}$.
2. The family $\{\sigma^\epsilon\}_{\epsilon\in I}$ is called *inner continuous at* $\epsilon_0 \in I$ if for any open $\mathcal{O} \subset \mathbb{R}$ such that $\mathcal{O} \cap \sigma^{\epsilon_0} \neq \emptyset$, there exists a neighborhood $V_{\mathcal{O}}^{\epsilon_0} \subset I$ of ϵ_0 with $\mathcal{O} \cap \sigma^\epsilon \neq \emptyset$, $\forall \epsilon \in V_{\mathcal{O}}^{\epsilon_0}$.

The spectral result

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3. The family $\{\sigma^\epsilon\}_{\epsilon\in I}$ is called *continuous at* $\epsilon_0 \in I$ if it is both inner and outer continuous.

The spectral result

Theorem A.M.P. *2010*

Suppose given a compact interval $I \subset \mathbb{R}$, a family of classical Hamiltonians $\{h^\epsilon\}_{\epsilon\in I}$ and a family of magnetic fields $\{B^\epsilon\}_{\epsilon\in I}$ satisfying the above Hypothesis.

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Suppose given a compact interval $I \subset \mathbb{R}$, a family of classical Hamiltonians $\{h^\epsilon\}_{\epsilon\in I}$ and a family of magnetic fields $\{B^\epsilon\}_{\epsilon\in I}$ satisfying the above Hypothesis.
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Then the spectra $\sigma^\epsilon := \sigma(H^\epsilon) \subset \mathbb{R}$ form a continuous family of subsets at any point $\epsilon \in I$.

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- It extends the known results of Nenciu [1986] and Iftimie [1993] to a large class of symbols of positive order, but with stronger regularity hypothesis on the magnetic field.
  Using our results in Contemp. Math. **307** (2002), the continuity result of Nenciu [1986] can be recovered.

The spectral result - Proof

Our proof is based on the following criterion:

Proposition

Suppose that $\{H^\epsilon\}_{\epsilon\in I}$ is a family of self-adjoint operators in the Hilbert space $\mathcal{H}$ such that for any $\mathfrak{z} \notin \mathbb{R}$ the map

$$I \ni \epsilon \mapsto \left\|(H^\epsilon - \mathfrak{z}1)^{-1}\right\| \in \mathbb{R}_+$$

is upper (resp. lower) semi-continuous in $\epsilon_0 \in I$.

Then the spectra $\{\sigma(H^\epsilon)\}_{\epsilon\in I}$ form an outer (resp. inner) continuous family of closed sets at the point $\epsilon_0 \in I$.

The spectral result - Proof

In connection with this Criterion, our main result is

Theorem A.M.P. *2010*

Suppose given a family of symbols $\{h^\epsilon\}_{\epsilon\in I}$ and a family of magnetic fields $\{B^\epsilon\}_{\epsilon\in I}$ satisfying our previous Hypothesis, then for any choice of vector potentials $\{A^\epsilon\}_{\epsilon\in I}$ associated to the magnetic fields B^ϵ ($B^\epsilon = dA^\epsilon$) and for any $\mathfrak{z}\in\mathbb{C}\setminus\mathbb{R}$ the map

$$I \ni \epsilon \mapsto \left\| \left(\mathfrak{Op}^{A^\epsilon}(h^\epsilon) - \mathfrak{z}1 \right)^{-1} \right\| \in \mathbb{R}_+$$

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In order to prove this statement we use an operator algebraic framework inspired by the work of G. Elliott and J. Bellissard.

A continuous field of twisted crossed-products

The twisted crossed-product structure

Let us consider the inverse partial Fourier transform

$$\mathfrak{F}^- : \mathcal{S}(\Xi) \rightarrow \mathcal{S}(\mathcal{X} \times \mathcal{X}), \quad \left[\mathfrak{F}^- f\right](x, x') := \int_{\mathcal{X}^*} d\xi \, e^{i\xi \cdot x'} f(x, \xi)$$

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We can transport the Moyal product $\sharp^B$ to a bilinear associative product on $\mathcal{S}(\mathcal{X}\times\mathcal{X})$ or $\mathfrak{F}^{-}\mathfrak{M}^B(\Xi)$ that we denote by $\diamond^B$:

$$\phi\diamond^B\psi:=\mathfrak{F}^{-}\left[(\mathfrak{F}\phi)\,\sharp^B(\mathfrak{F}\psi)\right].$$

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$$\phi \diamond^B \psi := \mathfrak{F}^- \left[(\mathfrak{F}\phi) \, \sharp^B (\mathfrak{F}\psi) \right].$$

$$\left[\phi \diamond^B \psi\right](x, x') =$$

$$= \int_{\mathcal{X}} dz \, \phi(x+(z-x')/2, z) \, \psi(x+z/2, x'-z) \, \exp\{(-i\gamma^B(x-x'/2; z, x'-z)\},$$

whith $\gamma^B(x, x', z)$ the flux of B through $< x, x+x', x+x'+z >$.

The $(1,\infty)$ norm

On $\mathcal{S}(\mathcal{X}\times\mathcal{X})$ let us consider the norm

$$\|\phi\|_{(1,\infty)} := \int_{\mathcal{X}} dx' \sup_{x\in\mathcal{X}} |\phi(x,x')|$$

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Let $\mathfrak{L} := L^1\big(\mathcal{X}; BC_u(\mathcal{X})\big)$ be the closure of $C_c\big(\mathcal{X}; BC_u(\mathcal{X})\big)$ under the $(1, \infty)$ norm.

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Let $\mathfrak{L} := L^1(\mathcal{X}; BC_u(\mathcal{X}))$ be the closure of $C_c(\mathcal{X}; BC_u(\mathcal{X}))$ under the $(1, \infty)$ norm.

It is a Banach *-algebra (with the involution $\phi^*(x, x') := \overline{\phi(x, -x')}$).

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Remark 1

Taking into acount the unitary equivalence (associated to a gauge A)

$$L^2(\mathcal{X}\times\mathcal{X})\overset{\mathfrak{Op}^A}{\equiv}\mathbb{B}_2\big(L^2(\mathcal{X})\big),$$

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so that $\|F\|_B = \|\mathfrak{R}^B(F)\|_{\mathbb{B}(L^2(\mathcal{X}\times\mathcal{X}))}$.

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Definition

Let $\mathfrak{B}^B \subset \mathfrak{C}^B(\Xi)$ be the completion of $\mathfrak{L}$ for the C^*-norm $\| \cdot \|_B$.

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One can also form the direct product

$$\prod_{\epsilon \in I} \mathfrak{B}^\epsilon := \left\{ \{a_\epsilon\}_{\epsilon \in I} \mid a_\epsilon \in \mathfrak{B}^\epsilon, \|a\|_* := \sup_{\epsilon \in I} \|a_\epsilon\|_{\mathfrak{B}^\epsilon} < \infty \right\}.$$

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$$(\Phi \diamond \Psi)\,(x; \epsilon, x') :=$$

$$= \int_{\mathcal{X}} dz\, \Phi(x + (z - x')/2; \epsilon, z)\, \Psi(x + z/2; \epsilon, x' - z)\, e^{-i\gamma^B_\epsilon(x - x'/2; z, x' - z)}$$

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Then we can define $\widetilde{\mathfrak{L}}:=L^1\big(\mathcal{X};C\big(I;BC_u(\mathcal{X})\big)\big)$
as the completion of $C_c\big(\mathcal{X};C\big(I;BC_u(\mathcal{X})\big)\big)$ for the above norm.

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and the evaluation maps (surjective and contractive)

$$e_\epsilon : C\big(I; C_c\big(\mathcal{X}; BC_u(\mathcal{X})\big)\big) \to C_c\big(\mathcal{X}; BC_u(\mathcal{X})\big), \quad \epsilon \in I,$$

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allow us to define evaluation maps

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and by glueing them together, a continuous injective map:

$$\mathfrak{e} : \widetilde{\mathfrak{L}} \to \prod_{\epsilon \in I} \mathfrak{B}^\epsilon.$$

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For each $\epsilon \in I$, the map $\mathfrak{e}_{\epsilon}$ extends by continuity to a contraction:

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Proposition

For any $\Phi \in \widetilde{\mathfrak{B}}$ the map:

$$I \ni \epsilon \mapsto \|\widetilde{\mathfrak{e}_\epsilon}(\Phi)\|_{B_\epsilon} \in \mathbb{R}_+$$

is upper semi-continuous.

Proof of our main result

An affiliation result

Proposition

Under our Hypothesis on $\{h^\epsilon\}_{\epsilon\in I}$ and $\{B^\epsilon\}_{\epsilon\in I}$, there exists some $a > 0$ large enough such that for any $\mathfrak{z} \in \mathbb{C} \setminus [a, +\infty)$ we have:

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1. for any $\epsilon\in I$, the function $h^\epsilon-\mathfrak{z}1\in S^m_1(\Xi)\subset\mathfrak{M}^\epsilon(\Xi)$ is invertible for the $\sharp^\epsilon$-product having an inverse $r^\epsilon_{\mathfrak{z}}\in\mathfrak{F}\big[\mathfrak{L}\big]$;

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2. moreover the function $I\times\Xi\ni(\epsilon,X)\mapsto\widetilde{r}_{\mathfrak{z}}(\epsilon,X):=r^\epsilon_{\mathfrak{z}}(X)$ belongs to the algebra $\mathfrak{F}\left[\widetilde{\mathcal{L}}\right]$ and $\mathfrak{e}_\epsilon\left(\widetilde{r}_{\mathfrak{z}}\right)=r^\epsilon_{\mathfrak{z}}$.

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Using this result and the previous Proposition
we obtain the upper semi-continuity part of our main result.

The lower semi-continuity

Proposition

Given a continuous function $I \ni \epsilon \mapsto \phi^\epsilon \in \mathfrak{L}$ and an element $\psi \in \mathcal{H}$, the map

$$I \ni \epsilon \mapsto \phi^\epsilon \diamond^\epsilon \psi \in \mathcal{H}$$

is continuous.

The lower semi-continuity

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is continuous.

Using this result and the well known fact that if a family $\{S^\epsilon\}_{\epsilon \in I}$ of bounded linear operators in a Hilbert space $\mathcal{H}$ is *strongly* continuous, then $\epsilon \mapsto \| S^\epsilon \|_{\mathbb{B}(\mathcal{H})}$ is lower semi-continuous
we obtain the lower semi-continuity part of our main result.