Propagation and controlled K-theory

(joint work with G. Yu)

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Propagation of indices

Data (Atiyah, Kasparov, Mishchenko)

- *M compact manifold ;*
- *D elliptic differential operator on M.*
- *$\widetilde{M} \stackrel{\Gamma}{\longrightarrow} M$ covering.*

- $\widetilde{D}$ equivariant lift of D to $\widetilde{M}$;
- Q parametrix suppported near the diagonal for D;
- $\widetilde{Q}$ equivariant lift of Q to a paramétrix for $\widetilde{D}$;
- $\widetilde{S_0} := Id - \widetilde{Q}\widetilde{D}$ and $\widetilde{S_1} := Id - \widetilde{D}\widetilde{Q}$ are Γ-invariant smooth kernel operators on $\widetilde{M} \times \widetilde{M}$ with support near the diagonal, i.e with finite propagation.

Equivariant Index

- $P = \begin{pmatrix} \widetilde{S_0}^2 & \widetilde{S_0}(Id + \widetilde{S_0})\widetilde{Q} \\ \widetilde{S_1}\widetilde{D} & Id - \widetilde{S_1}^2 \end{pmatrix}$ is an idempotent. Coefficients are Γ-invariant smooth kernels on $\widetilde{M} \times \widetilde{M}$ with finite propagation.
- The reduced convolution C^*- algebra associated to these kernels is Morita equ. to $C^*_r(\Gamma)$. This Morita equ. preserves propagation

Definition (Γ-invariant Index for D)

$$\mathrm{Ind}_\Gamma D \overset{def}{=\!=} [P] - \left[\begin{pmatrix} 0 & 0 \\ 0 & Id \end{pmatrix}\right] \in K_0(C^*_r(\Gamma)).$$

- K-theory for C^*-algebra is homotopy invariant but we loose track of the propagation (problem when defining higher order indices).
- The K-theory for algebras that keep track of propagation (smooth algebras) is not in general homotopy invariant.
- Can we keep track of the propagation within the C^*-algebra framework?

Propagation and filtered C^*-algebras

Definition

A filtered C^*-algebra A is a C^*-algebra equipped with a family $(A_r)_{r>0}$ of closed linear subspaces:

- $A_r \subset A_{r'}$ if $r \leqslant r'$;
- A_r is stable by involution;
- $A_r \cdot A_{r'} \subset A_{r+r'}$;
- the subalgebra $\bigcup_{r>0} A_r$ is dense in A.

If A is unital, we also require that the identity 1 is an element of A_r for every positive number r.
The elements of A_r are said to have propagation r.

Exemples

- Roe algebras:
  - Σ proper discrete metric space, H separable Hilbert space
  - $C[\Sigma]_r$: space of loc. cpct operators on $\ell^2(\Sigma)\otimes H$ with propagation less than r , i.e $T = (T_{x,y})_{(x,y)\in\Sigma^2}$ with
    - $T_{x,y}$ cpct operator on H;
    - $T_{x,y} = 0$ if $d(x,y) > r$.
  - The Roe algebra of Σ is $C^*(\Sigma) = \overline{\cup_{r>0} C[\Sigma]_r} \subset \mathcal{L}(\ell^2(\Sigma)\otimes H)$ (filtered by $(C[\Sigma]_r)_{r>0}$).
- C^*-algebras of groups and cross-products:
  - If Γ is a discrete group finitely generated group equipped with a word metric. Set

$$\mathbb{C}[\Gamma]_r = \{x \in \mathbb{C}[\Gamma] \text{ with support in } B(e,r)\}.$$

    Then $C^*_{red}(\Gamma)$ and $C^*_{max}(\Gamma)$ are filtered by $(\mathbb{C}[\Gamma]_r)_{r>0}$.
  - More generally, if Γ acts on a A by automorphisms, then $A \rtimes_{red} \Gamma$ and $A \rtimes_{max} \Gamma$ are filtered C^*-algebras.

Almost projections and almost unitaries

Let A be a unital filtered C^*-algebra, $r > 0$ (propagation) and $0 < \varepsilon < 1/4$ (defect):

- $p \in A$ is a ε-r-projection if $p \in A_r$, $p = p^*$ and $\|p^2 - p\| < \varepsilon$.
- $u \in A$ is a ε-r-unitary if $u \in A_r$, $\|u^* \cdot u - I_n\| < \varepsilon$ and $\|u \cdot u^* - I_n\| < \varepsilon$.
- $\mathrm{P}^{\varepsilon,r}(A)$ is the set of ε-r-projections of A.
- a ε-r proj. p gives rise by functional calculus to a projection $\kappa_0(p)$.
- $\mathrm{U}^{\varepsilon,r}(A)$ is the set of ε-r-unitaries of A.
- $\mathrm{P}^{\varepsilon,r}_\infty(A) = \bigcup_{n\in\mathbb{N}} \mathrm{P}^{\varepsilon,r}(M_n(A))$ for $\mathrm{P}^{\varepsilon,r}(M_n(A)) \hookrightarrow \mathrm{P}^{\varepsilon,r}(M_{n+1}(A))$; $x \mapsto \mathrm{diag}(x, 0)$.
- $\mathrm{U}^{\varepsilon,r}_\infty(A) = \bigcup_{n\in\mathbb{N}} \mathrm{U}^{\varepsilon,r}(M_n(A))$ for $\mathrm{U}^{\varepsilon,r}(M_n(A)) \hookrightarrow \mathrm{U}^{\varepsilon,r}(M_{n+1}(A))$; $x \mapsto \mathrm{diag}(x, 1)$.

Quantitative K-semi-groups

We define for a unital filtered C^*-algebra A, $r > 0$ and $0 < \varepsilon < 1/4$ the equiv. relations on $P_\infty^{\varepsilon,r}(A) \times \mathbb{N}$ and on $U_\infty^{\varepsilon,r}(A)$:

- $(p,l) \sim (q,l')$ if there is $k \in \mathbb{N}$ and $h \in P_\infty^{\varepsilon,r}(C([0,1],A))$ s.t $h(0) = \operatorname{diag}(p, I_{k+l'})$ and $h(1) = \operatorname{diag}(q, I_{k+l})$.
- $u \sim v$ if there is $h \in U_\infty^{\varepsilon,r}(C([0,1],A)$ s.t $h(0) = u$ and $h(1) = v$.

Definition

1. *$K_0^{\varepsilon,r}(A) = P^{\varepsilon,r}(A)/\sim$ and $[p,l]_{\varepsilon,r}$ is the class of (p,l) mod. $\sim$;*
2. *$K_1^{\varepsilon,r}(A) = U^{\varepsilon,r}(A)/\sim$ and $[u]_{\varepsilon,r}$ is the class of u mod. $\sim$.*

- $K_0^{\varepsilon,r}(A)$ is an abelian group for $[p,l]_{\varepsilon,r} + [p',l']_{\varepsilon,r} = [\operatorname{diag}(p,p'), l+l']_{\varepsilon,r}$;
- $K_1^{\varepsilon,r}(A)$ is an abelian semi-group for $[u]_{\varepsilon,r} + [v]_{\varepsilon,r} = [\operatorname{diag}(u,v)]_{\varepsilon,r}$;
- if u is a ε-r-unitary, then $[u]_{3\varepsilon,2r} + [u^*]_{3\varepsilon,2r} = [1]_{3\varepsilon,2r}$.

The non-unital case

Lemma

$K_0^{\varepsilon,r}(\mathbb{C}) \stackrel{\cong}{\to} \mathbb{Z}$; $[p,l]_{\varepsilon,r} \mapsto \textit{rank}\, \kappa_0(p) - l$; $\quad K_1^{\varepsilon,r}(\mathbb{C}) \cong \{0\}$.

Definition

If A is a non unital filtered C^*-algebra and $\tilde{A}$ the unitarization of A,

- $K_0^{\varepsilon,r}(A) = \ker : \ K_0^{\varepsilon,r}(\tilde{A}) \to K_0^{\varepsilon,r}(\mathbb{C}) \cong \mathbb{Z}$;
- $K_1^{\varepsilon,r}(A) = K_1^{\varepsilon,r}(\tilde{A})$;

Definition

If A and B are filtered C^*-algebras with respect to $(A_r)_{r>0}$ and $(B_r)_{r>0}$, a homomorphism $f : A \to B$ is filtered if $f(A_r) \subset B_r$.

- A filtered $f : A \to B$ induces $f_*^{\varepsilon,r} : K_*^{\varepsilon,r}(A) \to K_*^{\varepsilon,r}(B)$;
- $A \hookrightarrow A \otimes \mathcal{K}(H)$; $a \mapsto a \otimes e_{1,1}$ induces $K_*^{\varepsilon,r}(A) \stackrel{\cong}{\to} K_*^{\varepsilon,r}(A \otimes \mathcal{K}(H))$.

Controlled K-theory

We have for any filtered C^*-algebra A, $0 < \varepsilon \leqslant \varepsilon' < 1/4$ and $0 < r \leqslant r'$ natural semi-group homomorphisms

- $\iota_0^{\varepsilon,r} : K_0^{\varepsilon,r}(A) \longrightarrow K_0(A);\ [p,l]_{\varepsilon,r} \mapsto [\kappa_0(p)] - [I_l];$
- $\iota_1^{\varepsilon,r} : K_1^{\varepsilon,r}(A) \longrightarrow K_1(A);\ [u]_{\varepsilon,r} \mapsto [u];$
- $\iota_*^{\varepsilon,r} = \iota_0^{\varepsilon,r} \oplus \iota_1^{\varepsilon,r};$
- $\iota_0^{\varepsilon,\varepsilon',r,r'} : K_0^{\varepsilon,r}(A) \longrightarrow K_0^{\varepsilon',r'}(A);\ [p,l]_{\varepsilon,r} \mapsto [p,l]_{\varepsilon',r'};$
- $\iota_1^{\varepsilon,\varepsilon',r,r'} : K_1^{\varepsilon,r}(A) \longrightarrow K_1^{\varepsilon',r'}(A);\ [u]_{\varepsilon,r} \mapsto [u]_{\varepsilon',r'}.$
- $\iota_*^{\varepsilon,\varepsilon',r,r'} = \iota_0^{\varepsilon,\varepsilon',r,r'} \oplus \iota_1^{\varepsilon,\varepsilon',r,r'}.$

Definition

If A is a filtered C^*-algebra, the controlled K-theory for A is the family

$$\mathcal{K}_*(A) = (K_*^{\varepsilon,r}(A))_{0<\varepsilon<1/4,\, r>0}.$$

Controlled morphisms

A control pair is a pair (λ, h) with $\lambda > 1$ and $h : (0, \frac{1}{4\lambda}) \to (0, +\infty);\ \varepsilon \mapsto h_\varepsilon$ non-increasing.

Definition

Let (λ, h) be a control pair, and let A and B be filtered C^*-algebras. A **(λ, h)-controlled morphism** $\mathcal{F} : \mathcal{K}_*(\mathbf{A}) \to \mathcal{K}_*(\mathbf{B})$ *is a family* $\mathcal{F} = (F^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4\lambda}, r>0}$ *of semi-group homomorphisms*

$$F^{\varepsilon,r} : K_*^{\varepsilon,r}(A) \to K_*^{\lambda\varepsilon, h_\varepsilon r}(B)$$

s.t for any ε, ε', r and r' with $0 < \varepsilon \leqslant \varepsilon' < \frac{1}{4\lambda}$ and $h_\varepsilon r \leqslant h_{\varepsilon'} r'$, we have

$$F^{\varepsilon',r'} \circ \iota_*^{\varepsilon,\varepsilon',r,r'} = \iota_*^{\lambda\varepsilon,\lambda\varepsilon',h_\varepsilon r,h_{\varepsilon'} r'} \circ F^{\varepsilon,r}.$$

$\mathcal{F}$ induces $F : K_*(A) \to K_*(B)$ defined in a unique way by $F \circ \iota_*^{\varepsilon,r} = \iota_*^{\lambda\varepsilon,h_\varepsilon r} \circ F^{\varepsilon,r}$.

Controlled isomorphism, controlled exacness

Let (λ, h) be a control pair and let $\mathcal{F} : \mathcal{K}_*(A) \to \mathcal{K}_*(B)$ be a $(\alpha_\mathcal{F}, k_\mathcal{F})$-controlled morphism.

- $\mathcal{F}$ is (λ, h)-injective if $(\alpha_\mathcal{F}, k_\mathcal{F}) \leqslant (\lambda, h)$ and for any $0 < \varepsilon < \frac{1}{4\lambda}$, any $r > 0$ and any $x \in K_*^{\varepsilon,r}(A)$, then $F^{\varepsilon,r}(x) = 0$ in $K_*^{\alpha_\mathcal{F}\varepsilon, k_{\mathcal{F},\varepsilon} r}(B) \Longrightarrow \iota_*^{\varepsilon,\lambda\varepsilon,r,h_\varepsilon r}(x) = 0$ in $K_*^{\lambda\varepsilon,h_\varepsilon r}(A)$;
- $\mathcal{F}$ is (λ, h)-surjective, if for any $0 < \varepsilon < \frac{1}{4\lambda\alpha_\mathcal{F}}$, any $r > 0$ and any $y \in K_*^{\varepsilon,r}(B)$, there exists $x \in K_*^{\lambda\varepsilon,h_\varepsilon r}(A)$ s.t. $F^{\lambda\varepsilon,h_{\lambda\varepsilon} r}(x) = \iota_*^{\varepsilon,\alpha_\mathcal{F}\lambda\varepsilon,r,k_{\mathcal{F},\lambda\varepsilon} h_\varepsilon r}(y)$ in $K_*^{\alpha_\mathcal{F}\lambda\varepsilon,k_{\mathcal{F},\lambda\varepsilon} h_\varepsilon r}(B)$.
- $\mathcal{F}$ is a (λ, h)-isomorphism if $\mathcal{F}$ is (λ, h)-injective and (λ, h)-surjective (in this case there exists a controlled inverse).
- In the same way, we can define (λ, h)-exactness for a composition $\mathcal{K}_*(A) \stackrel{\mathcal{F}}{\to} \mathcal{K}_*(B_1) \stackrel{\mathcal{G}}{\to} \mathcal{K}_*(B_2)$.

Examples

If q is an ε-r-projection in A unital filtered C^*-algebra, then

$$z_q : [0,1] \to A;\ t \mapsto qe^{2\imath\pi t} + 1 - q$$

is 5ε-r-unitary in $\widetilde{SA}$ (with $SA = C_0((0,1), A)$). Hence

$$Z_A^{\varepsilon,r} : K_0^{\varepsilon,r}(A) \to K_1^{5\varepsilon,r}(SA);\ [q,k]_{\varepsilon,r} \to [z_q]_{5\varepsilon,r} + [y_k]_{5\varepsilon,r}$$

with $y_k(t) = e^{-2\pi\imath kt}$ defines a $(5,1)$-controlled morphism $\mathcal{Z}_{\mathcal{A}} = (Z_A^{\varepsilon,r})_{0<\varepsilon<1/20, r>0} : \mathcal{K}_0(A) \to \mathcal{K}_1(SA)$.

Theorem (O-Yu)

There exists a control pair (λ, h) such that $\mathcal{Z}_{\mathcal{A}}$ is a (λ, h)-isomorphism for any filtered C^*-algebra A.

If Γ is a discrete group acting by automorphisms on A and B, then elements in $KK_*^\Gamma(A,B)$ gives rise to a (λ, h)-controlled morphisms $\mathcal{K}_*(A \rtimes_{red} \Gamma) \to \mathcal{K}_*(B \rtimes_{red} \Gamma)$ (for some universal control pair (λ, h)) compatible with Kasparov products. In particular KK-equiv. provide controlled isomorphisms.

Extension of filtered C^*-algebra

Let A be a C^*-algebra filtered by $(A_r)_{r>0}$ and let J be an ideal of A. Then A/J is filtered by $((A/J)_r)_{r>0}$, where $(A/J)_r$ is the image of A_r in A/J.

Definition

An extension of C^*-algebras

$$0 \to J \to A \to A/J \to 0$$

is filtered and semi-split if there exists a completely positive cross-section $s : A/J \to A$ such that $s((A/J)_r) \subset A_r$ for any $r > 0$. In this case, J is filtered by $(J \cap A_r)_{r>0}$.

Example: If $0 \to J \to A \to A/J \to 0$ is a semi-split extension of Γ-C^*-algebras for a discrete group Γ, then $0 \to J \rtimes_{red} \Gamma \to A \rtimes_{red} \Gamma \to A/J \rtimes_{red} \Gamma \to 0$ is filtered and semi-split (the same holds for max. cross-products).

The six term (α, h)-exact sequence

Theorem

There exists a control pair (λ, h) such that for any semi-split extension of filtered C^*-algebras

$$0 \longrightarrow J \stackrel{\jmath}{\longrightarrow} A \stackrel{q}{\longrightarrow} A/J \longrightarrow 0,$$

there is a six-term (λ, h)-exact sequence

$$\begin{array}{ccccc} \mathcal{K}_0(J) & \xrightarrow{\jmath_*} & \mathcal{K}_0(A) & \xrightarrow{q_*} & \mathcal{K}_0(A/J) \\ \uparrow & & & & \downarrow \\ \mathcal{K}_1(A/J) & \xleftarrow{q_*} & \mathcal{K}_1(A) & \xleftarrow{\jmath_*} & \mathcal{K}_1(J) \end{array}.$$

Consequence : Suspension controlled isomorphism $\mathcal{K}_1(A) \stackrel{\cong}{\to} \mathcal{K}_0(SA)$, controlled Bott periodicity $\mathcal{K}_*(A) \stackrel{\cong}{\to} \mathcal{K}_*(S^2A)$...

Application to K-amenability

Definition

A discrete group Γ is K-amenable if $K^0(C^*_{red}(\Gamma)) \to K^0(C^*_{max}(\Gamma))$ (induced by the regular representation) is an isomorphism.

Examples :

- $\mathbb{F}_n$, $SL_2(\mathbb{Z})$ and group with Haagerup prop. (Higson-Kasparov);
- Γ satisfying the strong Baum-Connes conj. i.e with $\gamma = 1$ (Tu),
- π_1 of compact oriented 3-manifold (Matthey-O-Pitsch).

For any action of Γ on a C^*-algebra A, there is epimorphism $\lambda_{\Gamma,A} : A \rtimes_{max} \Gamma \to A \rtimes_{red} \Gamma$.

Theorem

There exists a control pair (λ, h) s.t for any K-amenable discr. group Γ,

$$\lambda_{\Gamma,A,*} : \mathcal{K}_*(A \rtimes_{max} \Gamma) \to \mathcal{K}_*(A \rtimes_{red} \Gamma)$$

is a (λ, h)-isomorphism for any Γ-algebra A